Sato Processes in Default Modelling

Thomas Kokholm*

Finance Research Group, Department of Business Studies, Aarhus School of Business, University of Aarhus, e-mail: thko@asb.dk

Elisa Nicolato*

Finance Research Group, Department of Business Studies, Aarhus School of Business, University of Aarhus, e-mail: eln@asb.dk

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Abstract

Classically, in reduced form default models the instantaneous default intensity λ is the modelling object and survival probabilities are given by the Laplace transform of $A_t = \int_0^t \lambda_s ds$. Instead, recent literature has shown a tendency towards specifying the process A directly. We will refer to A as the *cumulative hazard process*.

We present a new cumulative hazard based framework where survival probabilities are still obtained in closed form but where A belongs to the class of self-similar additive processes also termed Sato processes.

We analyze two specifications for the cumulative hazard process; Sato-Gamma and Sato-IG processes where the unit time distribution A_1 is described by a Gamma law and Inverse Gaussian law respectively. The models are calibrated to data on the single names included in the iTraxx Europe index and compared with two Ornstein-Uhlenbeck type intensity models. It is shown how the Sato models achieve similar calibration errors with fever parameters, and with more stable parameter estimates in time.

Key words: CDS, Credit Default Swap, Lévy process, reduced form model, cumulative hazard, Gamma process, Inverse Gaussian process, self-similar additive process, Sato process.

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1 Introduction

The market for credit derivatives has exploded during the last 10-15 years and along with this the academic research concerning the modelling of corporate default risk. The estimated total value of the credit default swap market in the US has risen from $900 billion in 2000 to the breathtaking number of more than $45.5 trillion today - roughly twice the size of the entire US stock market. In contrast to this growth in literature recent credit crises have shown that still much remain unsolved.

In this paper a new default modelling approach will be presented. The model belongs to the reduced form class first introduced in Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), Lando (1994, 1998), Duffie and Singleton (1997, 1999) and Madan and Unal (1998) among others. In this framework the time of default of a firm is modelled as the first jump time of an underlying counting process, classically described by a Cox process, with stochastic intensity rate λ. The survival probabilities up to time t are then given as

$$\mathbb{Q}(\tau > t) = \mathbb{E}\left[e^{-A_t}\right] \tag{1}$$

with the process A defined as the integrated instantaneous intensity

$$A_t = \int_0^t \lambda_s ds \tag{2}$$

and with intensity rate λ typically described by a positive and stationary affine process. This intensity based modelling approach has been quite successful in the pricing of univariate instruments such as credit default swaps. However problems are still present when considering parameter stability over time, leading to large variations in sensitivities and hedge parameters. This is evidently problematic from risk management perspectives.

Recent literature in the context of modelling default correlation has shown a tendency towards specifying directly the process A in (1), which we will refer to as the *cumulative hazard process*, as opposite to the instantaneous intensity rate λ. From the theoretical point of view, this modelling approach has been analyzed e.g. in Elliot and alt. (2000) and Jeanblanc and Le Cam (2007) while cumulative hazard processes displaying jumps have been adopted for concrete applications by Joshi and Stacey (2006), Di Graziano and Rogers (2006) and Hull and White (2007).

Within the cumulative hazard based framework, we present a new model where survival probabilities are still obtained as in (1) but where the cumulative hazard process A belongs to the class of self-similar additive processes also termed Sato processes as they were introduced and thoroughly studied in Sato (1991,1999). A Sato process is defined as an inhomogeneous Lévy process with the added feature of self-similarity, i.e. for a given exponent of self-similarity $\gamma > 0$ and any $\alpha > 0$ it satisfies

$$\{A_{\alpha t} : t > 0\} \stackrel{d}{=} \{\alpha^\gamma A_t : t > 0\} \tag{3}$$

implying that any change in the time-scale has the same effect of a change in the spatial scale. Sato processes have been successfully employed both in the context of equity option pricing and interest rate modelling (see Carr, Geman, Madan and Yor (2007) and Skovmand (2008)). This family of processes is indeed extremely convenient for a number of reasons. It displays enormous flexibility in terms of distribution-modelling as for any possible self-decomposable law X there exists an additive process satisfying (3) and having unit time law given by X. Sato processes are also analytically tractable as they allow for closed form expressions for default probabilities hereby enabling straightforward calibration to credit default swap prices. Finally they allow for more flexibility and non-linearity in the long term behavior of the cumulative hazard as opposite to the classical framework described in (2) where, if the intensity λ is ergodic, it holds that

$$t^{-1} \int_0^t \lambda_s ds \to \overline{\lambda} \quad \text{as} \quad t \to \infty$$

with $\overline{\lambda}$ denoting the long-run average of λ.

In this work we analyze two concrete specifications of Sato processes for the cumulative hazard A, namely the Sato-Gamma and the Sato-IG processes where the unit time distribution A_1 is described by a Gamma law and Inverse Gaussian law respectively. The models are calibrated to weekly observations on the single names included in the iTraxx Europe Series 8 index in the period from September 17, 2007 to March 14, 2008 (total of 26 weeks). For each observation the models are calibrated to spreads for maturities 1,3,5,7 and 10 years. Their performances are then compared with those of a number of benchmark processes listed below.

(1) Lévy-Gamma: Here A is given by a Gamma process, i.e. a subordinator with Gamma law at time 1.
(2) Lévy-IG: Here A is given by an Inverse Gaussian process, i.e. a subordinator with Inverse Gaussian law at time 1.
(3) Gamma-OU: Here the cumulative hazard process is given as in (2) with instantaneous intensity λ described by a Gamma-Ornstein Uhlenbeck type process.
(4) IG-OU: Here the cumulative hazard process is given as in (2) but with λ described by an Inverse Gaussian-Ornstein Uhlenbeck type process.

The choice of the perhaps unusual Integrated-OU processes as benchmark models has been dictated by the work of Cariboni and Schoutens (2006) who demonstrated their superior performance compared to the classical CIR specification for the intensity λ.

The Lévy-Gamma and the Lévy-IG specifications perform very poorly. This it not surprising since only constant spreads across maturities can be achieved using a time homogeneous cumulative hazard process. On the other hand, the calibration errors produced by the two Sato processes are comparable with those of the Ornstein Uhlenbeck models. However, this in itself favors the two Sato models since they are more parsimonious in the number of parameters. Moreover, the calibrated parameters in the Sato models display a more stable behavior in time.

The paper is structured as follows: Section 2 describes the general cumulative hazard modelling framework. Section 3 gives an introduction to Sato processes and two specifications for the cumulative hazard are described. In Section 4 the benchmark models are introduced. In Section 5 the models are calibrated on iTraxx Europe market data and last, Section 6 concludes.

2 The General Cumulative Hazard Modelling Framework

In the classical intensity based reduced-form modelling approach introduced by Lando (1994, 1998), the default time τ of a company is modelled as the first jump-time of a Cox process $\widetilde{N}$ with intensity rate λ

$$\tau = \inf\left\{t > 0 \mid \widetilde{N}_t > 0\right\}, \tag{4}$$

or equivalently as

$$\tau = \inf\left\{t > 0 \mid \int_0^t \lambda_s ds \geq E_1\right\}, \tag{5}$$

where E_1 is an exponential random variable with mean 1 independent of the process λ. The survival probabilities are then given by

$$\mathbb{Q}\left(\tau > t\right) = \mathbb{E}[e^{-\int_0^t \lambda_s ds}].$$

The intensity rate λ represents the instantaneous probability of default. Intuitively, this can be realized by noticing that, the probability of defaulting during the time interval $(t, t+\Delta t]$ given that default has not occurred at time t, can be approximated as

$$\mathbb{Q}\left(\tau \leq t + \Delta t \mid \tau > t\right) \approx \lambda_t \Delta t,$$

when Δt is small.

More recently, a number of authors have proposed reduced form models where the default time is described, similarly to the classical framework in

(5), as the first time an increasing process A reaches or is above a level of an independent exponential random variable, with the exception that A is no longer constrained to be absolutely continuous w.r.t. the Lebesgue measure. In what follows we describe and examine in some details this this new modelling approach, also known as Cox construction of default time. For a rigorous theoretical analysis of more general reduced form models we refer the reader to Jeanblac and Le Cam (2007).

We start by considering a filtered probability space $\left(\Omega, \mathcal{G}, (\mathcal{F})_{t\geq 0}, \mathbb{Q}\right)$ which is large enough to support a cadlag, adapted process A and a mean one exponential random variable E_1 which is independent of $\mathcal{F}_\infty$. Furthermore we assume that the process A, which we will refer to as the *cumulative hazard process*, has (strictly) increasing paths and is such that $A_0 = 0$. The default time τ is then defined as follows

$$\tau = \inf \{t > 0 \mid A_t \geq E_1\}, \tag{6}$$

and the conditional survival probabilities are then given by

$$\mathbb{Q}(\tau > t \mid \mathcal{F}_t) = e^{-A_t}.$$

Notice that the default time τ may still be related to the first jump-time of a process $\widetilde{N}$. In fact, if the probability space is large enough to support a standard unit rate Poisson process N independent of A and defining $\widetilde{N}$ as

$$\widetilde{N}_t = N_{A_t} \tag{7}$$

i.e. by subordination of N to A, we see that

$$\tau = \inf\left\{t > 0 : \widetilde{N}_t > 0\right\}. \tag{8}$$

The process $\widetilde{N}$ jumps only when the Poisson process N jumps, and that it displays jumps of size one (or zero) when the cumulative hazard A has continuous trajectories. However, under the rather general assumptions on A, we can only state that $\widetilde{N}$ takes values in the non negative integer numbers, but is not necessarily a counting process.

The directing filtration $(\mathcal{F})_{t\geq 0}$ can be seen as carrying the default-free information while the knowledge about whether default has occurred or not is contained in the enlarged filtration

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$$

where $\mathcal{H}_t = \sigma\{\mathbb{1}_{\{\tau \leq s\}} : 0 \leq s \leq t\}$ is the filtration generated by the default process ($H_t = \mathbb{1}_{\{\tau\leq t\}}$). If the compensator Λ^G (w.r.t the filtration $(\mathcal{G})_{t\geq 0}$) of the process H is absolutely continuous w.r.t the Lebesgue measure

$$\Lambda_t^G = \int_0^t \lambda_s^G ds,$$

the time of default is said to have *intensity rate* λ^G and it holds that

$$\lambda_t^G = \lim_{h\to 0} \frac{1}{h} \mathbb{Q}(t < \tau \leq t+h |\ \mathcal{G}_t)$$

(see e.g. Jeanblanc and Le Cam (2007)). The compensator Λ^G can be related to the cumulative hazard process A as follows

$$\Lambda_t^G = \int_0^{t\wedge\tau} \frac{dC_s}{e^{-A_{s-}}}$$

where the process C is the $(\mathcal{F})_{t\geq 0}$-compensator of the process $1 - e^{-A}$. Hence, if C is absolutely continuous w.r.t. the Lebesgue measure, $C_t = \int_0^t c_s ds$, the time of default has intensity

$$\lambda_t^G = \mathbb{1}_{\{t<\tau\}} \lambda_t^F \quad \text{with} \quad \lambda_t^F = \frac{c_t}{e^{-A_{t-}}} \tag{9}$$

and with a slight abuse of terminology, the $(\mathcal{F})_{t\geq 0}$-adapted process λ^F is also referred to as the intensity process. Notice that if the cumulative hazard A is specified as in the classical intensity based approach described in (5), i.e. $A_t = \int_0^t \lambda_s ds$, then $\lambda^F = \lambda$.

For the actual computations, it is more convenient to express the intensity process λ^F in terms of the compensator of the cumulative hazard process A itself. Few simple calculations, which can be found in Appendix A, lead to the following result.

Proposition 1 *Assume that the cumulative hazard process A is integrable and it has $(\mathcal{F})_{t\geq 0}$-compensator $Y_t = \int_0^t y_s ds$ which is absolutely continuous w.r.t the Lebesgue measure. Assume furthermore that the random measure $\mu(\omega, dx, dt)$ associated with the jumps of A has compensator*

$$\nu(\omega, dx, dt) = K(\omega, dx, t)dt$$

which is also absolutely continuous. Then the time of default defined in (6) *admits intensity λ^F which is given by*

$$\lambda_t^F = y_t - \int_0^{+\infty} (e^{-x} - 1 + x) K(\omega, dx, t) \tag{10}$$

When the cumulative hazard A is modelled as an integrable additive process, i.e. a process continuous in probability with independent but not necessarily homogeneous increments, then the random measure ν as well as the compensator Y are deterministic and absolutely continuous, with $y_t = \frac{d}{dt}\mathbb{E}[A_t]$. Moreover, by the Lévy-Khintchine formula we have that

$$\mathbb{E}[e^{-A_t}] = \exp\left(-\int_0^t y_s ds + \int_0^t \int_0^{+\infty} (e^{-x} - 1 + x) K(dx, s) ds \right) \tag{11}$$

from which it follows immediately that the (deterministic) intensity λ^F is given by

$$\lambda_t^F = -\frac{d}{dt} \log \mathbb{E}[e^{-A_t}].$$

We conclude this section by recalling the so-called *hazard based pricing rule* (see Elliot et al. (2000)): given a claim $X \in \mathcal{F}_T$, it holds that

$$\mathbb{E}[X \mathbb{1}_{\{T<\tau\}} |\ \mathcal{G}_t] = \mathbb{1}_{\{\tau>t\}} e^{A_t} \mathbb{E}[e^{-A_T} X |\ \mathcal{F}_t].$$

In particular, assuming that the instantaneous interest rate process r is adapted to the default-free filtration $(\mathcal{F})_{t\geq 0}$, we obtain that the price $B(0,T)$ of a corporate bond is given by

$$B(0,T) = \mathbb{E}\left[e^{-\left(A_T + \int_0^T r_s ds \right)} \right].$$

By assuming independence between A and the instantaneous interest rate process r, the expression further reduces to

$$\begin{aligned} B(0,T) &= p(0,T)\, \mathbb{E}\left[e^{-A_T} \right] \\ &= p(0,T)\, \mathbb{Q}(\tau > T), \end{aligned}$$

where $p(0,T)$ is the price of a risk-free zero coupon bond with maturity T.

3 Cumulative Hazard as Sato processes

In this section we introduce the main modelling idea of the paper, namely the choice of specifying the cumulative hazard as a self-similar additive process.

We start by recalling that a stochastic process $\{A_t : t \geq 0\}$ is called self-similar if for any $\alpha > 0$ there is a $\beta > 0$ such that

$$\{A_{\alpha t} : t > 0\} \stackrel{d}{=} \{\beta A_t : t > 0\}, \tag{12}$$

implying that any change of time scale has the same effect of a change in the spatial scale. From the definition (12) it follows that there exists a $\gamma > 0$, the *exponent of self-similarity*, such that $\beta = \alpha^\gamma$ and we say that A is γ-self-similar. A well known class of self-similar processes is given by the strictly stable processes, i.e. processes with independent and homogeneous increments with law at unit time described by a strictly stable distribution. Sato (1991) introduced the broader class of self-similar additive processes, here termed *Sato processes*, as processes satisfying (12) and with independent but not necessarily homogeneous increments. More precisely, Sato (1991) showed that for any given self-decomposable law X and an exponent $\gamma > 0$ there exists a γ-self-similar additive process A with law at unit time A_1 given by X. (For more information on Sato processes see Sato (1991, 1999)).

By choosing for the unit time a self-decomposable law X on the positive real line, we obtain that the corresponding γ-self-similar additive process A has increasing paths and therefore suitable to describe a cumulative hazard. As observed in Section 2, if X has finite first moment then the cumulative hazard has compensator

$$Y_t = \mathbb{E}[A_t] = \mathbb{E}[X]t^\gamma$$

allowing for a flexible behavior of the growth rate accordingly to the choice of exponent of self-similarity γ.

Denoting by

$$\kappa_X(u) = \log \mathbb{E}[e^{uX}]$$

the cumulant generating function of the unit time law, the unconditional survival probabilities are then given by the following simple expression

$$\mathbb{Q}\left(\tau > t\right) = \mathbb{E}\left[e^{-A_t}\right] = \mathbb{E}\left[e^{-t^\gamma X}\right] = e^{\kappa_X(-t^\gamma)}, \tag{13}$$

i.e. they are determined by the Laplace transform of the unit time law X computed at value t^γ. Therefore, if the cumulant generating function κ_X is known explicitly, the survival probabilities can be computed in closed-form enabling straightforward calibration of the model to credit default swaps. Notice that, by Proposition 1, the model allows for the existence of the intensity rate λ^F which is deterministic and given by the following expression

$$\lambda_t^F = \gamma t^{\gamma-1} \frac{d}{dt} \kappa_X(-t^\gamma).$$

In this paper we calibrate two concrete specifications of Sato processes for the cumulative hazard A to market data:

(1) *The Sato-Gamma process*, where the unit time distribution X is described by a Gamma law $\Gamma\left(a, b\right)$ with density function

$$f_X(x) = \frac{b^a}{\Gamma\left(a\right)} x^{a-1} e^{-bx} \quad x > 0 \ , \tag{14}$$

$\mathbb{E}[X] = \frac{a}{b}, Var(X) = \frac{a}{b^2}$ and cumulant generating function

$$\kappa_X(u) = \log\left(\frac{1}{\left(1 - u\, b^{-1}\right)^a}\right) . \tag{15}$$

(2) *The Sato-IG process* where the unit time distribution X is described by Inverse Gaussian law $IG(a, b)$ with density function

$$f_X(x) = \frac{a}{\sqrt{2\pi}}\, e^{a\,b} x^{-\frac{3}{2}}\, e^{-(b^2 x + a^2/x)/2} \qquad x > 0 , \tag{16}$$

$\mathbb{E}[X] = \frac{a}{b}, Var(X) = \frac{a}{b^3}$ and cumulant generating function

$$\kappa_X(u) = ab - a\sqrt{b^2 - 2u} . \tag{17}$$

Both the $\Gamma(a, b)$ and the $IG(a, b)$ distributions are self-decomposable (see e.g. Sato (1999)) and therefore the corresponding Sato process specifications are meaningful. Both these distributions are well known in the financial literature and have been widely used both in the context of econometrics and derivative pricing by Madan and Seneta (1990), Rydberg (1999), Barndorff-Nielsen and Shephard (2001), Nicolato and Venardos (2003) among others. Our choice to examine these distributions as specifications for self-similar additive cumulative hazards has mainly been motivated by the work of Cariboni and Schoutens (2006) who, as we shall see below, have introduced the Gamma and IG laws in the context of credit derivatives as building blocks for intensity rates of default times.

4 The Benchmark Models

In this section we introduce and briefly illustrate the models which will be used as benchmark in order to assess the performances of the model based on Sato processes illustrated above.

4.1 The Ornstein-Uhlenbeck type intensity rates

Cariboni and Schoutens (2006) have recently proposed a new default model where the intensity rate is governed by a positive process of the Ornstein-Uhlenbeck type (OU type process henceforth). Recall that a positive OU process $\{\lambda_t : t > 0\}$ is defined as the solution of the stochastic differential equation

$$d\lambda_t = -\theta \lambda_t dt + dZ_{\theta t} \qquad \lambda_0 > 0, \tag{18}$$

where θ is a positive real number and Z is a subordinator, i.e. a purely jumping process with independent, positive and stationary increments which is often termed the *Background Driving Lévy Process* (BDLP henceforth). The process λ is mean-reverting, it increases only by jumps and between jumps it decays exponentially. This might be considered a natural property in the credit market if we think of the jumps as the arrival of new bad information, which drives the intensity process up, while during periods with no arrival of bad information the intensity decays. Under (mild) integrability conditions on the BDLP Z, the OU process λ described in (18) is also stationary with invariant distribution X which is self-decomposable and, due to the unusual timing $Z_{\theta t}$, does not depend on the mean reversion parameter θ. In fact the converse statement is also true in the sense that for any given self-decomposable distribution X, there exists a Lévy process Z such that for any $\theta > 0$ the corresponding OU type process driven by Z as in (18) has invariant law given by X. Moreover the cumulant function $\kappa_X(u)$ of X and the cumulant function $\kappa_Z(u)$ of Z_1 are related by the formula

$$\kappa_Z(u) = u\,\frac{d\kappa_X}{du}(u). \tag{19}$$

For further details and a thorough analysis of OU processes see Sato (1999).

Cariboni and Schoutens (2006) examine the two concrete stationary OU processes, denoted by IG(a,b)-OU and Gamma(a,b)-OU, characterized by having invariant (self-decomposable) law given by the $\Gamma\,(a,b)$ and the $IG\,(a,b)$ distributions respectively.

From expressions (15) and (19) it follows that the Gamma(a,b)-OU specification is obtained by choosing the BDLP Z as a compound Poisson process with Lévy density

$$w(x) = a\,b\exp(-bx)\ , \tag{20}$$

i.e. Z is a process with jumps arriving at the same jump times of a Poisson process with intensity a and having jump-size distributed according to an exponential law with parameter b.

Analogous derivations imply that the IG(a,b)-OU process is obtained by specifying the Lévy density of Z as

$$w(x) = \frac{a}{2\sqrt{2\pi}} x^{-\frac{3}{2}}(1+b^2x)e^{-\frac{1}{2}b^2x}. \tag{21}$$

In this case Z is not a compound Poisson process, since the density in (21) is not integrable. In other words, Z (and therefore λ) jumps infinitely often in finite time intervals. For a complete derivation of the results above see Barndorff-Nielsen and Shephard (2001).

Both in the Gamma(a,b)-OU and the IG(a,b)-OU case, the Laplace transform of the cumulative hazard $A_t = \int_0^t \lambda_s ds$ is available in terms of elementary functions and thereby provides explicit expressions for the survival probabilities in (**??**). Having specified λ as a Gamma(a, b)-OU process with dynamics as in (18), one obtains

$$\mathbb{E}[e^{u\,A_t}] = \exp\left(\frac{u\lambda_0}{\theta}(1-e^{-\theta t}) + \frac{\theta a}{u-\theta b}\left(b\log\left(\frac{b}{b-u\theta^{-1}(1-e^{-\theta t})}\right) - ut\right)\right) \tag{22}$$

while for the IG(a, b)-OU specification for λ the Laplace transform takes the form

$$\mathbb{E}[e^{u\,A_t}] = \left(\frac{\lambda_0 u}{\theta}\left(1-e^{-\theta t}\right) + \frac{2au}{b\theta}B\left(u,t\right)\right) \tag{23}$$

where

$$\begin{aligned} B\left(u,t\right) &= \frac{1-\sqrt{1+v\left(1-e^{-\theta t}\right)}}{v} \qquad (24)\\ &\quad + \frac{1}{\sqrt{1+v}}\left(\operatorname{arctanh}\left(\frac{\sqrt{1+v\left(1-e^{-\theta t}\right)}}{\sqrt{1+v}}\right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{1+v}}\right)\right)\\ v &= \frac{-2u}{b^2\theta}\ . \end{aligned}$$

For the derivation, see Nicolato and Venardos (2003) or Cariboni and Schoutens (2006). However, Cariboni and Schoutens (2006) find that while both the Gamma(a, b)-OU and IG(a, b)-OU processes provide accurate fits to the inferred survival probabilities, the results obtained in the Gamma(a, b)-OU specification are more stable. Moreover, the Gamma(a, b)-OU process allows for exact and fast simulation, due to the fact that it is driven by a compound Poisson process Z. This is a very convenient feature if portfolios of credit derivatives has to be priced.

Finally, Cariboni and Schoutens (2006) compare the performance of OU type intensity processes to other intensity models and find that Ornstein-Uhlenbeck type processes exhibit the most stable parameters and are able to fit market prices as good as the classical CIR process. These findings motivate our choice to include only Ornstein-Uhlenbeck type processes in our analysis.

4.2 The simplest model with jumps

Specifying the cumulative hazard A as a subordinator gives rise to the simplest model with jumps. If A has finite first moment then the cumulative hazard has compensator

$$Y_t = \mathbb{E}[A_t] = \mathbb{E}[A_1]t$$

i.e. the growth rate of A is constant. Moreover, the default time admits an intensity which is also constant and, by Proposition 1, is given by

$$\lambda_t^F = -\kappa_A(-1)$$

with κ_A denoting the cumulant generating function of A_1, the subordinator at time one. The analogous model in the classical intensity based approach is where the default time is given by the first jump of a standard Poisson process.

The survival probability equals

$$\mathbb{Q}(\tau > t) = \mathbb{E}\left[e^{-A_t}\right] = \mathbb{E}\left[e^{t\cdot\kappa_A(-1)}\right] \ , \tag{25}$$

and we analyze the models where A is either a Gamma or an Inverse Gaussian process with unit time densities given by (14) respectively (16). For these two models the survival probabilities are given by (25) with the appropriate cumulant generating function taken from either (15) or (17).

5 Calibration of the models

In this section the models are calibrated and it is shown how well they match the credit default swap spread curves.

A credit default swap (CDS) provides the protection buyer insurance against default of the underlying company in exchange for a stream of payments to the protection seller. The payments continue until the maturity of the contract or the underlying company defaults. In case of default the contract is terminated prematurely and protection seller pays the face value of the corporate bond minus an eventually recovery on the bond to the protection buyer.

Calibration of the model is performed by matching the market inferred credit default swap spread on each name from liquidly traded credit default swaps to the model inferred spread.

Assuming independence between the cumulative hazard process and interest rates the price of a CDS with maturity T is given by the difference between the discounted protection payment and the discounted continuously paid CDS spread c

$$CDS = (1-R)\int_0^T p(0,s)\, d\mathbb{Q}(s) - c\int_0^T p(0,s)\,\mathbb{Q}(\tau > s)\, ds,$$

where $\mathbb{Q}(s) = \mathbb{Q}(\tau \le s)$ is the default probability up to time s, R is the recovery on the bond and $\{p(0,s), s \in [0,T]\}$ the observed default-free zero coupon bond prices. From this we get the par spread equal to

$$c^* = \frac{(1-R)\int_0^T p(0,s)\, d\mathbb{Q}(s)}{\int_0^T p(0,s)\, \mathbb{Q}(\tau > s)\, ds}. \tag{26}$$

For any of the models described in Section 3 and 4 the survival probabilities are given by an analytically closed expression and the parameters can be calibrated from the spreads observed in the market by inversion of (26).

In the calibration we minimize the average relative percentage error (arpe) given by

$$\text{arpe} = \frac{1}{\text{number of CDS prices}} \sum \frac{\mid \text{Market spread - Model Spread} \mid}{\text{Market Spread}}$$

using the Nelder-Mead simplex algorithm.

Standardly, the recovery is set to 0.4 and the interest rate r is chosen deterministic and constant equal to 4 percent (i.e. $p(0,s) = e^{-rs}$).

The models are calibrated to weekly observations on the single names consisting the iTraxx Europe Series 8 index in the period from September 17, 2007 to March 14, 2008 (total of 26 weeks) in order to check both the calibration/pricing capabilities of the models and the stability of the parameters over time. [1] The models are fitted to the weekly spreads for maturities 1,3,5,7 and 10 years, which gives a total of 25 market prices for each company per observation. The calibration is initialized each week with the estimated parameters from the previous week. A total of six models are compared: [2]

(1) Lévy-Gamma: Here the cumulative hazard process is given by a Gamma process.
(2) Lévy-IG: Here the cumulative hazard process is given by an Inverse Gaussian process.
(3) Gamma-OU: Here the cumulative hazard is continuous and given by an integrated Gamma-Ornstein Uhlenbeck type process.
(4) IG-OU: Here the cumulative hazard is continuous and given by an integrated Inverse Gaussian-Ornstein Uhlenbeck type process.
(5) Sato-Gamma: Here the cumulative hazard is given by a Sato process, where the self-decomposable law at unit time is given by a Gamma distribution.

[1] September 17th was the first Monday after the roll over of the index and March 14th the last Friday before the next roll over.

[2] For the two integrated intensity models the calibration for all the companies and for a given point in time (i.e. one week) takes less than one minute. The calibration is even faster for the two Sato models, approximately 30 seconds.

(6) Sato-IG: Here the cumulative hazard is given by a Sato process, where the self-decomposable law at unit time is given by an Inverse Gaussian distribution.

The performance of the models is illustrated by focusing on three companies: Banco Espirito Santo SA, Endesa SA and Deutsche Post AG.

In Table B.1 and in Figure C.1 the calibration performance of the models based on a single day is shown. Models 3 to 6 all perform satisfactorily, with an error below 2% for all the models. The Lévy-Gamma and the Lévy-IG specifications perform very poorly. This it not surprising since only constant spreads across maturities can be achieved using a time homogeneous cumulative hazard process. Based on this observation, we focus at the other models in the remaining.

In Figures C.2 to C.5 in Appendix C the calibrated parameters for the 26 weeks and the stability of the parameters can be seen for the four models. For all the models one of the parameters have been fixed on the basis of identification considerations. For example, in the Gamma-OU model an increase in the intensity a of the jumps can be offset by an increase in the exponential jump distribution parameter b. Even though the pricing errors are reduced slightly without this fixation, the stability of the parameters are severely reduced.

Comparing the two continuous Ornstein-Uhlenbeck models with the two Sato process models it is seen from both Table B.2 and Figure C.6, where all the 3250 $(26 \cdot 125)$ calibration errors have been collected in histograms for the different models, that the performance is similar with respect to the error. This in itself favors the two Sato process models since they can achieve this with only two parameters, where the two OU models have three free parameters. The Sato processes are favorable when the focus is on stability over time of the calibrated parameters, and especially the scaling parameter γ behaves very stable in time for all three companies. For both Sato models the Gamma parameter is roughly constant until week 11-12 where it drops to a lower level. From then onwards it displays a stable behavior again.

Similarly to Cariboni and Schoutens (2006) we conclude the stability analysis by comparing the lag-i autocorrelation of the parameter time series for the various models. Recall, that the lag-i autocorrelation for the time series $X = \{X_t\}_{t=1,\ldots,N}$ is given by

$$\rho_i = \frac{\mathbb{E}\left[\left(X_t - \mathbb{E}\left[X_t\right]\right)\left(X_{t+i} - \mathbb{E}\left[X_t\right]\right)\right]}{\sqrt{\mathbb{E}\left[\left(X_t - \mathbb{E}\left[X_t\right]\right)^2\right]\mathbb{E}\left[\left(X_{t+i} - \mathbb{E}\left[X_t\right]\right)^2\right]}} \ ,$$

showing that ρ_i is the correlation coefficient between X_t and X_{t+i}. The autocorrelation is thus a stability measure of the calibrated parameters. In Figure

C.7 the lag-1 autocorrelation coefficient for the 125 parameter time series are collected in histograms for each parameter and each model. As also indicated from Figures C.2 to C.5 it is observed that the two Sato models give rise to lag-1 autocorrelation distributions shifted to the right compared to the benchmark models. Finally, Figure C.8 shows that the autocorrelation is also higher for the two Sato models, when higher lags are considered.

All the Ornstein Uhlenbeck models share the property that $\frac{1}{T}\int_0^T \lambda_s ds \to \overline{y}$ This linearity in the long term behavior is not present in the Sato models. The natural question then, is if the data confirms that the cumulative hazard is best described by an integrated Ornstein-Uhlenbeck process. In Figure C.9 the speed of the convergence of $\frac{1}{T}\mathbb{E}\left[\int_0^T \lambda_s ds\right]$ towards the long run mean $\overline{\lambda}$ is depicted for the Gamma-OU model. As input the average of the 26 calibrated parameter values are taken for the three companies. The same graph is shown but with two different time horizons. It is only for Banco Espirito Santo that $\frac{1}{T}\mathbb{E}\left[\int_0^T \lambda_s ds\right]$ gets close to the long run mean within a horizon of ten years, which is the maximum maturity for credit default swaps in the market. For the other companies we need to go beyond approximately 20 years before it closes in on the long run mean. This stems from the fact that the speed of the mean reversion θ is relatively low for these companies. This indicates that mean-reverting intensities are not supported by the data. Furthermore, for all the 3250 calibrations not a single of them result in a starting intensity value λ_0 above the long run mean of the intensity. This stems from the increasing CDS spread property of the firms constituent in the iTraxx index, which again implies increasing risk neutral intensities.[3] This has the consequence that the classical intensity models will predict increasing risk neutral intensities because of the mean reversion. This behavior is naturally present for γ-parameters greater than 1 in the Sato models.

6 Conclusion

A framework was described in which the cumulative hazard A is modelled directly as opposed to the classical instantaneous intensity based models. This approach encompasses recently proposed models in the context of default modelling.

In this paper a new specification was proposed, where the cumulative hazard belongs to the class of Sato processes. Two specific Sato cumulative hazard models were analyzed and compared to four benchmark models; two Lévy models and two Ornstein Uhlenbeck models. All the models were calibrated

[3] Increasing spreads over time to maturity is not a general quality, but almost always the case for the firms that constitute the iTraxx index.

to weekly market data on credit default swaps constituting the iTraxx Europe series 8 index. The two time homogeneous Lévy models were immediately discarded, since they can only achieve constant spreads across maturities. With respect to fitting the market the remaining four models are comparable. However, the Sato models can achieve this with one less parameter.

Our stability analysis favors the two Sato process models, since the calibrated parameter estimates in these models display more stable behavior than in the two Ornstein Uhlenbeck models.

A possibly controversial issue in the Sato process models is the fact that the risk neutral dynamics of the intensity is not mean reverting as is the classical case in intensity models. Even though the mean reverting property of the intensity process is appealing it might not be the case implied by the data, since the calibrated parameter controlling the speed of the mean reversion is typically very low or even imply an explosive behavior. An analysis concerning this issue was performed and preliminary results indicate that the mean reversion property is not supported by the data.

References

[1] Barndorff-Nielsen, O. E., Nicolato, E. and Shephard, N. (2001). Some Recent Development in Stochastic Volatility Modelling, Quantitative Finance, 2, pp. 11-23.

[2] Barndorff-Nielsen, O. E., Pedersen, J. and Sato, K. (2001). Multivariate Subordination, Selfdecomposability and Stability, Advances in Applied Probability, 33, pp. 160-187.

[3] Barndorff-Nielsen, O. E. and Shephard, N. (2001). Non-Gaussian Ornstein-Uhlenbeck-Based Models and Some of Their Uses in Financial Economics, J. Royal Stat. Soc. B, 63, pp. 167-241.

[4] Cariboni, J. and Schoutens, W. (2006). Jumps in Intensity Models, Working Paper.

[5] Carr, P., Geman, H., Madan, D. and Yor, M. (2003). Stochastic Volatility for Lévy Processes, Mathematical Finance, 13, pp. 345-382.

[6] Carr, P., Geman, H., Madan, D. and Yor, M. (2007). Self-Decomposability and Option Pricing, Mathematical Finance, 17, pp. 31-57.

[7] Cont, R. and Tankov, P. (2004). Financial Modelling with Jump Processes, Chapman and Hall.

[8] Duffie, D. and Singleton, K. J. (1997). An Econometric Model of the Term Structure of Interest-Rate Swap Yields, Journal of Finance, 52, pp. 1287-1321.

[9] Duffie, D. and Singleton, K. J. (1999). Modelling Term Structures of Defaultable Bonds, The Review of Financial Studies 12, pp. 687-720.

[10] Eberlein, E., Kallsen, J., and Kristen, J. (2003). Risk Management Based on Stochastic Volatility. Journal of Risk, 5, pp. 19-44.

[11] Elliott, R. J., Jeanblanc, M. and Yor, M. (2000). On models of default risk, Mathematical Finance, 10, pp. 179-195.

[12] Feldhütter, P. (2007). An Empirical Investigation of an Intensity-Based Model for Pricing of CDO Tranches, Working Paper.

[13] Glasserman, P. (2004). Monte Carlo Methods in Financial Engineering, Springer.

[14] Graziano, G. and Rogers, L. (2006). A Dynamic Approach to the Modelling of Correlation Credit Derivatives using Markov Chains, working paper.

[15] Hull, J. and White, A. (2007). Dynamic Models of Portfolio Credit Risk: A Simplified Approach, Working Paper.

[16] Jarrow, R. and Turnbull, S. (1995). Pricing Derivatives on Financial Securities Subject to Credit Risk, Journal of Finance, 50, pp. 53-85.

[17] Jarrow, R., Lando, D. and Turnbull, S. (1997). A Markov Model for the Term Structure of Credit Risk Spreads, Review of Financial Studies, 10, pp. 481-523.

[18] Jeanblanc, M. and Le Cam, Y. (2007). Reduced Form Modelling for Credit Risk, working paper.

[19] Joshi, M. and Stacey, A. (2006). Intensity Gamma, Risk Magazine, July 2006, pp. 78-83.

[20] Lando, D. (1994). Three Essays on Contingent Claims Pricing, PhD Dissertation, Cornell University.

[21] Lando, D. (1998). On Cox Processes and Credit Risky Securities, Review of Derivatives Research, 2, pp. 99-120.

[22] Lando, D. (2004). Credit Risk Modeling, Princeton University Press.

[23] Li, D. (2000). On Default Correlation: A Copula Function Approach, Working Paper 99-07, The RiskMetrics Group.

[24] Luciano, E. and Schoutens, W. (2006). A Multivariate Jump-Driven Financial Asset Model, Quantitative Finance, 6, pp. 385-402.

[25] Madan, D. and Seneta, E. (1990). The Variance Gamma (V.G.) Model for Share Market Returns, The Journal of Business, 63, pp. 511-524.

[26] Madan, D. and Unal, H. (1998). Pricing the Risks of Default, Review of Derivatives Research, 2, pp. 121-160.

[27] Musiela, M. and Rutkowski, M. (2004). Martingale Methods in Financial Modelling, second edition., Springer.

[28] Nicolato, E. and Venardos, E. (2003). Option Pricing in Stochastic Volatility Models of the Ornstein-Uhlenbeck Type, Mathematical Finance, 13, pp. 445-466.

[29] Rydberg, T. H. (1999). Generalized Hyperbolic Diffusion Processes with Applications in Finance, Mathematical Finance, 9, pp. 183-201.

[30] Sato, K. (1991). Self-Similar Processes with Independent Increments, Probability Theory Related Fields, 89, pp. 285-300.

[31] Sato, K. (1999). Lévy Processes and Infinitely Divisible Distributions, Cambridge: Cambridge University Press.

[32] Schönbucher, P. (2003). Credit Derivatives Pricing Models, Wiley.

[33] Skovmand, D. (2008). Alternative Specifications for the Lévy LIBOR Market Model: An Empirical Investigation, working paper.

A Proof of Proposition 1

First, we compute the compensator C of the process $1 - e^{-A}$. By change of variable formula, we obtain that

$$1 - e^{-A_t} = e^{-A_{t-}} \cdot L_t$$

with

$$L_t = A_t - (e^{-x} - 1 + x) \star \mu.$$

By canonical representation (see Jacod and Shiryaev, II.2.38 and Lemma I.4.14), we can rewrite L as follows

$$\begin{aligned} L_t &= x \star (\mu - \nu) + \int_0^t y_s ds - (e^{-x} - 1 + x) \star \mu \\ &= -(e^{-x} - 1) \star (\mu - \nu) + \int_0^t y_s ds - (e^{-x} - 1 + x) \star \nu \end{aligned}$$

from which it follows that

$$C_t = \int_0^t e^{-A_{s-}} (y_s - \int_0^{+\infty} (e^{-x} - 1 + x) K(\omega, dx, s)) ds$$

and the result (10) now follows from expression (9) $\square$

B Tables

Company	Model	1YR	3YR	5YR	7YR	10YR	arpe
Banco Espirito	Market	37.90	45.00	51.50	54.20	55.00	
Santo SA	Lévy-Gamma	51.50	51.50	51.50	51.50	51.50	12.33
	Lévy-IG	51.50	51.50	51.50	51.50	51.50	12.33
	Gamma-OU	37.90	46.55	51.35	54.20	56.65	1.35
	IG-OU	37.90	46.79	51.50	54.19	56.45	1.33
	Sato-Gamma	37.90	46.88	51.33	54.20	57.04	1.65
	Sato-IG	37.90	46.55	51.07	54.20	57.59	1.80
Endesa SA	Market	44.40	63.70	74.50	81.30	88.30	
	Lévy-Gamma	63.70	63.70	63.70	63.70	63.70	21.49
	Lévy-IG	63.70	63.70	63.70	63.70	63.70	21.49
	Gamma-OU	44.40	63.26	74.50	81.50	87.77	0.31
	IG-OU	44.40	63.10	74.50	81.73	88.30	0.29
	Sato-Gamma	44.40	63.64	74.18	81.30	88.51	0.15
	Sato-IG	44.40	62.88	73.55	81.30	90.02	0.90
Deutsche Post AG	Market	12.50	24.80	37.20	45.30	55.00	
	Lévy-Gamma	24.80	24.80	24.80	24.80	24.80	46.38
	Lévy-IG	24.80	24.80	24.80	24.80	24.80	46.38
	Gamma-OU	12.50	25.10	35.63	44.43	55.00	1.47
	IG-OU	12.50	26.04	36.77	45.30	55.00	1.23
	Sato-Gamma	12.50	26.45	36.76	45.30	55.66	1.72
	Sato-IG	12.50	26.11	36.52	45.30	56.50	2.00

Table B.1
Calibration results of the models on Thursday, January 3, 2008.

Model	Lévy-Gamma	Lévy-IG	Gamma-OU	IG-OU	Sato-Gamma	Sato-IG
arpe	38.71	38.71	4.99	5.01	5.79	6.08

Table B.2
The average error in percent accross all firms and all weeks for the different models.

C Figures

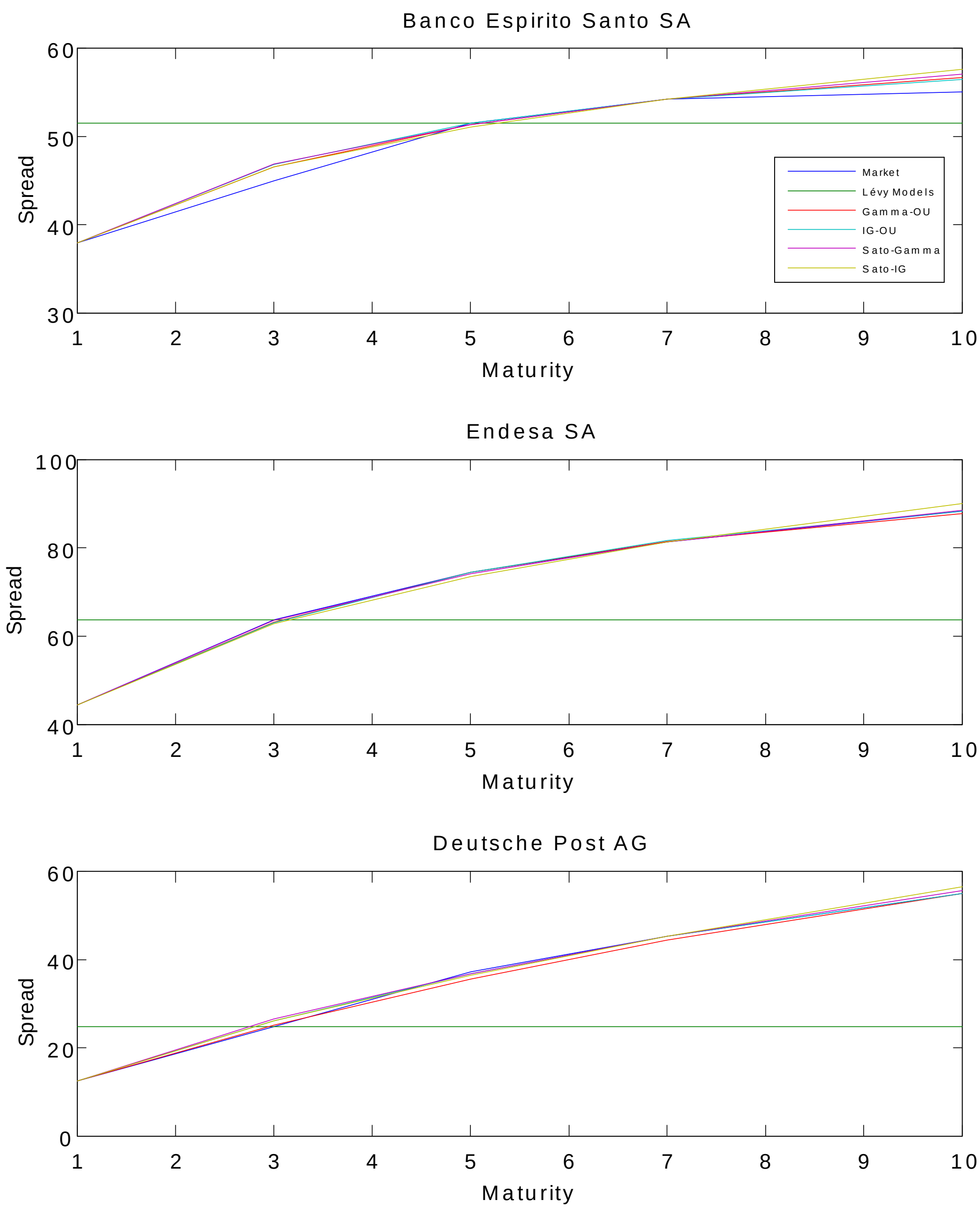

Fig. C.1. *Calibrated spreads for the various models to the market on January 3rd, 2008.*

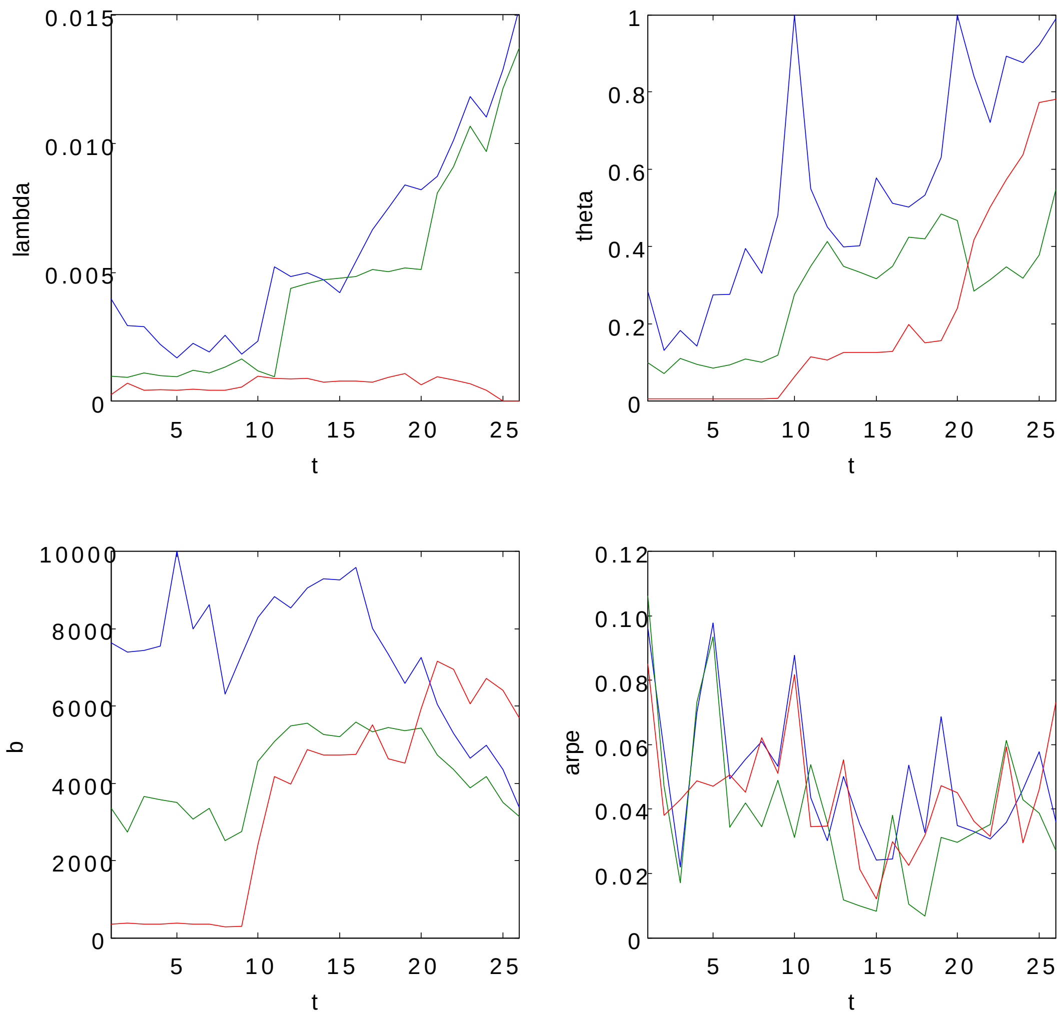

Fig. C.2. *The calibrated* λ_0, θ, b *parameters and the arpe in the Gamma-OU model for 3 different companies. The intensity of the jumps* a *is fixed to 100.*

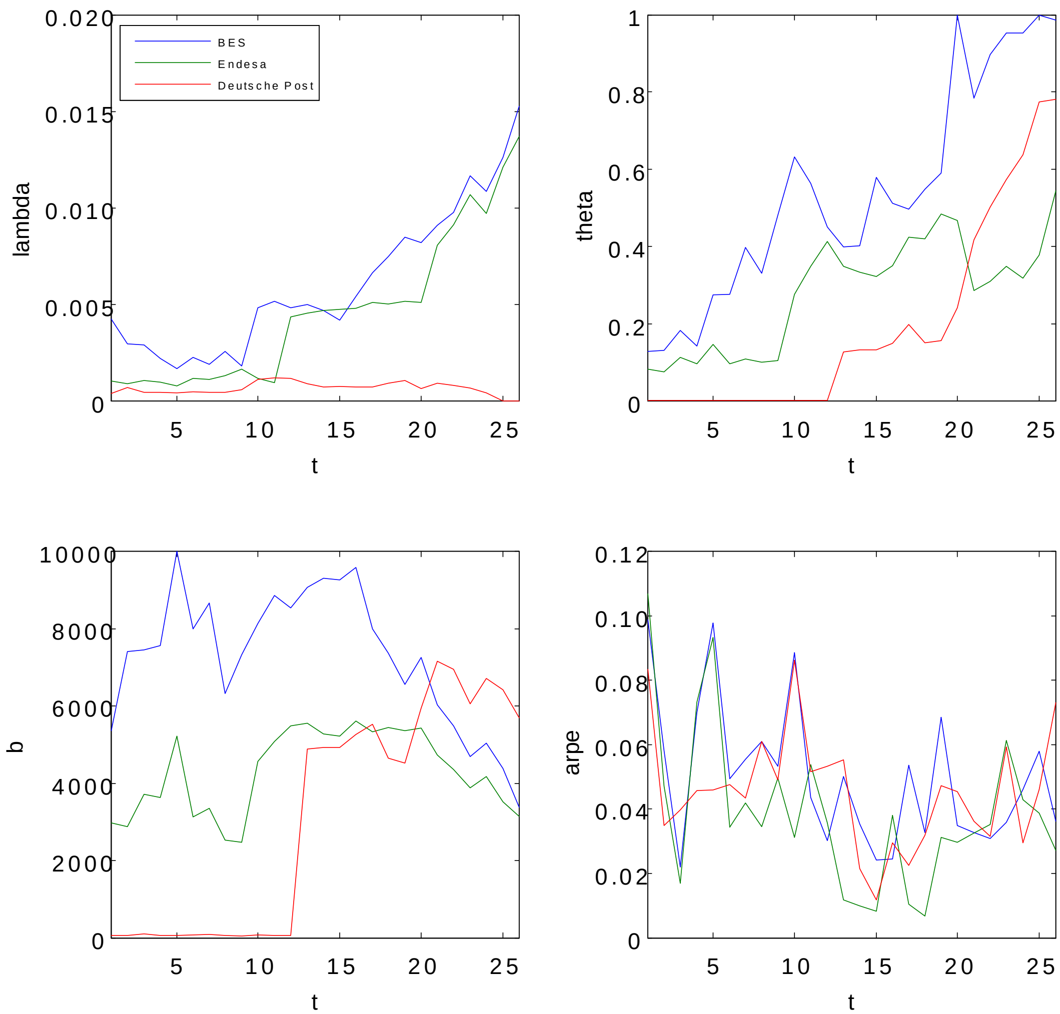

Fig. C.3. *The calibrated λ_0, θ, b parameters and the arpe in the IG-OU model for 3 different companies. The intensity of the jumps a is fixed to 100.*

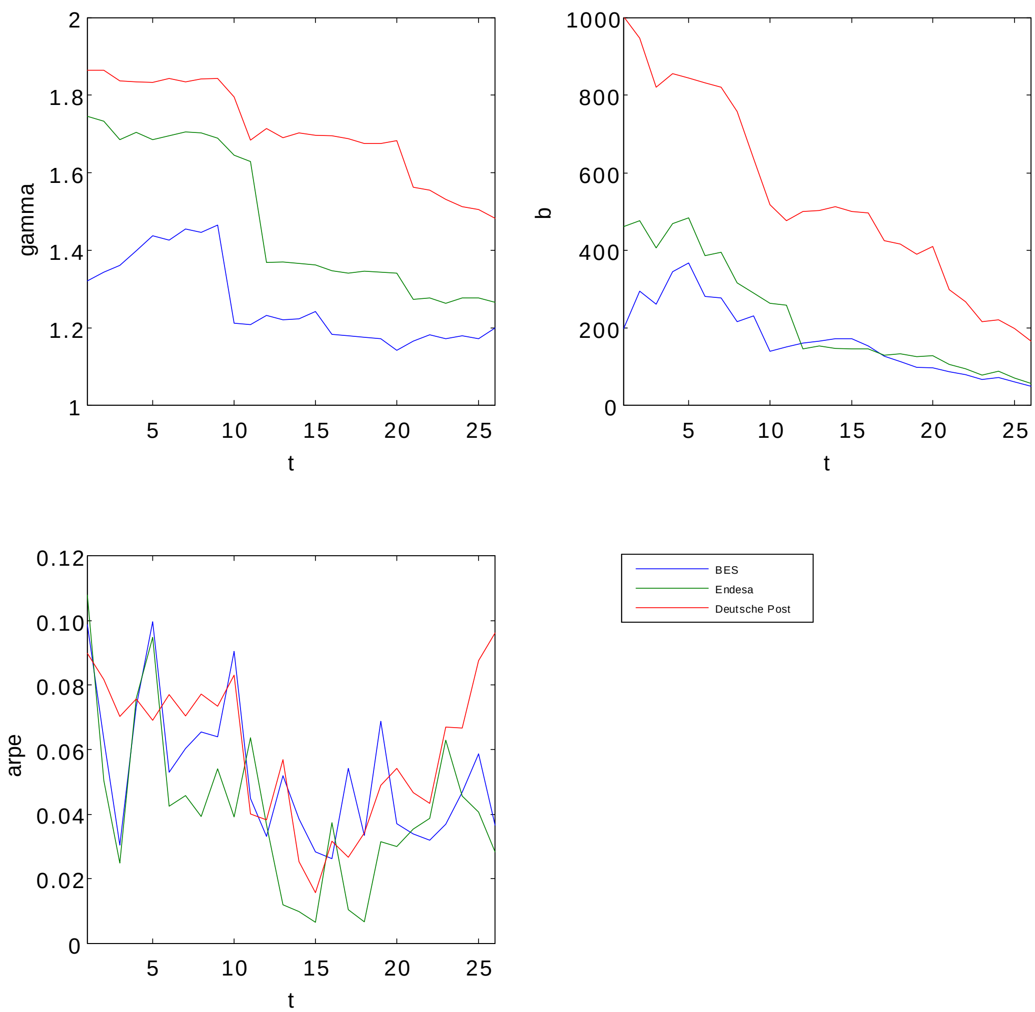

Fig. C.4. *The calibrated γ, b and resulting arpe in the Gamma self-similar additive model for 3 different companies. The parameter a is fixed to 1.*

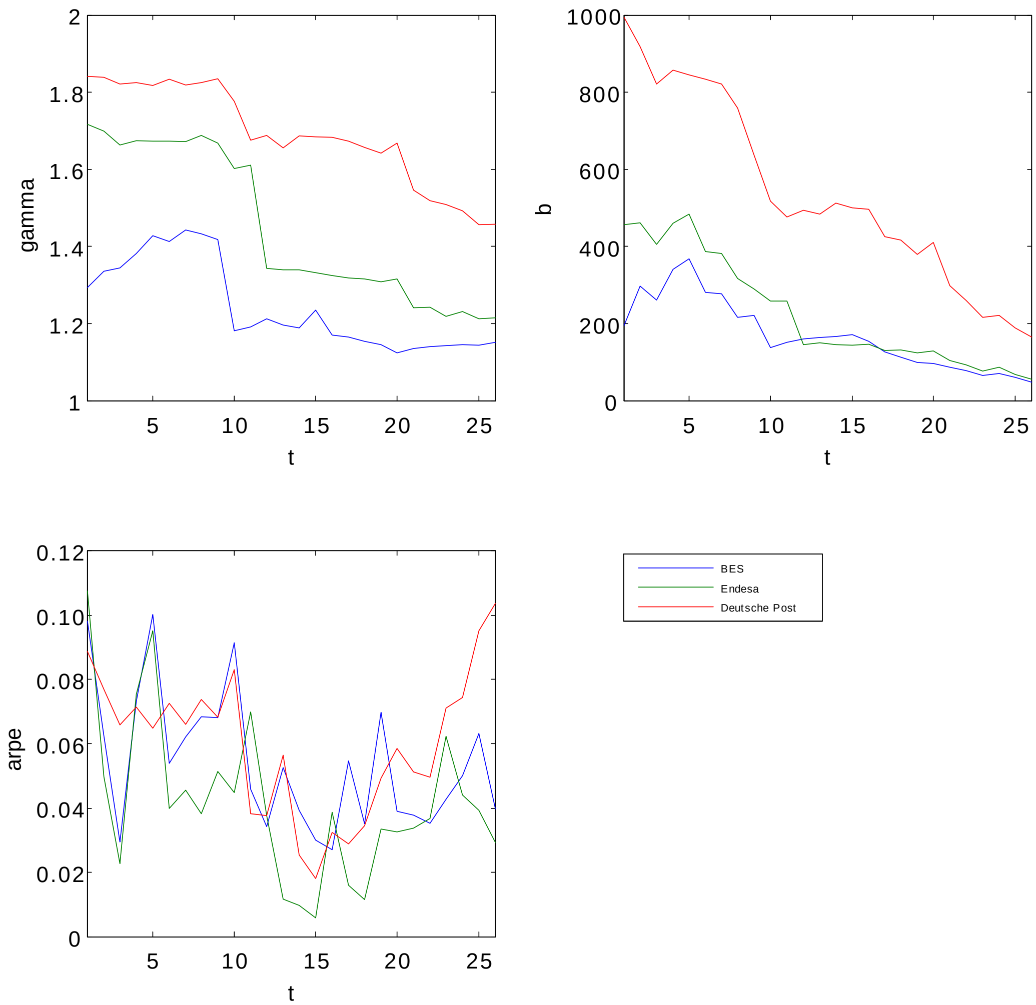

Fig. C.5. *The calibrated* γ, b *parameters and the arpe in the IG self-similar additive model for 3 different companies. The parameter* a *is fixed to* 1*.*

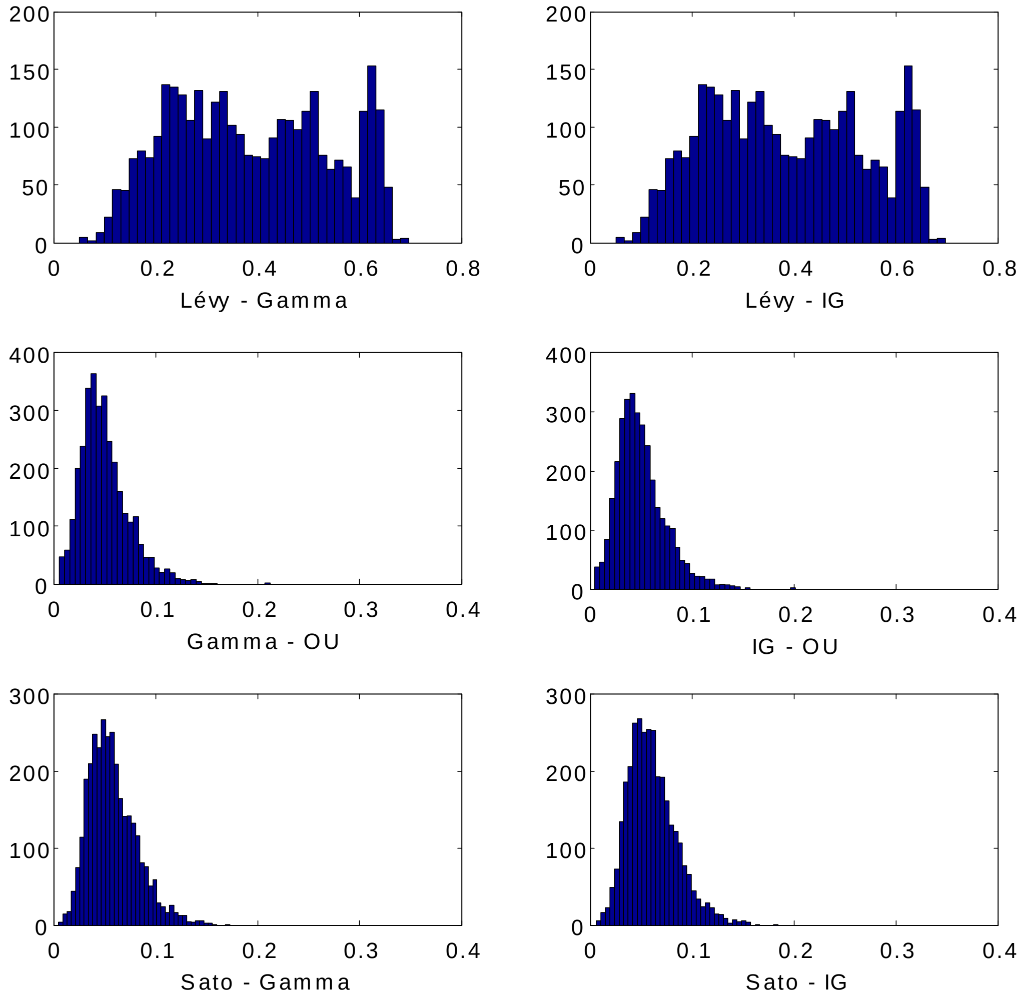

Fig. C.6. *Distributions of the arpe in percent across companies and weeks for the different models.*

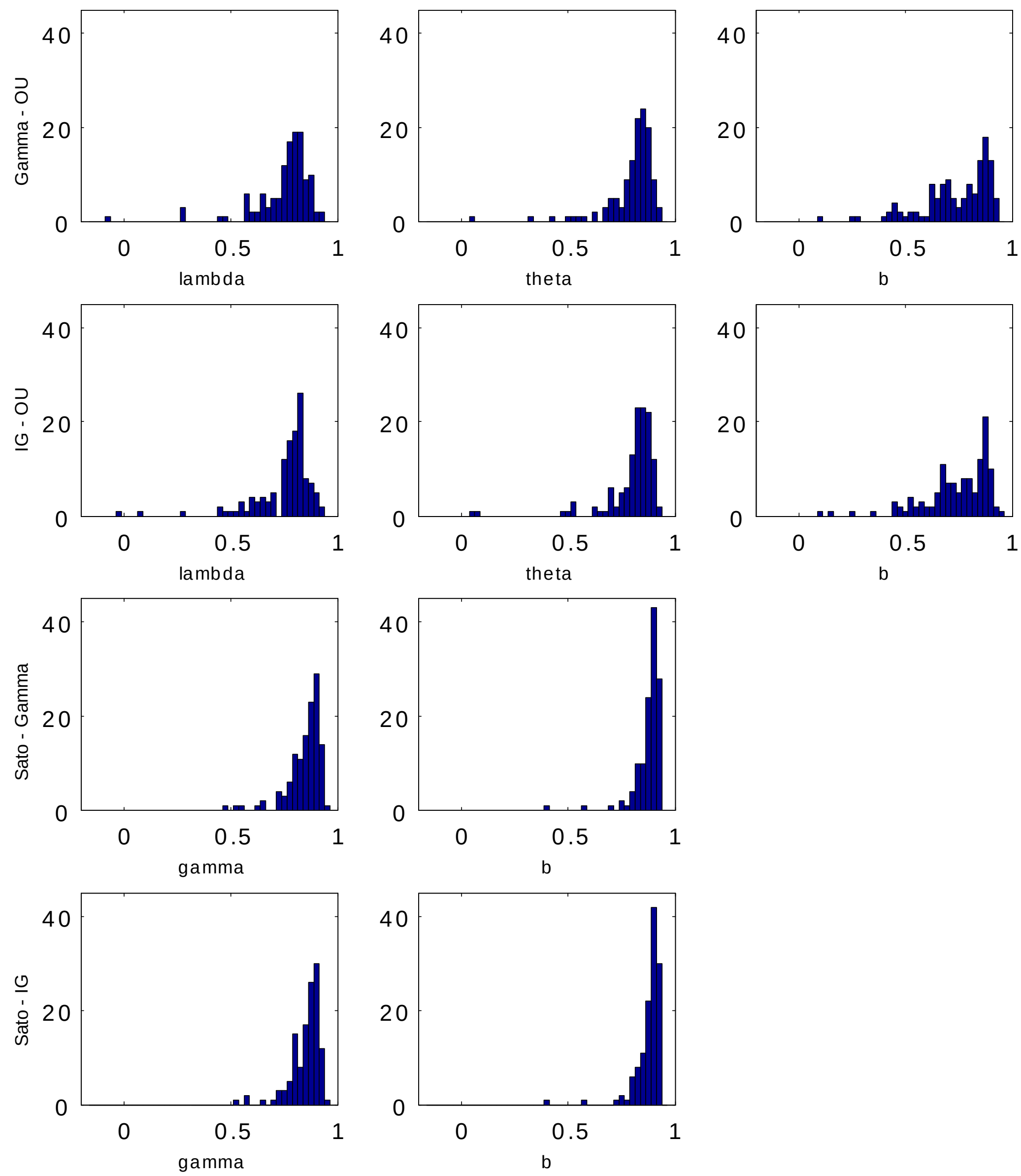

Fig. C.7. 1-*lag autocorrelation distributions for the parameters in the different models.*

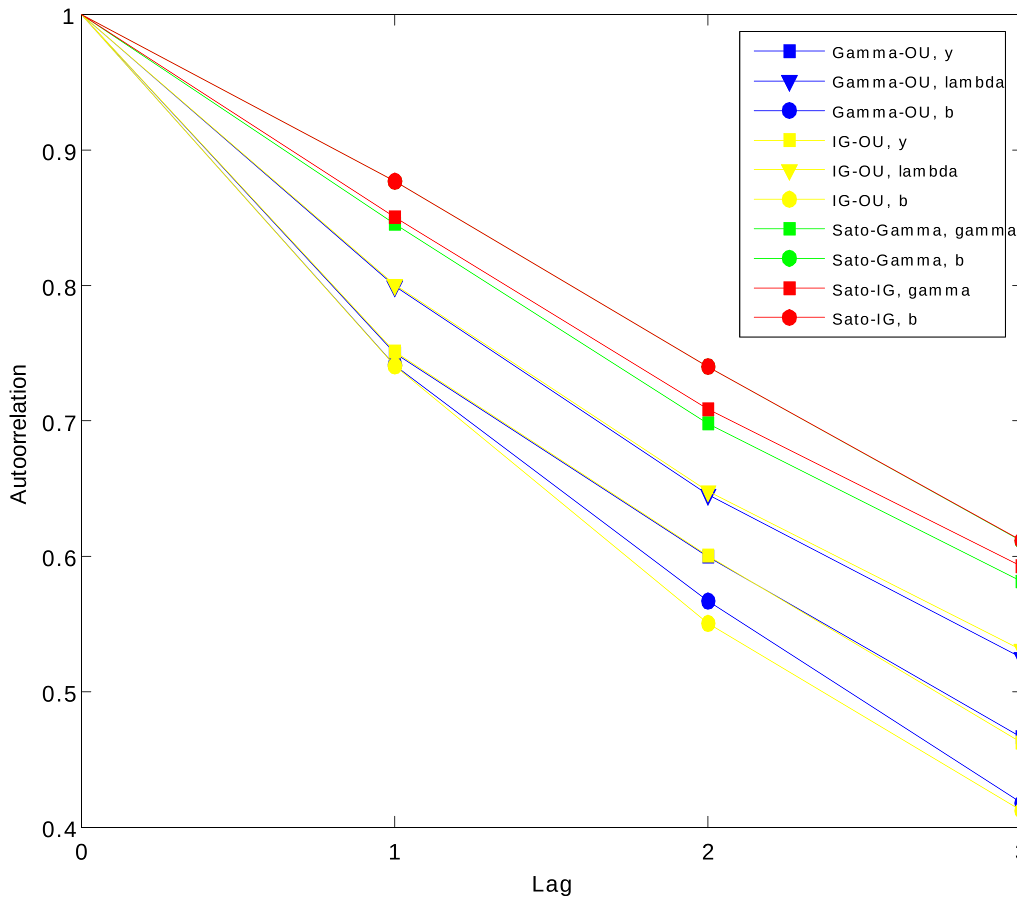

Fig. C.8. *The average across firms of the autocorrelation for each parameter as a function of lag for the different models.*

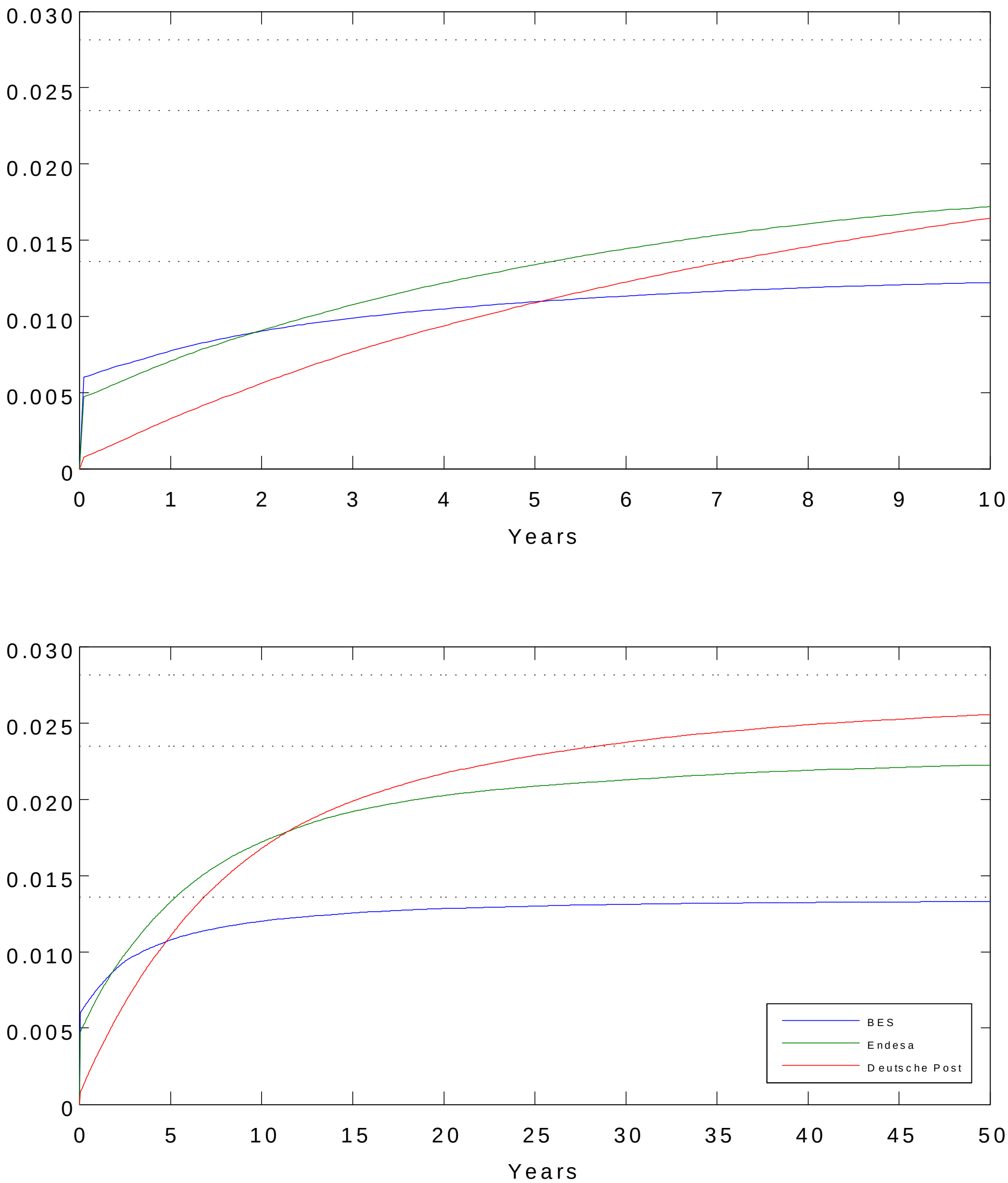

Fig. C.9. *The convergence of* $\frac{1}{T}\mathbb{E}\left[\int_0^T \lambda_s ds\right]$ *in the Gamma-OU model towards the long run mean given by* $\frac{a}{b}$. *The average of all the calibrated parameters for the three companies are used as input. The parameters are*
Banco Espirito Santo SA: $\lambda_0 = 0.0059, \theta = 0.5499, b = 7350, a = 100.$
Endesa SA: $\lambda_0 = 0.0046, \theta = 0.2789, b = 4256, a = 100.$
Deutsche Post AG: $\lambda_0 = 0.00062, \theta = 0.2027, b = 3553, a = 100.$