Monte Carlo Computation of Small Loss Probabilities

René Carmona

Bendheim Center for Finance
ORFE, Princeton University

joint work with Stéphane Crépey

Evry, June 26, 2008
\{L(t)\}_{t \geq 0} \text{ loss process for a credit portfolio}

Typically

\[ L(t) = V(X_t) \]

where

- \( X_t = (X_t^1, \cdots, X_t^d) \) is a Markov process
- \( V: \mathbb{R} \rightarrow V(\mathbb{R}) \) deterministic

**Typical Computations**

- \( \mathbb{P}\{L(T) = x\} \) for \( x \) large
- \( \mathbb{P}\{L(T_2) = x_2 | L(T_1) = x_1\} \) for \( T_1 < T_2 \) and \( x_1 < x_2 \)
\{L(t)\}_{t \geq 0} \text{ loss process for a credit portfolio}

Typically

\[ L(t) = V(X_t) \]

where

- \( X_t = (X_t^1, \ldots, X_t^d) \) is a Markov process
- \( V : \mathbf{x} \mapsto V(\mathbf{x}) \) deterministic

**Typical Computations**

- \( \mathbb{P}\{L(T) = x\} \) for \( x \) large
- \( \mathbb{P}\{L(T_2) = x_2 | L(T_1) = x_1\} \) for \( T_1 < T_2 \) and \( x_1 < x_2 \)
\{L(t)\}_{t \geq 0} \text{ loss process for a credit portfolio}

Typically

\[ L(t) = V(X_t) \]

where

- \( X_t = (X^1_t, \ldots, X^d_t) \) is a Markov process
- \( V : x \mapsto V(x) \) deterministic

**Typical Computations**

- \( \mathbb{P}\{L(T) = x\} \) for \( x \) large
- \( \mathbb{P}\{L(T_2) = x_2 | L(T_1) = x_1\} \) for \( T_1 < T_2 \) and \( x_1 < x_2 \)
Three Related Papers


- **D. Vestal, R. Carmona and J.-P. Fouque.** Interacting Particle Systems for the Computation of CDO Tranche Spreads with Rare Defaults. *Submitted.*
As in Del Moral - Garnier

Choice of the potential function for mutation stage

\[ G_i^\alpha(y_i) = G_i^\alpha(x_0, x_1, \cdots, x_i) \]
\[ = G^\alpha(x_{i-1}, x_i) \]
\[ = e^{\alpha(V(x_i) - V(x_{i-1}))} \]

for a free parameter \( \alpha \)
**Importance Sampling**

- Use Girsanov to *twist* the distribution in path space

\[ I = \mathbb{E}^Q \left\{ f(X_T) \exp\left[ \int_0^T \nabla h(X_s) dW_s + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right] \right\} \]

- Generate Monte Carlo samples from twisted distribution \( Q \) and compute

\[ I \approx \frac{1}{M} \sum_{m=1}^{M} f(X_T(\omega_m)) \exp\left[ \int_0^T \nabla h(X_s) ds + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right](\omega_m) \]

**Interacting Particles System**

- Use Feynman-Kac *twisted distributions*

\[ I = \mathbb{E}^Q \{ f(X_T) e^{\int_0^T V(X_s) ds} \} \quad \text{with} \quad \frac{dQ}{dP} = \frac{e^{-\int_0^T V(X_s) ds}}{\mathbb{E}^P \{ e^{-\int_0^T V(X_s) ds} \}} \]

- Generate **ALL** Monte Carlo samples from **ORIGINAL** distribution \( P \)
IS versus IPS to Compute $I = \mathbb{E}^P\{ f(X_T) \}$

- **Importance Sampling**
  - Use Girsanov to *twist* the distribution in path space
    
    $$I = \mathbb{E}^Q \left\{ f(X_T) \exp \left[ \int_0^T \nabla h(X_s) dW_s + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right] \right\}$$

  - Generate Monte Carlo samples from twisted distribution $Q$ and compute
    
    $$I \approx \frac{1}{M} \sum_{m=1}^M f(X_T(\omega_m)) \exp \left[ \int_0^T \nabla h(X_s) ds + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right](\omega_m)$$

- **Interacting Particles System**
  - Use Feynman-Kac *twisted distributions*
    
    $$I = \mathbb{E}^Q \{ f(X_T) e^{\int_0^T V(X_s) ds} \} \quad \text{with} \quad \frac{dQ}{dP} = \frac{e^{-\int_0^T V(X_s) ds}}{\mathbb{E}^P \{ e^{-\int_0^T V(X_s) ds} \}}$$

  - Generate **ALL** Monte Carlo samples from **ORIGINAL** distribution $P$
IS versus IPS to Compute \( I = \mathbb{E}^{\mathbb{P}} \{ f(X_T) \} \)

- **Importance Sampling**
  - Use Girsanov to *twist* the distribution in path space

  \[
  I = \mathbb{E}^{\mathbb{Q}} \left\{ f(X_T) \exp\left[ \int_0^T \nabla h(X_s) dW_s + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right] \right\}
  \]

  - Generate Monte Carlo samples from twisted distribution \( \mathbb{Q} \) and compute

  \[
  I \approx \frac{1}{M} \sum_{m=1}^M f(X_T(\omega_m)) \exp\left[ \int_0^T \nabla h(X_s) ds + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right](\omega_m)
  \]

- **Interacting Particles System**
  - Use Feynman-Kac *twisted distributions*

  \[
  I = \mathbb{E}^{\mathbb{Q}} \{ f(X_T) e^{\int_0^T V(X_s) ds} \} \quad \text{with} \quad \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{-\int_0^T V(X_s) ds}}{\mathbb{E}^{\mathbb{P}} \{ e^{-\int_0^T V(X_s) ds} \}}
  \]

  - Generate **ALL** Monte Carlo samples from **ORIGINAL** distribution \( \mathbb{P} \)
IS versus IPS to Compute $I = \mathbb{E}^P\{f(X_T)\}$

- **Importance Sampling**
  - Use Girsanov to *twist* the distribution in path space
    $$I = \mathbb{E}^Q\left\{ f(X_T) \exp\left[ \int_0^T \nabla h(X_s) dW_s + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right] \right\}$$
  - Generate Monte Carlo samples from twisted distribution $\mathbb{Q}$ and compute
    $$I \approx \frac{1}{M} \sum_{m=1}^{M} f(X_T(\omega_m)) \exp\left[ \int_0^T \nabla h(X_s) ds + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right](\omega_m)$$

- **Interacting Particles System**
  - Use Feynman-Kac *twisted distributions*
    $$I = \mathbb{E}^Q\{f(X_T)e^{\int_0^T V(X_s)ds}\} \quad \text{with} \quad \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{-\int_0^T V(X_s)ds}}{\mathbb{E}^P\{e^{-\int_0^T V(X_s)ds}\}}$$
  - Generate *ALL* Monte Carlo samples from *ORIGINAL* distribution $\mathbb{P}$
IS versus IPS to Compute $I = \mathbb{E}_P^P \{ f(X_T) \}$

**Importance Sampling**
- Use Girsanov to *twist* the distribution in path space

$$I = \mathbb{E}_Q^Q \left\{ f(X_T) \exp \left[ \int_0^T \nabla h(X_s) dW_s + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right] \right\}$$

- Generate Monte Carlo samples from twisted distribution $Q$ and compute

$$I \approx \frac{1}{M} \sum_{m=1}^M f(X_T(\omega_m)) \exp \left[ \int_0^T \nabla h(X_s) ds + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right](\omega_m)$$

**Interacting Particles System**
- Use Feynman-Kac *twisted distributions*

$$I = \mathbb{E}_Q^Q \{ f(X_T) e^{\int_0^T V(X_s) ds} \} \quad \text{with} \quad \frac{dQ}{dP} = \frac{e^{-\int_0^T V(X_s) ds} \mathbb{E}_P^P \{ e^{-\int_0^T V(X_s) ds} \}}{\mathbb{E}_P^P \{ e^{-\int_0^T V(X_s) ds} \}}$$

- Generate *ALL* Monte Carlo samples from *ORIGINAL* distribution $P$
IS versus IPS to Compute $I = \mathbb{E}^P\{f(X_T)\}$

- **Importance Sampling**
  - Use Girsanov to *twist* the distribution in path space
    
    $$I = \mathbb{E}^Q \left\{ f(X_T) \exp\left[ \int_0^T \nabla h(X_s) dW_s + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right] \right\}$$

  - Generate Monte Carlo samples from twisted distribution $Q$ and compute
    
    $$I \approx \frac{1}{M} \sum_{m=1}^{M} f(X_T(\omega_m)) \exp\left[ \int_0^T \nabla h(X_s) ds + \frac{1}{2} \int_0^T |\nabla h(X_s)|^2 ds \right](\omega_m)$$

- **Interacting Particles System**
  - Use Feynman-Kac *twisted distributions*
    
    $$I = \mathbb{E}^Q \{ f(X_T) e^{\int_0^T V(X_s) ds} \} \quad \text{with} \quad \frac{dQ}{dP} = \frac{e^{-\int_0^T V(X_s) ds}}{\mathbb{E}^P \{ e^{-\int_0^T V(X_s) ds} \}}$$

  - Generate **ALL** Monte Carlo samples from **ORIGINAL** distribution $P$
A First Example: Poisson Probabilities

Standard Monte Carlo versus IPS approach.
<table>
<thead>
<tr>
<th>$i$</th>
<th>$p(i)$</th>
<th>$MC(i)$</th>
<th>$\alpha(i)$</th>
<th>$IPS(i)$</th>
<th>$errIPS(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.737947e-03</td>
<td>6.850000e-03</td>
<td>0</td>
<td>–</td>
<td>1.6630140</td>
</tr>
<tr>
<td>1</td>
<td>3.368973e-02</td>
<td>3.381000e-02</td>
<td>0</td>
<td>–</td>
<td>0.3569782</td>
</tr>
<tr>
<td>2</td>
<td>8.422434e-02</td>
<td>8.287000e-02</td>
<td>0</td>
<td>–</td>
<td>1.6080120</td>
</tr>
<tr>
<td>3</td>
<td>1.403739e-01</td>
<td>1.411600e-01</td>
<td>0</td>
<td>–</td>
<td>0.5600074</td>
</tr>
<tr>
<td>4</td>
<td>1.754674e-01</td>
<td>1.762900e-01</td>
<td>0</td>
<td>–</td>
<td>0.4688223</td>
</tr>
<tr>
<td>5</td>
<td>1.754674e-01</td>
<td>1.741600e-01</td>
<td>0</td>
<td>–</td>
<td>0.7450786</td>
</tr>
<tr>
<td>6</td>
<td>1.462228e-01</td>
<td>1.465800e-01</td>
<td>0</td>
<td>–</td>
<td>0.2442792</td>
</tr>
<tr>
<td>7</td>
<td>1.044449e-01</td>
<td>0.10586</td>
<td>1</td>
<td>9.642260e-02</td>
<td>7.6808635</td>
</tr>
<tr>
<td>8</td>
<td>6.527804e-02</td>
<td>0.06519</td>
<td>1</td>
<td>7.269477e-02</td>
<td>11.3617485</td>
</tr>
<tr>
<td>9</td>
<td>3.626558e-02</td>
<td>0.03548</td>
<td>1</td>
<td>3.832182e-02</td>
<td>5.6699651</td>
</tr>
<tr>
<td>10</td>
<td>1.813279e-02</td>
<td>0.01884</td>
<td>2</td>
<td>1.706726e-02</td>
<td>5.8762642</td>
</tr>
<tr>
<td>11</td>
<td>8.242177e-03</td>
<td>0.00761</td>
<td>2</td>
<td>7.025229e-03</td>
<td>14.7648865</td>
</tr>
<tr>
<td>12</td>
<td>3.434240e-03</td>
<td>0.00331</td>
<td>2</td>
<td>3.798202e-03</td>
<td>10.5980232</td>
</tr>
<tr>
<td>13</td>
<td>1.320862e-03</td>
<td>0.00135</td>
<td>2</td>
<td>1.301592e-03</td>
<td>1.4588951</td>
</tr>
<tr>
<td>14</td>
<td>4.717363e-04</td>
<td>0.00039</td>
<td>2</td>
<td>4.948383e-04</td>
<td>4.8972237</td>
</tr>
<tr>
<td>15</td>
<td>1.572454e-04</td>
<td>0.00014</td>
<td>3</td>
<td>1.315650e-04</td>
<td>16.3314126</td>
</tr>
</tbody>
</table>

**Computation of** $\mathbb{P}\{N = i\}$. 
<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>4.913920e-05</td>
<td>0.00007</td>
<td>3</td>
<td>5.158852e-05</td>
<td>4.9844475</td>
</tr>
<tr>
<td>17</td>
<td>1.445271e-05</td>
<td>0.00004</td>
<td>3</td>
<td>1.453860e-05</td>
<td>0.5943091</td>
</tr>
<tr>
<td>18</td>
<td>4.014640e-06</td>
<td>0</td>
<td>3</td>
<td>3.570278e-06</td>
<td>11.0685371</td>
</tr>
<tr>
<td>19</td>
<td>1.056484e-06</td>
<td>0</td>
<td>3</td>
<td>9.457888e-07</td>
<td>10.4777238</td>
</tr>
<tr>
<td>20</td>
<td>2.641211e-07</td>
<td>0</td>
<td>3</td>
<td>2.389309e-07</td>
<td>9.5373474</td>
</tr>
<tr>
<td>21</td>
<td>6.288597e-08</td>
<td>0</td>
<td>4</td>
<td>6.775490e-08</td>
<td>7.7424802</td>
</tr>
<tr>
<td>22</td>
<td>1.429227e-08</td>
<td>0</td>
<td>4</td>
<td>1.578648e-08</td>
<td>10.4547036</td>
</tr>
<tr>
<td>23</td>
<td>3.107014e-09</td>
<td>0</td>
<td>4</td>
<td>3.165438e-09</td>
<td>1.8803674</td>
</tr>
<tr>
<td>24</td>
<td>6.472947e-10</td>
<td>0</td>
<td>4</td>
<td>6.513996e-10</td>
<td>0.6341628</td>
</tr>
<tr>
<td>25</td>
<td>1.294589e-10</td>
<td>0</td>
<td>4</td>
<td>1.149943e-10</td>
<td>11.1731226</td>
</tr>
<tr>
<td>26</td>
<td>2.489595e-11</td>
<td>0</td>
<td>4</td>
<td>2.563031e-11</td>
<td>2.9497091</td>
</tr>
<tr>
<td>27</td>
<td>4.610361e-12</td>
<td>0</td>
<td>4</td>
<td>2.805776e-12</td>
<td>39.1419521</td>
</tr>
<tr>
<td>28</td>
<td>8.232787e-13</td>
<td>0</td>
<td>4</td>
<td>6.662209e-13</td>
<td>19.0771085</td>
</tr>
<tr>
<td>29</td>
<td>1.419446e-13</td>
<td>0</td>
<td>5</td>
<td>1.453048e-13</td>
<td>2.3672480</td>
</tr>
<tr>
<td>30</td>
<td>2.365743e-14</td>
<td>0</td>
<td>5</td>
<td>2.589671e-14</td>
<td>9.4654196</td>
</tr>
<tr>
<td>31</td>
<td>3.815715e-15</td>
<td>0</td>
<td>5</td>
<td>4.039209e-15</td>
<td>5.8571942</td>
</tr>
<tr>
<td>32</td>
<td>5.962055e-16</td>
<td>0</td>
<td>5</td>
<td>4.274102e-16</td>
<td>28.3115984</td>
</tr>
<tr>
<td>33</td>
<td>9.033417e-17</td>
<td>0</td>
<td>5</td>
<td>5.303823e-17</td>
<td>41.2866355</td>
</tr>
<tr>
<td>34</td>
<td>1.328444e-17</td>
<td>0</td>
<td>6</td>
<td>1.845935e-18</td>
<td>86.1045282</td>
</tr>
</tbody>
</table>
Losses as function of $\alpha$ (left), and log-probabilities (right).
Credit Portfolio Loss Process

Typical example

- $\tilde{X} = (\tilde{X}^1, \cdots, \tilde{X}^d)$ continuous-time $d$-variate Markov Chain
- Components $\tilde{X}^i_t$ in $\{0, 1, \cdots, \nu\}$ for some fixed integer $\nu$.
- Exclude simultaneous jumps of the $\tilde{X}^i$'s.
- $\tilde{X}_0 = 0 = (0, \cdots, 0)$.
- Particular case $d = 1$ class of $\nu = 125$ names
- Generator matrix is $(\nu + 1)^d \times (\nu + 1)^d$ TOO LARGE!
  
  (e.g. $(5 + 1)^{25} = 2.843029e + 19$)
One Single Group  $d = 1$

- **Local intensity model** in a pure **Top-Down Approach**
  - $L$ is modeled as a Markov point process stopped at level $\nu = 125$
  - **Pure birth process** with *local intensity* $\lambda(t, L_t)$
  - **Infinitesimal generator**

\[
A_t = \begin{pmatrix}
-\lambda(t,0) & \lambda(t,0) & 0 & 0 & 0 \\
0 & -\lambda(t,1) & \lambda(t,1) & 0 & 0 \\
0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & -\lambda(t,n-1) & \lambda(t,n-1) \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
One Single Group $d = 1$

- **Local intensity model** in a pure **Top-Down Approach**
- $L$ is modeled as a Markov point process stopped at level $\nu = 125$
- Pure birth process with **local intensity** $\lambda(t, L_t)$
- Infinitesimal generator

$$A_t = \begin{pmatrix}
-\lambda(t, 0) & \lambda(t, 0) & 0 & 0 & 0 \\
0 & -\lambda(t, 1) & \lambda(t, 1) & 0 & 0 \\
0 & 0 & \cdots & -\lambda(t, n-1) & \lambda(t, n-1) \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
One Single Group $d = 1$

- **Local intensity model** in a pure **Top-Down Approach**
- $L$ is modeled as a Markov point process stopped at level $\nu = 125$
- **Pure birth process** with *local intensity* $\lambda(t, L_t)$
- **Infinitesimal generator**

$$A_t = \begin{pmatrix}
-\lambda(t, 0) & \lambda(t, 0) & 0 & 0 & 0 \\
0 & -\lambda(t, 1) & \lambda(t, 1) & 0 & 0 \\
0 & 0 & \ldots & -\lambda(t, n-1) & \lambda(t, n-1) \\
0 & 0 & 0 & \ldots & -\lambda(t, n-1) & \lambda(t, n-1) \\
0 & 0 & 0 & 0 & \ldots & -\lambda(t, n-1) & \lambda(t, n-1)
\end{pmatrix}$$
Local intensity model in a pure Top-Down Approach

$L$ is modeled as a Markov point process stopped at level $\nu = 125$

Pure birth process with local intensity $\lambda(t, L_t)$

Infinitesimal generator

$$A_t = \begin{pmatrix}
-\lambda(t, 0) & \lambda(t, 0) & 0 & 0 & 0 \\
0 & -\lambda(t, 1) & \lambda(t, 1) & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & -\lambda(t, n-1) & \lambda(t, n-1) \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
- \( L_0 = 0 \)
- \( L \) jumps by one at some (increasing) \((0, +\infty)\)-valued random times \( \tilde{t}_i \)
- Loss probabilities \( p_i(t) = \mathbb{P}\{L(t) = i\} \), satisfy **forward Kolmogorov equation** (system of ODEs)
  \[
  (\partial_t - A^*_t) p = 0 \text{ on } (0, T] \quad p(0) = \delta_0,
  \]
  \( A^* \) represents the adjoint (transpose) of \( A \)
- For (time) homogeneous case
  \[
  p(t) = \exp(tA^*)\delta_0
  \]
$L_0 = 0$

$L$ jumps by one at some (increasing) $(0, +\infty)$-valued random times $\tilde{t}_i$

Loss probabilities $p_i(t) = \mathbb{P}\{L(t) = i\}$, satisfy forward Kolmogorov equation (system of ODEs)

$$(\partial_t - A_t^*)p = 0 \text{ on } (0, T] \quad p(0) = \delta_0,$$

$A^*$ represents the adjoint (transpose) of $A$

For (time) homogeneous case

$$p(t) = \exp(tA^*)\delta_0$$
$L_0 = 0$

$L$ jumps by one at some (increasing) $(0, +\infty)$-valued random times $\tilde{t}_i$

Loss probabilities $p_i(t) = \mathbb{P}\{L(t) = i\}$, satisfy **forward Kolmogorov equation** (system of ODEs)

$$(\partial_t - A_t^*)p = 0 \text{ on } (0, T] \quad p(0) = \delta_0,$$

$A^*$ represents the adjoint (transpose) of $A$

For (time) homogeneous case

$$p(t) = \exp(tA^*)\delta_0$$
- \( L_0 = 0 \)
- \( L \) jumps by one at some (increasing) \((0, +\infty)\)-valued random times \( \tilde{t}_i \)
- Loss probabilities \( p_i(t) = \mathbb{P}\{L(t) = i\} \), satisfy forward Kolmogorov equation (system of ODEs)

\[
(\partial_t - A^*_t)p = 0 \text{ on } (0, T] \quad p(0) = \delta_0,
\]

\( A^* \) represents the adjoint (transpose) of \( A \)
- For (time) homogeneous case

\[
p(t) = \exp(tA^*)\delta_0
\]
$T > 0$ fixed maturity (5yr)

$\tilde{t}_i$ time of the $i^{th}$ jump of $\tilde{X}$, ($\tilde{t}_0 = 0$)

Loss process $L_t = V(\tilde{X}_t) = \tilde{X}_t^1 + \cdots + \tilde{X}_t^d$

Introduce

$$n = \inf\{i \in \mathbb{N} ; \tilde{t}_i \geq T\}.$$  

and assume $n$ is finite almost surely.

Set, for $i \in \mathbb{N}$,

$$t_i = \tilde{t}_i \wedge n \wedge T \quad X_i = \tilde{X}_{t_i}.$$  

So $t_i = T$ iff $i \geq n$, and for $i \geq n$ we have that $X_i = \tilde{X}_T$ and $V(X_i) = L_T$. 
### Numerical Results for Homogeneous Groups

<table>
<thead>
<tr>
<th>$T$</th>
<th>$n = d \times \nu$</th>
<th>$\lambda^I(t, \nu)$</th>
<th>$\mu$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5y</td>
<td>$125 = 5 \times 25$</td>
<td>$\frac{\nu - i_l}{n}$</td>
<td>11</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Parameter Values.
Losses as function of $\alpha$ (left) and log-probabilities (right) for $m = 10^5$. 
Losses as function of $\alpha$ for $m = 20000$ (left) and $m = 5000$ (right) samples.
Corresponding log-probabilities ($m = 20000, 5000$).
Simple Importance Sampling Twist

Simple Twist Transformation:

- **Speed Up Arrivals** equivalently **Scale Up the Intensity** by $\alpha$

- Use (simple form of) Girsanov

$$I = \mathbb{E}^{Q^\alpha} \left\{ f(X_T) \exp[(\alpha - 1)L(T) - (\alpha - 1) \int_0^T \lambda(L_s)ds] \right\}$$

- Generate Monte Carlo samples from intensity $\lambda_t^\alpha = \alpha \lambda(L_t)$ and compute

$$I \approx \frac{1}{M} \sum_{m=1}^{M} f(L_T(\omega_m)) e^{(\alpha - 1)L(T,\omega_m) - (\alpha - 1) \int_0^T \lambda(L(s,\omega_m))ds}$$

- Analogous formula for the Homogeneous Group Model

Carmona Small Loss Probabilities
Simple Importance Sampling Twist

Simple Twist Transformation:

**Speed Up Arrivals**  equivalently  **Scale Up the Intensity** by $\alpha$

- Use (simple form of) Girsanov

$$I = \mathbb{E}^{Q^\alpha} \left\{ f(X_T) \exp[(\alpha - 1)L(T) - (\alpha - 1) \int_0^T \lambda(L_s) ds] \right\}$$

- Generate Monte Carlo samples from intensity $\lambda^\alpha_t = \alpha \lambda(L_t)$ and compute

$$I \approx \frac{1}{M} \sum_{m=1}^{M} f(L_T(\omega_m)) e^{(\alpha-1)L(T,\omega_m) - (\alpha-1) \int_0^T \lambda(L(s,\omega_m)) ds}$$

- Analogous formula for the Homogeneous Group Model
Simple Importance Sampling Twist

Simple Twist Transformation:

**Speed Up Arrivals**  equivalently  **Scale Up the Intensity** by $\alpha$

- Use (simple form of) Girsanov

$$I = \mathbb{E}^{Q^\alpha} \left\{ f(X_T) \exp[(\alpha - 1)L(T) - (\alpha - 1) \int_0^T \lambda(L_s)ds] \right\}$$

- Generate Monte Carlo samples from intensity $\lambda_t^\alpha = \alpha \lambda(L_t)$ and compute

$$I \approx \frac{1}{M} \sum_{m=1}^{M} f(L_T(\omega_m)) e^{(\alpha - 1)L(T,\omega_m) - (\alpha - 1) \int_0^T \lambda(L(s,\omega_m))ds}$$

- Analogous formula for the Homogeneous Group Model

Carmona  Small Loss Probabilities
Simple Importance Sampling Twist

Simple Twist Transformation:

**Speed Up Arrivals**  equivalently  **Scale Up the Intensity** by $\alpha$

- Use (simple form of) Girsanov

$$I = \mathbb{E}_Q^{\alpha} \left\{ f(X_T) \exp[(\alpha - 1) L(T) - (\alpha - 1) \int_0^T \lambda(L_s)ds] \right\}$$

- Generate Monte Carlo samples from intensity $\lambda_t^\alpha = \alpha \lambda(L_t)$ and compute

$$I \approx \frac{1}{M} \sum_{m=1}^{M} f(L_T(\omega_m)) e^{(\alpha - 1) L(T,\omega_m) - (\alpha - 1) \int_0^T \lambda(L(s,\omega_m))ds}$$

- Analogous formula for the Homogeneous Group Model
Explicit IS Losses and Values \((m = 5000)\).
IPS Computation of $\mathbb{P}\{L(T_2) = j|L(T_1) = i\}$

Range of $(i, j)$ for which Plain MC (left) and IPS (right) can compute with $m = 5000$ simulated paths.
In order to compute small probabilities (e.g. for credit portfolios)

- **Simple MC explicit IP can do wonders when available**
- **When is IPS Competitif?**
  - When explicit/deterministic methods are prohibitive
- **When is IPS a must?**
  - When no obvious Girsanov change of measure is explicit
  - When MC simulations are based on a Black Box one cannot open
Conclusions

In order to compute small probabilities (e.g. for credit portfolios)

- **Simple MC explicit IP can do wonders when available**
- **When is IPS Competitif?**
  - When explicit/deterministic methods are prohibitive
- **When is IPS a must?**
  - When no obvious Girsanov change of measure is explicit
  - When MC simulations are based on a Black Box one cannot open
Conclusions

In order to compute small probabilities (e.g. for credit portfolios)

- Simple MC explicit IP can do wonders when available
- When is IPS Competitif?
  - When explicit/deterministic methods are prohibitive
- When is IPS a must?
  - When no obvious Girsanov change of measure is explicit
  - When MC simulations are based on a Black Box one cannot open
Conclusions

In order to compute small probabilities (e.g. for credit portfolios)

- Simple MC explicit IP can do wonders when available
- When is IPS Competitif?
  - When explicit/deterministic methods are prohibitive
- When is IPS a must?
  - When no obvious Girsanov change of measure is explicit
  - When MC simulations are based on a Black Box one cannot open
Conclusions

In order to compute small probabilities (e.g. for credit portfolios)

- Simple MC explicit IP can do wonders when available
- When is IPS Competitif?
  - When explicit/deterministic methods are prohibitive
- When is IPS a must?
  - When no obvious Girsanov change of measure is explicit
  - When MC simulations are based on a Black Box one cannot open
In order to compute small probabilities (e.g. for credit portfolios)

- **Simple MC explicit IP can do wonders when available**
- **When is IPS Competitif?**
  - When explicit/deterministic methods are prohibitive
- **When is IPS a must?**
  - When no obvious Girsanov change of measure is explicit
  - When MC simulations are based on a Black Box one cannot open