Advanced credit portfolio modeling and CDO pricing

Ernst Eberlein[1], Rüdiger Frey[2],
Ernst August v. Hammerstein[1]

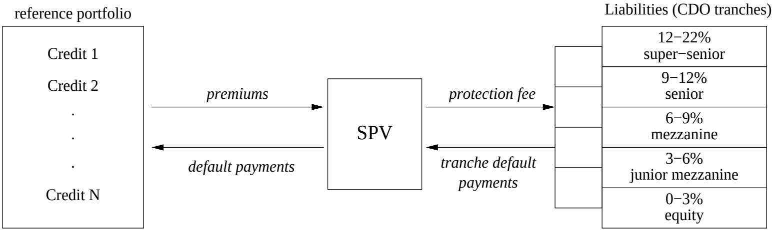
[1] Department of Mathematical Stochastics and Freiburg Center for Data Analysis and Modeling, University of Freiburg

[2] Department of Mathematics, University of Leipzig

International Workshop: CREDIT RISK

Université d'Évry Val d'Essone June 25–27, 2008

Schematic representation of the payments in a CDO structure



Assume credits of the same nominal amount L

Modeling of individual defaults

Exponentially distributed default times $T_1, \ldots, T_N$

$$Q_i(t) = P(T_i \leq t) = 1 - e^{-\lambda t}$$

with default intensity λ.

Risk neutral estimation of λ

$$\lambda = \frac{s_a}{(1 - R) \cdot 10000}$$

s_a = average CDX or iTraxx spread in basis points

R = recovery rate

Modeling of portfolio dependence

X_i state variable for credit i $\quad (1 \leq i \leq N)$

$$X_i = \sqrt{\rho}\, M + \sqrt{1-\rho}\, Z_i, \qquad 0 \leq \rho \leq 1,$$

where M and $Z_1, \ldots, Z_N$ are iid

M systematic factor, $\quad Z_i$ idiosyncratic factors

classical approach: $\qquad M, Z_i \sim N(0,1)$

$$\longrightarrow \qquad X_i \sim N(0,1), \quad \text{Corr}(X_i, X_j) = \rho$$

Time dependent threshold

$$d(t) = \Phi^{-1}(Q(t))$$

Individual default time

$$T_i := \inf\{\, t \geq 0 \mid X_i \leq d(t)\,\}$$

$$\longrightarrow \; P(T_i \leq t) = P\big(X_i \leq \Phi^{-1}(Q(t))\big) = P\big(\Phi(X_i) \leq Q(t)\big) = Q(t)$$

Portfolio loss distribution under normality (1)

Conditional on M default times are independent

$$P(T_i < t\,|M) = P(X_i < d(t)\,|M) = \Phi\left(\frac{d(t) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right)$$

D_t number of defaults up to time t

$$D_t|M \sim B\big(N, P(T_i < t\,|M)\big)$$

Consequently $Z_t = \frac{D_t}{N}$ has distribution function

$$F_{Z_t}(q) = \int_{-\infty}^{\infty} \sum_{k=0}^{[Nq]} \binom{N}{k} \Phi\left(\frac{d(t)-\sqrt{\rho}\,u}{\sqrt{1-\rho}}\right)^k \left(1 - \Phi\left(\frac{d(t)-\sqrt{\rho}\,u}{\sqrt{1-\rho}}\right)\right)^{N-k} \mathrm{d}P_M(u)$$

Portfolio loss distribution under normality (2)

Define $p_t(M) := \Phi\left(\frac{d(t)-\sqrt{\rho}\,M}{\sqrt{1-\rho}}\right)$ with df G_{p_t}, then

$$F_{Z_t}(q) = \int_0^1 \sum_{k=0}^{[Nq]} \binom{N}{k} s^k (1-s)^{N-k} \, \mathrm{d}G_{p_t}(s)$$

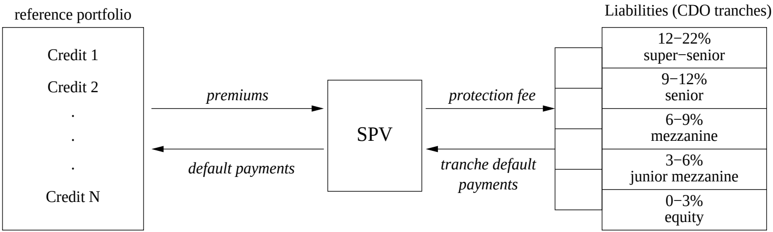
By the strong law of large numbers

$$\sum_{k=0}^{[Nq]} \binom{N}{k} s^k (1-s)^{N-k} \underset{N\to\infty}{\longrightarrow} \mathbb{1}_{[0,q]}(s)$$

which allows the approximation

$$F_{Z_t}(q) \approx P\left(-\frac{\sqrt{1-\rho}\,\Phi^{-1}(q)-d(t)}{\sqrt{\rho}} \leq M\right) = \Phi\left(\frac{\sqrt{1-\rho}\,\Phi^{-1}(q)-d(t)}{\sqrt{\rho}}\right)$$

Schematic representation of the payments in a CDO structure



Assume credits of the same nominal amount L

Structure of CDO tranches

Attachment points $0 = K_0 < K_1 < \ldots < K_n \leq 1$

$K_{i-1},\ K_i$	lower/upper attachment points of tranche i
Z_t	relative number of defaults up to time t
L_t^i	$= \min[(1-R)Z_t, K_i] - \min[(1-R)Z_t, K_{i-1}]$ loss of tranche i, expressed as a fraction of the total notional value NL
r_i	annual fair spread of tranche i
$0 = t_0$	$< t_1 < \ldots < t_n$ payment dates
$\beta(0, t_k)$	discount factor for time t_k

Derivation of fair spread

Premium leg of tranche i

$$PL_i(r_i) = \sum_{k=1}^{n} (t_k - t_{k-1})\beta(0, t_k)\, r_i\, \mathrm{E}\big[\big(K_i - K_{i-1} - L^i_{t_k}\big) NL\big]$$

Default leg of tranche i

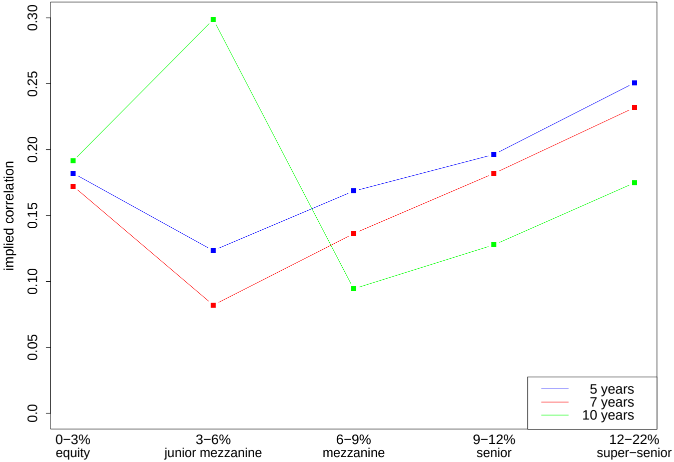
$$D_i = \sum_{k=1}^{n} \beta(0, t_k)\mathrm{E}\big[\big(L^i_{t_k} - L^i_{t_{k-1}}\big) NL\big]$$

Fair spread: $\quad PL_i(r_i) = D_i$

Consequently

$$r_i = \frac{\sum_{k=1}^{n} \beta(0, t_k)\big(\mathrm{E}\big[L^i_{t_k}\big] - \mathrm{E}\big[L^i_{t_{k-1}}\big]\big)}{\sum_{k=1}^{n} (t_k - t_{k-1})\beta(0, t_k)\big(K_i - K_{i-1} - \mathrm{E}\big[L^i_{t_k}\big]\big)}$$

Implied iTraxx correlations



Implied correlations from DJ iTraxx Europe tranche prices at November 13, 2006

Advanced distributions

Generalized hyperbolic distributions

$$d_{GH(\lambda,\alpha,\beta,\delta,\mu)}(x) = a(\lambda,\alpha,\beta,\delta,\mu)\left(\delta^2+(x-\mu)^2\right)^{(\lambda-\frac{1}{2})/2} e^{\beta(x-\mu)} \times K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)$$

with norming constant

$$a(\lambda,\alpha,\beta,\delta,\mu) = \frac{(\alpha^2-\beta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\,\alpha^{\lambda-\frac{1}{2}}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}$$

Parameters: $\alpha > 0, \quad 0 \leq |\beta| < \alpha, \quad \delta > 0, \quad \mu \in \mathbb{R}, \quad \lambda \in \mathbb{R}$

Special cases

$\lambda = 1$	Hyperbolic distributions (HYP)
$\lambda = -1/2$	Normal Inverse Gaussian distributions (NIG)

Limiting cases

$\lambda > 0, \delta \to 0$	Variance Gamma distributions (VG)
$\lambda < 0, \alpha, \beta \to 0$	scaled t distributions (t)

Hyperbolic distribution ($\lambda = 1$)

$$d_{HYP}(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} \exp\left(-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)\right)$$

Eb., Keller (1995); Eb., Keller, Prause (1998)

Normal inverse Gaussian ($\lambda = -1/2$)

$$d_{NIG}(x) = \frac{\alpha\delta}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\right) \frac{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}}$$

O. E. Barndorff-Nielsen (1997)

Hyperbolic distribution ($\lambda = 1$)

$$d_{HYP}(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} \exp\left(-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)\right)$$

Eb., Keller (1995); Eb., Keller, Prause (1998)

Normal inverse Gaussian ($\lambda = -1/2$)

$$d_{NIG}(x) = \frac{\alpha\delta}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\right) \frac{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}}$$

O. E. Barndorff-Nielsen (1997)

Variance-Gamma distribution ($\lambda > 0$, $\delta \to 0$)

$$d_{VG}(x) = \frac{(\alpha^2 - \beta^2)^\lambda \, |x-\mu|^{\lambda - \frac{1}{2}}}{\sqrt{\pi}(2\alpha)^{\lambda-\frac{1}{2}} \Gamma(\lambda)} K_{\lambda-\frac{1}{2}}(\alpha|x-\mu|) \, e^{\beta(x-\mu)}$$

Madan, Seneta (1990); Madan, Carr, Chang (1998)

scaled t distribution ($\lambda < 0$, $\alpha, \beta \to 0$)

$$d_t(x) = \frac{\Gamma(-\lambda+\frac{1}{2})}{\delta\sqrt{\pi}\Gamma(-\lambda)} \left(1 + \frac{(x-\mu)^2}{\delta^2}\right)^{\lambda-\frac{1}{2}}$$

Variance-Gamma distribution ($\lambda > 0,\ \delta \to 0$)

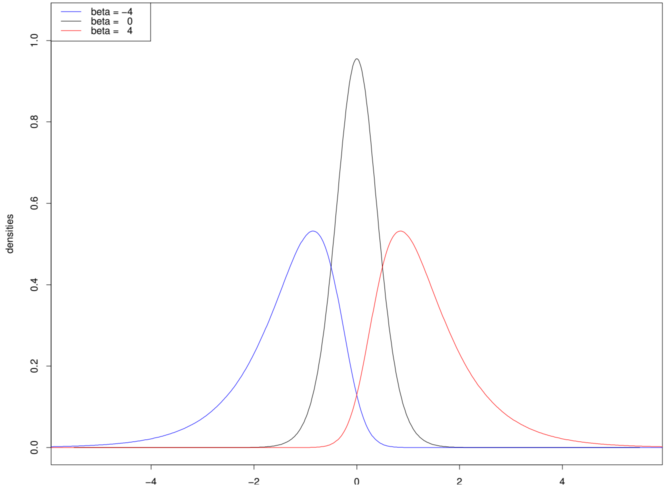
$$d_{VG}(x) = \frac{(\alpha^2 - \beta^2)^\lambda \, |x-\mu|^{\lambda - \frac{1}{2}}}{\sqrt{\pi}(2\alpha)^{\lambda-\frac{1}{2}}\Gamma(\lambda)} \, K_{\lambda-\frac{1}{2}}(\alpha|x-\mu|)\, e^{\beta(x-\mu)}$$

Madan, Seneta (1990); Madan, Carr, Chang (1998)

scaled t distribution ($\lambda < 0,\ \alpha, \beta \to 0$)

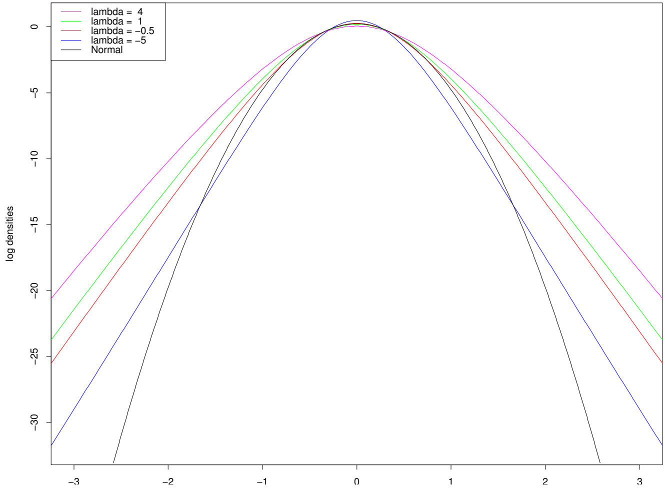
$$d_t(x) = \frac{\Gamma(-\lambda + \frac{1}{2})}{\delta\sqrt{\pi}\Gamma(-\lambda)} \left(1 + \frac{(x-\mu)^2}{\delta^2}\right)^{\lambda - \frac{1}{2}}$$

Influence of β



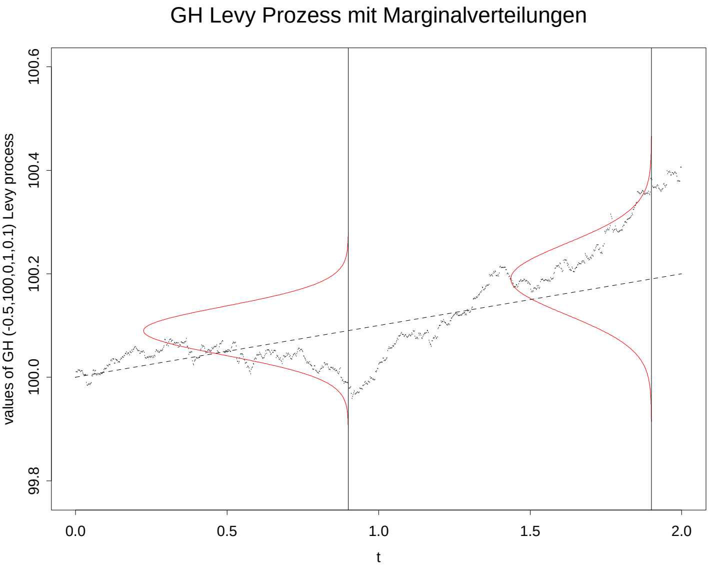
densities of $GH(-0.5, 5, \beta, 1, 0)$ distributions

Influence of λ



densities of $GH(\lambda, 10, 0, 1, 0)$ distributions

GH Lévy process with marginal densities



Generalized factor approach

State variable for credit i

$$X_i = \sqrt{\rho}\, M + \sqrt{1-\rho}\, Z_i, \qquad 0 \leq \rho \leq 1,$$

$M, Z_1, \ldots, Z_N$ independent,

$$M \sim GH(\lambda_M, \alpha_M, \beta_M, \delta_M, \mu_M), \quad Z_i \sim GH(\lambda_Z, \alpha_Z, \beta_Z, \delta_Z, \mu_Z)$$

Distribution of the relative number of defaults

$$F_{Z_t}(q) \approx 1 - F_M\left(\frac{F_X^{-1}(Q(t)) - \sqrt{1-\rho}\, F_Z^{-1}(q)}{\sqrt{\rho}}\right)$$

Computation of quantiles of X_i via Fourier inversion

Generalized factor approach (2)

Special case NIG

$$NIG(\alpha,\beta,\delta_1,\mu_1) * NIG(\alpha,\beta,\delta_2,\mu_2) = NIG(\alpha,\beta,\delta_1+\delta_2,\mu_1+\mu_2)$$

Assumption: Both M and Z_i are NIG.

$$\text{Then choose} \quad \alpha_Z := \alpha_M \frac{\sqrt{1-\rho}}{\sqrt{\rho}}, \quad \beta_Z := \beta_M \frac{\sqrt{1-\rho}}{\sqrt{\rho}}$$

$$\longrightarrow \; X_i \sim NIG\left(\frac{\alpha_M}{\sqrt{\rho}}, \frac{\beta_M}{\sqrt{\rho}}, \frac{\bar{\delta}_M}{\sqrt{\rho}}, \frac{\bar{\mu}_M}{\sqrt{\rho}}\right)$$

Similar for VG case

$$VG(\lambda_1,\alpha,\beta,\mu_1) * VG(\lambda_2,\alpha,\beta,\mu_2) = VG(\lambda_1+\lambda_2,\alpha,\beta,\mu_1+\mu_2)$$

Conclusion: The distribution of the idiosyncratic factor is uniquely determined by the systematic factor.

Factor scaling

State variable of credit i

$$X_i = \sqrt{\rho}\, M + \sqrt{1-\rho}\, Z_i, \qquad 0 \leq \rho \leq 1,$$

Preserve the role of ρ as correlation parameter:

$$\begin{aligned} \mathrm{Var}(M) &= \mathrm{Var}(Z_i) = 1 \\ \mathrm{E}(M) &= \mathrm{E}(Z_i) = 0 \end{aligned}$$

Choose λ, α, β as free parameters (tail behaviour, shape, skewness)

Derive $\bar{\delta}$ and $\bar{\mu}$ by

$$\mathrm{Var}\big(GH(\lambda, \alpha, \beta, \bar{\delta}, \bar{\mu})\big) = 1 \quad \text{and} \quad \mathrm{E}\big(GH(\lambda, \alpha, \beta, \bar{\delta}, \bar{\mu})\big) = 0$$

Factor scaling (2)

To obtain $\bar{\delta}$, solve numerically the equation

$$1 = \mathrm{Var}(GH) = \frac{\delta^2}{\zeta}\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} + \beta\,\frac{\delta^4}{\zeta^2}\left(\frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \frac{K^2_{\lambda+1}(\zeta)}{K^2_\lambda(\zeta)}\right)$$

where $\zeta := \delta\sqrt{\alpha^2 - \beta^2}$.

Then choose $\bar{\mu}$ such that

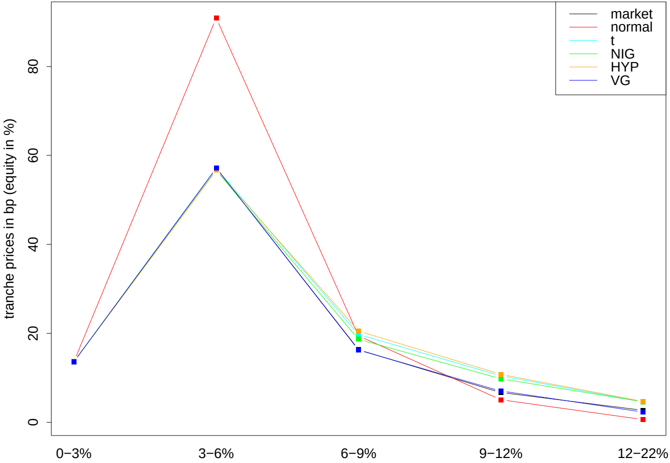
$$0 = \mathrm{E}(GH) = \bar{\mu} + \frac{\beta\bar{\delta}^2}{\bar{\zeta}}\frac{K_{\lambda+1}(\bar{\zeta})}{K_\lambda(\bar{\zeta})} \qquad \text{where} \quad \bar{\zeta} := \bar{\delta}\sqrt{\alpha^2 - \beta^2}.$$

For NIG distributions this simplifies to

$$1 = \mathrm{Var}(NIG) = \frac{\delta\alpha^2}{\sqrt{\alpha^2-\beta^2}(\alpha^2-\beta^2)}$$

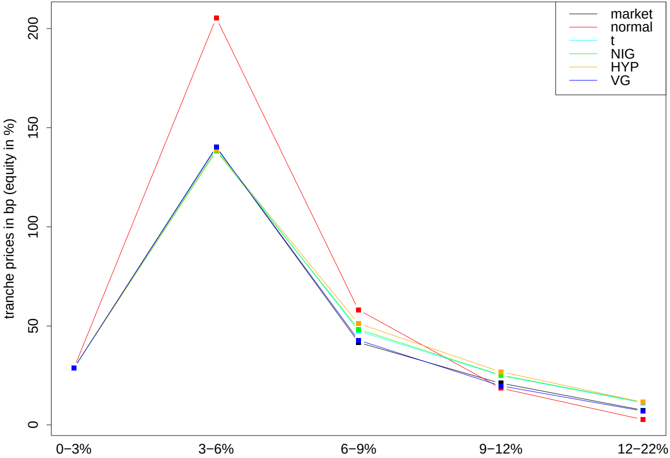
$$0 = \mathrm{E}(NIG) = \mu + \frac{\beta(\alpha^2-\beta^2)}{\alpha^2}$$

Calibration of the 5y iTraxx



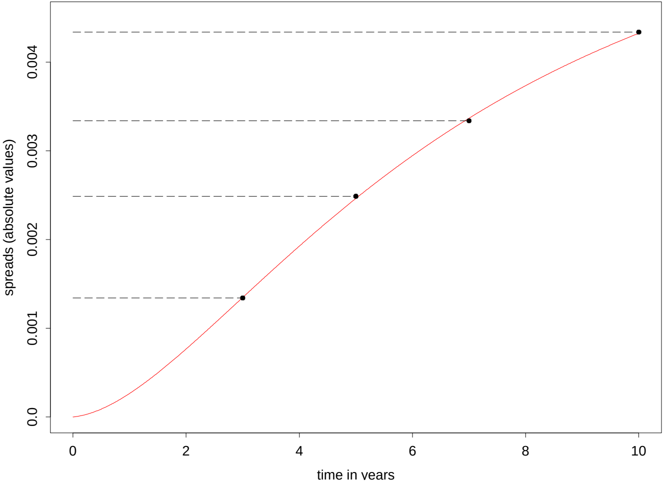
Comparison of calibrated model prices and market prices of the 5y iTraxx at November 13, 2006

Calibration of the 7y iTraxx



Comparison of calibrated model prices and market prices of the 7y iTraxx at November 13, 2006

Nelson–Siegel spread curve (1)



Constant iTraxx spreads of November 13, 2006, and fitted Nelson–Siegel curve with parameters $\hat{\beta}_0 = 0.0072, \hat{\beta}_1 = -0.0072, \hat{\beta}_2 = -0.0069, \hat{\tau}_1 = 2.0950$

Nelson–Siegel spread curve (2)

Consistent calibration over several maturities:

Introduce time-dependency of default intensity

$$\lambda = \lambda(t) = \frac{\hat{r}_{NS}(t)}{(1-R)\cdot 10000}$$

$\hat{r}_{NS}(t)$ = Nelson–Siegel spot rate curve fitted to average iTraxx spreads

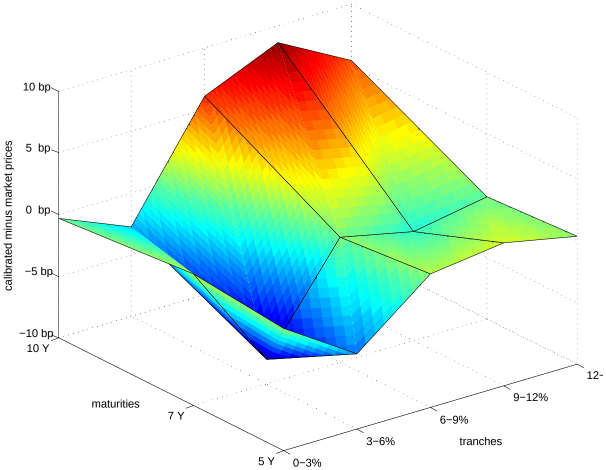
$$r_{NS}(t) = \beta_0 + (\beta_1 + \beta_2)\frac{\tau_1}{t}\left(1 - e^{-\frac{t}{\tau_1}}\right) - \beta_2 e^{-\frac{t}{\tau_1}}$$

Simultaneous calibration over all maturities (1)

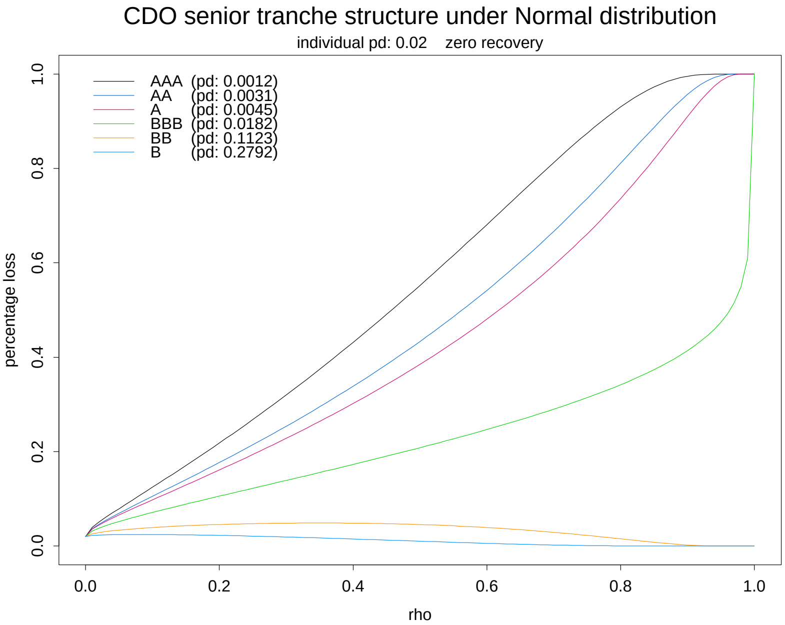
Tranches	Market	VG	Market	VG	Market	VG
	5Y		7Y		10Y	
0–3%	13.60%	13.60%	28.71%	28.72%	42.67%	42.67%
3–6%	57.16bp	53.30bp	140.27bp	132.27bp	360.34bp	357.60bp
6–9%	16.31bp	17.19bp	41.64bp	41.83bp	105.08bp	111.17bp
9–12%	6.65bp	8.23bp	21.05bp	19.90bp	43.33bp	52.00bp
12–22%	2.67bp	3.05bp	7.43bp	7.34bp	13.52bp	18.97bp

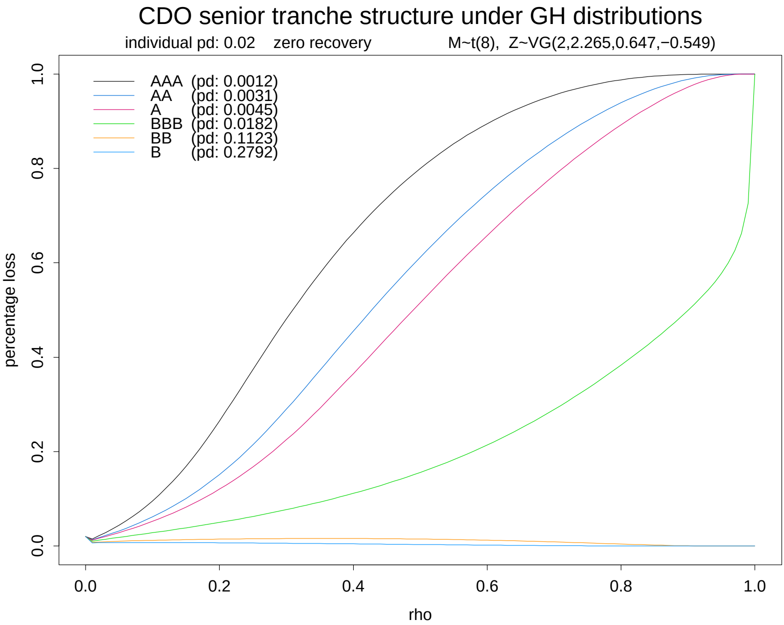
Estimated parameters: $\lambda_M = 0.920$, $\alpha_M = 5.553$, $\beta_M = 1.157$,
$\lambda_Z = 2.080$, $\alpha_Z = 2.306$, $\beta_Z = -0.753$,
$\rho = 0.321$

Simultaneous calibration over all maturities (2)



Graphical representation of the differences between model and market prices


CDO senior tranche structure under Normal distribution
individual pd: 0.02 zero recovery
AAA (pd: 0.0012)
AA (pd: 0.0031)
A (pd: 0.0045)
BBB (pd: 0.0182)
BB (pd: 0.1123)
B (pd: 0.2792)
percentage loss
0.0
0.2
0.4
0.6
0.8
1.0
rho
0.0
0.2
0.4
0.6
0.8
1.0



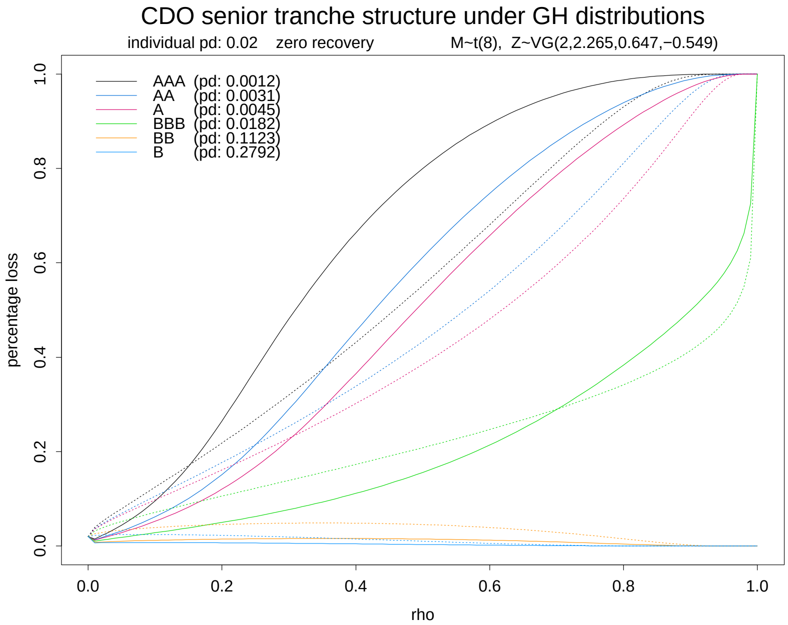
CDO structure and classical modeling

Modeling with advanced distributions

Calibration of the DJ iTraxx Europe

Pricing n^{th} to default CDSs

Conclusion and References


CDO senior tranche structure under GH distributions
individual pd: 0.02 zero recovery M~t(8), Z~VG(2,2.265,0.647,−0.549)
AAA (pd: 0.0012)
AA (pd: 0.0031)
A (pd: 0.0045)
BBB (pd: 0.0182)
BB (pd: 0.1123)
B (pd: 0.2792)
percentage loss
rho
0.0
0.2
0.4
0.6
0.8
1.0

Modeling of individual defaults (N small)

Exponentially distributed default times $T_1, \ldots, T_N$

$$Q_i(t) = P(T_i \leq t) = 1 - e^{-\lambda_i t}$$

with individual default intensities λ_i.

Risk neutral estimation of λ_i

$$\lambda_i = \frac{s_i}{(1 - R) \cdot 10000}$$

s_i = individual CDS spread in basis points

R = recovery rate

Modeling of portfolio dependence

X_i state variable for credit i $\qquad (1 \leq i \leq N)$

$$X_i = \sqrt{\rho}\, M + \sqrt{1-\rho}\, Z_i, \qquad 0 \leq \rho \leq 1,$$

$M, Z_1, \ldots, Z_N$ independent,

$$M \sim GH(\lambda_M, \alpha_M, \beta_M, \delta_M, \mu_M), \quad Z_i \sim GH(\lambda_Z, \alpha_Z, \beta_Z, \delta_Z, \mu_Z)$$

Time dependent thresholds

$$d_i(t) = F_X^{-1}(Q_i(t))$$

Individual default times

$$T_i := \inf\{\, t \geq 0 \mid X_i \leq d_i(t) \,\}$$

D_t number of defaults up to time t

pdf of the n^{th} default time

The conditional probabilities of exactly k defaults up to time t are

$$P(D_t = k|M), \qquad k = 0, \dots, N.$$

Let $S_N^k := \{\sigma \in S_N \,|\, \sigma(1) < \cdots < \sigma(k) \text{ and } \sigma(k+1) < \cdots < \sigma(N)\}$, then

$$\{D_t = k\} = \bigcup_{\sigma \in S_N^k} \left(\bigcap_{i=1}^{k} \{T_{\sigma(i)} < t\} \cap \bigcap_{i=k+1}^{N} \{T_{\sigma(i)} \geq t\} \right), \quad k = 1, \dots, N.$$

By the definition of T_i we have

$$\begin{aligned} P(T_i < t|M) &= P(X_i < F_X^{-1}(Q_i(t))|M) \\ &= F_Z\left(\frac{F_X^{-1}(Q_i(t)) - \sqrt{\rho} M}{\sqrt{1-\rho}} \right) =: H_i(t, M) \end{aligned}$$

pdf of the n^{th} default time (2)

The conditional independence of the T_i implies

$$P(D_t = 0|M) = P\left(\bigcap_{i=1}^{N} \{T_i \geq t\} \,\middle|\, M \right) = \prod_{i=1}^{N} (1 - H_i(t, M))$$

$$\begin{aligned} P(D_t = k|M) &= P\left[\bigcup_{\sigma \in S_N^k} \left(\bigcap_{i=1}^{k} \{T_{\sigma(i)} < t\} \cap \bigcap_{i=k+1}^{N} \{T_{\sigma(i)} \geq t\} \right) \,\middle|\, M \right] \\ &= P(D_t = 0|M) \cdot \sum_{\sigma \in S_N^k} \left(\prod_{i=1}^{k} \frac{H_{\sigma(i)}(t, M)}{1 - H_{\sigma(i)}(t, M)} \right) \end{aligned}$$

If all default intensities λ_i and hence all Q_i coincide, then

$$P(D_t = k|M) = \binom{N}{k} H(t, M)^k \left(1 - H(t, M)\right)^{N-k}$$

pdf of the n^{th} default time (3)

The time of the n^{th} default is given by

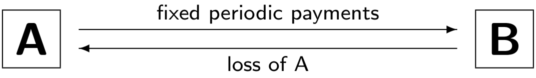
$$Y_n(\omega) := \inf \left\{ t \in \mathbb{R}_+ \,\middle|\, \sum_{i=1}^{N} \mathbb{1}_{\{T_i < t\}}(\omega) \geq n \right\}, \quad n = 1, \ldots, N$$

For $t \in \mathbb{R}_+$ we have

$$\begin{aligned} F_{Y_n}(t) &= P(Y_n < t) = P\left(\sum_{i=1}^{N} \mathbb{1}_{\{T_i < t\}} \geq n \right) \\ &= \sum_{k=n}^{N} P(D_t = k) = \int_{\mathbb{R}} \sum_{k=n}^{N} P(D_t = k | M = x)\, f_M(x)\, \mathrm{d}x \end{aligned}$$

n^{th} to default CDS

Consider a portfolio of N bonds with nominal L and recovery rate R.



Assume the riskless interest rate r to be constant over $[0, T]$. Let

$u(t)$ the discounted value of the premiums paid up to time t,

$f_{Y_n}(t)$ the density of the n^{th} default time Y_n,

r_n the swap rate of the n^{th} to default CDS.

If the n^{th} credit event occurs within $[0, T]$, A receives $(1 - R)L$.

A pays a fixed premium of $0.25 r_n L$ quarterly and in case of the n^{th} default an accrual premium to cover the time since the last fixed payment.

The swap rate r_n

The discounted value of the premium leg is

$$E_n(r_n) = r_n\, L \int_0^T u(t) f_{Y_n}(t)\, \mathrm{d}t + r_n\, L\, u(T)(1 - F_{Y_n}(T))$$

For the discounted value of the default leg we have

$$D_n = (1 - R)L \int_0^T f_{Y_n}(t) e^{-rt}\, \mathrm{d}t$$

The no-arbitrage condition $E_n(r_n) = D_n$ implies

$$r_n = \frac{(1 - R) \int_0^T f_{Y_n}(t) e^{-rt}\, \mathrm{d}t}{\int_0^T u(t) f_{Y_n}(t)\, \mathrm{d}t + u(T)(1 - F_{Y_n}(T))} = \frac{D_n}{E_n(1)}$$

Numerical calculation of r_n

Divide $]0, T]$ in equidistant parts $]t_{i-1}, t_i]$ where $t_i = 0.25 \cdot i$, $i = 1, \ldots, 4T$. At the end of each subinterval A pays 0.25 (assume $L = 1$).

If the n^{th} default occurs at time $t \in]t_{i-1}, t_i]$, A additionally has to pay a final premium of $(t - t_i) \cdot 0.25$.

Therefore $u(t)$ is given by

$$u(t) = \sum_{i=1}^{4T} \mathbb{1}_{]t_{i-1},t_i]}(t) \cdot \left(\sum_{j=1}^{i} 0.25 \cdot e^{-r \cdot 0.25 \cdot j} + (t - t_{i-1})\, e^{-rt} \right)$$
$$u(0) = 0$$

The density f_{Y_n} is discretized by $f^s_{Y_n}$, where $s \in \{\frac{1}{m} \mid m \in \mathbb{N}\}$, $z = \frac{T}{s}$ and

$$f^s_{Y_n}(t) := \frac{1}{s} \sum_{k=0}^{z-1} [F_{Y_n}((k+1)s) - F_{Y_n}(ks)] \cdot \mathbb{1}_{]ks,(k+1)s]}(t), \quad t \in [0, T]$$

Computational assumptions

We consider an n^{th} to default CDS with time to maturity $T = 5$ years.

The underlying portfolio consists of $N = 10$ firms (bonds, references).

The intensity parameter λ_i is the same for all underlyings.

The recovery rate is $R = 0.4$ for all firms.

The riskless interest rate is $r = 0.05$.

The discretization parameter is $s = \frac{1}{12}$, i.e. the densities f_{Y_n} are evaluated on a monthly time grid.

Numerical results for NIG

$\rho = 0.3,\ \lambda_i \equiv \lambda = 0.01$

α_M/β_M α_Z/β_Z	Normal	25/0 8/7.5	8/7.5 25/0	8/7.5 8/7.5	8/ − 7.5 8/ − 7.5	1/ − 0.7 1/ − 0.7
n	swap rates (basis points p.a.)					
1	441.12	198.03	551.16	391.97	467.60	447.70
2	139.45	133.35	122.13	162.97	95.29	84.99
3	53.33	98.87	23.50	67.59	32.83	30.66
4	21.42	71.67	3.51	25.22	18.94	20.81
5	8.56	49.93	0.40	8.11	13.94	17.53
6	3.28	32.49	0.02	2.17	11.31	15.64
7	1.15	19.01	0.00	0.47	9.62	14.21
8	0.35	9.42	0.00	0.05	8.39	12.89
9	0.08	3.54	0.00	0.01	7.36	11.40
10	0.01	0.76	0.00	0.00	0.32	9.14

Conclusion

Implementation of heavier tailed and skewed distributions in the factor model

Completely different distributions for systematic and idiosyncratic factors possible

Much more flexible pricing formulas for n^{th} to default CDSs and CDOs

Consequences for CDO calibration:

Perfect fit to market data

Flat correlation structure over all CDO tranches and even maturities

No base correlation framework needed

More complex dependence structure

References

- E. Eberlein, R. Frey, and E. A. v. Hammerstein: Advanced credit portfolio modeling and CDO pricing. In: W. Jäger, H.-J. Krebs (Eds.), *Mathematics – Key technology for the future*, Springer Verlag (2008) 253–280
- E. Eberlein, E. A. v. Hammerstein: Generalized hyperbolic and inverse Gaussian distributions: limiting cases and approximation of processes. In: R. C. Dalang, M. Dozzi, and F. Russo (Eds.), *Seminar on Stochastic Analysis, Random Fields and Applications IV*, Progress in Probability 58, Birkhäuser Verlag (2004) 221–264
- E. Eberlein: Application of generalized hyperbolic Lévy motions to finance. In: O. E. Barndorff-Nielsen, T. Mikosch, and S. Resnick (Eds.), *Lévy processes: Theory and Applications*, Birkhäuser Verlag (2001) 319–337
- E. Eberlein, U. Keller: Hyperbolic distributions in finance. Bernoulli 1 (1995) 281–299