Credit Default Swaps and Swaptions in a Hazard Process Model

Marek Rutkowski
School of Mathematics and Statistics
University of New South Wales

Joint work with Tomasz Bielecki and Monique Jeanblanc

Université d’Évry Val d’Essonne
January 29, 2009
Outline

1. Credit Default Swaps
2. Credit Default Swaptions
3. Pre-Default Volatilities
4. CIR Default Intensity Model
References on Hedging


D. Brigo and A. Alfonsi: Credit default swaps calibration and option pricing with the SSRD stochastic intensity and interest-rate model. *Finance and Stochastics* 9 (2005), 29-42.

D. Brigo and N. El-Bachir: An exact formula for default swaptions’ pricing in SSRJD stochastic intensity model. Forthcoming in *Mathematical Finance*. 
Credit Default Swaps
Hazard Process Set-up

Terminology and notation:

1. The **default time** is a strictly positive random variable $\tau$ defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2. We define the **default indicator process** $H_t = \mathbb{1}_{\{\tau \leq t\}}$ and we denote by $\mathcal{H}$ its natural filtration.

3. We assume that we are given, in addition, some auxiliary filtration $\mathcal{F}$ and we write $\mathcal{G} = \mathcal{H} \vee \mathcal{F}$, meaning that $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$.

4. The filtration $\mathcal{F}$ is termed the **reference filtration**.

5. The filtration $\mathcal{G}$ is called the **full filtration**.
The underlying market model is arbitrage-free, in the following sense:

1. Let the savings account $B$ be given by

$$B_t = \exp \left( \int_0^t r_u \, du \right), \quad \forall \, t \in \mathbb{R}_+,$$

where the short-term rate $r$ follows an $\mathbb{F}$-adapted process.

2. A spot martingale measure $\mathbb{Q}$ is associated with the choice of the savings account $B$ as a numéraire.

3. The underlying market model is arbitrage-free, meaning that it admits a spot martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. Uniqueness of a martingale measure is not postulated.
Let us summarize the main features of the hazard process approach:

1. Let us denote by
   \[ G_t = \mathbb{Q}(\tau > t \mid \mathcal{F}_t) \]
   the survival process of \( \tau \) with respect to the reference filtration \( \mathcal{F} \). We postulate that \( G_0 = 1 \) and \( G_t > 0 \) for every \( t \in [0, T] \).

2. We define the hazard process \( \Gamma = -\ln G \) of \( \tau \) with respect to the filtration \( \mathcal{F} \).

3. For any \( \mathbb{Q} \)-integrable and \( \mathcal{F}_T \)-measurable random variable \( Y \), the following classic formula is valid
   \[ \mathbb{E}_\mathbb{Q}(1_{\{T<\tau\}} Y \mid G_t) = 1_{\{t<\tau\}} G_t^{-1} \mathbb{E}_\mathbb{Q}(G_T Y \mid \mathcal{F}_t). \]
1. Assume that the supermartingale $G$ is continuous.
2. We denote by $G = \mu - \nu$ its Doob-Meyer decomposition.
3. Let the increasing process $\nu$ be absolutely continuous, that is, $d\nu_t = \nu_t \, dt$ for some $\mathbb{F}$-adapted and non-negative process $\nu$.
4. Then the process $\lambda = G_t^{-1} \nu_t$ is called the $\mathbb{F}$-intensity of default time.

**Lemma**

The process $M$, given by the formula

$$M_t = H_t - \int_0^{t \wedge \tau} \lambda_u \, du = H_t - \int_0^t (1 - H_u) \lambda_u \, du,$$

is a $(\mathbb{Q}, \mathbb{G})$-martingale.
Defaultable Claim

A generic defaultable claim \((X, A, Z, \tau)\) consists of:

1. A promised contingent claim \(X\) representing the payoff received by the holder of the claim at time \(T\), if no default has occurred prior to or at maturity date \(T\).
2. A process \(A\) representing the dividends stream prior to default.
3. A recovery process \(Z\) representing the recovery payoff at time of default, if default occurs prior to or at maturity date \(T\).
4. A random time \(\tau\) representing the default time.

**Definition**

The dividend process \(D\) of a defaultable claim \((X, A, Z, \tau)\) maturing at \(T\) equals, for every \(t \in [0, T]\),

\[
D_t = X 1_{\{\tau > T\}} 1_{[T, \infty)}(t) + \int_0^t (1 - H_u) dA_u + \int_0^t Z_u dH_u.
\]
Ex-dividend Price

Recall that:
- The process $B$ represents the **savings account**.
- A probability measure $\mathbb{Q}$ is a **spot martingale measure**.

**Definition**

The **ex-dividend price** $S$ associated with the dividend process $D$ equals, for every $t \in [0, T]$,

$$S_t = B_t \mathbb{E}_\mathbb{Q}\left( \int_{[t, T]} B_u^{-1} dD_u \middle| G_t \right) = 1_{\{t < \tau\}} \tilde{S}_t$$

where $\mathbb{Q}$ is a spot martingale measure.

- The ex-dividend price represents the (market) **value** of a defaultable claim.
- The $\mathcal{F}$-adapted process $\tilde{S}$ is termed the **pre-default value**.
Lemma

The value of a defaultable claim \((X, A, Z, \tau)\) maturing at \(T\) equals

\[
S_t = 1_{\{t < \tau\}} \frac{B_t}{G_t} \mathbb{E}_Q \left( B_T^{-1} G_T X 1_{\{t < T\}} + \int_t^T B_u^{-1} G_u Z_u \lambda_u \, du + \int_t^T B_u^{-1} G_u \, dA_u \bigg| \mathcal{F}_t \right)
\]

where \(Q\) is a martingale measure.

- Recall that \(\mu\) is the martingale part in the Doob-Meyer decomposition of \(G\).
- Let \(m\) be the \((Q, \mathcal{F})\)-martingale given by the formula

\[
m_t = \mathbb{E}_Q \left( B_T^{-1} G_T X + \int_0^T B_u^{-1} G_u Z_u \lambda_u \, du + \int_0^T B_u^{-1} G_u \, dA_u \bigg| \mathcal{F}_t \right).
\]
Proposition

The dynamics of the value process $S$ on $[0, T]$ are

$$dS_t = -S_t \, dM_t + (1 - H_t) (\lambda_t S_t - r_t) dt + dA_t$$

$$+ (1 - H_t) G_t^{-1} (B_t \, dm_t - S_t \, d\mu_t)$$

$$+ (1 - H_t) G_t^{-2} (S_t \, d\langle \mu \rangle_t - B_t \, d\langle \mu, m \rangle_t).$$

The dynamics of the pre-default value $\tilde{S}$ on $[0, T]$ are

$$d\tilde{S}_t = ((\lambda_t + r_t) \tilde{S}_t - \lambda_t Z_t) \, dt + dA_t + G_t^{-1} (B_t \, dm_t - \tilde{S}_t \, d\mu_t)$$

$$+ G_t^{-2} (\tilde{S}_t \, d\langle \mu \rangle_t - B_t \, d\langle \mu, m \rangle_t).$$
Definition

A forward CDS issued at time $s$, with start date $U$, maturity $T$, and recovery at default is a defaultable claim $(0, A, Z, \tau)$ where

$$dA_t = -\kappa \mathbb{1}_{U,T}(t) \, dL_t, \quad Z_t = \delta_t \mathbb{1}_{U,T}(t).$$

- An $\mathcal{F}_s$-measurable rate $\kappa$ is the CDS rate.
- An $\mathcal{F}$-adapted process $L$ specifies the tenor structure of fee payments.
- An $\mathcal{F}$-adapted process $\delta: [U, T] \to \mathbb{R}$ represents the default protection.

Lemma

The value of the forward CDS equals, for every $t \in [s, U]$,

$$S_t(\kappa) = B_t \mathbb{E}_Q \left( \mathbb{1}_{U<\tau\leq T} B^{-1}_\tau Z_\tau \bigg| \mathcal{G}_t \right) - \kappa B_t \mathbb{E}_Q \left( \int_{t \wedge U}^{\tau \wedge T} B^{-1}_u \, dL_u \bigg| \mathcal{G}_t \right).$$
Lemma

The value of a credit default swap started at $s$, equals, for every $t \in [s, U]$,

$$S_t(\kappa) = \mathbb{1}_{\{t < \tau\}} \frac{B_t}{G_t} \mathbb{E}_Q \left( - \int_U^T B_u^{-1} \delta_u \, dG_u - \kappa \int_{]U,T]} B_u^{-1} G_u \, dL_u \bigg| \mathcal{F}_t \right).$$

Note that $S_t(\kappa) = \mathbb{1}_{\{t < \tau\}} \tilde{S}_t(\kappa)$ where the $\mathbb{F}$-adapted process $\tilde{S}(\kappa)$ is the pre-default value. Moreover

$$\tilde{S}_t(\kappa) = \tilde{P}(t, U, T) - \kappa \tilde{A}(t, U, T)$$

where

- $\tilde{P}(t, U, T)$ is the pre-default value of the protection leg,
- $\tilde{A}(t, U, T)$ is the pre-default value of the fee leg per one unit of $\kappa$. 
The forward CDS rate is defined similarly as the forward swap rate for a default-free interest rate swap.

Definition

The forward market CDS at time $t \in [0, U]$ is the forward CDS in which the $\mathcal{F}_t$-measurable rate $\kappa$ is such that the contract is valueless at time $t$.

The corresponding pre-default forward CDS rate at time $t$ is the unique $\mathcal{F}_t$-measurable random variable $\kappa(t, U, T)$, which solves the equation

$$\tilde{S}_t(\kappa(t, U, T)) = 0.$$

Recall that for any $\mathcal{F}_t$-measurable rate $\kappa$ we have that

$$\tilde{S}_t(\kappa) = \tilde{P}(t, U, T) - \kappa \tilde{A}(t, U, T).$$
**Lemma**

*For every* $t \in [0, U]$, 

$$
\kappa(t, U, T) = \frac{\tilde{P}(t, U, T)}{\tilde{A}(t, U, T)} = -\frac{\mathbb{E}_Q\left(\int_U^T B^{-1}_u \delta_u \, dG_u \bigg| \mathcal{F}_t\right)}{\mathbb{E}_Q\left(\int_{[U, T]} B^{-1}_u G_u \, dL_u \bigg| \mathcal{F}_t\right)} = \frac{M^P_t}{M^A_t}
$$

*where the* $(Q, \mathbb{F})$-martingales $M^P$ and $M^A$ are given by

$$
M^P_t = -\mathbb{E}_Q\left(\int_U^T B^{-1}_u \delta_u \, dG_u \bigg| \mathcal{F}_t\right)
$$

*and*

$$
M^A_t = \mathbb{E}_Q\left(\int_{[U, T]} B^{-1}_u G_u \, dL_u \bigg| \mathcal{F}_t\right).
$$
Credit Default Swaptions
A credit default swaption is a call option with expiry date $R \leq U$ and zero strike written on the value of the forward CDS issued at time $0 \leq s < R$, with start date $U$, maturity $T$, and an $\mathcal{F}_s$-measurable rate $\kappa$.

The swaption’s payoff $C_R$ at expiry equals $C_R = (S_R(\kappa))^+$. 

**Lemma**

For a forward CDS with an $\mathcal{F}_s$-measurable rate $\kappa$ we have, for every $t \in [s, U]$, 

$$S_t(\kappa) = \mathbb{1}_{\{t < \tau\}} \tilde{A}(t, U, T)(\kappa(t, U, T) - \kappa).$$

It is clear that 

$$C_R = \mathbb{1}_{\{R < \tau\}} \tilde{A}(R, U, T)(\kappa(R, U, T) - \kappa)^+. $$

A credit default swaption is formally equivalent to a call option on the forward CDS rate with strike $\kappa$. This option is knocked out if default occurs prior to $R$. 

T. Bielecki, M. Jeanblanc and M. Rutkowski  Credit Default Swaps and Swaptions
Lemma

The price at time $t \in [s, R]$ of a credit default swaption equals

$$C_t = 1_{\{t < \tau\}} \frac{B_t}{G_t} E_Q \left( \frac{G_R}{B_R} \tilde{A}(R, U, T)(\kappa(R, U, T) - \kappa)^+ \mid \mathcal{F}_t \right).$$

Define an equivalent probability measure $\hat{Q}$ on $(\Omega, \mathcal{F}_R)$ by setting

$$\frac{d\hat{Q}}{dQ} = \frac{M^A_R}{M^A_0}, \quad Q\text{-a.s.}$$

Proposition

The price of the credit default swaption equals, for every $t \in [s, R]$,

$$C_t = 1_{\{t < \tau\}} \tilde{A}(t, U, T) E_{\hat{Q}} \left( (\kappa(R, U, T) - \kappa)^+ \mid \mathcal{F}_t \right) = 1_{\{t < \tau\}} \tilde{C}_t.$$

The forward CDS rate $(\kappa(t, U, T), t \leq R)$ is a $(\hat{Q}, \mathcal{F})$-martingale.
Brownian Case

- Let the filtration $\mathbb{F}$ be generated by a Brownian motion $W$ under $\mathbb{Q}$.
- Since $M^P$ and $M^A$ are strictly positive $(\mathbb{Q}, \mathbb{F})$-martingales, we have that
  
  $$dM^P_t = M^P_t \sigma^P_t \, dW_t, \quad dM^A_t = M^A_t \sigma^A_t \, dW_t,$$

  for some $\mathbb{F}$-adapted processes $\sigma^P$ and $\sigma^A$.

**Lemma**

*The forward CDS rate* $(\kappa(t, U, T), \ t \in [0, R])$ *is* $(\hat{\mathbb{Q}}, \mathbb{F})$*-martingale and*

$$d\kappa(t, U, T) = \kappa(t, U, T) \sigma^\kappa_t \, d\hat{W}_t$$

*where* $\sigma^\kappa = \sigma^P - \sigma^A$ *and the* $(\hat{\mathbb{Q}}, \mathbb{F})$*-Brownian motion* $\hat{W}$ *equals*

$$\hat{W}_t = W_t - \int_0^t \sigma^A_u \, du, \quad \forall \ t \in [0, R].$$
Let \( \varphi = (\varphi^1, \varphi^2) \) be a trading strategy, where \( \varphi^1 \) and \( \varphi^2 \) are \( \mathcal{G} \)-adapted processes.

The wealth of \( \varphi \) equals, for every \( t \in [s, R] \),

\[
V_t(\varphi) = \varphi^1_t S_t(\kappa) + \varphi^2_t A(t, U, T)
\]

and thus the pre-default wealth satisfies, for every \( t \in [s, R] \),

\[
\tilde{V}_t(\varphi) = \varphi^1_t \tilde{S}_t(\kappa) + \varphi^2_t \tilde{A}(t, U, T).
\]

It is enough to search for \( \mathbb{F} \)-adapted processes \( \tilde{\varphi}^i, \ i = 1, 2 \) such that the equality

\[
1_{\{t < \tau\}} \varphi^i_t = \tilde{\varphi}^i_t
\]

holds for every \( t \in [s, R] \).
The next result yields a general representation for hedging strategy.

**Proposition**

Let the Brownian motion $W$ be one-dimensional. The hedging strategy $\tilde{\varphi} = (\tilde{\varphi}^1, \tilde{\varphi}^2)$ for the credit default swaption equals, for $t \in [s, R]$,

$$
\tilde{\varphi}^1_t = \tilde{\xi}_t \kappa(t, U, T) \sigma_t, \quad \tilde{\varphi}^2_t = \frac{\tilde{C}_t - \tilde{\varphi}^1_t \tilde{S}_t(\kappa)}{\tilde{A}(t, U, T)}
$$

where $\tilde{\xi}$ is the process satisfying

$$
\frac{\tilde{C}_R}{\tilde{A}(R, U, T)} = \frac{\tilde{C}_0}{\tilde{A}(0, U, T)} + \int_0^R \tilde{\xi}_t \, d\tilde{W}_t.
$$

The main issue is an explicit computation of the process $\tilde{\xi}$. 
Proposition

Assume that the volatility $\sigma^\kappa = \sigma^P - \sigma^A$ of the forward CDS spread is deterministic. Then the pre-default value of the credit default swaption with strike level $\kappa$ and expiry date $R$ equals, for every $t \in [0, U]$,

$$\tilde{C}_t = \tilde{A}_t \left( \kappa_t N(d_+(\kappa_t, U - t)) - \kappa N(d_-(\kappa_t, U - t)) \right)$$

where $\kappa_t = \kappa(t, U, T)$ and $\tilde{A}_t = \tilde{A}(t, U, T)$. Equivalently,

$$\tilde{C}_t = \tilde{P}_t N(d_+(\kappa_t, t, R)) - \kappa \tilde{A}_t N(d_-(\kappa_t, t, R))$$

where $\tilde{P}_t = \tilde{P}(t, U, T)$ and

$$d_\pm(\kappa_t, t, R) = \frac{\ln(\kappa_t/\kappa) \pm \frac{1}{2} \int_t^R (\sigma^\kappa(u))^2 \, du}{\sqrt{\int_t^R (\sigma^\kappa(u))^2 \, du}}.$$
Pre-Default Volatilities
Assumption 1

Definition
For any $u \in \mathbb{R}^+$, we define the $\mathbb{F}$-martingale $G_t^u = \mathbb{Q}(\tau > u \mid \mathcal{F}_t)$ for $t \in [0, T]$.

- We set $G_t = G_t^t$. Then the process $(G_t, t \in [0, T])$ is an $\mathbb{F}$-supermartingale.
- We also assume that $G$ is a strictly positive process.

Assumption
There exists a family of $\mathbb{F}$-adapted processes $(f_t^x; t \in [0, T], x \in \mathbb{R}^+)$ such that, for any $u \in \mathbb{R}^+$,

$$G_t^u = \int_u^\infty f_t^x \, dx, \quad \forall \ t \in [0, T].$$
Default Intensity

- For any fixed $t \in [0, T]$, the random variable $f_t$ represents the conditional density of $\tau$ with respect to the $\sigma$-field $\mathcal{F}_t$, that is,

$$\int f_t^x \, dx = \mathbb{Q}(\tau \in dx \mid \mathcal{F}_t).$$

- We write $f_t^t = f_t$ and we define $\hat{\lambda}_t = G_t^{-1} f_t$.

**Lemma**

*Under Assumption 1, the process $(M_t, t \in [0, T])$ given by the formula*

$$M_t = H_t - \int_0^t (1 - H_u) \hat{\lambda}_u \, du$$

*is a $\mathbb{G}$-martingale.*

- It can be deduced from the lemma that $\hat{\lambda} = \lambda$ is the default intensity.
Assumption 2

**Assumption**

*The filtration $\mathbb{F}$ is generated by a one-dimensional Brownian motion $W$.*

We now work under Assumptions 1-2. We have that

- For any fixed $u \in \mathbb{R}_+$, the $\mathbb{F}$-martingale $G^u$ satisfies, for $t \in [0, T]$,

  $$
  G^u_t = G^u_0 + \int_0^t g^u_s \, dW_s
  $$

  for some $\mathbb{F}$-predictable, real-valued process $(g^u_t, t \in [0, T])$.

- For any fixed $x \in \mathbb{R}_+$, the process $(f^x_t, t \in [0, T])$ is an $(\mathbb{Q}, \mathbb{F})$-martingale and thus there exists an $\mathbb{F}$-predictable process $(\sigma^x_t, t \in [0, T])$ such that, for $t \in [0, T]$,

  $$
  f^x_t = f^x_0 + \int_0^t \sigma^x_s \, dW_s.
  $$
Survival Process

- The following relationship is valid, for any \( u \in \mathbb{R}_+ \) and \( t \in [0, T] \),

\[
g_t^u = \int_u^\infty \sigma_t^x \, dx.
\]

- By applying the Itô-Wentzell-Kunita formula, we obtain the following auxiliary result, in which we denote \( g_s^s = g_s \) and \( f_s^s = f_s \).

**Lemma**

*The Doob-Meyer decomposition of the survival process \( G \) equals, for every \( t \in [0, T] \),*

\[
G_t = G_0 + \int_0^t g_s \, dW_s - \int_0^t f_s \, ds.
\]

*In particular, \( G \) is a continuous process.*
Volatility of Pre-Default Value

Under the assumption that $B$, $Z$ and $A$ are deterministic, the volatility of the pre-default value process can be computed explicitly in terms of $\sigma_t^u$. Recall that, for $t \in [0, T]$,

$$f_t^\times = f_0^\times + \int_0^t \sigma_s^\times dW_s, \quad g_t^u = \int_u^\infty \sigma_t^\times dx.$$

Corollary

If $B$, $Z$ and $A$ are deterministic then we have that, for every $t \in [0, T]$,

$$d\tilde{S}_t = \left( (r(t) + \lambda_t)\tilde{S}_t - \lambda_t Z(t) \right) dt + dA(t) + \zeta_t^T dW_t$$

with $\zeta_t^T = G_t^{-1} B(t)\nu_t^T$ where

$$\nu_t^T = B^{-1}(T)XG_t^T + \int_t^T B^{-1}(u)Z(u)\sigma_t^u du + \int_t^T B^{-1}(u)g_t^u dA(u).$$
Lemma

If $B$, $\delta$ and $L$ are deterministic then the forward CDS rate satisfies under $\hat{Q}$

$$d \kappa(t, U, T) = \kappa(t, U, T) (\sigma_t^P - \sigma_t^A) \, d \hat{W}_t$$

where the process $\hat{W}$, given by the formula

$$\hat{W}_t = W_t - \int_0^t \sigma_u^A \, du, \quad \forall \, t \in [0, R],$$

is a Brownian motion under $\hat{Q}$ and

$$\sigma_t^P = \left( \int_U^T B^{-1}(u) \delta(u) \sigma_t^U \, du \right) \left( \int_U^T B^{-1}(u) \delta(u) f_t^u \, du \right)^{-1},$$

$$\sigma_t^A = \left( \int_U^Y B^{-1}(u) g_t^u \, du \right) \left( \int_U^T B^{-1}(u) G_t^u \, du \right)^{-1}.$$
CIR Default Intensity Model
We make the following standing assumptions:

1. The default intensity process $\lambda$ is governed by the CIR dynamics

$$d\lambda_t = \mu(\lambda_t) \, dt + \nu(\lambda_t) \, dW_t$$

where $\mu(\lambda) = a - b\lambda$ and $\nu(\lambda) = c\sqrt{\lambda}$.

2. The default time $\tau$ is given by

$$\tau = \inf \left\{ t \in \mathbb{R}_+ : \int_0^t \lambda_u \, du \geq \Theta \right\}$$

where $\Theta$ is a random variable with the unit exponential distribution, independent of the filtration $\mathbb{F}$. 

From the martingale property of $f^u$ we have, for every $t \leq u$,

$$f^u_t = \mathbb{E}_Q(f_u | \mathcal{F}_t) = \mathbb{E}_Q(\lambda_u G_u | \mathcal{F}_t).$$

The immersion property holds between $\mathbb{F}$ and $\mathbb{G}$ so that $G_t = \exp(-\Lambda_t)$, where $\Lambda_t = \int_0^t \lambda_u \, du$ is the hazard process. Therefore

$$f^s_t = \mathbb{E}_Q(\lambda_s e^{-\Lambda_s} | \mathcal{F}_t).$$

Let us denote

$$H^s_t = \mathbb{E}_Q\left(e^{-(\Lambda_s - \Lambda_t)} | \mathcal{F}_t\right) = \frac{G^s_t}{G_t}.$$ 

It is important to note that for the CIR model

$$H^s_t = e^{m(t,s) - n(t,s)\lambda_t} = \hat{H}(\lambda_t, t, s)$$

where $\hat{H}(\cdot, t, s)$ is a strictly decreasing function when $t < s$. 

**Model Properties**
We assume that:

1. The tenor structure process $L$ is deterministic.
2. The savings account is $B$ is deterministic. We denote $\beta = B^{-1}$.
3. We also assume that $\delta$ is constant.

**Proposition**

The volatility of the forward CDS rate satisfies $\sigma^\kappa = \sigma^P - \sigma^A$ where

$$
\sigma^P_t = \nu(\lambda_t) \frac{\beta(T)H^T_t n(t, T) - \beta(U)H^U_t n(t, U) + \int_U^T r(u)\beta(u)H^U_t n(t, u) \, du}{\beta(U)H^U_t - \beta(T)H^T_t - \int_U^T r(u)\beta(u)H^U_t \, du}
$$

and

$$
\sigma^A_t = \nu(\lambda_t) \frac{\int_{[U,T]} \beta(u)H^U_t n(t, u) \, dL(u)}{\int_{[U,T]} \beta(u)H^U_t \, dL(u)}.
$$
Equivalent Representations

- One can show that

\[ C_R = 1_{\{ R < \tau \}} \left( \delta \int_U^T B(R, u) \lambda_R^u \, du - \kappa \int_{[U,T]} B(R, u) H_R^u \, dL(u) \right)^+. \]

- Straightforward computations lead to the following representation

\[ C_R = 1_{\{ R < \tau \}} \left( \delta B(R, U) H_R^U - \int_{[U,T]} B(R, u) H_R^u \, d\chi(u) \right)^+ \]

where the function \( \chi : \mathbb{R}_+ \rightarrow \mathbb{R} \) satisfies

\[ d\chi(u) = -\delta \frac{\partial \ln B(R, u)}{\partial u} \, du + \kappa \, dL(u) + \delta \, 1_{[T, \infty]}(u). \]
Auxiliary Functions

We define auxiliary functions $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}_+$ by setting

$$
\zeta(x) = \delta B(R, U) \hat{H}(x, R, U)
$$

and

$$
\psi(y) = \int_{]U, T]} B(R, u) \hat{H}(y, R, u) \, d\chi(u).
$$

There exists a unique $\mathcal{F}_R$-measurable random variable $\lambda_R^*$ such that

$$
\zeta(\lambda_R) = \delta B(R, U) \hat{H}(\lambda_R, R, U) = \int_{]U, T]} B(R, u) \hat{H}(\lambda_R^*, R, u) \, d\chi(u) = \psi(\lambda_R^*).
$$

It suffices to check that $\lambda_R^* = \psi^{-1}(\zeta(\lambda_R))$ is the unique solution to this equation.
The payoff of the credit default swaption admits the following representation

\[ C_R = \mathbb{1}_{\{ R < \tau \}} \int_{[u, T]} B(R, u) (\hat{H}(\lambda^*_R, R, u) - \hat{H}(\lambda_R, R, u))^+ \, d\chi(u). \]

Let \( D^0(t, u) \) be the price at time \( t \) of a unit defaultable zero-coupon bond with zero recovery maturing at \( u \geq t \) and let \( B(t, u) \) be the price at time \( t \) of a (default-free) unit discount bond maturing at \( u \geq t \).

If the interest rate process \( r \) is independent of the default intensity \( \lambda \) then \( D^0(t, u) \) is given by the following formula

\[ D^0(t, u) = \mathbb{1}_{\{ t < \tau \}} B(t, u) H_t^u. \]
Let \( P(\lambda_t, U, u, K) \) stand for the price at time \( t \) of a put bond option with strike \( K \) and expiry \( U \) written on a zero-coupon bond maturing at \( u \) computed in the CIR model with the interest rate modeled by \( \lambda \).

**Proposition**

Assume that \( R = U \). Then the payoff of the credit default swaption equals

\[
C_U = \int_{[U,T]} (K(u)D^0(U, U) - D^0(U, u))^+ d\chi(u)
\]

where \( K(u) = B(U, u)\hat{H}(\lambda^*_U, U, u) \) is deterministic, since \( \lambda^*_U = \psi^{-1}(\delta) \).

The pre-default value of the credit default swaption equals

\[
\widetilde{C}_t = \int_{[U,T]} B(t, u)P(\lambda_t, U, u, \hat{K}(u)) d\chi(u)
\]

where \( \hat{K}(u) = K(u) / B(U, u) = \hat{H}(\lambda^*_U, U, u) \).
The price $P^u_t := P(\lambda_t, U, u, \hat{K}(u))$ of the put bond option in the CIR model with the interest rate $\lambda$ is known to be

$$P^u_t = \hat{K}(u) H^U_t \mathbb{P}_U(H^u_U \leq \hat{K}(u) | \lambda_t) - H^u_t \mathbb{P}_u(H^u_U \leq \hat{K}(u) | \lambda_t)$$

where $H^u_t = \hat{H}(\lambda_t, t, u)$ is the price at time $t$ of a zero-coupon bond maturing at $u$.

Let us denote $Z_t = H^u_t / H^U_t$ and let us set, for every $u \in [U, T]$,

$$\mathbb{P}_u(H^u_U \leq \hat{K}(u) | \lambda_t) = P_u(t, Z_t).$$

Then the pricing formula for the bond put option becomes

$$P^u_t = \hat{K}(u) H^U_t P_U(t, Z_t) - H^u_t P_u(t, Z_t)$$
Let us recall the general representation for the hedging strategy when $\mathbb{F}$ is the Brownian filtration.

**Proposition**

The hedging strategy $\tilde{\varphi} = (\tilde{\varphi}^1, \tilde{\varphi}^2)$ for the credit default swaption equals, for $t \in [s, U]$,

$$\tilde{\varphi}^1_t = \frac{\tilde{\xi}_t}{\kappa(t, U, T)\sigma_t^\kappa}, \quad \tilde{\varphi}^2_t = \frac{\tilde{C}_t - \tilde{\varphi}^1_t \tilde{S}_t(\kappa)}{\tilde{A}(t, U, T)}$$

where $\tilde{\xi}$ is the process satisfying

$$\frac{\tilde{C}_U}{\tilde{A}(U, U, T)} = \frac{\tilde{C}_0}{\tilde{A}(0, U, T)} + \int_0^U \tilde{\xi}_t d\tilde{W}_t.$$
Recall that we are searching for the process \( \tilde{\xi} \) such that

\[
d(\tilde{C}_t/\tilde{A}(t, U, T)) = \tilde{\xi}_t \, d\tilde{W}_t.
\]

**Proposition**

Assume that \( R = U \). Then we have that, for every \( t \in [0, U] \),

\[
\tilde{\xi}_t = \frac{1}{\tilde{A}_t} \left( \int_{[U, T]} B(t, u) \left( \vartheta_t H_u^u (b_u^u - b_t^u) - P_t^u b_t^u \right) d\chi(u) - \tilde{C}_t \sigma_t^A \right)
\]

where

\[
\tilde{A}_t = \tilde{A}(t, U, T), \quad H_u^u = \tilde{H}(\lambda_t, t, u), \quad b_u^u = cn(t, u) \sqrt{\lambda_t}, \quad P_u^u = P(\lambda_t, U, u, \tilde{K}(u))
\]

and

\[
\vartheta_t = \tilde{K}(u) \frac{\partial P_u}{\partial z}(t, Z_t) - P_u(t, Z_t) - Z_t \frac{\partial P_u}{\partial z}(t, Z_t).
\]
For $R = U$, we obtain the following final result for hedging strategy.

**Proposition**

Consider the CIR default intensity model with a deterministic short-term interest rate. The replicating strategy $\tilde{\varphi} = (\tilde{\varphi}^1, \tilde{\varphi}^2)$ for the credit default swaption maturing at $R = U$ equals, for any $t \in [0, U]$,

\[
\tilde{\varphi}^1_t = \frac{\tilde{\xi}_t}{\kappa(t, U, T)\sigma^\kappa_t}, \quad \tilde{\varphi}^2_t = \frac{\tilde{C}_t - \tilde{\varphi}^1_t \tilde{S}_t(\kappa)}{\tilde{A}(t, U, T)},
\]

where the processes $\sigma^\kappa$, $\tilde{C}$ and $\tilde{\xi}$ are given in previous results.

Note that for $R \leq U$ the problem remains open, since a closed-form solution for the process $\tilde{\xi}$ is not readily available in this case.
Summary

1. We analyzed pricing formulae and dynamics of asset prices in a hazard process framework.
2. A generic representation for the hedging strategy for the credit default swaption in a hazard process model was derived.
3. We obtained formulae for the volatility of the forward CDS rate in terms of quantities related to conditional distributions of default time.
4. A complete solution to the valuation and hedging problem for the credit default swaption in the CIR default intensity model was given for $R = U$.
5. The case where $R < U$, that is, the swaption’s maturity precedes the start date of the underlying forward swap is more difficult to handle and a closed-form solution is not yet available.