Modelling of Successive Defaults Events with density

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Introduction

Motivation

:

⇒ Understand the **correlation structure** of several default times : spread correlation, contagious defaults and correlation dynamics

⇒ Develop a dynamic framework for portfolio credit derivatives, integrating the information given by the succesive default.

Several approaches

:

- Copula methods : static modelling and dynamic property analysis
- Intensity processes correlation and incorporation of contagious jumps
- A recent new approach : top-down approach and cumulative loss modelling

Contribution of this work

⇒ Propose to study the **successive defaults**

– sufficient for k^{th}-to-default swaps and CDOs

– simplification $n+1$ default scenarios instead of 2^n

– valid for the general case

⇒ Propose a new approach based on the **density** of conditional survival probabilities

⇒ For single-credit, study respectively the before-default set $\{\tau > t\}$ and the after-default set $\{\tau \leq t\}$

– For multi-credits, a natural extension and a general framework for the pricing

Single credit

Single-credit : Information and Filtrations

Different types of information on the market $(\Omega, \mathcal{G}, \mathbb{P})$:

- The minimal one : $\mathbb{F} = (\mathcal{F}_t)$
- Default information $\mathbb{D} = (\mathcal{D}_\mathbf{t})_{\mathbf{t}\geq\mathbf{0}}$ where $\mathcal{D}_t = \sigma(\mathbb{1}_{\{\tau \leq s\}}, s \leq t)$

  In general τ is not a $\mathbb{F}$-stopping time
- The global market information $\mathbb{G} = (\mathcal{G}_\mathbf{t})_{\mathbf{t}\geq\mathbf{0}}$ contains at least $\mathbb{F}$ and $\mathbb{D}$.

For the purpose of pricing :

– Conditional expectations w.r.t $\mathbb{G}$ and some sub-filtration

– $\mathbb{G}$-martingale characterization

But essentially, the main question is,

"What happens after the default ?"

Before the default $\{t < \tau\}$

Minimal Assumption (Jeulin, Yor)

$(\mathbb{F}, \mathbb{G}, \tau)$ satisfy the **minimal assumption** if $\forall\, t \geq 0$ and $U^{\mathbb{G}} \in \mathcal{G}_t$, $\exists\, U^{\mathbb{F}} \in \mathcal{F}_t$ such that

$$U^{\mathbb{G}} \cap \{\tau > t\} = U^{\mathbb{F}} \cap \{\tau > t\}.$$

– If $\mathbb{G}^\tau = \mathbb{F} \vee \mathbb{D}$, then $(\mathbb{F}, \mathbb{G}^\tau, \tau)$ enjoys MA.

– Classical result (Elliott-Jeanblanc-Yor) : under MA, for any $\mathcal{G}$-measurable r.v. $Y^{\mathbb{G}}$,

$$1\!\!1_{\{\tau>t\}}\mathbb{E}\left(\left[\right) Y^{\mathbb{G}}|\mathcal{G}_t\right] = 1\!\!1_{\{\tau>t\}} \frac{\mathbb{E}\left(\left[\right) Y^{\mathbb{G}} 1\!\!1_{\{\tau>t\}}|\mathcal{F}_t\right]}{\mathbb{P}(\tau > t|\mathcal{F}_t)} \quad a.s.$$

on the set $A := \{\mathbb{P}(\tau > t|\mathcal{F}_t) > 0\}$.

After the default $\{t \geq \tau\}$

After-Default Hypothesis

1. Stronger hypothesis on the filtrations : $\mathbb{G} = \mathbb{G}^\tau = \mathbb{F} \vee \mathbb{D}$
2. **J-Hypothesis** (Jacod, existence of conditional density) :

   The regular conditional distribution of τ, given $\mathcal{F}_t$ is absolutely continuous w.r.to some deterministic measure η (with no atom), that is $\forall$ $\mathbf{t} \geq \mathbf{0}$, $\exists$ a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^{+})$ r.v. $\alpha_t(\theta)$ s.t.

   $$\mathbb{P}(\tau \in d\eta(\theta)|\mathcal{F}_\mathbf{t}) := \alpha_\mathbf{t}(\theta)d\eta(\theta) \quad a.e.\, \theta \geq \mathbf{0},\ \mathbb{P} - a.s.$$

3. Under the above hypotheses, we can compute $\mathcal{G}_t$-conditional expectations on the set $\{\tau \leq t\}$
4. A weak version of J-hypothesis : $0 \leq t \leq \theta$, useful for the before-default, but not sufficient for the after-default

After the default $\{\tau \le t\}$

Proposition (computation of $\mathcal{G}_t$ conditional expectation)

- Bounded $\mathcal{F}_\mathbf{T} \otimes \mathcal{B}(\mathbb{R})$ r.v. $Y_T(\theta)\ \forall\ T, \theta \ge 0$.
- Computation by $\mathcal{F}_t$ conditional expectation $\forall\ \theta \le t$:

$$Y_t^{ad}(T,\theta) := \frac{\mathbb{E}[Y_T(\theta)\alpha_\mathbf{T}(\theta)\big|\mathcal{F}_t]}{\alpha_\mathbf{t}(\theta)}\mathbb{1}_{\{\alpha_t(\theta)>0\}} \quad d\eta(\theta)\otimes d\mathbb{P}\ a.s.$$

- Then $\forall\ t \le T$, on the set $\mathcal{A}(\tau) = \{\alpha_t(\tau) > 0\}$,

$$\mathbf{E}[\mathbf{Y_T}(\tau)|\mathcal{G}_\mathbf{t}]\,\mathbb{1}_{\{\tau\le \mathbf{t}\}} = Y_t^{ad}(T,\tau)\mathbb{1}_{\{\tau\le t\}} \quad d\mathbb{P}\ a.s. \tag{1}$$

H-hypothesis, or Immersion

– under H-hypothesis, $\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$, so $\alpha_t(\theta) = \alpha_\theta(\theta)$ for any $t \ge \theta$.

– default probability conditioned on future information

Dynamic Point of View

Density and martingale w.r.t. $\mathbb{F}$ Filtration

(AD)-Hypothesis : Conditional density and enlargement of filtration

Many papers : Jeulin, Yor,Jacod (1985) (enlargement of filtration), Credit framework,Duffie, Lando, Jeanblanc and al, Jarrow and all, Schoenbucher and all,....Y.Le Cam (2008)

• Thanks to old Jacod's results which show the existence of "universal"(in θ) càdlàg version of all (super)martingales processes that we need :

– Conditional (supermartingale) **survival process** $(S_t = \mathbb{P}(\tau > t|\mathcal{F}_t), t \geq 0)$

– Conditional probability process $(S_t(\theta) = \mathbb{P}(\tau > \theta|\mathcal{F}_t), t \geq 0)$.
Useful to make the distinction between $t \leq \theta$ since then $S_t(\theta) = \mathbb{E}[S_\theta|\mathcal{F}_t]$, and $t \leq \theta$.

– Martingale density $(\alpha_{\mathbf{t}}(\theta), \mathbf{t} \geq \mathbf{0})$, density of $S_t(\theta)$

Decompositions of Survival processes

Up to the default : Classical context in credit framework.

Survival process

Let $M^{\mathbb{F}} - A^{\mathbb{F}}$ be the **Doob-Meyer decomposition** of the survival process (supermartingale) S

– the increasing process $A^{\mathbb{F}}_t = \int_0^t \alpha_u(u)\,du$,

– the $\mathbb{F}$-martingale $M^{\mathbb{F}}_t = -\int_0^t(\alpha_t(u) - \alpha_u(u))du$

The **multiplicative decomposition** is

–

$$S_t = L^{\mathbb{F}}_t \exp\big(-\int_0^t \lambda^{\mathbb{F}}_s ds\big),\ \ \lambda^{\mathbb{F}}_t = \frac{\alpha_t(t)}{S_t}$$

on $\{S_t > 0\}$

Density and $\mathbb{G}$-Intensity

– Classical intensity approach

  – **$\mathbb{G}$-compensator** $\Lambda^{\mathbb{G}}$: the $\mathbb{G}$-predictable increasing process such that $(\mathbb{1}_{\{\tau \le t\}} - \Lambda^{\mathbb{G}}_t, t \ge 0)$ is a $\mathbb{G}$-martingale.

  – **$\mathbb{G}$-intensity** $\lambda^{\mathbb{G}}$: if $\Lambda^{\mathbb{G}}$ is a.c., $\Lambda^{\mathbb{G}}_t = \int_0^t \lambda^{\mathbb{G}}_{\mathbf{s}} ds$

– Under the **weak version** of J-Hypothesis,

$$\lambda^{\mathbb{G}}_t = \mathbb{1}_{\{\tau > t\}} \lambda^{\mathbb{F}}_t = \mathbb{1}_{\{\tau > t\}} \frac{\alpha_t(t)}{S_{t-}} \quad a.s..$$

$\Rightarrow$ For any $T \ge t$,

$$\alpha_t(T) = \mathbb{E}\left(\left[\right)\lambda^{\mathbb{G}}_T | \mathcal{F}_t\right] \quad a.s..$$

Remark :

– existence of intensity $\leftrightarrow$ weak version of J-hypothesis.

– intensity approach — **not enough** for the after-default case.

$\mathbb{G}$-martingales stopped at τ

All $\mathbb{G}$-cadlag martingale can be written as

$$M_t^X = X_t \mathbb{1}_{\{\tau > t\}} + X_t(\tau)\mathbb{1}_{\{\tau \le t\}}$$

where X is a $\mathbb{F}$-adapted cadlag process and $(X_t(\theta))$ has some "universal" regularity.

The resultats on stopped martingales are well-known.

Proposition A $\mathbb{G}$-stopped process M^X is a $\mathbb{G}$- martingale bounded iff

$\Rightarrow$ $(\mathbf{X_t S_t} + \int_0^t \mathbf{X_s(s)}\alpha_\mathbf{s}(\mathbf{s})\,\mathbf{ds}, t \ge 0)$ is a $\mathbb{F}$-martingale.

– Equivalently, $\mathbf{L_t^{\mathbb{F}}}(\mathbf{X_t} - \int_0^t (\mathbf{X_s} - \mathbf{X_s(s)})\lambda_\mathbf{s}^{\mathbb{F}}\,\mathbf{ds}$ is an $\mathbb{F}$-local martingale.

$\Rightarrow$ $M_t^d = (X_\tau - X_\tau(\tau)\mathbf{1}_{\tau \le t} - \int_0^t (X_s - X_s(s))\mathbf{1}_{s<\tau}\lambda_s^{\mathbb{F}}\,ds$ is a pure jump martingale with one jump at time τ.

– $M_t^X - M_t^d$ is the $\mathbb{G}$-martingale $M_t^Y = Y_{t\vee\tau}$ where $Y_t = X_t - \int_0^t (X_s - X_s(s))\lambda_s^{\mathbb{F}}\,ds$.

$\Rightarrow$ Then, formally, a stopped $\mathbb{G}$-martingale, conitinuous at time τ $Y_{t\vee\tau}$ is characterized by $L^{\mathbb{F}}Y_t$ is a local martingale,

$\Rightarrow$ Equivalently, formally, Y is a local martingale w.r. to the probability measure $P^L = L^{\mathbb{F}}_T.P$.

– As an induitive consequence, any $\mathbb{F}$-martingale, stopped at time τ is a $\mathbb{G}$ semimartingale such that

$$M^{\mathbb{F}}_{t\vee\tau} = M^{\mathbb{G}}_t + < M^{\mathbb{F}}_t, (1/L^F_.).L^F$$

Starting after the default

$\mathcal{G}_{t\wedge\tau}$-Martingale

Let $X_{t\wedge\tau} = X_{t\wedge\tau}\tau)$ a $\mathbb{G}$-optional process. Then this process is a $\mathbb{G}$- martingale iff

$\Rightarrow$ $(X_t(\theta)\alpha_t(\theta), t \geq \theta)$ is an $\mathbb{F}$-martingale

$\Rightarrow$ Formally, let $\mathbb{P}^\theta$ the probability with density the martingale $\alpha_t(\theta)$. Then, $(X_t(\theta)$ ia $\mathbb{P}^\theta$ local martingale.

$\Rightarrow$ A $\mathbb{F}$-martingale $X^\mathbb{F}$ starting after τ is a martingale iff $X^\mathbb{F}_{t\vee\tau}$ is a local martingale on $\mathbb{P}^\theta$.

Remark :

1. Under (H)-hypothesis, $L^\mathbb{F}$ is a constant martingale and the characterization becomes :
2. $X_t(\theta)$ is a $\mathbb{F}$-martingale on $[\theta, \infty)$.
3. $X_t - \int_0^t (X_s - X_s(s))\lambda^\mathbb{F}_s \, ds$ is an $\mathbb{F}$-martingale

Girsanov Theorem

Assume that $\Omega, \mathbb{F}, \mathbb{G}, \tau, \mathbb{P})$ satisfies the (AD)-hypothesis, with parameters $(S_t, (\alpha_t(\theta),\ \theta \leq t)$ with the compatibilitÃ© condition that $S_t + \int_0^t \alpha_s(s)ds$ is a martingale.

Let $Q_t^{\mathbb{G}} = q_t \mathbf{1}_{\{\tau < t\}} + q_t(\tau)\mathbf{1}_{\{\tau \geq t\}}$ be a cadla positive integrable martingale with $Q_0 = 1$ and Q the probability equivalente to $\mathbb{P}$ with density $Q^{\mathbb{G}}$. The $\mathbb{F}$-projection of the martingale $Q^{\mathbb{G}}$ is $Q^{\mathbb{F}}$.

Then, $\Omega, \mathbb{F}, \mathbb{G}, \tau, \mathsf{Q})$ satisfies the (AD)-hypothesis, with parameters

$$\alpha_t^Q(\theta) = \frac{q_t(\theta)\alpha_t(\theta)}{Q_t^{\mathbb{F}}}, S_t^Q = \frac{S_t\, q_t}{Q_t^F},\ \ \lambda_t^Q = \frac{q_t(t)}{q_t}{}_t^F$$

.

Modelling the density process

Modelling density process

Two possible solutions :

– model firstly $S_t(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t)$ and then take derivatives

– model directly the density $\alpha_t(\theta)$

Remarks :

– both processes are positive $\mathbb{F}$ martingales

– reference to the interest models

– distinction between $\theta \geq t$ (classical part) and $\theta < t$ (non-classical part)

$\mathbb{F}$-martingale representation and HJM framework

Model the $\mathbb{F}$-martingale $S_t(\theta)$ in the HJM framework

– Classically for the bond $\theta \geq t$, but extendable to $\theta \geq 0$.

HJM results

Suppose

$$\frac{dS_t(\theta)}{S_t(\theta)} = \Psi_t(\theta) dM_t, \quad t, \theta \geq 0$$

where M is a continuous multi-dimensional $\mathbb{F}$-martingale, then

- $S_t(\theta) = S_0(\theta) \exp\left(\int_0^t \Psi(s,\theta) dM_s - \frac{1}{2} \int_0^t |\Psi(s,\theta)|^2 d\langle M \rangle_s \right)$;
- $S_t = \exp\left(- \int_0^t \lambda_s^{\mathbb{F}} ds + \int_0^t \boldsymbol{\Psi}(\mathbf{s},\mathbf{s}) dM_s - \frac{1}{2} \int_0^t |\boldsymbol{\Psi}(\mathbf{s},\mathbf{s})|^2 d\langle M \rangle_s \right)$.

Examples of martingale density process

- Compatibility between martingale and probability properties
- **A Generalized exponential model :** $\forall\ t, \theta \geq 0$, let

$$S_t(\theta) = \exp\big(- \theta M_t - \frac{1}{2}\,\theta^2 \,\langle M\rangle_t \,\big)$$

where M is an $\mathbb{F}$-martingale.

- Exponential law $S_0(\theta) = \mathbb{P}(\tau > \theta) = \exp(-\theta M_0)$.
- Probability condition : $\theta M_t + \frac{1}{2}\theta^2 \langle M\rangle_t^2 \geq 0 \Rightarrow$ the martingale M should be **non-negative**.

Comparison with interest rate modelling

– Interest short rate models for $\lambda_t^{\mathbb{F}}$ (Schönbucher) $\longrightarrow$ deduce $S_t(T)$, but only for $T \geq t$

– Zero-coupon $B(t,T) = \mathbb{E}[e^{-\int_t^T r_s ds}|\mathcal{F}_t]$.

Short rate $\mathbf{r_t} = -\partial_T|_{T=t} \log B(t,T)$.

– Defaultable zero-coupon without actualization

$$\mathbb{E}[1\!\!1_{\{\tau>T\}}|\mathcal{G}_t] = 1\!\!1_{\{\tau>t\}}\mathbb{E}[S_T/S_t|\mathcal{F}_t] := 1\!\!1_{\{\tau>t\}}B^\tau(t,T).$$

Intensity $\lambda_{\mathbf{t}}^{\mathbb{F}} = -\partial_T|_{T=t} \log B^\tau(t,T)$.

Density process in a CIR model

– Let $\lambda^{\mathbb{F}}$ be given by a CIR model, that is

$$d\lambda_t^{\mathbb{F}} = \kappa(\theta - \lambda_t^{\mathbb{F}})dt + \sigma\sqrt{\lambda_t^{\mathbb{F}}}dW_t$$

– Conditional probability

$$S_t(T) = \mathbb{E}[e^{-\int_0^T \lambda_s^{\mathbb{F}} ds}|\mathcal{F}_t] = S_t\, A(t,T)e^{-\lambda_t^{\mathbb{F}} C(t,T)}, \quad \mathbf{T \geq t}$$

where $C(t,T) = C(T-t)$ and $A(t,T) = A(T-t)$ depend on $T-t$ and are given by

$$\begin{cases} C(t) & = \frac{2(e^{\rho t}-1)}{(\rho+\kappa)(e^{\rho t}-1)+2\rho} \quad \text{where } \rho = \sqrt{\kappa^2 + 2\sigma^2}, \\ A(t) & = \phi(t)^{2\kappa\theta/\sigma^2} \quad \text{where } \phi(t) = \frac{2\rho\exp((\rho+\kappa)t/2)}{(\rho+\kappa)(e^{\rho t}-1)+2\rho}. \end{cases}$$

Density process and H-hypothesis

- Interest models provide examples for $S_t(\theta)$ and $\alpha_t(\theta)$ where $\theta \geq t$
- For the non-classical part where $\theta < t$, the case with H-hypothesis is easy since $S_t(\theta) = S_\theta$
- In the general case without H-hypothesis, we propose a general construction for $\alpha_t(\theta)$ where $\theta \leq t$, by **change of probability measure from a model with H-hypothesis**

A general construction of density process

1. Start with $\mathbb{P}_0$ with H-hypothesis, and τ with density process $(\alpha_t^0(\theta), t \geq 0)$ constant in time after θ

2. Let $\mathbf{Q_t^{\mathbb{G}}} = Q_t 1\!\!1_{\{\tau > t\}} + Q_t(\tau) 1\!\!1_{\{\tau \leq t\}}$ a positive $(\mathbb{G}, \mathbb{P}_0)$-martingale with expectation 1

3. Define $\mathbf{d\mathbb{P} = Q_T^{\mathbb{G}} d\mathbb{P}_0}$ on $\mathbb{G}$. The martingale $\mathbf{Q_t^{\mathbb{F}}} = Q_t S_t + \int_0^t Q_s(s) \alpha_s^0(s) ds$ is the density of $\mathbb{P}$ on $\mathbb{F}$.

4. Then the **density process of τ under $\mathbb{P}$** is

$$\alpha_t^{\mathbb{P}}(\theta) = \alpha_\theta^0(\theta) \frac{Q_t(\theta)}{Q_t^{\mathbb{F}}} \text{ if } \theta < t$$

$$\alpha_t^{\mathbb{P}}(\theta) = \frac{\mathbb{E}[Q_\theta(\theta) \alpha_\theta^0(\theta) | \mathcal{F}_t]}{Q_t^{\mathbb{F}}} \text{ if } \theta \geq t.$$

Two ordered default times

Setup

- Before-default and after-default analysis extended naturally to the ordered defaults
- Two $\mathbb{G}$-stopping times :

$$\tau = \tau^{(1)} := \min(\tau_1, \tau_2) \quad \text{and} \quad \sigma = \tau^{(2)} := \max(\tau_1, \tau_2).$$

- **Filtrations :** $\mathbb{D}^{(1)}$ for τ and $\mathbb{D}^{(2)}$ for σ respectively. Let

$$\mathbb{G}^{(\mathbf{1})} = \mathbb{F} \vee \mathbb{D}^{(\mathbf{1})} \quad \text{and} \quad \mathbb{G}^{(\mathbf{2})} = \mathbb{F} \vee \mathbb{D}^{(1)} \vee \mathbb{D}^{(2)} = \mathbb{G}^{(\mathbf{1})} \vee \mathbb{D}^{(\mathbf{2})}.$$

- For τ — if suffices to apply directly the previous studies
- For σ — a recursive procedure

The second default σ

- Global filtration $\mathbb{G} \to \mathbb{G}^{(2)}$, sub-filtration $\mathbb{F} \to \mathbb{G}^{(1)}$
- Before and After the second default $\{t < \sigma\}$ and $\{t \geq \sigma\}$
- $\mathbb{G}^{(1)}$-conditional survival probability of σ

$$S_t^{\sigma|\mathbb{G}^{(1)}}(\theta) = \mathbb{P}(\sigma > \theta|\mathcal{G}_t^{(1)})$$

$$= \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{P}(\tau > t, \sigma > \theta|\mathcal{F}_t)}{\mathbb{P}(\tau > t|\mathcal{F}_t)} + \mathbb{1}_{\{\tau\leq t\}} \frac{\partial_s \mathbb{P}(\sigma > \theta, \tau > s|\mathcal{F}_t)}{\partial_s \mathbb{P}(\tau > s|\mathcal{F}_t)}\bigg|_{s=\tau}$$

- Hypothesis on $\mathbb{F}$-joint density $\alpha_t^{\tau,\sigma}$ of $(\tau, \sigma) \Longrightarrow$ **J-hypothesis** on the $\mathbb{G}^{(1)}$-density $\alpha_t^{\sigma|\mathbb{G}^{(1)}}(\theta)$ of σ
- $\alpha_t^{\tau,\sigma}(u_1, u_2) = 0$, $\forall u_1 \geq u_2$.

Recursive method

- $\mathbb{G}^{(2)}$-intensity $\lambda^{\sigma,(2)}$ of σ :

$$\lambda_t^{\sigma,(2)} = 1\!\!1_{\{\sigma>t\}} \frac{\alpha_t^{\sigma|\mathbb{G}^{(1)}}(t)}{S_{t-}^{\sigma|\mathbb{G}^{(1)}}} = \mathbf{1}_{\{\sigma>\mathbf{t}\geq\tau\}} \frac{\alpha_t^{\tau,\sigma}(\tau,t)}{\int_t^\infty \alpha_t^{\tau,\sigma}(\tau,v)dv}$$

  Since $\alpha_t^{\tau,\sigma}(\tau,t)$ for $t<\tau$.

- $\mathbb{G}^{(2)}$-conditional expectations $\mathbb{E}\big[Y_T(\tau,\sigma)\,|\,\mathcal{G}_t^{(2)}\big]$
  - Distinguish three sets $\{t<\tau\}$**,** $\{\tau\leq t<\sigma\}$ **and** $\{\sigma\leq t\}$
  - Easy on $\{t<\tau\}$, minimal assumption for $(\mathbb{F},\mathbb{G}^{(2)},\tau)$
  - Recursive method on $\{\tau\leq t<\sigma\}$ and on $\{\sigma\leq t\}$: $\mathcal{G}_t^{(1)}$-conditional expectations, using $\alpha_t^{\sigma|\mathbb{G}^{(1)}}(\theta)$ and $\mathcal{F}_t$-conditional expectations, using the $\mathbb{F}$-joint density $\alpha_t^{\tau,\sigma}(\theta_1,\theta_2)$ of (τ,σ)

Calculation of $\mathbb{G}^{(2)}$-conditional expectations

Explicit formulas similar on the three sets :

– on $\{t < \tau\}$,

$$\frac{\mathbb{E}\big[\mathbb{1}_{\{\tau>t\}} Y_T(\tau,\sigma)\,|\,\mathcal{F}_t\big]}{\mathbb{E}[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]} = \frac{\mathbb{E}\big[\int_t^\infty du_1 \int_{u_1}^\infty du_2\, Y_T(u_1,u_2)\alpha_{\mathbf{T}}^{\tau,\sigma}(\mathbf{u_1},\mathbf{u_2})|\mathcal{F}_t\big]}{\int_t^\infty du_1 \int_{u_1}^\infty du_2 \alpha_{\mathbf{t}}^{\tau,\sigma}(\mathbf{u_1},\mathbf{u_2})}$$

– on $\{\tau \le t < \sigma\}$,

$$\frac{\mathbb{E}\big[\int_t^\infty du_2\, Y_T(u_1,u_2)\alpha_{\mathbf{T}}^{\tau,\sigma}(\mathbf{u_1},\mathbf{u_2})|\mathcal{F}_t\big]}{\int_t^\infty du_2 \alpha_{\mathbf{t}}^{\tau,\sigma}(\mathbf{u_1},\mathbf{u_2})}\Big|_{u_1=\tau}$$

– on $\{\sigma \le t\}$,

$$\frac{\mathbb{E}\big[Y_T(u_1,u_2)\alpha_{\mathbf{T}}^{\tau,\sigma}(\mathbf{u_1},\mathbf{u_2})\,|\,\mathcal{F}_t\big]}{\alpha_{\mathbf{t}}^{\tau,\sigma}(\mathbf{u_1},\mathbf{u_2})}\Big|_{\substack{u_1=\tau\\ u_2=\sigma}}$$

$\mathbb{F}$-joint density process

- Information of individual names
- Ordered $\mathbb{F}$ joint density $\alpha^{\tau,\sigma}$ of (τ,σ) deduced from the non-ordered one $\alpha^{1,2}$ of (τ_1,τ_2) by statistics orders :

$$\alpha_t^{\tau,\sigma}(u_1,u_2) = 1\!\!1_{\{u_1 \leq u_2\}} \big(\alpha_t^{1,2}(u_1,u_2) + \alpha_t^{1,2}(u_2,u_1)\big), \; \forall u_1,u_2 \geq 0.$$

- Model the non-ordered density via the $\mathbb{F}$-joint conditional survival probability $\mathbf{S_t^{1,2}}(\theta_\mathbf{1},\theta_\mathbf{2}) = \mathbb{P}(\tau_\mathbf{1} > \theta_\mathbf{1}, \tau_\mathbf{2} > \theta_\mathbf{2} | \mathcal{F}_\mathbf{t})$

A backward example

- Cox process model (Lando) for τ_1 and τ_2 :

$$\tau_i = \inf\{t : \Phi^i_t \geq \xi_i\}$$

  Φ^i an $\mathbb{F}$-adapted process, $\Phi_0 = 0$, $\lim_{t\to\infty} \Phi_t = +\infty$

  ξ_i a $\mathcal{G}$-measurable r.v. independent of $\mathcal{F}_\infty$, $\xi_i \sim \exp(1)$.

- Marginal survival process $S^i_t = \mathbb{P}(\tau_i > t | \mathcal{F}_\infty) = e^{-\Phi^i_t}$, **H-hypothesis** satisfied for $\mathbb{F}$ and $\mathbb{G}^i$.

- (Schönbucher-Schubert) Correlation of defaults by a copula function $C(x_1, x_2)$:

$$\mathbb{P}(\tau_1 > \theta_1, \tau_2 > \theta_2 | \mathcal{F}_\infty) = C(S^1_{\theta_1}, S^2_{\theta_2}).$$

  Then

$$S^{1,2}_t(\theta_1, \theta_2) = \mathbb{E}\big[C(S^1_{\theta_1}, S^2_{\theta_2}) | \mathcal{F}_t\big].$$

A copula diffusion example

- Joint survival probability

$$S_0(\theta_1, \theta_2) = \mathbb{P}(\tau_1 > \theta_1, \tau_2 > \theta_2) = \exp\big(- (\theta_1^2 + \theta_2^2)^{\frac{1}{2}} \big)$$

a c.d.f of two exponential r.v. with unit parameter linked by a **Clayton copula**.

- Diffuse the copula function as a martingale : $\forall t, \theta_1, \theta_2 \geq 0$, let

$$S_t(\theta_1, \theta_2) = \exp\Big(- \big(\theta_1^2 M_t^1 + \theta_2^2 M_t^2\big)^{\frac{1}{2}} - A_t \Big)$$

where

$$A_t = \frac{1}{8} \int_0^t \frac{1 + X_s^{\frac{1}{2}}}{X_s^{\frac{2}{3}}} d\langle X \rangle_s \quad \text{and} \quad X_s = \theta_1^2 M_s^1 + \theta_2^2 M_s^2$$

where M^1, M^2 positive $\mathbb{F}$-martingales s.t. $\langle M^1, M^2 \rangle_t > 0$.

Exponential diffusion model

– **A two-dimensional exponential example** :

$$\exp\Big(-\theta_1 M_t^1 - \theta_2 M_t^2 - \frac{1}{2}\theta_1^2 \langle M^1 \rangle_t - \frac{1}{2}\theta_2^2 \langle M^2 \rangle_t - \theta_1\theta_2 \Big(\langle \mathbf{M^1}, \mathbf{M^2} \rangle_{\mathbf{t}} + \mathbf{a} \Big) \Big)$$

M^1, M^2 positive $\mathbb{F}$ martingales s.t. $\langle M^1, M^2 \rangle_t \geq 0$

– At $t = 0$, $S_0(\theta_1, \theta_2) = \exp(-\theta_1 M_0^1 - \theta_2 M_0^2 - \mathbf{a}\,\theta_1\theta_2)$.

– Dependence at $t > 0$ characterized by $\langle M^1, M^2 \rangle_t$

– Probability condition

$$M_t^1 M_t^2 - \langle M^1, M^2 \rangle_t > a > 0$$

not yet simple examples for positive martingales

Generalizations

– Generalization to n successive defaults $\sigma_1 \leq \cdots \leq \sigma_n$ by a recursive method

– Representation of conditional expectation with respect to $\mathcal{G}_t^{(1,\cdots,n)} = \mathcal{F}_t \vee \mathcal{D}_t^{\sigma_1} \vee \cdots \vee \mathcal{D}_t^{\sigma_n}$

$\Rightarrow$ Let $Y_t(u_1, \cdots, u_n)$ be a family of r.v. $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^n)$-measurable where $t, u_1, \cdots, u_n \geq 0$. Then

$$\mathbb{E}\left[Y_T(\sigma_1, \cdots, \sigma_n)|\mathcal{G}_t^{(1,\cdots,n)}\right] = \sum_{i=0}^{n} \mathbb{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}}\, q_t^i(T, \sigma_1, \cdots, \sigma_i, Y_T)$$

where $q_t^i(T, s_1, \cdots, s_i, Y_T)$ is a quotient of $\mathcal{F}_t$ conditional expectations, $\sigma_0 = 0$ and $\sigma_{n+1} = \infty$.

Pricing of the portfolio credit derivatives

– **k^{th}-to-default swap** depends on the k^{th} default time of the underlying portfolio :

$$\mathbb{E}\left[1\!\!1_{\{\sigma_k>T\}}Y_T|\mathcal{G}_t^{(1,\cdots,n)}\right]=\sum_{i=0}^{k-1}1\!\!1_{\{\sigma_i\leq t<\sigma_{i+1}\}}q^i_{t,\mathsf{Q}}(T,\sigma_1,\cdots,\sigma_i,Y_TS_T^{(k)})$$

where $S_T^{(k)}=\mathbb{P}(\sigma_k>T|\mathcal{F}_t)$.

– For **a CDO tranche**, total loss $l_T=\sum_{i=1}^n 1\!\!1_{\{\tau_i\leq t\}}$ and key term to calculate :

$$\mathbb{E}\left[(K-l_T)^+|\mathcal{G}_t^{(1,\cdots,n)}\right]=\int_{-\infty}^K du\,\mathbb{E}\left[1\!\!1_{\{\sigma_{\lfloor u\rfloor+1}>T\}}|\mathcal{G}_t^{(1,\cdots,n)}\right]$$

$$=\int_{-\infty}^K du\sum_{i=0}^{\lfloor u\rfloor}1\!\!1_{\{\sigma_i\leq t<\sigma_{i+1}\}}q^i_{t,\mathsf{Q}}\Big(T,\sigma_1,\cdots,\sigma_i,S_T^{\lfloor u\rfloor}\Big).$$

Perspectives

A general framework for portfolio of defaultable names :

– explicit model studies for the joint density process

– application to the pricing

– calibration of parameters

– dynamic hedging