

Sato Processes in Default Modelling

International Workshop: Credit Risk
Evrý University

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joint work with
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Development of project



*Thomas: Hey Elisa. I have this idea to a project. Can we talk about it?
Good, now listen up...*

Elisa: That sounds interesting Thomas,
but why don't you do it in this way instead....?

*Thomas: Okay, That could be cool, but I am not sure I can work out the
math now....*

Elisa: No problem. I will do the math
if you do the implementation.

Thomas: Great! Now, let's get to work!

Four months later.... (after submission to Evry).

Thomas: I just found this paper by Jeanblanc and Le Cam. We need to change the paper....!

Elisa: No worries. Leave it to me...

Two months later.... Now

- 1 Reduced form modelling (Hazard process models)
 - 1 (General) framework
 - 2 Classical intensity based framework
 - 3 New directions in the literature
- 2 Cumulative hazards as Sato processes
- 3 Calibration and performance

Reduced form modelling (Hazard process models)

Consider a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{F})_{t \geq 0}, \mathbb{Q})$, with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ and $\mathcal{G} = \mathcal{G}_\infty$, where

- \mathcal{F}_t carries the default-free information
- $\mathcal{H}_t = \sigma \{1_{\{\tau \leq s\}} : 0 \leq s \leq t\}$ is the filtration generated by the default process $1_{\{\tau \leq s\}}$. (τ not a $\{\mathcal{F}_t\}_{t \geq 0}$ -stopping time)
- The default time τ is defined by what is known as a Cox construction

$$\tau = \inf \{t > 0 \mid A_t \geq E_1\}, \quad (1)$$

with E_1 a mean-one exponential RV independent of \mathcal{F}_∞ , and the *cumulative hazard process* A_t increasing.

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- In this setup the conditional survival probabilities are given by

$$\mathbb{Q}(\tau > t \mid \mathcal{F}_t) = e^{-A_t}. \quad (2)$$

and a claim $X \in \mathcal{F}_T$ can be valued by the *hazard based pricing rule* (see Elliot, Jeanblanc and Yor (2000)):

$$\mathbb{E}[X 1_{\{T < \tau\}} \mid \mathcal{G}_t] = 1_{\{\tau > t\}} e^{A_t} \mathbb{E}[e^{-A_\tau} X \mid \mathcal{F}_t].$$

- If the compensator Λ^G (w.r.t the filtration $(\mathcal{G})_{t \geq 0}$) of the process H is absolutely continuous w.r.t the Lebesgue measure

$$\Lambda_t^G = \int_0^t \lambda_s^G ds,$$

the time of default is said to have *intensity rate* λ^G and it holds that

$$\lambda_t^G = \lim_{h \rightarrow 0} \frac{1}{h} \mathbf{Q}(t < \tau \leq t + h | \mathcal{G}_t)$$

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- The compensator Λ^G can be related to the cumulative hazard process A as follows

$$\Lambda_t^G = \int_0^{t \wedge \tau} \frac{dC_s}{e^{-A_{s-}}}$$

where the process C is the $(\mathcal{F})_{t \geq 0}$ -compensator of the process $1 - e^{-A}$. Hence, if C is absolutely continuous w.r.t. the Lebesgue measure, $C_t = \int_0^t c_s ds$, the time of default has intensity

$$\lambda_t^G = 1_{\{t < \tau\}} \lambda_t^F \quad \text{with} \quad \lambda_t^F = \frac{c_t}{e^{-A_{t-}}} \quad (3)$$

Intensity based framework

In the intensity based framework as introduced by Lando (1998) default happens at the first jump of a Cox process \tilde{N}_t with intensity process λ_t (i.e. a standard Poisson process N_t subordinated by the *continuous* process $A = \int_0^t \lambda_s ds$).

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- Given an \mathcal{F}_T measurable claim X , the price at time t is given by

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- Clearly, the tractability of the model depends on the analytical tractability of the intensity process λ_t and the survival probability given by the Laplace transform (2) of the cumulative process $\int_0^t \lambda_s ds$.

This approach has been quite successful and widely used in practice.

Following the explosion of the credit derivatives market it has been readily extended to the multivariate setting, e.g. by specifying single name intensities as the sum of an idiosyncratic component and a common component affecting the default of all the firms

$$\lambda_t^i = y_t^i + a_i y_t^c$$

(see e.g. Duffie and Garleanu (2001)).

However, few flaws have emerged:

- At single-name level, parameter estimates do not retain stability over time, leading to large variations in sensitivities.

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However, few flaws have emerged:

- At single-name level, parameter estimates do not retain stability over time, leading to large variations in sensitivities.
- At multivariate level, it has been argued that intensity-based models cannot generate enough default correlation to match market prices.

New directions

Recent literature (although in the context of multivariate modelling) has shown a tendency to specifying the *cumulative hazard process (CHP)* A_t directly, with A_t not necessarily of the form $\int_0^t \lambda_s ds$ and possibly displaying jumps.

Specifications of CHP displaying jumps are presented in Di Graziano and Rogers (2006), Joshi and Stacey (2006) and Hull and White (2007)

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- More general Cumulative Hazard based models have started to appear in the context of default correlation modelling.
- However their performances on single-name derivatives have not really been empirically investigated, focus being in matching market-inferred correlations.
- Classical Intensity based models exhibit instability in parameter-estimates over time.
- **Our Objective:** To address bullet point 2 and find a model that fits data with more stable parameter estimates.

Intensity when Cumulative Hazards display jumps

When the CHP displays jumps the actual intensity in (3) is given by the following proposition.

Proposition

Assume that the cumulative hazard process A is integrable and it has $(\mathcal{F})_{t \geq 0}$ -compensator $Y_t = \int_0^t y_s ds$ which is absolutely continuous w.r.t the Lebesgue measure. Assume furthermore that the random measure $\mu(\omega, dx, dt)$ associated with the jumps of A has compensator

$$\nu(\omega, dx, dt) = K(\omega, dx, t)dt$$

which is also absolutely continuous. Then the time of default defined in (1) admits intensity λ^F which is given by

$$\lambda_t^F = y_t - \int_0^{+\infty} (e^{-x} - 1 + x)K(\omega, dx, t) \quad (4)$$

So why use Sato processes instead of the classical intensity based framework?

- In the classical intensity framework $A_t = \int_0^t \lambda_s ds$, if the intensity process λ_t is ergodic (as in CIR) then it holds that

$$t^{-1}A_t \rightarrow \bar{\lambda} \quad \text{as } t \rightarrow +\infty$$

with $\bar{\lambda}$ denoting the long-run average of λ_t , implying a linear behavior of cumulative hazard for long time-horizon.

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- Sato processes display enormous flexibility in terms of distribution-modelling (more on this later). Sato processes are also analytically tractable as they allow for closed form expressions for default probabilities hereby enabling straightforward calibration to credit default swap prices. Finally they allow for more flexibility and non-linearity in the long term behavior of the cumulative hazard.

Cumulative Hazards as Sato processes

Self-similar processes

- A stochastic process $\{A_t : t \geq 0\}$ is called self-similar if for any $\alpha > 0$ there is a $\beta > 0$ such that

$$\{A_{\alpha t} : t > 0\} \stackrel{d}{=} \{\beta A_t : t > 0\}, \quad (5)$$

implying that any change of time scale has the same effect of a change in the spatial scale.

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- It follows that there exists a $\gamma > 0$, the *exponent of self-similarity*, such that $\beta = \alpha^\gamma$ and we say that A is γ -self-similar ($A_t \simeq t^\gamma A_1$).

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- It follows that there exists a $\gamma > 0$, the *exponent of self-similarity*, such that $\beta = \alpha^\gamma$ and we say that A is γ -self-similar ($A_t \simeq t^\gamma A_1$).
- A well known class of self-similar processes is given by the strictly stable processes, i.e. processes with independent and homogeneous increments with law at unit time described by a strictly stable distribution.

Cumulative Hazards as Sato processes

Sato processes

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- More precisely, Sato (1991) showed that for any given self-decomposable law X (i.e. $X \simeq cX + X^c$, where X and X^c independent) and an exponent $\gamma > 0$ there exists a γ -self-similar additive process A with law at unit time A_1 given by X .

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- By choosing for the unit time a self-decomposable law X on the positive real line, we obtain that the corresponding γ -self-similar additive process A has increasing paths and therefore suitable to describe a cumulative hazard.

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- By choosing for the unit time a self-decomposable law X on the positive real line, we obtain that the corresponding γ -self-similar additive process A has increasing paths and therefore suitable to describe a cumulative hazard.
- Sato processes have been successfully employed both in the context of equity option pricing and interest rate modelling (see Carr, Geman, Madan and Yor (2007) and Skovmand (2008)).

Some properties

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$$\kappa_X(u) = \log \mathbb{E}[e^{uX}]$$

the cumulant generating function of the unit time law, the survival probabilities are then given by

$$\mathbb{Q}(\tau > t) = \mathbb{E}\left[e^{-A_t}\right] = \mathbb{E}\left[e^{-t^\gamma X}\right] = e^{\kappa_X(-t^\gamma)}. \quad (6)$$

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- Therefore, if the cumulant generating function κ_X is known explicitly, the survival probabilities can be computed in closed-form enabling straightforward calibration of the model to credit default swaps.
- Notice that, by Proposition 1 the intensity rate λ^F given by

$$\lambda_t^F = \gamma t^{\gamma-1} \frac{d}{dt} \kappa_X(t) \Big|_{-t^\gamma}.$$

Cumulative Hazards as Sato processes

Concrete specifications

We consider two concrete specifications of Sato processes for the cumulative hazard A_t :

- 1 *The Sato-Gamma process*, where X is described by a Gamma law $\Gamma(a, b)$ with cumulant generating function

$$\kappa_X(u) = \log \left(\frac{1}{(1 - u b^{-1})^a} \right) .$$

- 2 *The Sato-IG process* where the unit time distribution X is described by Inverse Gaussian law $IG(a, b)$ with cumulant generating function

$$\kappa_X(u) = ab - a\sqrt{b^2 - 2u} .$$

Both the $\Gamma(a, b)$ and the $IG(a, b)$ distributions are self-decomposable and therefore the corresponding Sato process specifications are meaningful.

Both are well known distributions in the financial literature (see Madan and Seneta (1990), Rydberg (1999), Barndorff-Nielsen and Shephard (2001), Nicolato and Venardos (2003) among (many) others).

Benchmark models

- Cariboni and Schoutens (2006) consider intensities of the Ornstein-Uhlenbeck type.

$$d\lambda_t = -\theta\lambda_t dt + dZ_{\theta t} \quad \lambda_0 > 0, \quad (7)$$

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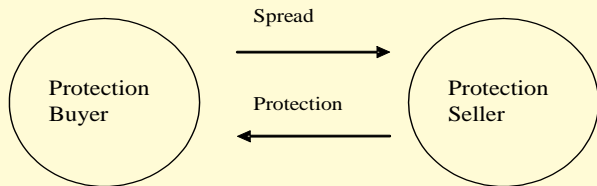
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- Cariboni and Schoutens (2006) compare the performance of the two OU type intensity processes to other intensity models and find that OU type intensities exhibit the most stable parameters and are able to fit market prices as good as the classical CIR process.
- These findings motivate our choice to include only Ornstein-Uhlenbeck type processes in our analysis.

Calibration

A credit default swap (CDS) provides protection buyer insurance against default of the underlying company in exchange for a stream of payments to protection seller. The payments continue until the maturity of the contract or the underlying company defaults. In case of default the contract is terminated and protection seller pays face value minus recovery $(1 - R)$ to protection buyer.



The Credit Default Spread is calculated as the premium that makes the value of the contract zero.

Calibration of the model is performed by matching the market inferred spread from liquidly traded credit derivatives to the model inferred spread on these.

- The price of a CDS with maturity T is given by

$$CDS = (1 - R) \int_0^T e^{-rs} dQ(s) - c \int_0^T e^{-rs} Q(\tau > s) ds,$$

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- Then the model inferred par spread equals

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- Standardly, the recovery is set to 0.4 and the interest rate is chosen deterministic and constant equal to 4 percent.
- Performance is illustrated by focusing on: Kingfisher PLC, Altadis SA and E. ON AG.

- In the calibration we minimize the average relative percentage error (arpe) given by

$$\text{arpe} = \frac{1}{\text{number of CDS prices}} \sum \frac{|\text{Market spread} - \text{Model Spread}|}{\text{Market Spread}}$$

using the Nelder-Mead simplex method.

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- The models are calibrated to weekly observations on the single names consisting the iTraxx Europe index in the period from September 17, 2007 to March 14, 2008 (total of 26 weeks) in order to check both the calibration/pricing capabilities of the models and stability of the parameters over time. For each week the models are calibrated to spreads for maturities 1, 3, 5, 7 and 10 years.

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- For all the models one of the parameters have been fixed on the basis of identification considerations. Even though the pricing errors are reduced slightly without this fixation, the stability of the parameters are severely reduced for all models.
- Speed: 60 sec. for the OU-type models for all 125 names and 30 sec. for the Sato models.

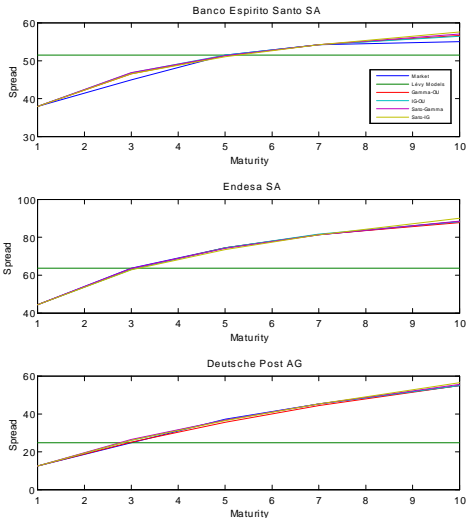


Figure: Calibrated spreads for the various models to the market on January 3rd, 2008.

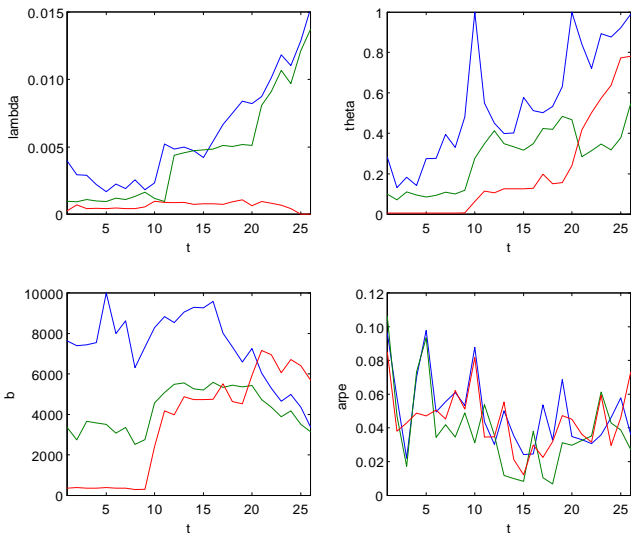


Figure: The calibrated λ_0, θ, b parameters and the $arpe$ in the Gamma-OU model for 3 different companies. The intensity of the jumps a is fixed to 100.

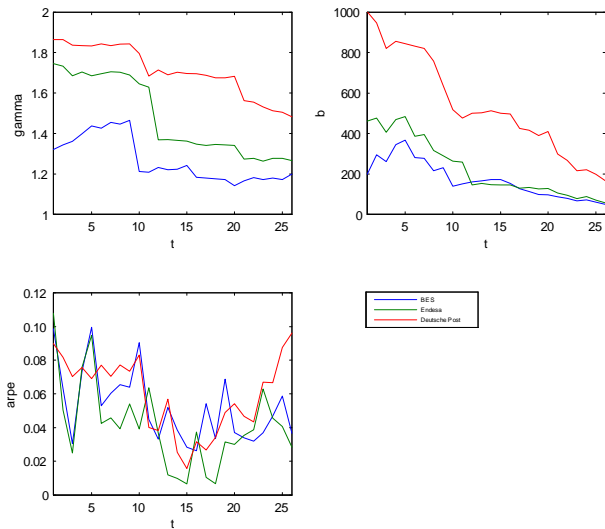


Figure: The calibrated γ , b and resulting $arpe$ in the Sato-Gamma model for 3 different companies. The parameter a is fixed to 1.

The average error in percent across all firms and all weeks for the different models.

Model	Gamma-OU	IG-OU	Sato-Gamma	Sato-IG
arpe	4.99	5.01	5.79	6.08

- The performance of the two model classes is similar with respect to the error. However, it has to be noticed that the two Sato process models employ only 2 parameters, as opposite to the two OU models displaying three free parameters.
- The Sato processes seem to be favorable when the focus is on stability over time of the calibrated parameters. Especially the scaling parameter γ behaves very stable in time for all three companies.

More on stability

- Similar to Cariboni and Schoutens (2006) we conclude the stability analysis by comparing the lag- i autocorrelation of the parameter time series for the various models.

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- Recall, that the lag- i autocorrelation for the time series $X = \{X_t\}_{t=1, \dots, N}$ is given by

$$\rho_i = \frac{\mathbb{E}[(X_t - \mathbb{E}[X_t])(X_{t+i} - \mathbb{E}[X_{t+i}])]}{\sqrt{\mathbb{E}[(X_t - \mathbb{E}[X_t])^2] \mathbb{E}[(X_{t+i} - \mathbb{E}[X_{t+i}])^2]}}$$

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- The autocorrelation might be considered a stability measure of the calibrated parameters.

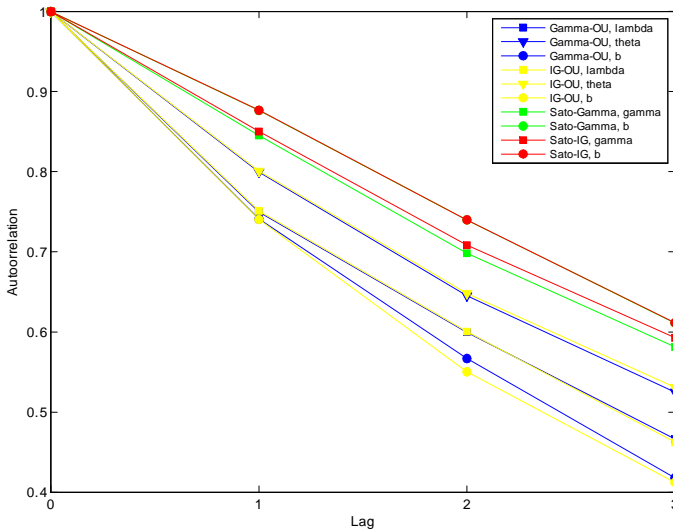














Figure: The average across firms of the autocorrelation for each parameter as a function of lag for the different models.




Coming up (maybe)

- Examination of other specifications of the cumulative hazard with same intensity and compensator as Sato-CH, but not self-similar. To check if self-similarity is really important...
- Multivariate setting via time-change...
- Stochastic recovery rates?

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