# Recovering portfolio default intensities implied by CDO quotes: solution of an inverse problem via intensity control

Rama CONT
Columbia University, New York
Andreea MINCA (Ecole Polytechnique, France)
Ioana SAVESCU (Merrill Lynch)

Based on:

R Cont & A Minca (2008) Recovering portfolio default intensities implied by CDO quotes,
Columbia Financial Engineering Report No. 2008-01.

R Cont & I Savescu : Forward equations for portfolio credit derivatives, in:
*New Frontiers in Quantitative Finance: Credit risk and volatility modeling*, Wiley (2008).

Available on: **www.ssrn.com**

## Outline

- CDOs and portfolio credit derivatives
- Top-down pricing models for portfolio credit derivatives
- Loss process and default intensity
- Reconstruction of the default intensity by relative entropy minimization under constraints
- Interpretation of dual problem as intensity control problem
- Nonlinear representation as expectation
- Numerical solution and implementation
- Application to ITRAXX CDO data

# Portfolio credit derivatives

Contracts whose payoffs depend on the losses due to defaults in some underlying reference portfolio (of loans, bonds or credit default swaps).

Most common example: Collateralized Debt Obligations (CDOs).

Challenge: develop stochastic models for portfolio credit losses which can be

- computationally tractable: easy pricing of simple instruments (index default swap, single tranche CDOs,..)
- can match market observation: efficient/ implementable *calibration* algorithms
- capture *dynamics* of the portfolio $\rightarrow$ scenario simulation, hedging

## ITRAXX CDO tranches

| Maturity | Low | High | Bid\ Upfront | Ask\ Upfront |
|---|---|---|---|---|
| 5Y | 0% | 3% | 11.75% | 12.00% |
| | 3% | 6% | 53.75 | 55.25 |
| | 6% | 9% | 14.00 | 15.50 |
| | 9% | 12% | 5.75 | 6.75 |
| | 12% | 22% | 2.13 | 2.88 |
| | 22% | 100% | 0.80 | 1.30 |
| 7Y | 0% | 3% | 26.88% | 27.13% |
| | 3% | 6% | 130 | 132 |
| | 6% | 9% | 36.75 | 38.25 |
| | 9% | 12% | 16.50 | 18.00 |
| | 12% | 22% | 5.50 | 6.50 |
| | 22% | 100% | 2.40 | 2.90 |

| Maturity | Low | High | Bid\ Upfront | Ask\ Upfront |
|---|---|---|---|---|
| 10Y | 0% | 3% | 41.88% | 42.13% |
| | 3% | 6% | 348 | 353 |
| | 6% | 9% | 93 | 95 |
| | 9% | 12% | 40 | 42 |
| | 12% | 22% | 13.25 | 14.25 |
| | 22% | 100% | 4.35 | 4.85 |

Table 1: ITRAXX tranche spreads, in bp. For the equity tranche the periodic spread is 500bp and figures represent upfront payments.

## Ingredients

- Portfolio (index) with $n$ names ( e.g. $n = 125$)
- Number of defaults $N_t$
- Default dates $\tau_i, i = 1..n$
- Recovery rate $R_i$
- Risk-free discount factor $B(t, T)$
- Portfolio loss (as percentage of total nominal):
  $L_t = \frac{1}{N} \sum_{i=1}^{n} (1 - R_i) 1_{\tau_i \leq t}$
- Tranche attachment/detachment points $0 \leq a < b \leq 1$.
- Tranche loss: $L_{a,b}(t) = (L(t) - a)^+ - (L(t) - b)^+$

## Cash flow structure of a CDO tranche

Default leg: tranche loss due to defaults between $t_{j-1}$ and $t_j$

$$
\begin{aligned}
\text{Cash flow at } t_j \qquad & N[L_{a,b}(t_j) - L_{a,b}(t_{j-1})] \\
\text{Value at } t = 0 \qquad & N \sum_{j=1}^{J} B(0,t_j) E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1})\ ] \qquad (1) \\
= \qquad & N \sum_{j=1}^{J} B(0,t_j) E^{\mathbb{Q}}[(L(t_j) - a)^+ - (L(t_j) - b)^+ \\
& \qquad -(L(t_{j-1}) - a)^+ \ + (L(t_{j-1}) - b)^+]
\end{aligned}
$$

Similar to pricing of a portfolio of calls on $L(t)$.

Requires knowledge of the risk neutral distribution of total portfolio loss $L(t)$

Premium leg: pays fixed spread S(a,b) at dates $t_j$ on remaining principal

Cash flow at $t_j$ $\qquad S(a,b)N(t_j - t_{j-1})[(b - L(t_j))^+ - (a - L(t_j))^+]$

Value at $t = 0$ $\qquad S(a,b)N \sum_{j=1}^{J} B(0,t_j)(t_j - t_{j-1})$

$$E^{\mathbb{Q}}[(b - L(t_j))^+ - (a - L(t_j))^+]$$

Computation of $E^{\mathbb{Q}}[(L(t_j) - K)^+]$ requires knowledge of the (risk neutral) distribution of total loss $L(t_j)$ which depends on dependence among defaults

Fair spread of a CDO tranche swap with attachment point $a$ and detachment $b$ initiated at $t = 0$:

$$S_0(a,b) = \frac{\sum_{j=1}^{J} B(0,t_j) E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1})\ ]}{\sum_{j=1}^{J} B(0,t_j)(t_j - t_{j-1}) E^{\mathbb{Q}}[(b - L(t_j))^+ - (a - L(t_j))^+]}$$

Computation of CDO spread involves $E^{\mathbb{Q}}[(L(t_j) - K)^+]$ which requires knowledge of the (risk neutral) distribution of total loss $L(t_j)$

Mark to market value of the value of a protection seller on the tranche: premium leg- default leg

$$
\begin{aligned}
MTM(t) &= NS_0(a,b)\sum_{t_j>t} B(t,t_j)\delta_j E^{\mathbb{Q}}[(b-L(t_j))^+ - (a-L(t_j))^+|\mathcal{F}_t] \\
&\quad -N\sum_{t_j>t} B(t,t_j)E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1})|\mathcal{F}_t] \\
&= N(b-a)\sum_{t_j>t} B(t,t_j)[S_0(a,b)\delta_j(1-P_{a,b}(t,t_j))-
\end{aligned}
$$

$$
= [S_0(a,b) - S_t(a,b)]\ N\sum_{t_j>t} B(t,t_j)\delta_j E^{\mathbb{Q}}[(b-L(t_j))^+ - (a-L(t_j))^+|\mathcal{F}_t]
$$

where $\delta_j = t_j - t_{j-1}$.

## Bottom-up approach in credit portfolio modeling

Idea:

- calibrate implied default probabilities for portfolio components to credit default swap term structures
- add extra ingredient (copula, dependence structure) to obtain joint distribution $F(t_1, .., t_n)$ of default times ($n$-dimensional probability distribution)
- Use numerical procedure to compute the risk–neutral distribution of portfolio loss $L_t$ from $F$: recursion methods, FFT, quadrature, Monte Carlo,...
- Imply correlation parameters from tranche spreads

Issues:

- High dimensional models: $n \simeq 100 - -500$.
- Need to separate joint distribution into copula + marginals and parameterize them separately otherwise calibration to CDS and CDO tranches cannot be separated $\rightarrow$ high-dimensional nonlinear optimization problem
- scarcity of data $\rightarrow$ crude parametrization of joint distribution/copula $\rightarrow$ restrictions on default dependence structure.

## Default time copula models

Copula models

- are unable to reproduce implied correlations for quoted CDO tranches in a simple manner ("correlation skew").
- are static: no dynamics for state variables, no way to update prices as time goes on, no self-consistent hedge.
- do not tell us how to compute conditional default probabilities, forward tranche prices,...

## Dynamic bottom-up models

Prototype: Duffie & Garleanu (2005)

Default in each of $i = 1..N \sim 100$ names driven by a random intensity process $\lambda^i(t)$ modeled as an affine jump-diffusion

$$\lambda^i(t) = \sqrt{\rho}\lambda^0(t) + \sqrt{1-\rho}\lambda_i(t) \tag{2}$$

$$d\lambda_i(t) = (a + b\lambda_i)dt + c\sqrt{\lambda_i(t)}dW_t^i + dJ_i(t) \tag{3}$$

Parameter $\rho$ is difficult to calibrate: it cannot be calibrated separately from parameters describing dynamics of $N$ individual spreads.

As a result, getting market-consistent prices is a challenge...

Recall the expression for a CDO tranche spread

$$S_t(a,b) = \frac{\sum_{j=1}^{m} B(0,t_j) E^{\mathbb{Q}}[L_{a,b}(t_j) - L_{a,b}(t_{j-1}) \ |\mathcal{F}_t]}{\sum_{j=1}^{m} B(0,t_j)\delta_j E^{\mathbb{Q}}[(b - L(t_j))^+ - (a - L(t_j))^+ |\mathcal{F}_t]}$$

Key observation: only involves the (conditional) distribution of total portfolio loss $L_t$:

$$p_{t,T}(x) = \mathbb{Q}(\ L_T \leq x|\mathcal{F}_t) \tag{4}$$

## The top-down approach

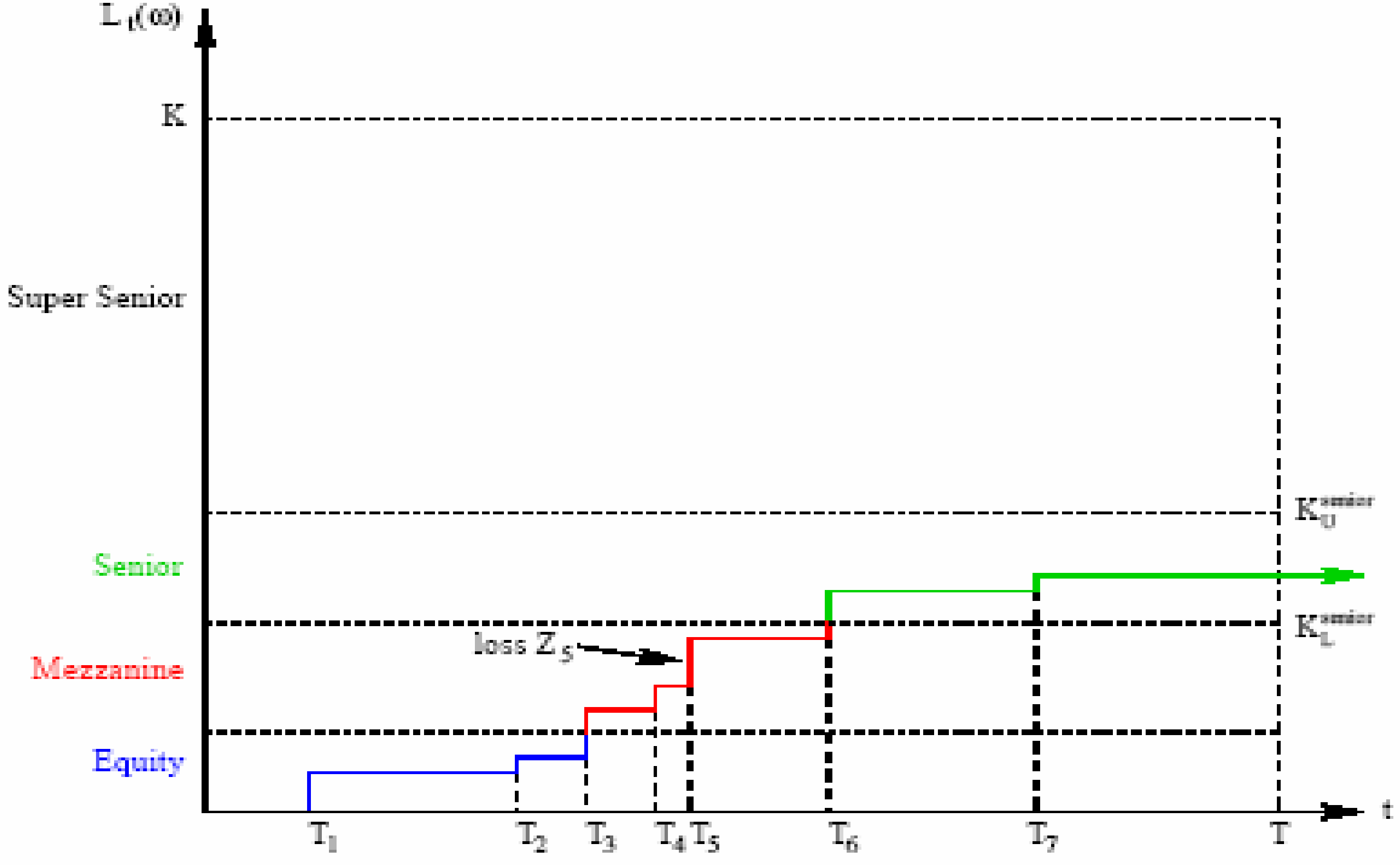
Idea: model risk–neutral/ market–implied dynamics of portfolio loss $L_t$.

Loss $L_t$ is a jump process with increasing, piecewise constant sample paths, whose jump times $T_j$ are the default events and whose jump sizes $L_j$ are default losses:

$$L_t = \frac{1}{N}\sum_{i=1}^{n} N_i(1 - R_i)1_{\tau_i \leq t} = \sum_{j=1}^{N_t} L_j \tag{5}$$

where $N_t = \sum_{i=1}^{n} 1_{\tau_i \leq t}$ is the number of defaults in portfolio before $t$ and $L_j$ is loss at $j$-th default event.

# Sample path of the loss process



Default intensity Idea: model the occurrence of jumps via the *aggregate default intensity* $\lambda_t$.

$N_t$ is said to have $(\mathcal{F}_t)_{t\in[0,T^*]}-$ intensity $\lambda_t$ under $\mathbb{Q}$ if

$$N_t - \int_0^t \lambda_u du$$

is a $\mathbb{Q}$-local martingale.

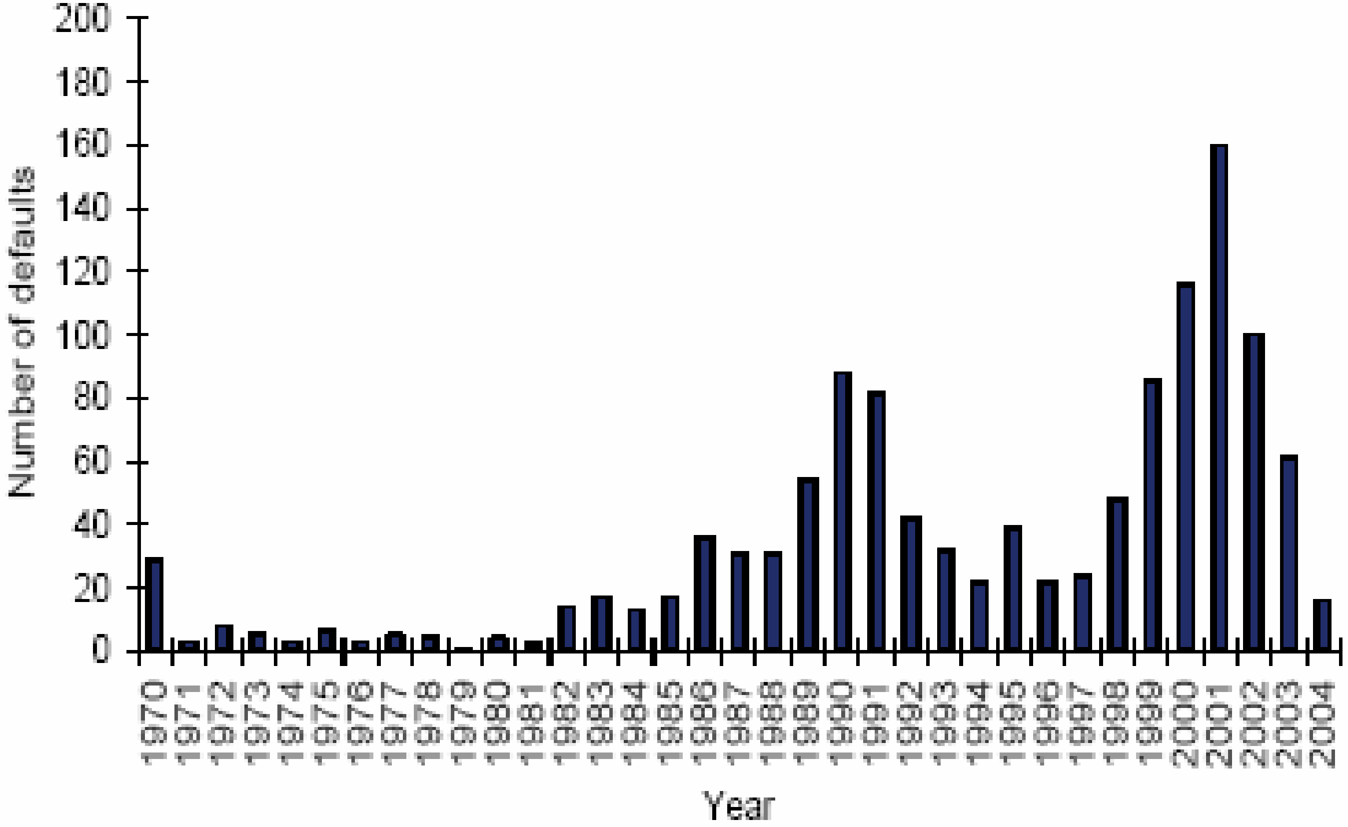
Intuitively: probability per unit time of the next default conditional on current market information

$$\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{Q}[N_{t+\Delta t} = N_t + 1 | \mathcal{F}_t]$$

Market convention: $L_j = (1-R)/N$ is constant.

Important: no specific assumption on filtration.

# Clustering of defaults



Wide variety of specifications for portfolio default intensity:

- Intensity $\lambda_t$ of (next) default event:

$$\lambda_t(\omega) = \lim_{\Delta t \to 0} \frac{\mathbb{Q}(N(t+\Delta t) = N(t-) + 1 | \mathcal{F}_t)}{\Delta t}$$

  - Compound Poisson: $\lambda_t = f(t)$ (Brigo & Pallavicini 05)
  - Cox process: default intensity driven by other "market factors", not by default itself (Longstaff & Rajan)

$$d\lambda_t = \mu(t, \lambda_t)dt + \sigma(\lambda_t)dW_t$$

  - Continuous-time Markov process: $\lambda_t = f(t, N_t)$
    Example: Herbertsson model $\lambda_t = (n - N_t) \sum_{k=1}^{N_t} b_k$
  - Dependence on history of defaults/ losses (Hawkes process, Giesecke & Goldberg):

$$\lambda_t = g(t_j, L_j, j = 1..N_t - 1)$$

Which one to pick? how to choose its parameters (loss intensity) consistently with market observations of CDO spreads?

## Information content of credit portfolio derivatives

Market observations consist of fair spreads for (index) CDO tranches. These can be represented in terms of expected tranche notionals

$$C(t_j, K_i) = C_i = E^{\mathbb{Q}}[(K_i - L_{t_j})^+] \tag{6}$$

Common procedure is to "strip" CDO spreads to get expected tranche notionals $C(t_j, K_i)$ and then calibrate these using a model.

Problem: we need $C(t_j, K_i)$ for all payment dates $t_j$: many more than data observed! Ill-posed linear problem $\rightarrow$ parametrization of $C(.,.)$ / interpolation usually used

Here we will avoid this step and use a nonparametric approach

## Information content of credit portfolio derivatives

**Proposition 1.** *Consider any non-explosive jump process $(L_t)_{t\in[0,T^*]}$ with a intensity process $(\lambda_t(\omega))_{t\in[0,T^*]}$ and IID jumps with distribution $F$. Define $(\tilde{L}_t)_{t\in[0,T^*]}$ as the Markovian jump process with jump size distribution $F$ and intensity*

$$\lambda_{\text{eff}}(t,l) = E^{\mathbb{Q}}[\lambda_t | L_{t-} = l, \mathcal{F}_0] \tag{7}$$

*Then, for any $t \in [0,T^*]$, $L_t$ and $\tilde{L}_t$ have the same distribution conditional on $\mathcal{F}_0$. In particular, the flow of marginal distributions of $(L_t)_{t\in[0,T^*]}$ only depends on the intensity $(\lambda_t)_{t\in[0,T^*]}$ through its conditional expectation $\lambda_{\text{eff}}(.,.)$.*

Analogy with *local volatility*.

*Proof.* Consider any bounded measurable function $f(.)$. Using the pathwise decomposition of $L_T$ into the sum of its jumps we can write

$$f(L_T) = f(L_0) + \sum_{0<s\leq T} (f(L_{s-} + \Delta L_s) - f(L_{s-})\ ) \qquad (8)$$

so

$$\begin{aligned} E[f(L_T)|\mathcal{F}_0] &= f(L_0) + E[\sum_{0<s\leq T} (f(L_{s-} + \Delta L_s) - f(L_{s-})\ )|\mathcal{F}_0] \\ &= f(L_0) + \int_0^T dt \quad E[(f(L_{t-} + \Delta L_t) - f(L_{t-})\ )\lambda_t|\mathcal{F}_0] \end{aligned}$$

Denote

$$\mathcal{G}_t = \sigma(\mathcal{F}_0 \vee L_{t-})$$

the information set obtained by adding the knowledge of $L_{t-}$ to the current information set $\mathcal{F}_0$. Define the *local intensity* function

$$\lambda_{\text{eff}}(t, l) = E^{\mathbb{Q}}[\lambda_t|\mathcal{F}_0, L_{t-} = l]. \qquad (9)$$

Noting that $\mathcal{F}_0 \subset \mathcal{G}_t$ we have

$$E[(f(L_{t-} + \Delta L_t) - f(L_{t-})\ )\lambda_t|\mathcal{F}_0]$$
$$= E[E[(f(L_{t-} + \Delta L_t) - f(L_{t-})\ )\lambda_t|\mathcal{G}_t]|\mathcal{F}_0]$$
$$= E[\int_0^1 F(dy)\ (f(L_{t-} + y) - f(L_{t-})\ E[\lambda_t|\mathcal{G}_t]|\mathcal{F}_0]$$
$$= E[\lambda_{\text{eff}}(t, L_{t-}) \int F(dy)\ (f(L_{t-} + y) - f(L_{t-})\ |\mathcal{F}_0] \qquad \text{so}$$
$$E[f(L_T)|\mathcal{F}_0] = f(L_t) +$$
$$E[\int_0^T dt\ \lambda_{\text{eff}}(t, L_{t-}) \int F(dy)\ (f(L_{t-} + y) - f(L_{t-})\ |\mathcal{F}_0]$$

The above equality shows that $E[f(L_T)|\mathcal{F}_0] = E[f(\tilde{L}_T)|\mathcal{F}_0]$ where $(\tilde{L}_t)_{0 \leq t \leq T}$ is the Markovian loss process with intensity $\gamma_t = \lambda_{\text{eff}}(t, \tilde{L}_{t-})$ and jump size distribution $F$ hence $\tilde{L}_t =^d L_t$. $\square$

**Corollary 1** (Information content of non-path dependent portfolio credit derivatives). *The value* $E^{\mathbb{Q}}[f(L_T)|\mathcal{F}_0]$ *at* $t = 0$ *of any derivative whose payoff depends on the aggregate loss* $L_T$ *of the portfolio at on a fixed grid of dates, only depends on the default intensity* $(\lambda_t)_{t\in[0,T^*]}$ *through its risk-neutral conditional expectation with respect to the current loss level:*

$$\lambda_{\text{eff}}(t,l) = E^{\mathbb{Q}}[\lambda_t | L_{t-} = l, \mathcal{F}_0] \tag{10}$$

*In particular, CDO tranche spreads and mark-to-market value of CDO tranches only depends on the transition rate* $(\lambda_t)_{t\in[0,T^*]}$ *through the effective default intensity* $\lambda_{\text{eff}}(.,.)$*.*

Forward equation for expected tranche loss (Cont & Savescu 2007)

Proposition The expected tranche loss $C(T,K) = E^{\mathbb{Q}^\lambda}[(K - L_T)^+]$ solves a (Dupire-type) forward equation

$$\begin{aligned} &\frac{\partial C(T,K)}{\partial T} - C(T, K-\delta)\lambda_k(T) + \lambda_{k-1}(T) C(T,K) \\ &+ \sum_{j=1}^{k-2} [\lambda_{j+1}(T) - 2\lambda_j(T) + \lambda_{j+1}(T)]\, C(T, j\delta) = 0 \end{aligned} \tag{11}$$

where $\lambda_k(T) = \lambda_{\text{eff}}(T, k\delta)$ and $\delta = (1-R)/N$.

This bidiagonal system of ODEs can be solved efficiently with high-order time stepping schemes (e.g. Runge Kutta).

**Problem 1** (Calibration problem)**.** *Given a set of observed CDO tranche spreads* $(S_0(K_i, K_{i+1}, T_k), i = 1..I-1, k = 1..m)$ *for a reference portfolio, construct a (risk–neutral) default rate/ loss intensity* $\lambda = (\lambda_t)_{t\in[0,T]}$ *such that the spreads computed under the model* $\mathbb{Q}^\lambda$ *match the market observations*

$$S_0(K_i, K_{i+1}, T_k) = \frac{E^{\mathbb{Q}^\lambda} \sum_{t_j \le T_k} B(0,t_j)[L_{K_i,K_{i+1}}(t_j) - L_{K_i,K_{i+1}}(t_{j-1})\ ]}{E^{\mathbb{Q}^\lambda} \sum_{t_j \le T_k} B(0,t_j)(t_j - t_{j-1})[(K_{i+1} - L(t_j))^+ - (K_i - L(t_j))^+]}$$

## Calibration by Relative entropy minimization under constraints

One period case: Buchen & Kelly, Avellaneda 1998

Diffusion models: Avellaneda Friedman Holmes Samperi 1997

Monte Carlo setting: Avellaneda et al 2001

Lévy processes: Cont & Tankov 2004, 2006)

Given market prices $C(K_i)$ of tranche payoffs and a prior guess $\lambda^0$ for the loss intensity process, the reconstruction of the default intensity process $(\lambda_t)_{t\in[0,T^*]}$ can be formalized as

$$\inf_{\mathbb{Q}^\lambda \in \Lambda} E^{\mathbb{Q}_0}[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0} \ln \frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}] \tag{12}$$

under the constraint that the model $\mathbb{Q}^\lambda$ prices correctly the observed CDO tranches, where $\mathbb{Q}^\lambda$ is the law of the point process with intensity process $\lambda$ and $\mathbb{Q}_0$ is the law of the point process with intensity $\lambda^0$.

**Problem 2** (Calibration via relative entropy minimization)**.** *Given a prior loss process with law $\mathbb{Q}_0$, find a default intensity $(\lambda_t)_{t\in[0,T^*]}$ which minimizes*

$$\inf_{\mathbb{Q}^\lambda\in\Lambda} E^{\mathbb{Q}_0}[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}\ln\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}] \quad \text{under} \quad E^{\mathbb{Q}^\lambda}[H_{i,k}]=0 \tag{13}$$

$$H_{ik} = S_0(K_i, K_{i+1}, T_k)\sum_{t_j\leq T_k} B(0,t_j)(t_j - t_{j-1})[(K_{i+1}-L(t_j))^+ - (K_i - L(t_j))^+]$$
$$-\sum_{t_j\leq T_k} B(0,t_j)[(K_{i+1}-L(t_j))^+ - (K_i-L(t_j))^+ - (K_{i+1}-L(t_{j-1}))^+ + (K_i - L(t_{j-1}))^+))\,] \tag{14}$$

*and $\mathbb{Q}^\lambda$ denotes the law of the point process with intensity $(\lambda_t)_{t\in[0,T^*]}$ and $\mathbb{Q}_0$ is the law of the point process with intensity $\lambda^0$.*

Using the previous result we can restrict $\Lambda$ to *Markovian intensities* $\lambda(t, L_t)$.

## Computation of entropy

Equivalent change of measure for point processes (Jacod 1980, Bremaud 1981)

**Proposition 2.** *Let $N_t$ be a Poisson process with intensity $\gamma_0$ on $(\Omega, \mathcal{F}_t, \mathbb{Q}_0)$ and $\lambda = (\lambda_t)_{t\in[0,T]}$ be an $\mathcal{F}_t$-predictable process with*

$$\int_0^t \lambda_s ds < \infty \quad \mathbb{Q}_0 - a.s. \tag{15}$$

*Define the probability measure $\mathbb{Q}^\lambda$ on $\mathcal{F}_T$ by*

$$\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0} = Z_T \quad \text{where} \quad Z_t = \left( \prod_{\tau_j \le t} \frac{\lambda_{\tau_j}}{\gamma_0} \right) \exp\left\{ \int_0^t (\gamma_0 - \lambda_s)\, ds \right\}$$

*Then $N_t$ is a point process with $\mathcal{F}_t$ intensity $(\lambda_t)_{t\in[0,T]}$ under $\mathbb{Q}^\lambda$.*

**Proposition 3** (Computation of relative entropy)**.** *Denote by*

- $\mathbb{Q}_0$ *the law on* $[0,T]$ *of a (standard unit intensity) Poisson process and*
- $\mathbb{Q}^\lambda$ *the law on* $[0,T]$ *of the point process with intensity* $(\lambda_t)_{t\in[0,T]}$ *verifying* $\int_0^t \lambda_s ds < \infty \quad \mathbb{Q}_0 - a.s.$

*The relative entropy of* $\mathbb{Q}^\lambda$ *with respect to* $\mathbb{Q}_0$ *is given by:*

$$E^{\mathbb{Q}_0}[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0} \ln \frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}] = E^{\mathbb{Q}^\lambda}[\int_0^T \lambda_t \ln \lambda_t dt + T - \int_0^T \lambda_t dt] \qquad (16)$$

# Duality

Define the Lagrangian

$$\mathcal{L}(\lambda, \mu) = E^{\mathbb{Q}^\lambda}[\int_0^T \lambda_s \ln \lambda_s ds + T - \int_0^T \lambda_s ds - \sum_{i=1}^{I} \sum_{k=1}^{m} \mu_{i,k} H_{ik}]$$

Using convex duality arguments, the primal problem:

$$\inf_{\mathbb{Q}^\lambda \in \Lambda} E^{\mathbb{Q}_0}[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0} \ln \frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}] \quad \text{under} \quad E^{\mathbb{Q}^\lambda}[H_{ik}] = 0 \tag{17}$$

is equivalent to the dual problem

$$\sup_{\mu \in \mathbb{R}^{m.I}} \inf_{\lambda \in \Lambda} E^{\mathbb{Q}^\lambda}[\int_0^T \lambda_s \ln \lambda_s ds + T - \int_0^T \lambda_s ds - \sum_{i=1}^{I} \sum_{k=1}^{m} \mu_{i,k} H_{ik}] \tag{18}$$

# Intensity control problem

An *intensity control* problem is an optimization problem with a criterion of the type

$$E^{\mathbb{Q}^\lambda}[\int_0^T \varphi(t,\lambda_t,L_t)dt + \sum_{j=1}^{J} \Phi_j(t_j,L_{t_j})],$$

where $\varphi(t,\lambda_t,N_t)$ is a *running cost* and $\Phi_j(t_j,L_{t_j})$ represents the terminal cost. Here

$$\varphi(t,\lambda,L) = \lambda \ln \lambda + 1 - \lambda \qquad \text{and} \quad \Phi_j(t_j,L_{t_j}) = \sum_{i=1}^{I} M_{ij}(K_i - L_{t_j})^+$$

where $M_{ij}$ are constants depending on contract features and the initial discount curve.

## Single horizon case

$$\inf_{\lambda\in\Lambda([0,T])} E^{\mathbb{Q}^\lambda}[\int_0^T (\lambda_t \ln \lambda_t + 1 - \lambda_t)dt + \Phi(T, L_T)],$$

Solution by dynamic programming: introduce the value function

$$V(t,k) = E^{\mathbb{Q}^\lambda}[\int_0^T \varphi(t, \lambda_t, L_t)dt + \Phi(T, L_T)|N_t = k]$$

The value function can be characterized in terms of a Hamilton Jacobi equation (Bismut 1975, Bremaud 1982).

**Proposition 4.** *(Hamilton-Jacobi equations) Suppose there exists a bounded function* $V : [0, T^*] \times N \to V(t, n)$ *differentiable in* $t$*, such that*

$$\frac{\partial V}{\partial t}(t,k) + \inf_{\lambda \in ]0, infty[} \{\lambda [V(t,k+1) - V(t,k)] + \lambda \ln \lambda - \lambda + 1\} = 0 \quad (19)$$

$$\text{for} \quad t \in [0,T] \quad \text{and} \quad V(T,k) = \Phi(T, k\delta) \quad (20)$$

*and suppose there exists for each* $n \in N^+$ *an* $\mathcal{F}_t$*-predictable mapping* $t \to u^*(t, N_t)$ *such that for each* $n \in N^+$, $t \in [t_0, T]$

$$\lambda^*(t,k) = \underset{\lambda \in ]0,\infty[}{argmin} \{\lambda [V(t,k+1) - V(t,k)] + \lambda \ln \lambda - \lambda + 1\} \quad (21)$$

*Then* $\lambda_t^* = \lambda^*(t, N_t)$ *is an optimal control. Moreover* $V(t_0, N_{t_0}) = \inf_{\lambda \in \Lambda_t} E^{\mathbb{Q}_\lambda}[\int_{t_0}^T \varphi(t, \lambda_t, L_t) ds + \Phi_T(\lambda) | \mathcal{F}_{t_0}]$.

In our problem, in the case of a single maturity, the dual problem is an intensity control problem with running cost

$$(\ln \lambda(t, N_t) - 1)\lambda(t, N_t) + 1$$

and terminal cost is of the type $\Phi_j(L) = \sum M_{ij}(K_i - L)^+$.

The Hamilton Jacobi equations are given by

$$\frac{\partial V}{\partial t}(t,n) + \inf_{\lambda \in \Lambda} \{\lambda[V(t,n+1) - V(t,n)] + (\ln \lambda(t,n) - 1)\lambda(t,n) + 1) = 0$$

which is a system of $n = 125$ coupled nonlinear ODEs.

The maximum in the nonlinear term can be explicitly computed:

$$\lambda^*(t,n) = e^{-[V(t,n+1)-V(t,n)]} \tag{22}$$

$$\frac{\partial V}{\partial t}(t,n) + 1 - e^{-[V(t,n+1)-V(t,n)]} = 0 \tag{23}$$

$$V(T,k) = \Phi(T,k) \tag{24}$$

**Proposition 5** (Value function)**.** *Consider any terminal condition $\Phi$ such that $\Phi(x) = 0$ for $x > n_0\delta$. Then the solution of (23)-24 is given by*

$$V(t,k) = -\ln[1 + \sum_{j=0}^{n_0-k} e^{-(T-t)} \frac{(T-t)^j}{j!} (e^{-\Phi(T,(k+j)\delta)} - 1)] \tag{25}$$

The key is to note that if we consider the exponential change of variable $u(t,k) = e^{-V(t,k)}$ then $u$ solves a *linear* equation

$$\frac{\partial u(t,k)}{\partial t} + u(t,k+1) - u(t,k) = 0 \qquad \text{with} \quad u(T,k) = \exp(-\Phi(T,k\delta))$$

which is recognized as the backward Kolmogorov equation associated with the Poisson process (i.e. the prior process, with law $\mathbb{Q}_0$). The solution is thus given by the Feynman-Kac formula

$$u(t,k;\mu) = E^{\mathbb{Q}_0}[e^{-\Phi(T,\delta N_T)}|N_t = k] = E^{\mathbb{Q}_0}[e^{-\Phi(T,k\delta+\delta N_{T-t})}]$$

using the Markov property and the independence of increments of the Poisson process. The expectation is easily computed using the Poisson distribution, where the sum over jumps can be truncated using the fact that $\Phi(x) = 0$ for $x \geq n\delta$:

$$
\begin{aligned}
u(t,k;\mu) &= \sum_{j=0}^{n-k} e^{-(T-t)} \frac{(T-t)^j}{j!} e^{-\Phi(T,(k+j)\delta)} + \sum_{j>n-k} e^{-(T-t)} \frac{(T-t)^j}{j!} \\
&= \sum_{j=0}^{n-k} e^{-(T-t)} \frac{(T-t)^j}{j!} e^{-\Phi(T,(k+j)\delta)} + 1 - \sum_{j=0}^{n-k} e^{-(T-t)} \frac{(T-t)^j}{j!} \\
&= 1 + \sum_{j=0}^{n-k} e^{-(T-t)} \frac{(T-t)^j}{j!} [e^{-\Phi(T,(k+j)\delta)} - 1] \qquad (26)
\end{aligned}
$$

which leads to (25).

# Case of several maturities

Recursive algorithm via dynamic programming principle

1. Start from the last payment date $j = J$ and set $F_J(k) = \Phi_J(t_J, \delta k)$.
2. Solve the Hamilton–Jacobi equations (23) on $]t_{j-1}, t_j]$ backwards starting from the terminal condition

$$V(t_j, k) = F_j(k) \tag{27}$$

   which can be explicitly solved to yield $V(t, k; \mu)$ on $t \in ]t_{j-1}, t_j]$ using (25).
3. Set $F_{j-1}(k) = V(t_{j-1}, k) + \Phi_{j-1}(t_{j-1}, k\delta)$
4. Go to step 2 and repeat.

Discontinuities may appear in value function at junction dates.

## Calibration algorithm

1. Solve the dynamic programming equations (23)–(24) $\mu \in \mathbb{R}^I$ to compute $V(0,0,\mu)$.
2. Optimize $V(0,0,\mu)$ over $\mu \in \mathbb{R}^{I\times J}$ using a gradient–based method:

$$\inf_{\mu\in\mathbb{R}^I} V(0,0,\mu) = V(0,0,\mu^*) = V^*(0,0)$$

3. Compute the calibrated default intensity (optimal control) as follows:

$$\lambda^*(t,k) = e^{V^*(t,k)-V^*(t,k+1)} \tag{28}$$

4. Compute the term structure of loss probabilities by solving the Fokker-Planck equations.
5. The calibrated default intensity $\lambda^*(.,.)$ can then be used to compute CDO spreads for different tranches, forward tranches

etc. in a straightforward manner: first we compute the expected tranche loss $C(T,K)$ by solving the forward equation:

$$\frac{\partial C(T,K)}{\partial T} - C(T,K-\delta)\lambda_k(T) + \lambda_{k-1}(T)C(T,K) + \sum_{j=1}^{k-2}\left[\lambda_{j+1}(T) - 2\lambda_j(T) + \lambda_{j+1}(T)\right] C(T,j\delta) = 0 \qquad (29)$$

where $\lambda_k(T) = \lambda_{\text{eff}}(T,k\delta)$. In particular the calibrated default intensity can be used to "fill the gaps" in the base correlation surface in an arbitrage-free manner, by first computing the expected tranche loss for all strikes and then computing the spread/"base correlation" for that strike.

# Empirical results: ITRAXX

| Maturity | Low | High | Bid\ Upfront | Ask\ Upfront |
|---|---|---|---|---|
| 5Y | 0% | 3% | 11.75% | 12.00% |
| | 3% | 6% | 53.75 | 55.25 |
| | 6% | 9% | 14.00 | 15.50 |
| | 9% | 12% | 5.75 | 6.75 |
| | 12% | 22% | 2.13 | 2.88 |
| | 22% | 100% | 0.80 | 1.30 |
| 7Y | 0% | 3% | 26.88% | 27.13% |
| | 3% | 6% | 130 | 132 |
| | 6% | 9% | 36.75 | 38.25 |
| | 9% | 12% | 16.50 | 18.00 |
| | 12% | 22% | 5.50 | 6.50 |
| | 22% | 100% | 2.40 | 2.90 |

| Maturity | Low | High | Bid\ Upfront | Ask\ Upfront |
|---|---|---|---|---|
| 10Y | 0% | 3% | 41.88% | 42.13% |
| | 3% | 6% | 348 | 353 |
| | 6% | 9% | 93 | 95 |
| | 9% | 12% | 40 | 42 |
| | 12% | 22% | 13.25 | 14.25 |
| | 22% | 100% | 4.35 | 4.85 |

Table 2: ITRAXX tranche spreads, in bp. For the equity tranche the periodic spread is 500bp and figures represent upfront payments.

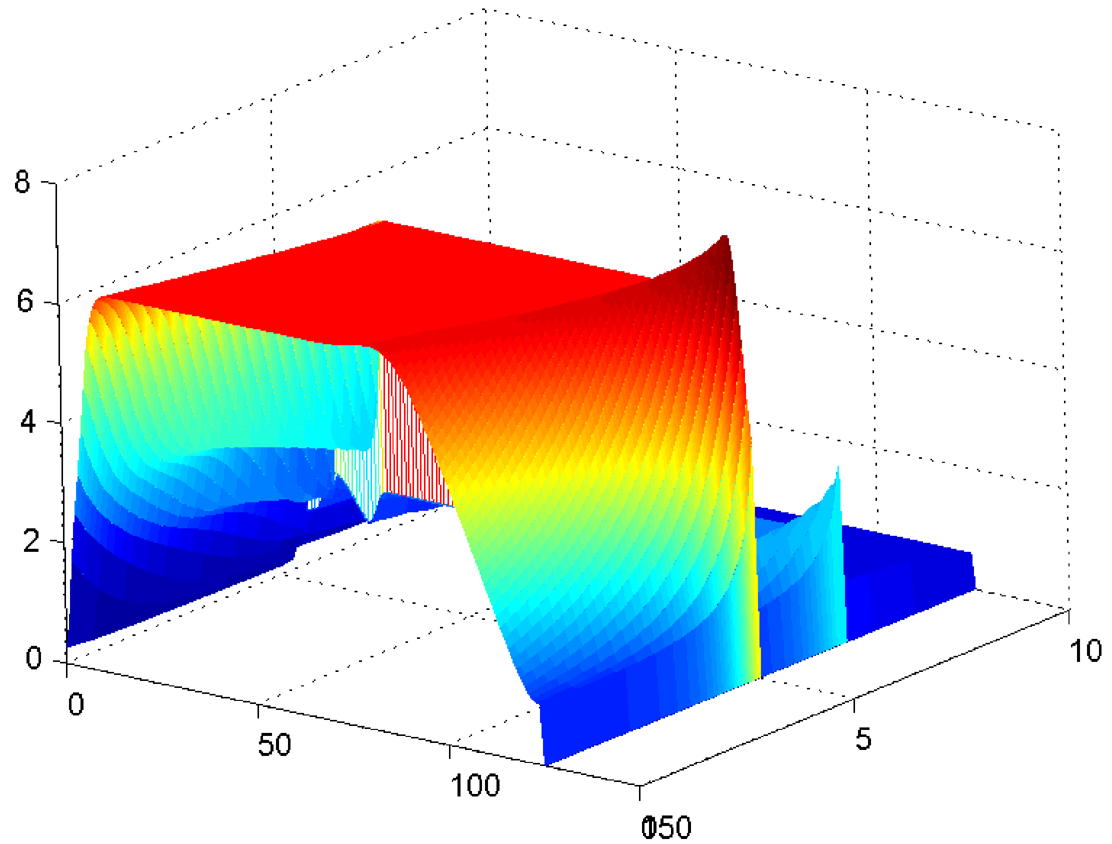



Figure 1: Calibrated intensity function $\lambda(t, L)$: ITRAXX Europe Series 6, March 15 2007.

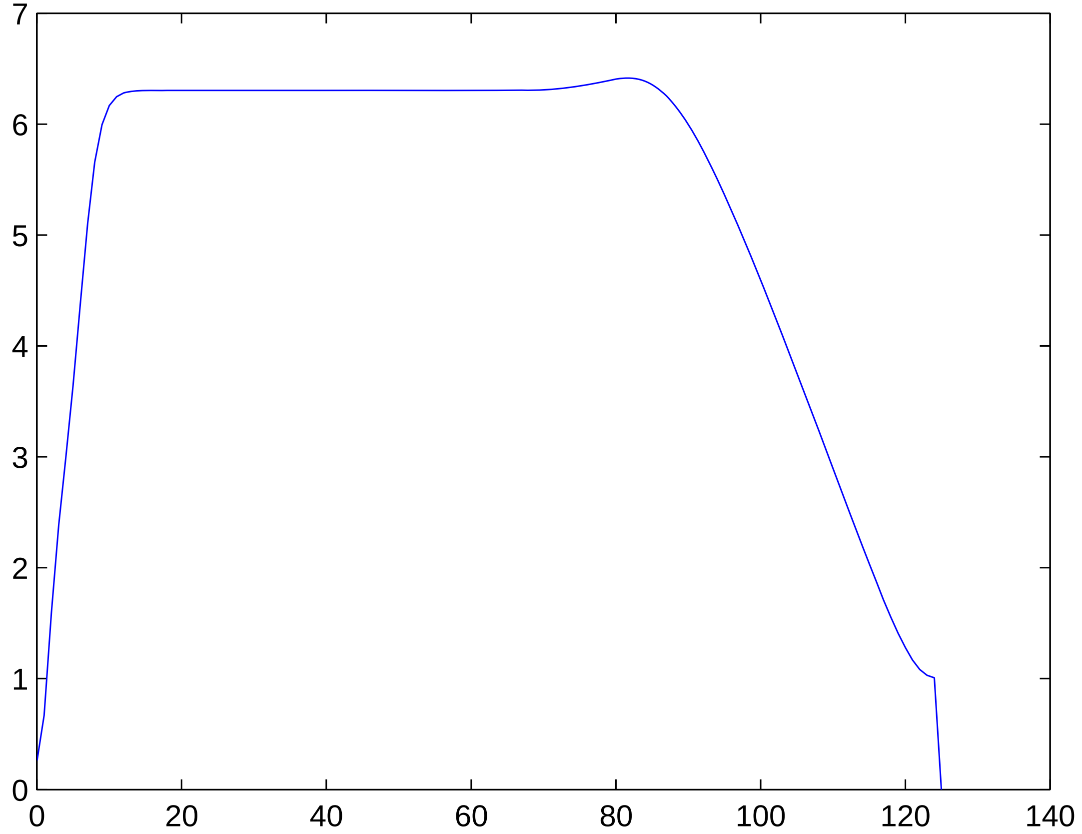



Figure 2: Dependence of default intensity on number of defaults for $t = 1$ year: ITRAXX Europe Series 6, March 15 2007..

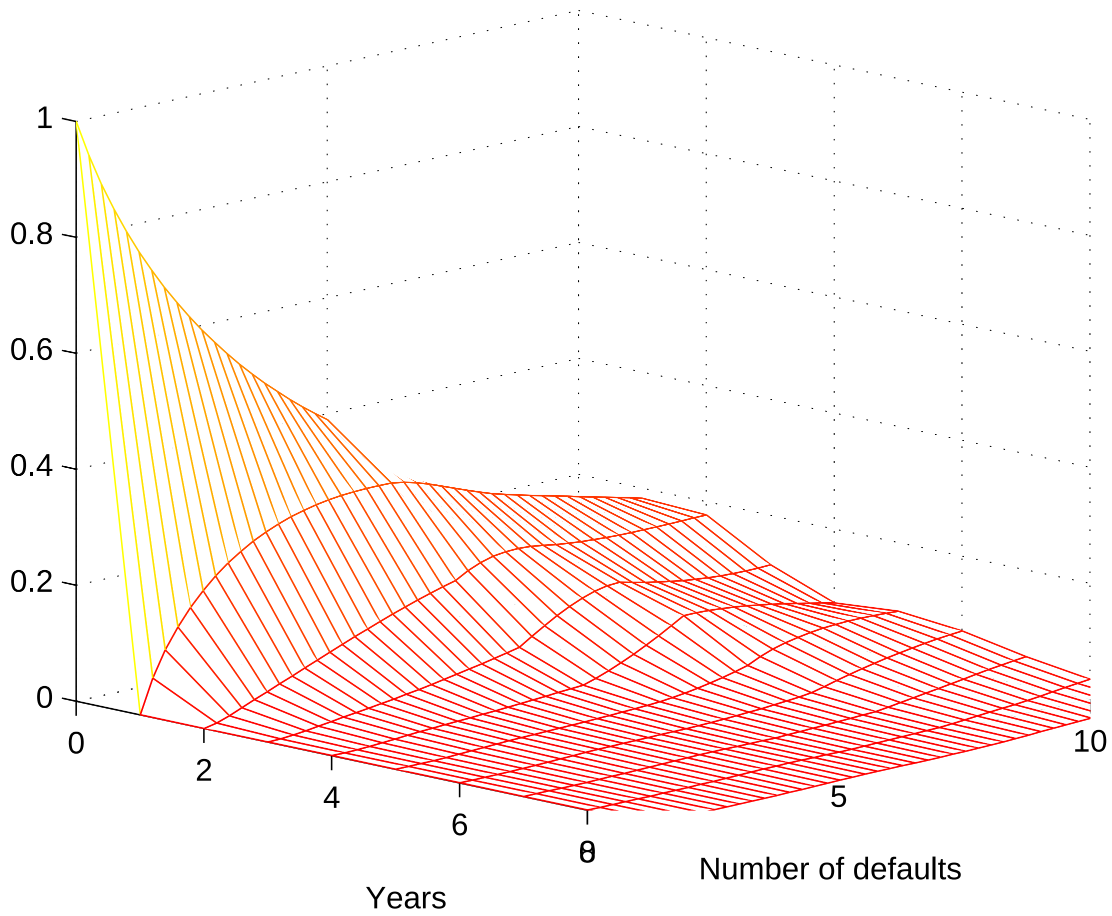



Figure 3: Term structure of loss distributions computed from calibrated default intensity: ITRAXX Europe Series 6, March 15 2007..

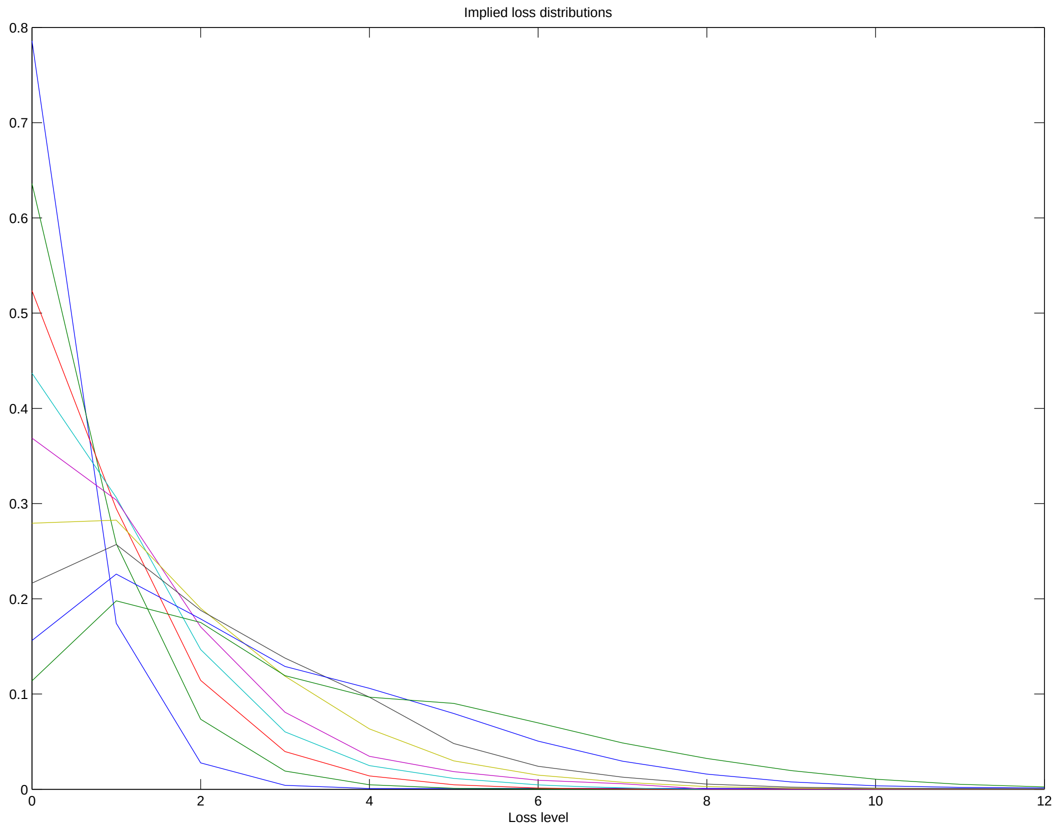



Figure 4: Implied loss distributions at various maturities: ITRAXX Europe Series 6, March 15 2007.

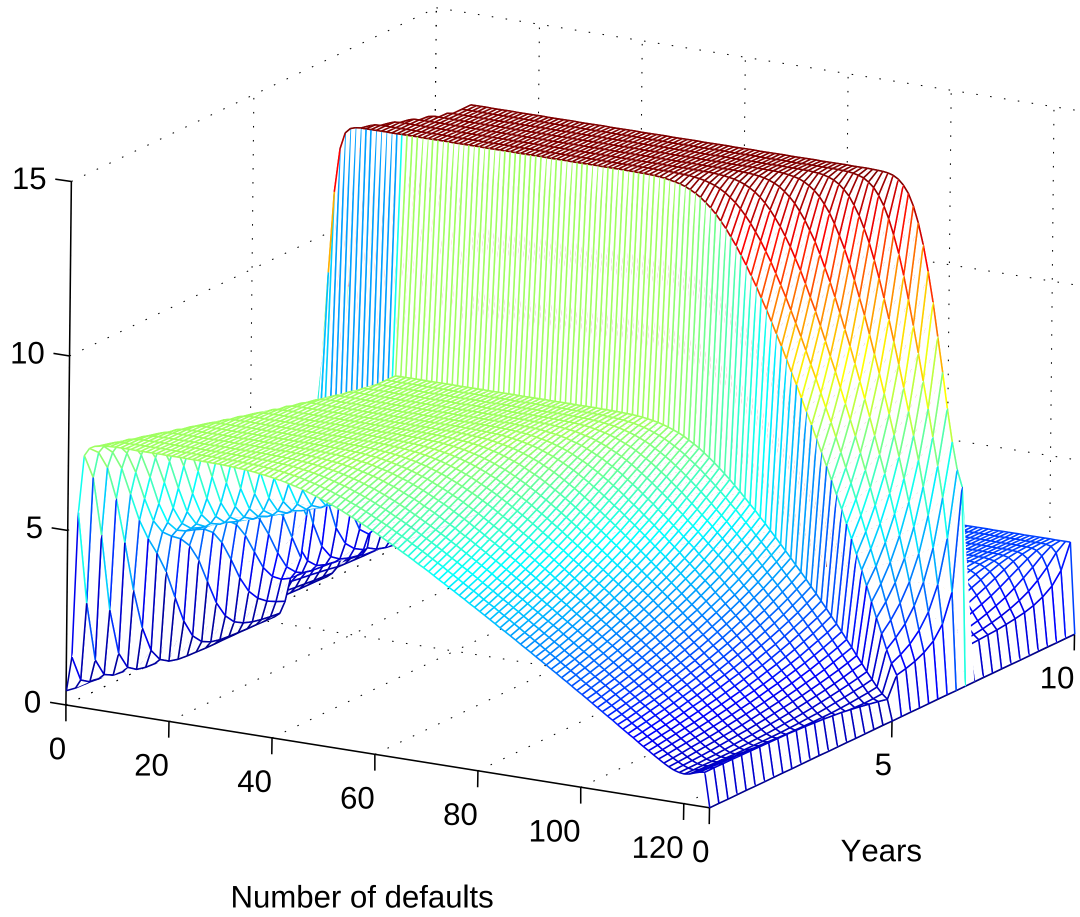



Figure 5: Calibrated intensity function $\lambda(t, L)$: ITRAXX September 26, 2005

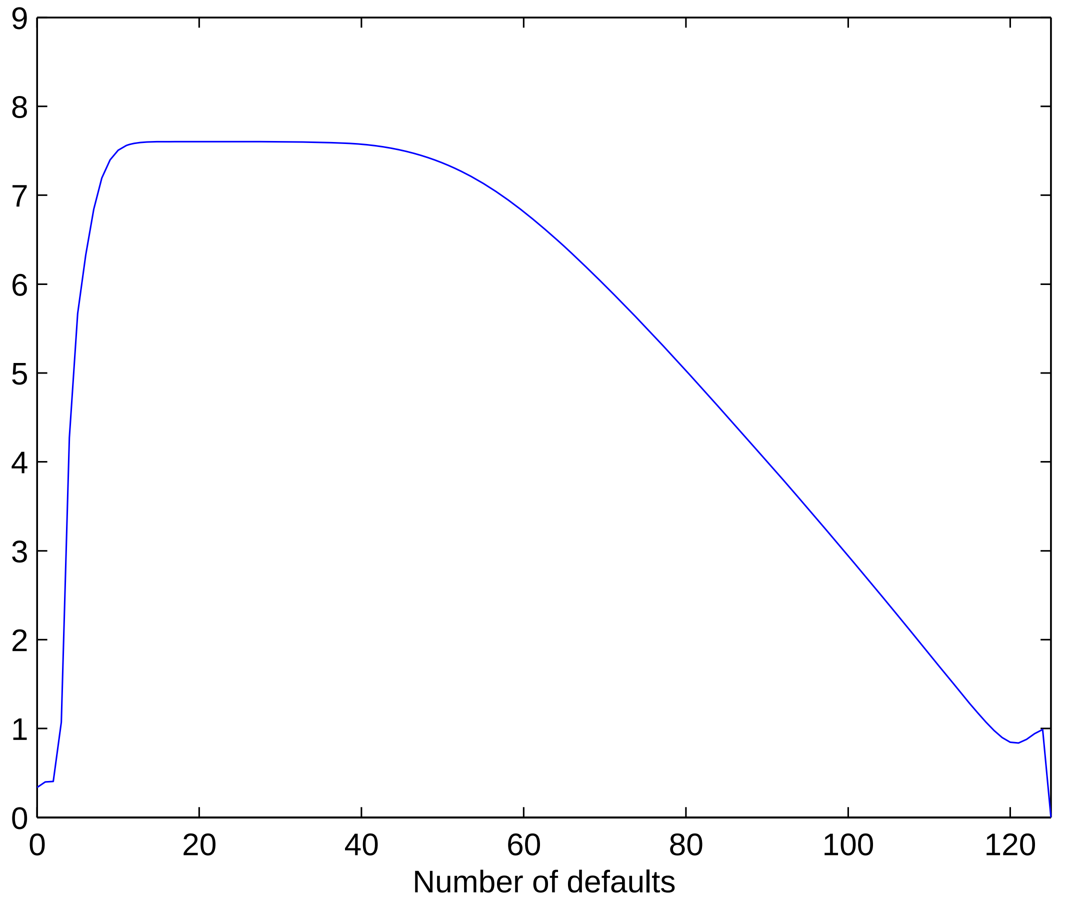



Figure 6: Dependence of default intensity on number of defaults for $t = 1 year$: ITRAXX September 26, 2005.

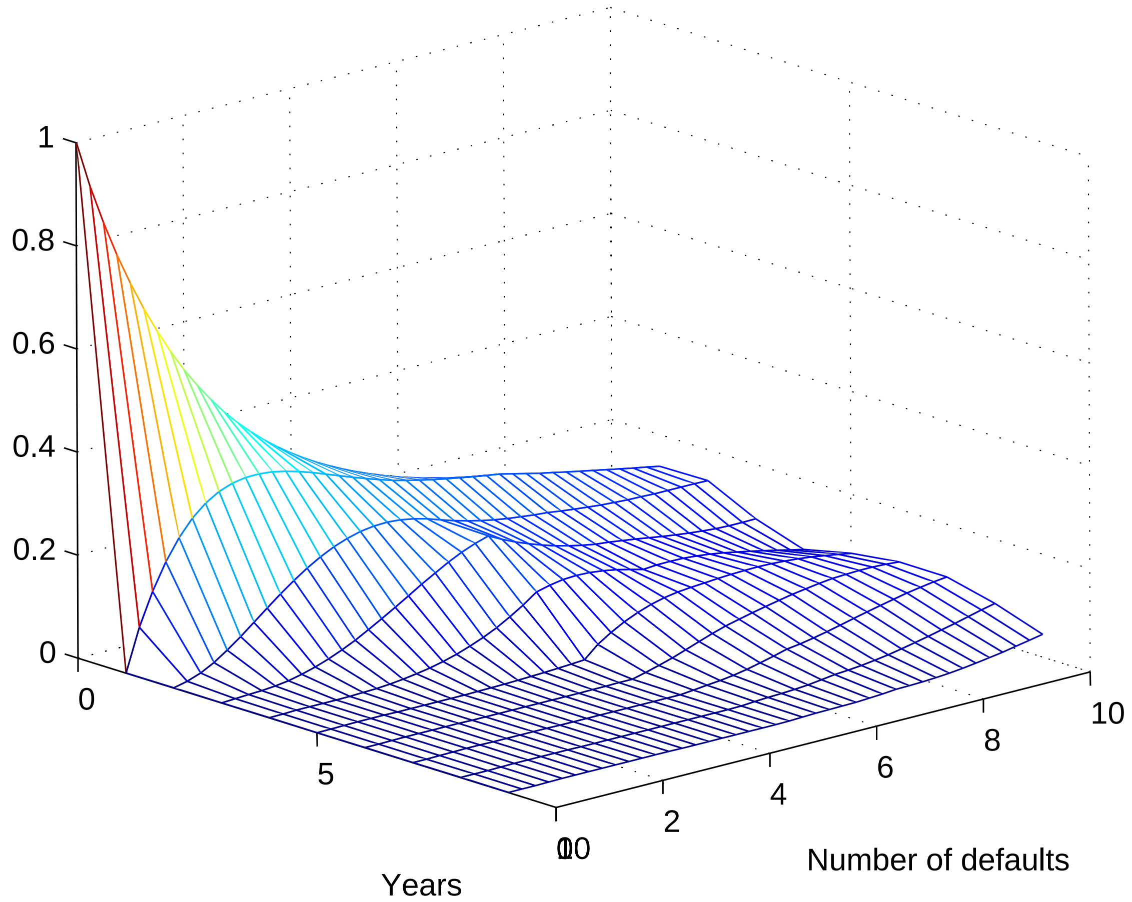



Figure 7: Term structure of loss distributions computed from calibrated default intensity: ITRAXX September 26, 2005.

- Default intensity non-monotonic observed number of defaults.
- Low initial default rate but sharp increase as soon as one default occurs.
- Equity tranche is priced using very different default intensities than higher tranches.
- Portfolio credit risk appears to be segmented into 3 parts: most risky names ($\leq 5\%$), high quality names (last to default, $\leq 10\%$), main body of the portfolio (plateau of default intensity).
- Shape of the function $\lambda(t, N_t)$ is qualitatively similar to the Herbertsson model.

# Conclusion

- Stochastic control method for solving a model calibration problem.
- Rigorous methodology for calibrating a top-down CDO pricing model to market data.
- Stable calibration algorithm based on intensity control method.
- No black box optimization.
- Nonparametric: no arbitrary functional form for the default intensity.
- No need to interpolate CDO data in maturity or strike!
- Unconstrained convex minimization in dimension $\simeq 20$.
- Results point to default contagion effects in the riskneutral loss process.

- Risk minimizing hedging strategies for portfolio credit derivatives can be computed explicitly in the calibrated model $\rightarrow$ joint work with Yu Hang KAN (Columbia).