Default contagion in large homogeneous portfolios

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Content

- Introduce default contagion in an intensity based credit risk model.
- Present expressions for multivariate default and survival distributions, both for ordered and unordered default times.
- Present formulas for CDO tranche spreads in our model.
- Show some numerical results implied by market data from the calibrated model.

Research for portfolio credit models

- Exists huge amount of research in portfolio credit risk
- Our contribution:
  - Intensity based model where dynamic default dependencies among obligors are expressed in an intuitive, direct and compact way, by using default contagion.
  - Enables fast computationally tractable closed-form expressions for multivariate default and survival distributions, credit derivatives, and much more.....
- Default contagion models clustering of defaults. It has been treated in many recent articles.
- Our approach close to papers by for example Davis et al, Frey et al., Jeanblanc et al., Laurent et al.

Default contagion in intensity models

- The general case (i.e. inhomogeneous portfolio): for default times $\tau_1, \tau_2 \ldots, \tau_m$, define $N_{t,i} = 1_{\{\tau_i \leq t\}}$ and $\mathcal{F}_{t,i} = \sigma(N_{s,i}; s \leq t)$, $\mathcal{F}_t = \bigvee_{i=1}^{m} \mathcal{F}_{t,i}$.

- Let $\lambda_{t,i}$ be the $\mathcal{F}_t$-intensity of the point processes $N_{t,i}$ given by

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j \leq t\}}, \qquad \tau_i \geq t,$$

  where $a_i \geq 0$ and $b_{i,j}$ such that $\lambda_{t,i} \geq 0$. Note that $\lambda_{t,i} = 0$, if $\tau_i \leq t$.

- Intensity for obligor i jumps by an amount $b_{i,j}$ at default of obligor j
  - $b_{i,j} > 0$ means that i is put at higher risk by the default of j.
  - $b_{i,j} < 0$ means that i benefits from the default of j.
  - $b_{i,j} = 0$ means that i is unaffected by the default of j.

Intensity based models

- The intensities $\{\lambda_{t,i}\}$ uniquely determines the multivariate distribution for $\tau_1, \tau_2 \dots, \tau_m$.
- Not obvious how to go from $\{\lambda_{t,i}\}$ to distribution of $\{\tau_i\}$.
- Solution: Reformulate as a Markov jump process.
- In a nonsymmetric portfolio, only practical for m up to 20, say.
- If portfolio homogeneous, the Markov approach works for large portfolios (CDO's). Then $m = 125$ is no problem.
- Symmetry implies that $\lambda_{t,i} = \lambda_t$, for $\tau_i > t$, where

$$\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \le t\}} \tag{1}$$

where $\{T_k\}$ ordering of $\{\tau_i\}$. Recall that $\lambda_{t,i} = 0$, for $\tau_i \le t$.

Translate into Markov jump process

Proposition 1

There exists a Markov jump process $(Y_t)_{t\geq 0}$ on a finite state space $\mathbf{E} = \{0, 1, 2, \ldots, m\}$, such that the stopping times

$$T_k = \inf\{t > 0 : Y_t = k\}, \quad k = 1, \ldots, m$$

are the ordering of m exchangeable stopping times $\tau_1, \ldots, \tau_m$ with intensities $\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}}$. The generator $\mathbf{Q}$ to Y_t is given by

$$\mathbf{Q}_{k,k+1} = (m-k)\left(a + \sum_{j=1}^{k} b_j\right) \quad \text{and} \quad \mathbf{Q}_{k,k} = -\mathbf{Q}_{k,k+1} \quad \text{for} \quad k = 0, 1, \ldots, m-1$$

where the other entries in $\mathbf{Q}$ are zero. The Markov process starts in $\{0\}$ so the initial distribution is given by $\boldsymbol{\alpha} = (1, 0, 0, \ldots, 0)$.

Due to **Proposition 1**, we can use matrix-analytic methods to find compact, computationally tractable closed-form expressions for many quantities.

Example: multivariate distributions

Consider m obligors with default intensities (1).

Proposition 2

Let $k_1 < \ldots < k_q$ be an increasing subsequence in $\{1, \ldots, m\}$ where $1 \leq q \leq m$. Furthermore, let $t_1 < t_2 < \ldots < t_q$. Then,

$$\mathbb{P}\left[T_{k_1} \leq t_1, \ldots, T_{k_q} \leq t_q\right] = \boldsymbol{\alpha} \left(\prod_{i=1}^{q} e^{\mathbf{Q}(t_i - t_{i-1})} \mathbf{N}_{k_i} \right) \mathbf{1}$$

where $\mathbf{N}_k$ is $(m+1) \times (m+1)$ diagonal matrix, $(\mathbf{N}_k)_{j,j} = 1_{\{j \geq k\}}$

Proposition 3

Let $t_1 < t_2$. Then,

$$\mathbb{P}\left[\tau_1 \leq t_1, \tau_2 \leq t_2\right] = \frac{(m-2)!}{m!} \left(\boldsymbol{\alpha} e^{\mathbf{Q}t_1} \mathbf{n} + \sum_{k_1=1}^{m} \sum_{k_2=k_1+1}^{m} \boldsymbol{\alpha} e^{\mathbf{Q}t_1} \mathbf{N}_{k_1} e^{\mathbf{Q}(t_2 - t_1)} \mathbf{N}_{k_2} \mathbf{1} \right)$$

where $\mathbf{n}$ is column vectors in $\mathbb{R}^{m+1}$ such that $\mathbf{n}_j = \frac{j(j-1)}{2}$.

Other examples

Proposition 4

Let q be a integer where $1 \leq q \leq m$ and let $t > 0$. Then,

$$\mathbb{P}\left[\tau_1 \leq t, \ldots, \tau_q \leq t\right] = \alpha e^{\mathbf{Q}t}\mathbf{d}^{(q)} \quad \text{where} \quad \mathbf{d}_j^{(q)} = \frac{\binom{j}{q}}{\binom{m}{q}} 1_{\{j \geq q\}}.$$

By using similar techniques as in the previous propositions, we can also find

- $\mathbb{P}\left[T_{k_1} > t_1, \ldots, T_{k_q} > t_q\right]$,
- $\mathbb{P}\left[\tau_1 > t_1, \tau_2 > t_2\right]$
- $\mathbb{P}\left[\tau_1 > t, \ldots, \tau_q > t\right]$
- $\mathbb{E}\left[T_k\right]$ and $\mathrm{Corr}(1_{\{\tau_1 \leq t\}}, 1_{\{\tau_2 \leq t\}})$ and $\mathbb{E}\left[\tau_1\right]$
- Formulas for credit derivatives, such as CDS and CDO tranche spreads.

The CDO tranche spreads

- Consider a synthetic CDO consisting of m credit default swaps on obligors with default times $\tau_1, \tau_2 \ldots, \tau_m$ and losses $\ell_1, \ell_2, \ldots, \ell_m$.

- The credit loss L_t for this portfolio at time t is $L_t = \sum_{i=1}^{m} \ell_i 1_{\{\tau_i \leq t\}}$.

- The $[a, b]$-tranche loss is $L_t^{(a,b)} = (L_t - a)\, 1_{\{L_t \in [a,b]\}} + (b - a)\, 1_{\{L_t > b\}}$.

- The CDO tranche spread $S_{(a,b)}(T)$ for tranche $[a, b]$ up to time T is

$$S_{(a,b)}(T) = \frac{B_T \mathbb{E}\left[L_T^{(a,b)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(a,b)}\right] dt}{\sum_{n=1}^{4T} B_{t_n} \left(b - a - \mathbb{E}\left[L_{t_n}^{(a,b)}\right]\right) \frac{1}{4}} \qquad \text{if } a > 0$$

where r_t deterministic, $B_t = \exp\left(-\int_0^t r_s ds\right)$ and $t_n = \frac{n}{4}$.

The CDO tranche spreads in the model

Proposition 5

Consider a synthetic CDO on a portfolio with m obligors that satisfy (1) and assume that the interest rate r is constant. Then,

$$S_{(a,b)}(T) = \frac{\left(\boldsymbol{\alpha} e^{\mathbf{Q}T} e^{-rT} + \boldsymbol{\alpha}\mathbf{R}(0,T) r\right) \boldsymbol{\ell}^{(a,b)}}{\sum_{n=1}^{4T} e^{-rt_n}\left(b - a - \boldsymbol{\alpha} e^{\mathbf{Q}t_n} \boldsymbol{\ell}^{(a,b)}\right)\frac{1}{4}} \qquad \text{if} \quad a > 0$$

for $t_n = \frac{n}{4}$, where

$$\mathbf{R}(0,T) = \int_0^T e^{(\mathbf{Q}-r\mathbf{I})t} dt = \left(e^{\mathbf{Q}T} e^{-rT} - \mathbf{I}\right)\left(\mathbf{Q} - r\mathbf{I}\right)^{-1}$$

and $\boldsymbol{\ell}^{(a,b)}$ is a column vector in $\mathbb{R}^{m+1}$, defined by

$$\ell_k^{(a,b)} = \begin{cases} 0 & \text{if} \quad k < \lceil am/\ell \rceil \\ k\ell/m - a & \text{if} \quad \lceil am/\ell \rceil \leq k \leq \lfloor bm/\ell \rfloor \\ b - a & \text{if} \quad k > \lfloor bm/\ell \rfloor \end{cases}$$

Calibrating the model

- Let $\mathbf{a} = (a, b_1, b_2, \ldots, b_{m-1})$ denote the m parameters describing λ_t.
- Let $\{C_j(T; \mathbf{a})\}$ be the model spreads for the instruments used in the calibration (=CDO tranche spreads, index CDS spread and average CDS spread) and $\{C_{j,M}(T)\}$ are the corresponding market spreads.
- We have 7 instruments: 5 tranches, the index and the average CDS.
- The vector $\mathbf{a}$ is obtained as $\mathbf{a} = \underset{\hat{\mathbf{a}}}{\operatorname{argmin}} \sum_{j=1}^{7} \left(C_j(T; \hat{\mathbf{a}}) - C_{j,M}(T) \right)^2$.
- To reduce the number of unknown parameters in $\mathbf{a}$ we assume that

$$b_k = \begin{cases} b^{(1)} & \text{if} \quad 1 \leq k < \mu_1 \\ b^{(2)} & \text{if} \quad \mu_1 \leq k < \mu_2 \\ \vdots & \\ b^{(6)} & \text{if} \quad \mu_5 \leq k < \mu_6 = m \end{cases}$$

where $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_4, \mu_6\} = \{7, 13, 19, 25, 46, 125\}$.

Good calibration for $T = 5$ (before subprime crises)

iTraxx Europe Series. Left: August 4^{th}, 2004. Right: November 28^{th}, 2006. The market and model spreads and the corresponding absolute errors (in bp). The $[0,3]$ spread is quoted in %. Maturities are for five years, $r = 3\%, \ell = 60\%$.

	Market	Model	error (bp)	Market	Model	error (bp)
$[0,3]$	27.6	27.6	0.0000385	14.5	14.5	0.008273
$[3,6]$	168	168	0.000316	62.5	62.48	0.02224
$[6,9]$	70	70	0.000498	18	18.07	0.07275
$[9,12]$	43	43	0.0005563	7	6.872	0.1282
$[12,22]$	20	20	0.0004006	3	3.417	0.4169
index	42	42.02	0.01853	26	26.15	0.1464
avg CDS	42	41.98	0.01884	26.87	26.13	0.7396
Σ abs.cal.err			0.03918 bp			1.534 bp

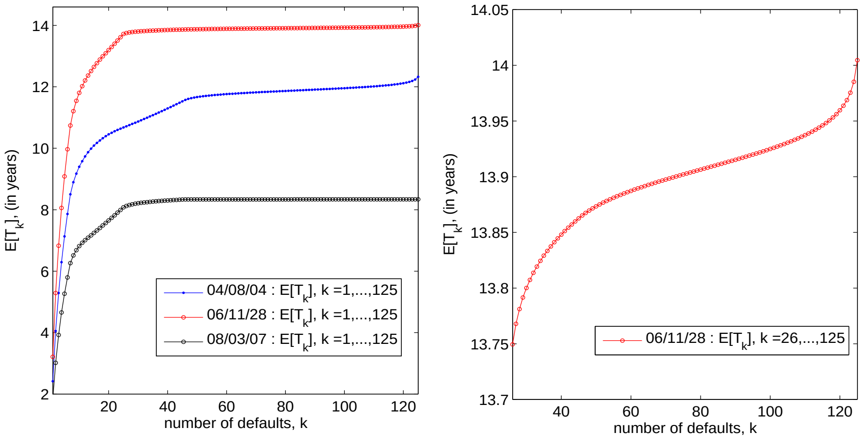
Good calibration for $T = 5$ (during subprime crises)

Table: iTraxx Europe, **March 7^{th} 2008**. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The $[0, 3]$ spread is quoted in %. Maturities are for five years, $r = 3\%, \ell = 60\%$.

	Market	Model	error (bp)	error (%)
$[0, 3]$	46.5	46.5	0.0505	0.001086
$[3, 6]$	567.5	568	0.4742	0.08356
$[6, 9]$	370	370	0.04852	0.01311
$[9, 12]$	235	234	1.035	0.4404
$[12, 22]$	145	149.9	4.911	3.387
index	150.3	144.3	5.977	3.978
avg CDS	145.1	143.8	1.296	0.8933
Σ error			13.79 bp	

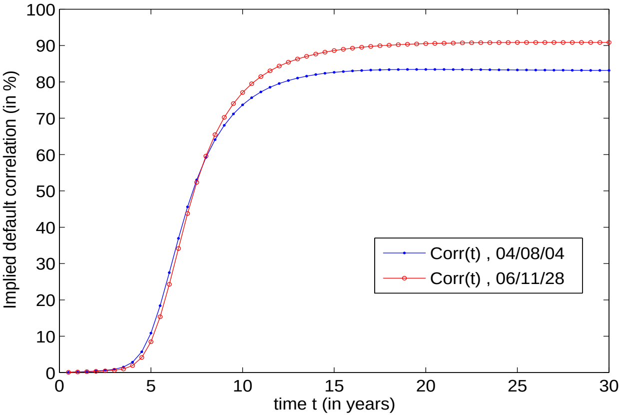
Market imply extreme clustering

The implied expected ordered default times $\mathbb{E}[T_k]$ for the 2004-08-04 and 2006-11-28 and **2008-03-07** portfolios (under risk-neutral measure).



The implied default correlation

The implied default correlation $\rho(t) = \text{Corr}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}})$, $i \neq j$ as function of time for the 2004-08-04 and 2006-11-28 portfolios.



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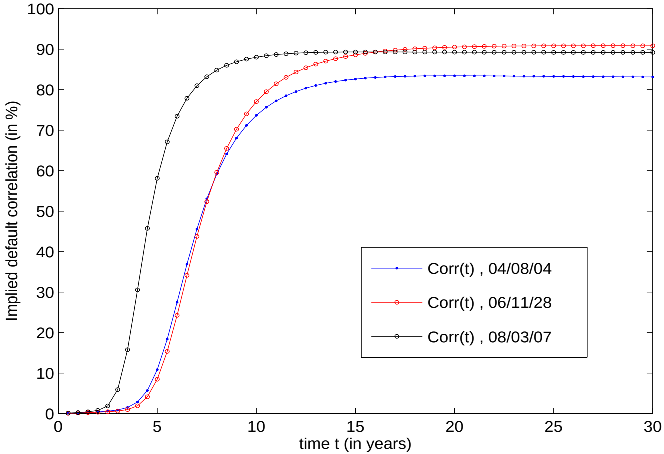
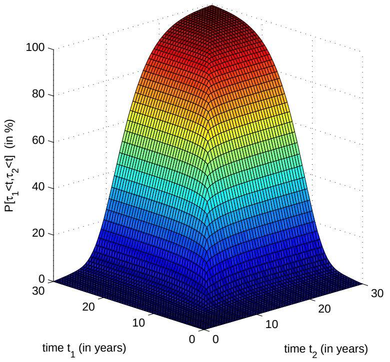
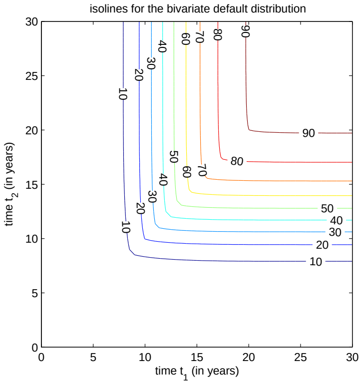


The implied bivariate default distribution

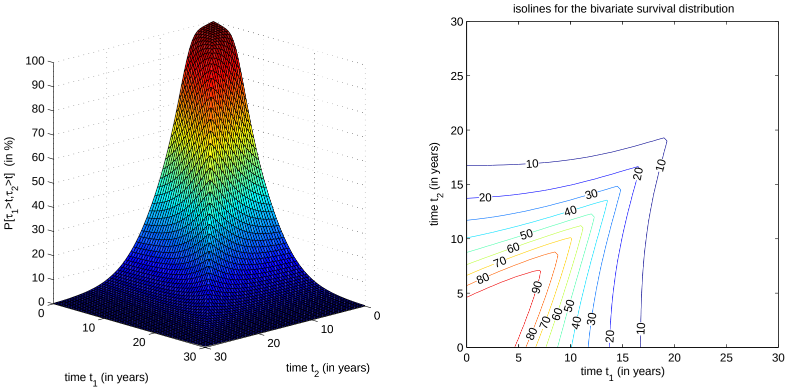
The implied bivariate default distribution (left) and its isolines (right), 06-11-28.





The implied bivariate survival distribution

The implied bivariate survival distribution (left) and its isolines (right), 06-11-28.



Thank you for your attention!