Counterparty risk valuation for CDS.

Frédéric Patras, CNRS et Université de Nice

Credit Risk Workshop EVRY

Evry, 27 Juin 2008

Joint work with Christophette Blanchet-Scalliet (Ecole Centrale de Lyon).

Contents of the talk.

- General Introduction to multiname default risk
- Counterparty Risk (C.R.)
- Modelling Counterparty Risk for CDS
- Default probabilities for two-dimensional processes
- Some closed formulas

General Introduction.

- Default risk computations are directly involved in three, closely connected, but still separated areas:
- First, the pricing of single and multiname credit derivatives (ABS, CDS, CDOs...)
- Second, risk valuation for regulatory, risk management and economic capital valuation purposes
- Third, rating assignements.
- The present presentation is focused mainly on the first two aspects of risk valuation. Notice however that the next generation of rating methodologies will most probably have to include advanced correlation tools such as the ones considered here.

General Introduction (2).

- Although deriving new formulas for credit derivative transactions on two assets, our main interest will be on counterparty risk.
- Once again, this is a central argument, both for regulatory purposes (counterparty risk was already a key issue in the Basel 1 agreements, but its relevance and complexity was emphasized by the later improvements of the set of rules), and also for practical risk management.

Lessons form the subprime crisis.

- Concretely, the "subprime" crisis has emphasized how a mispricing of counterparty risk could be dangerous. Before the crisis, financial institutions engaged heavily in credit derivatives transactions for hedging purposes.
- One of the main side effects of the crisis has been to reveal that credit risk counterparties were not reliable: monoline insurers (who act traditionaly as guarantees for municipalities-type bond emissions but engaged more recently in the ABS business) faced downgrades and/or bankruptcy, whereas some companies offering CDS protection have also been on the wedge of bankruptcy due to spread widenings and collateral agreements.
- In spite of these facts, in the present state of the art quantitative counterparty risk assessment is still largely in infancy.

Some classical insights on C.R.

- The example of Basel 1
- Ratings, netting agreements and Basel 2
- Time-dependence of expected exposure for swaps
- Emphasizing Correlation issues

Modelling C.R. on single-name credit derivatives

- We label 1 and 2 the two default-prone entities. The counterparty will be labelled 2.
- We write τ_1 (resp. τ_2) for the random time when the first (resp. second) entity defaults.
- We work in a structural model: defaults are triggered when two log-normal processes V_1 and V_2 fall below a barrier process $v_i(t) := K_i e^{\gamma_i t}$ where
$$\frac{dV_i(t)}{V_i(t)} = (r - k_i)dt + \sigma_i dB_i(t)$$
- r is the constant short-term interest rate and $B_i,\ i = 1,2$ are two correlated Brownian motions $Cov(B_1(t), B_2(t)) = \varrho t$. The coefficient k_i is a payout ratio representing net payouts/inflows by the firm.

Modelling C.R.

- We follow standard market practices. Namely, we assume that the CDS contract has a notional value C and that, if the underlying entity defaults, the buyer of protection receives $(1 - R_u) \cdot C$ from the seller of protection, where R_u stands for the recovery rate of the underlying entity.

- Similarly, if the counterparty (the protection seller) defaults at t on the CDS contract, we assume that the buyer of protection receives $(1 - R_c) \cdot V_t^+$, where $V_t^+ = \sup(0, V_t)$ stands for the positive part of the market value of the CDS contract at t, and where R_c stands for the counterparty recovery rate.

Counterparty default leg

- The coupon payments rate (CDS spread) by the protection buyer is written s. We assume continuous payments of the fees.
- First, the theoretical counterparty default leg D_c is given by

$$D_c \;=\; (1 - R_c) \cdot E[e^{-r\tau_2} \cdot \sup(0, p(V_1(\tau_2), \tau_2)) 1_{\tau_2 < (T \wedge \tau_1)}]$$

where $p(V_1(\tau_2), \tau_2)$ is the market price of the CDS contract at $t = \tau_2$ (when $\tau_1 \geq \tau_2$).

Counterparty default leg (2)

- The value $p(V_1(\tau_2), \tau_2)$ is obtained as the difference between the default and fee legs, that is:

  $$p(V_1(\tau_2), \tau_2) = D_l(V_1(\tau_2), \tau_2) - \frac{sC}{r} \cdot E[(1 - e^{-r((T \wedge \tau_1) - \tau_2)})1_{\tau_1 \geq \tau_2} | \mathcal{F}_{\tau_2}]$$

  where we write $\mathcal{F}_t$, as usual, for the natural filtration of the probability space underlying the two Brownian motions $B_1(t), B_2(t)$.
- Remark : a Monte Carlo approach would require approximating a conditional distribution w.r. to default time τ_2 !!

Counterparty default leg (3)

- Now, the value of $D_l(V_1(\tau_2),\tau_2)$ is given by:

$$D_l(V_1(\tau_2),\tau_2) \;=\; E[C(1-R_u)e^{-r(\tau_1-\tau_2)}1_{\tau_2\le\tau_1\le T}|\mathcal{F}_{\tau_2}]$$

- The counterparty default leg D_c is given by:

$$D_c = C(1-R_c)$$

$$*E\Big[1_{\tau_2<(T\wedge\tau_1)}\Big(e^{-r\tau_2}(1-R_u+\frac{s}{r})\big(e^{-\mu_{\tau_2}(\beta-\alpha)}N(\frac{-\mu_{\tau_2}-\alpha(T-\tau_2)}{\sqrt{T-\tau_2}})$$

$$+e^{-\mu_{\tau_2}(\beta+\alpha)}N(\frac{-\mu_{\tau_2}+\alpha(T-\tau_2)}{\sqrt{T-\tau_2}}))$$

Counterparty default leg (4)

$$-\frac{s}{r}\Big(1-e^{-r(T-\tau_2)}(1-N(\frac{-\mu_{\tau_2}-\beta(T-\tau_2)}{\sqrt{T-\tau_2}})$$

$$-e^{-2\mu_{\tau_2}\beta}N(\frac{-\mu_{\tau_2}+\beta(T-\tau_2)}{\sqrt{T-\tau_2}})))\Big)_+\Big]$$

where

$$\nu_1 := r - k_1 - \gamma_1 - \frac{1}{2}\sigma_1^2$$

$$\alpha := \sqrt{\frac{\nu_1^2}{\sigma_1^2}+2r},\ \beta := \frac{\nu_1}{\sigma_1}$$

$$\mu_{\tau_2} := \frac{\ln(\frac{V_1(\tau_2)}{v_1(\tau_2)})}{\sigma_1}$$

Explicit formulas (see article for other formulas)

- Recall that $\tau_i = \inf\{t, V_i(t) \leq K_i e^{\gamma_i t}\}$.
- The condition $V_i(t) \leq K_i e^{\gamma_i t}$ can be rewritten: $W_i(t) \geq y_0^i$, where $W_i(t) = \ln(\frac{K_i e^{\gamma_i t}}{V_i(t)}) - \ln(\frac{K_i}{V_i(0)})$ and $y_0^i = \ln V_i(0) - \ln K_i$.
- Equivalently, $W_i(t)$ is the diffusion process:

$$dW_i(t) = -\nu_i t - \sigma_i dB_i(t),$$

with $W_i(0) = 0$ and $\nu_i := r - k_i - \gamma_i - \frac{1}{2}\sigma_i^2$.

Explicit formulas (2)

Let us define $\mathbf{Z}(t)$ by:

$$\mathbf{Z}(t) = (Z_1(t), Z_2(t))^* = \frac{1}{\sqrt{1-\varrho^2}} \begin{pmatrix} \sigma_1^{-1} & -\varrho\sigma_2^{-1} \\ 0 & \sqrt{1-\varrho^2}\sigma_2^{-1} \end{pmatrix} \begin{pmatrix} y_0^1 - W_1(t) \\ y_0^2 - W_2(t) \end{pmatrix}.$$

We get:

$$dZ_1(t) = \phi_1 dt + dX_1(t),\ dZ_2(t) = \phi_2 dt + dX_2(t),$$

where $\mathbf{X}(t)$ is a standard planar Brownian motion and

$$\phi_1 = \frac{\nu_1\sigma_2 - \nu_2\sigma_1\varrho}{\sigma_1\sigma_2\sqrt{1-\varrho^2}}, \phi_2 = \frac{\nu_2}{\sigma_2}.$$

Explicit formulas (3)

- In particular, $\mathbf{Z}(t)$ is a 2-dim. Brownian motion with drift and the barrier conditions $V_i(t) = v_i(t)$ now read: $Z_2(t) = 0$ and $\sqrt{1-\varrho^2}Z_1(t) + \varrho Z_2(t) = 0$.

  Applying the Girsanov theorem, $(Z_1(t), Z_2(t))^*$ is a classical Brownian motion for the probability law $\mathbf{Q}$:

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = e^{-\phi_1 X_1(T) - \phi_2 X_2(T) - [\frac{\phi_1^2}{2} + \frac{\phi_2^2}{2}]T} \ \mathbf{P} a.s.$$

Explicit formulas (4)

Let $r_0 e^{i\theta_0} := Z_1(0) + iZ_2(0) = \frac{y_0^1\sigma_2 - \varrho y_0^2\sigma_1}{\sigma_1\sigma_2\sqrt{1-\varrho^2}} + i\,\frac{y_0^2}{\sigma_2}$.

Lemma 1 *We have, for* $(a, 0) \in \mathbf{R}^2$ *s.t.* $a > 0$ *:*

$$\mathbf{P}(\tau_2 \in dt, \tau_2 = \tau_2 \wedge \tau_1, Z_1(\tau_2) \in da) = e^{\phi_1(a - r_0\cos(\theta_0)) - \phi_2 r_0 \sin(\theta_0) - \frac{||\vec{\phi}||^2 t}{2}}$$

$$\frac{\pi}{\alpha^2 ta} e^{-(a^2+r_0^2)/2t} \sum_{n=0}^{\infty} n \sin\frac{n\pi\theta_0}{\alpha} I_{n\pi/\alpha}(\frac{ar_0}{t}) dadt.$$

where $||\vec{\phi}||^2 := \phi_1^2 + \phi_2^2$*,* $\alpha := \arcsin(\varrho) + \frac{\pi}{2}$ *and* $I_{n\pi/\alpha}$ *is the modified Bessel function of index* $n\pi/\alpha$*.*

Explicit formulas (5)

Key argument: computation of the density of a 2-dimensional BM in a wedge, with an absorbing boundary together with computation of corresponding absorption probabilities (to be explained later):

$$f(a,b,t)dadb = \mathbf{Q}(\mathbf{Z}(t) \in (da,db), \tau_1 \wedge \tau_2 > t)$$

$$= \frac{2\mu}{\alpha t} e^{-(\mu^2+r_0^2)/2t} \sum_{n=0}^{\infty} \sin\frac{n\pi\theta}{\alpha} \sin\frac{n\pi\theta_0}{\alpha} I_{n\pi/\alpha}(\frac{\mu r_0}{t}) d\mu d\theta,$$

where $\theta := arctg(\frac{b}{a}),\ \mu := \sqrt{a^2+b^2}$.

Explicit formulas (6)

Theorem 2 *The counterparty default leg* D_c *of the CDS is given by:*

$$
\begin{aligned}
D_c \;=\; & C(1-R_c)\int_0^T\int_0^{+\infty}\tilde{h}(\mu,t)e^{\phi_1(\mu-r_0\cos(\theta_0))-\phi_2 r_0\sin(\theta_0)-\frac{||\vec{\phi}||^2 t}{2}} \\
& \frac{\pi}{\alpha^2 t\mu}e^{-(\mu^2+r_0^2)/2t}\sum_{n=0}^{\infty} n\sin\frac{n\pi\theta_0}{\alpha} I_{n\pi/\alpha}\left(\frac{\mu r_0}{t}\right)dt\,d\mu .
\end{aligned}
$$

Explicit formulas (7)

Where the function $\tilde{h}$ is obtained from the computation of the theoretical counterparty default leg D_c.

$h(x,t) :=$

$$\Big(e^{-rt}(1 - R_u + \frac{s}{r})(e^{-\mu_{x,t}(\beta-\alpha)}N(\frac{-\mu_{x,t} - \alpha(T-t)}{\sqrt{T-t}})$$

$$+e^{-\mu_{x,t}(\beta+\alpha)}N(\frac{-\mu_{x,t} + \alpha(T-t)}{\sqrt{T-t}}))$$

$$-\frac{s}{r}(1 - e^{-r(T-t)}(1 - N(\frac{-\mu_{x,t} - \beta(T-t)}{\sqrt{T-t}})$$

$$-e^{-2\mu_{x,t}\beta}N(\frac{-\mu_{x,t} + \beta(T-t)}{\sqrt{T-t}})))\Big)_+,$$

$$\mu_{x,t} := \sigma_1^{-1}(\nu_1 t + \sigma_1 x + \ln V_1(0) - \ln K_1)$$

and

$$\tilde{h}(z,t) = h(\frac{-y_0^1 - \nu_1 t}{\sigma_1} + \sqrt{1-\rho^2} \cdot z, t).$$