Credit Risk Models with Filtered Market Information

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Introduction

Aim: find a suitable model for pricing and hedging portfolio credit derivatives

Criteria for a good model

- Realistic dynamics of credit spread allowing for spread risk and contagion.
- Realistic dependence structure of default times in order to capture observed properties of credit derivative prices (in particular the correlation-skew on CDO markets)
- Tractability. computation of prices/hedge ratios and calibration with reasonable computational effort.

The information-based approach

Notation

- Portfolio of m firms
- τ_i denotes the default time of firm i
- $Y_{t,i} = 1_{\{\tau_i \leq t\}}$ and $Y_t = (Y_{t,1}, \ldots, Y_{t,m})$ current default state
- default history is $\mathbb{F}^Y$. We work directly under risk-neutral measure Q.

The approach contains three layers of information:

1. Fundamental Model. (Full information)

τ_i are conditionally independent doubly stochastic random times, driven by a finite-state Markov chain X with state space $S^X = \{1, \ldots, K\}$.

The fundamental model is a theoretical device for model-construction.

Example

- $X \in \{1, \ldots, K\}$ models the state of economy. The default intensity of company i is
$$\lambda_i : \{1, \ldots, K\} \to (0, \infty)$$
and we have to filter X from available information.
- One-factor structure: Consider
$$X_t = (X_t^1, \ldots, X_t^{m+1}) \in \{0,1\}^m \times \{1, \ldots, K\}.$$
Then $X^1, \ldots, X^m$ are firm-specific components (good/bad state) and X^{m+1} is the general state of economy.

The second layer of information:

2. Market information.

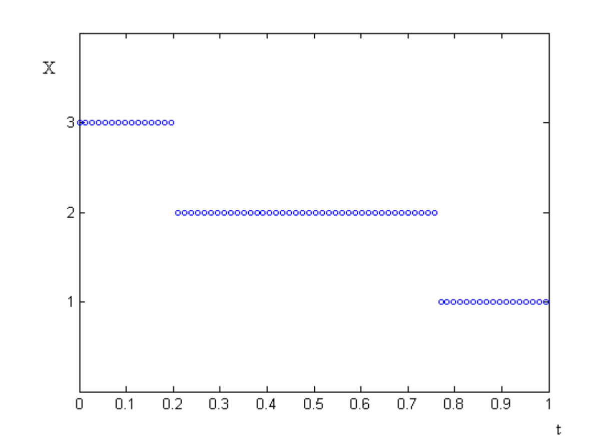
Prices of traded credit derivatives are determined by informed market-participants.

These investors observe the default history and some process Z giving X in additive Gaussian noise:

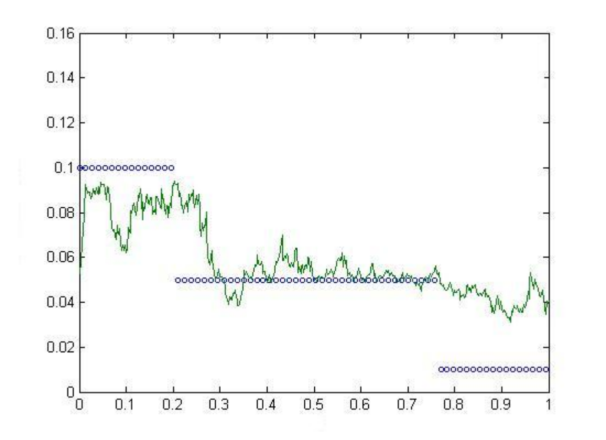
$$Z_t = \int_0^t \mathbf{a}(X_s)ds + dB_t$$

Discounted prices of traded securities will be martingales wrt so-called market information $\mathbb{F}^M := \mathbb{F}^Y \vee \mathbb{F}^Z$

$\Rightarrow$ Filtering results wrt $\mathbb{F}^M$ will be used to obtain asset price dynamics.



$X \in \{1, 2, 3\}$ denotes riskiness of the company. Given a realization.
In this example $X = (X_t)_{t \geq 0}$ represents riskiness of the company.



The estimated $\mathbb{E}(\lambda(X_t)|\mathcal{F}_t^M)$ from Market information

The third layer is motivated by practical applicability:

3. Investor information.

Z represents some kind of 'insider information' and is not directly observable

$\Rightarrow$ We study pricing and hedging of (non-traded) credit derivatives from the viewpoint of secondary-market investors with information $\mathbb{F}^I \subset \mathbb{F}^M$ (investor information).

We assume that $\mathbb{F}^I$ contains the default history $\mathbb{F}^Y$ and (noisily observed) prices of traded credit derivatives. Example:

$$U_t = \int_0^t \widehat{p}_s ds + vW_t.$$

v is a parameter which adjusts the calibration

Advantages

- Prices are weighted averages of full-information value (the theoretical price wrt $\mathbb{F}^X \vee \mathbb{F}^Y$) so that most computations are done in the full-information model. Since the latter has a simple structure, computations become straightforward.
- Rich credit-spread dynamics with spread risk (as credit spreads fluctuate in response to fluctuations in Z) and default contagion (as defaults of firms in the portfolio lead to an update of the conditional distribution of X given $\mathcal{F}_t^M$).
- Model has has a natural factor structure with factors given by the conditional probabilities $\pi_t^k = Q(X_t = k \mid \mathbb{F}^M)$, $1 \leq k \leq K$.
- Great flexibility for calibration. In particular, we may view observed prices as noisy observation of the state X_t and apply calibration via filtering.

Model and Notation

- We work on probability space $(\Omega, \mathcal{F}, Q)$ with filtration $\mathbb{F}$.
- Consider portfolio of m firms with default state $Y_t = (Y_{t,1}, \dots, Y_{t,m})$ for $Y_{t,i} = 1_{\{\tau_i \le t\}}$
- Default-free interest rate r independent of X and Y. Wlog $r(t) \equiv 0$.

The fundamental model

Consider a finite-state Markov chain X with generator Q^X and $S^X := \{1, \dots, K\}$ We assume that

A1 The default times have $(Q, \mathbb{F})$-default intensity $(\lambda_i(X_t))$, i.e. there are functions $\lambda_j : S^X \mapsto (0, \infty)$, such that the processes

$$M_{t,j} := Y_{t,j} - \int_0^{t \wedge \tau_j} \lambda_j(X_{s-}) ds \tag{1}$$

are $\mathbb{F}$-martingales, $1 \leq j \leq m$. Moreover, $\tau_1, \dots, \tau_m$ are conditionally independent given $\mathcal{F}^X_\infty = \sigma(X_s : s \geq 0)$.

Define the full-information value of a $\mathcal{F}^Y_T$-measurable claim H by

$$\mathbb{E}^Q \Big(H \mid \mathcal{F}_t \Big) =: h_t(X_t)\,; \tag{2}$$

where h_t is a $\mathcal{F}^Y_t$-measurable function.

Market information

Recall that the informational advantage of informed market participants is modelled via observations of a process Z. Formally,

A2 $\mathbb{F}^M = \mathbb{F}^Y \vee \mathbb{F}^Z$, where the l-dim. process Z solves the SDE

$$dZ_t = \mathbf{a}(X_t)dt + dB_t.$$

Here, B is an l-dim standard $\mathbb{F}$-Brownian motion independent of X and Y, and $\mathbf{a}(\cdot)$ is a function from S^X to $\mathbb{R}^l$.

Notation

Given a generic process U, we denote by $\widehat{U}$ the optional projection of U w.r.t. the market filtration $\mathbb{F}^M$; recall that $\widehat{U}$ is a right continuous process and $\widehat{U}_t = \mathbb{E}(U_t|\mathcal{F}_t^M)$ for all $t \geq 0$.

Traded securities.

We consider N liquidly traded credit derivatives (eg. corporate bonds) with maturity T and $\mathcal{F}^I_T$-measurable payoff $P_{T,1}, \dots, P_{T,N}$.

A3 (Martingale modelling) The observed prices of traded securities are given by $E\big(P_{T,i}|\mathcal{F}^M_t\big) =: \widehat{p}_{t,i}$ (expectation wrt. Q).

Market-pricing as a nonlinear filtering problem.

Denote by $p_i(t, X_t, Y_t)$ the full-information value of security i. Iterated conditional expectation $\Rightarrow$

$$\widehat{p}_{t,i} = \mathbb{E}\big(\mathbb{E}(P_{T,i}|\mathcal{F}_t) \mid \mathcal{F}^M_t\big) = \mathbb{E}\big(p_i(t, X_t, Y_t)|\mathcal{F}^M_t\big). \tag{3}$$

Hence we need to obtain the conditional distribution of X given $\mathcal{F}^M_t$ (a nonlinear filtering problem).

Security-price dynamics

We introduce the innovations processes as follows:

$$\widehat{M}_{t,j} := Y_{t,j} - \int_0^{t\wedge\tau_j} \widehat{\lambda_j(X_{s-})} ds \qquad \text{for } j = 1, \cdots, m$$
$$\mu_{t,i} := \widehat{B}_{t,i} = Z_{t,i} - \int_0^t \widehat{a_i(X_s)}\, ds \qquad \text{for } i = 1, \cdots, l.$$

Note that $\widehat{M}_j$ is an $\mathbb{F}^M$-martingale and μ is $\mathbb{F}^M$-Brownian motion.

Lemma

Every square integrable $\mathbb{F}^M$-martingale $(\widehat{U_t})_{t\in[0,T]}$ has the representation

$$\widehat{U}_T = \widehat{U}_0 + \int_0^T \gamma_s^\top d\widehat{M}_s + \int_0^T \boldsymbol{\alpha}_s^\top d\widehat{\mu}_s, \tag{4}$$

for $\mathbb{R}^m$ respectively $\mathbb{R}^l$-valued $\mathbb{F}^M$-predictable processes γ and $\boldsymbol{\alpha}$ such that $E\int_0^T |\gamma_s|^2 ds + E\int_0^T |\boldsymbol{\alpha}_s|^2 ds < \infty$.

General filtering equations

Proposition (General filtering equations)

Consider a real-valued $\mathbb{F}$-semimartingale $\xi_t = \xi_0 + \int_0^t A_s ds + \tilde{M}_t$, where $\tilde{M}_t$ is an $\mathbb{F}$-martingale with $[\tilde{M}, B] = 0$. Then the optional projection $\widehat{\xi}_t$ has the following representation

$$\widehat{\xi}_t = \widehat{\xi}_0 + \int_0^t \widehat{A}_s ds + \int_0^t \gamma_s^\top d\widehat{M}_s + \int_0^t \boldsymbol{\alpha}_s^\top d\mu_s. \tag{5}$$

The square-integrable predictable processes γ and $\boldsymbol{\alpha}$ are given by

$$\boldsymbol{\alpha}_t = \widehat{\xi_t \mathbf{a}(X_t)} - \widehat{\xi}_t \widehat{\mathbf{a}(X_t)}, \tag{6}$$

$$\gamma_{t,j} = (1 - Y_{t-,j}) \left(\mathbb{E}(\xi_t | \mathcal{F}^M_{t-} \vee \{\tau_j = t\}) - \mathbb{E}(\xi_t | \mathcal{F}^M_{t-}) \right). \tag{7}$$

The proof uses the standard arguments from the innovations approach to nonlinear filtering.

Security-price dynamics

Theorem

Under **A1** - **A3** *the price process of the traded securities has the martingale representation*

$$\begin{aligned} \widehat{p}_{t,i} &= \widehat{p}_{0,i} + \int_0^t \gamma_s^{\widehat{p}_i,\top} d\widehat{M}_s + \int_0^t \alpha_s^{\widehat{p}_i,\top} d\mu_s, \quad \textit{with} \\ \alpha_t^{\widehat{p}_i} &= \widehat{p_{t,i}\cdot \mathbf{a}_t} - \widehat{p}_{t,i}\,\widehat{\mathbf{a}}_t \\ \gamma_{t,j}^{\widehat{p}_i} &= (1-Y_{t-,j})\Big(\mathbb{E}\big(p_i(t,X_t,Y^j_{t-})|\mathcal{F}^M_{t-}\big) - \mathbb{E}\big(p_i(t,X_t,Y_{t-})|\mathcal{F}^M_{t-}\big)\Big). \end{aligned}$$

The predictable quadratic variations of the asset prices with respect to the market information $\mathbb{F}^M$ *satisfy* $d\langle \widehat{p}_i, \widehat{p}_j\rangle^M_t = v^{ij}_t\,dt$ *with*

$$v_t^{ij} = \sum_{n=1}^m \gamma_{t,n}^{\widehat{p}_i}\gamma_{t,n}^{\widehat{p}_j}\,\widehat{\lambda}_{t,n} + \sum_{n=1}^l \alpha_{t,n}^{\widehat{p}_i}\alpha_{t,n}^{\widehat{p}_j}. \tag{8}$$

Filtering

Define the conditional probability vector $\boldsymbol{\pi}_t = (\pi_t^1, \dots, \pi_t^K)^\top$ with $\pi_t^k := Q(X_t = k | \mathcal{F}_t^M)$. $\boldsymbol{\pi}_t$ is the natural state variable; under market information $\mathbb{F}^M$ all quantities of interest are functions of $\boldsymbol{\pi}_t$.

Dynamics of $\boldsymbol{\pi}_t$.

- Updating at a default time τ_i. One has

$$Q(X_t = k | \mathcal{F}_{t-}^M \vee \{\tau_i = t\}) = \frac{\lambda_i(k)\pi_{t-}^k}{\sum_{n=1}^K \lambda_i(n)\pi_{t-}^n}.$$

- Kushner-Stratonovich equation (K-dim SDE-system for $\boldsymbol{\pi}_t$)

$$d\pi_t^k = \sum_{\iota=1}^K q(\iota, k)\pi_{t-}^\iota dt + \boldsymbol{\gamma}_t^{\boldsymbol{\pi},\top} d\widehat{M}_t + \boldsymbol{\alpha}_t^{\boldsymbol{\pi},\top} d\mu_t, \text{ with} \tag{9}$$

$$\gamma_{t,i}^{\boldsymbol{\pi}} = \pi_{t-}^k \left(\frac{\lambda_i(k)}{\sum_{n=1}^K \lambda_i(n)\pi_{t-}^n} - 1 \right), \ \boldsymbol{\alpha}_t^{\boldsymbol{\pi}} = \pi_{t-}^k \left(\mathbf{a}(k) - \sum_{i=1}^K \pi_{t-}^i \mathbf{a}(i) \right).$$

Recall that secondary market investors do not observe Z. Their information set is given by $\mathbb{F}^I \subset \mathbb{F}^M$; typically $\mathbb{F}^I$ contains default history and noisy price information. Put $\nu_t^k := Q(X_t = k|\mathcal{F}_t^I)$, $1 \leq k \leq K$.

Pricing.

Consider non-traded $\mathcal{F}_T^Y$-measurable claim H. Define its secondary-market value as $\mathbb{E}(H|\mathcal{F}_t^I)$. Let $h_t(X_t) = E(H \mid \mathcal{F}_t)$ (full-information value of H). Iterated conditional expectations $\Rightarrow$

$$\mathbb{E}(H|\mathcal{F}_t^I) = \mathbb{E}\big(\mathbb{E}(H|\mathcal{F}_t^M)\big|\mathcal{F}_t^I\big) = \mathbb{E}\Big(\sum_{k=1}^K \pi_t^k h_t(k)\Big|\mathcal{F}_t^I\Big) = \sum_{k=1}^K \nu_t^k h_t(k),$$

i.e. pricing for secondary-market investors reduces to finding $\boldsymbol{\nu}_t$.

Hedging.

We look for risk-minimizing strategies under restricted information in the sense of Schweizer (1994).

- Quadratic criterion combines well with incomplete information
- On credit markets it is natural to minimize risk wrt martingale measure Q as historical default intensities are hard to determine.

- Risk-minimizing strategy θ^H can be computed by suitably projecting the $\mathbb{F}^M$-risk-minimizing hedging strategy ξ_t^H on the set of $\mathbb{F}^I$-predictable strategies.
- I.e. with only one traded asset, θ_t is left-continuous version of

$$E(v_t \xi_t^H \mid \mathcal{F}_t^I) \,/\, E(v_t \mid \mathcal{F}_t^I) \,.$$

- Recall that v_t and ξ_t are nonlinear functions of $\boldsymbol{\pi}_t$. $\Rightarrow$ We need to determine conditional distribution of $\boldsymbol{\pi}_t$ given $\mathcal{F}_t^I$.

Calibration via filtering

Assume that $\mathbb{F}^I = \mathbb{F}^Y \vee \mathbb{F}^U$ where the N-dim process U solves the SDE

$$dU_t = \widehat{p}_t dt + dW_t = \mathbf{p}(t, Y_t)\boldsymbol{\pi}_t dt + dW_t$$

for a Brownian motion W independent of X, Y, Z. U can be viewed as cumulative noisy price information of the traded assets $\widehat{p}_1, \ldots, \widehat{p}_N$; noise reflects observation errors and model errors.

Recall that $\boldsymbol{\pi}$ solves the KS-equation (9). Hence computation of the conditional distribution of $\boldsymbol{\pi}_t$ given $\mathcal{F}_t^I$ is a standard nonlinear filtering problem with signal process $\boldsymbol{\pi}$ and observation U.

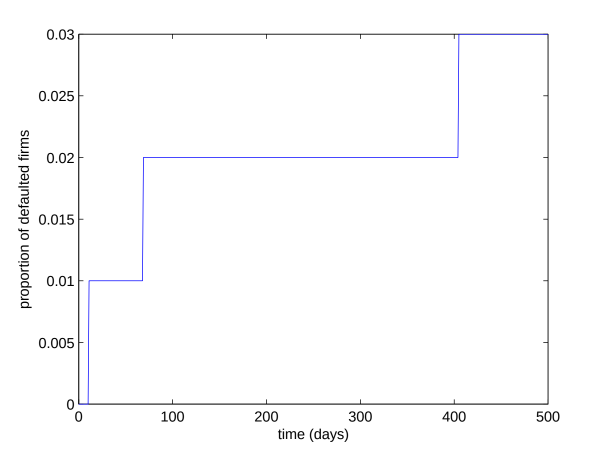
Note that $\boldsymbol{\pi}$ is typically high-dimensional; $\Rightarrow$ particle filtering might be used.

Numerical example

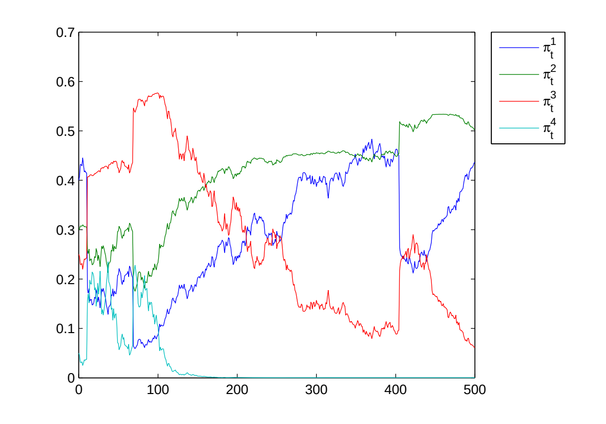
Assume that $X_t \equiv X$ takes values in $\{1, 2, 3, 4\}$. The observation $\mathbb{F}^M$ contains

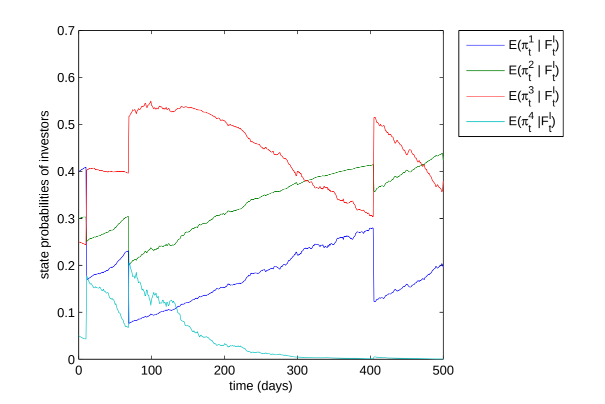
$$Z_t = Xt + B_t$$

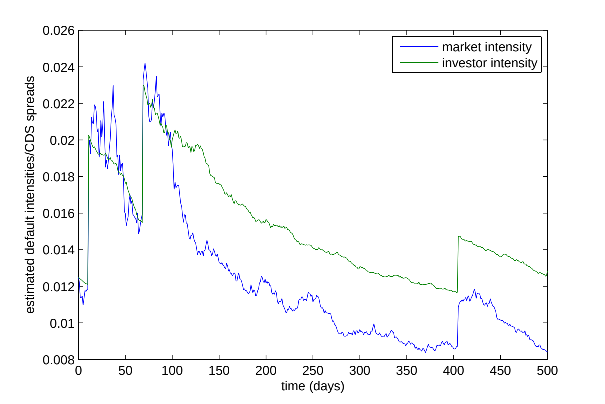
and default information.


0.03
0.025
0.02
0.015
0.01
0.005
0
proportion of defaulted firms
0
100
200
300
400
500
time (days)

Market participants filter the conditional distribution of X from Z. In our simulation $X(\omega) = 2$. The resulting $\pi = (\pi^1, \ldots, \pi^4)$ are




0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
100
200
300
400
500
state probabilities of investors
time (days)
$E(\pi_t^1 \mid F_t^I)$
$E(\pi_t^2 \mid F_t^I)$
$E(\pi_t^3 \mid F_t^I)$
$E(\pi_t^4 \mid F_t^I)$



market intensity
investor intensity
estimated default intensities/CDS spreads
time (days)
0.026
0.024
0.022
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0
50
100
150
200
250
300
350
400
450
500

Many Thanks!

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