Optimal prepayment rule for mortgage-backed securities

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- Preliminaries on mortgages and implied options.

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- Numerical results.

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In discrete time, instead, the borrower repays the capital with N equal rates A_N at dates kT/N, $k = 1, \ldots, N$ at the nominal rate ρ_N prevailing for each small period $[kT/N, (k+1)T/N]$: then

$$P = \sum_{i=1}^{N} \frac{A_N}{(1+\rho_N)^i} = \frac{A_N}{\rho_N} \left(1 - \frac{1}{(1+\rho_N)^N} \right).$$

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$$F_n := \sum_{i=n+1}^{N} \frac{A_N}{(1+\rho_N)^{i-n}} = \frac{A_N}{\rho_N} \left(1 - \frac{1}{(1+\rho_N)^{N-n}} \right).$$

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Once that this option is exercised, the contract terminates. This prepayment option has thus the character of a contingent claim of American type.

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$$V_t := \int_t^T B(t,u)A \, du = A \int_t^T B(t,u) \, du \qquad (2)$$

while in discrete time its market value at time nT/N is

$$V_n := \sum_{i=n+1}^{N} A_N B\left(n\frac{T}{N}, i\frac{T}{N}\right) = A_N \sum_{i=n+1}^{N} B\left(n\frac{T}{N}, i\frac{T}{N}\right)$$

where $B(t,s)$ is the value at time t of a zero-coupon bond with maturity $s \geq t$.

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where $B(t,s)$ is the value at time t of a zero-coupon bond with maturity $s \geq t$.
Thus, the lender is exposed to the risk of early exercise at time $t < T$ of an American option to exchange V_t for F_t.

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The options of prepayment and of surrendering are alternative to each other: once that the borrower exercises one of the two, the contracts expires and (s)he cannot exercise the other.

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- an IO-holder does not receive anymore interest for that single mortgage,
- while a PO-holder receives the principal of that mortgage before the natural maturity,
- a pass-through holder receives a lump sum immediately instead of a fixed stream of payments in the future, which usually have similar market values.

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In the sequel, we (first) concentrate on the problem of pricing a GNMA/FNMA type of MBS.

The pricing of (GNMA-FNMA) "rational" MBS

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$$V_t - \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^{\hat{\tau}} r_u \, du}(V_{\hat{\tau}} - F_{\hat{\tau}}) \mid \mathcal{F}_t]$$

where F (residual debt) and V (market value of the residual debt) are respectively given by Equations (1) and (2), $(\mathcal{F}_t)_t$ is a suitable filtration representing the amount of information up to time t (typically the filtration generated by the short rate r), $\mathbb{Q}$ is a suitable equivalent martingale measure and $\hat{\tau}$ is an "optimal" $(\mathcal{F}_t)_t$-stopping time.

Irrationality of the borrowers

If the borrowers were all rational, then

$$\mathbb{E}_{\mathbb{Q}}[e^{-\int_t^{\hat{\tau}} r_u \, du}(V_{\hat{\tau}} - F_{\hat{\tau}}) \mid \mathcal{F}_t] = \operatorname{ess}\sup_{\tau \in [t,T]} \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^{\tau} r_u \, du}(V_{\tau} - F_{\tau}) \mid \mathcal{F}_t] \tag{3}$$

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borrower could be not acquainted with rational economic reasoning, not financial market professional and/or not fully informed about the financial markets.
Even if they are, there could be other non-optimal reasons for exercising their prepayment options in a non-optimal time (for example, the house underlying the mortgage has been sold, or the borrower sees the optimality of prepayment with some delay).

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Let us call τ the prepayment time of a "typical" single borrower.
Define its cumulative distribution function and density with respect to $\mathbb{Q}$, conditional to a state variable θ:

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Define the *prepayment function* (analogous of the *default intensity* in credit risk) as

$$\pi(t|\theta) := \lim_{\Delta t \to 0^+} \frac{\mathbb{Q}\{t < \tau \leq t + \Delta t \mid \tau > t, \theta\}}{\Delta t} = \frac{f(t|\theta)}{1 - F(t|\theta)}$$

Pricing MBS with the prepayment function

The prepayment function $\pi(t|\theta)$ gives the density of prepayment at time t conditioned to the fact that the borrower has not yet prepaid (and to the state variable θ). It is well known that

$$\begin{aligned} F(t|\theta) &= 1 - \exp\left(-\int_0^t \pi(s|\theta)\, ds\right), \\ f(t|\theta) &= \pi(t|\theta) \exp\left(-\int_0^t \pi(s|\theta)\, ds\right) \end{aligned}$$

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Once that we specify the hazard function $\pi(\cdot|\theta)$ and the interest rates dynamics, we can obtain the price of a MBS. For example, the price of a pass-through MBS at time t would be

$$\begin{aligned} & V_t - \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^\tau r_u\ du}(V_\tau - F_\tau) \mid \mathcal{F}_t] = \\ &= \quad V_t - \mathbb{E}_{\mathbb{Q}}\left[\int_t^T \pi(u|\theta)e^{-\int_t^u (r_v + \pi(v|\theta))dv}(V_u - F_u)\ du \,\middle|\, \mathcal{F}_t\right] \end{aligned}$$

Prepayment function and optimal prepayment

In many models, π depends on the fact that at a given time t it is or not optimal to prepay: for $t \in [0, T]$, define

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with $\lambda, \rho \in \mathbb{R}^+$, $\lambda(t) \in C^0([0, T])$: an exogenous (i.e. non-optimal but happening due to exogenous reasons) prepayment has a hazard function λ (or $\lambda(t)$), while when it is optimal to prepay the hazard function is augmented by ρ (or goes to $+\infty$, i.e. all the borrowers always prepay immediately when it is optimal).

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$\rightarrow$ solving the problem (3), while not sufficient to price a MBS, can be necessary for many models.

Optimal rational prepayment

Assume that the short rate r is Markov, so that $(B(t,s))_{0\le t\le s\le T}$ can be obtained from r, and has stochastic differential

$$dr_t = \mu(t, r_t)\ dt + \sigma(t, r_t)\ dW_t \tag{4}$$

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$$\begin{cases} L\tilde{B} = 0, & (t, r) \in (0, T) \times \mathbb{R}, \\ \tilde{B}(T, T, r) = 1, & r \in \mathbb{R} \end{cases}$$

and L is the infinitesimal generator of r

$$Lf := f_t + \mu f_r + \frac{1}{2}\sigma^2 f_{rr} - rf$$

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Recall that we want to solve

$$\operatorname*{ess\,sup}_{\tau\in[t,T]} \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^\tau r_u\ du}(V_\tau - F_\tau) \mid \mathcal{F}_t]$$

Optimal prepayment and free boundary problems

It is well known that the solution to this problem is linked to the free boundary value problem (or variational inequality)

$$\begin{cases} \max(Lf, \psi - f) = 0, & (t, r) \in (0, T) \times \mathbb{R}, \\ f(T, r) = \psi(T, r)(\equiv 0), & r \in \mathbb{R} \end{cases} \tag{5}$$

where $\psi(t, r) = V(t, r) - F(t)$ is given by

$$\psi(t, r) \;\; = \;\; A\left(\int_t^T \tilde{B}(t, u, r)\, du - \frac{1 - e^{-\rho(T-t)}}{\rho}\right)$$

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$$\psi(t, r) = A \left(\int_t^T \tilde{B}(t, u, r)\, du - \frac{1 - e^{-\rho(T-t)}}{\rho} \right)$$

An optimal stopping time is given by

$$\tau := \inf\{t \le T \mid f(t, r_t) = \psi(t, r_t)\}$$

Optimal prepayment and free boundary problems

It is well known that the solution to this problem is linked to the free boundary value problem (or variational inequality)

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Unfortunately (5) has an explicit solution in very few cases!

Numerical solution

We now concentrate ourselves on numerical methods involving weak convergence of discrete time processes to solve directly problem (3).

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Consider N subdivisions of the interval $[0, T]$: then (3) becomes

$$\operatorname{ess}\sup_{\tau \in [n,N]} \mathbb{E}_{\mathbb{Q}}[e^{-\frac{1}{N}\sum_{i=n}^{\tau} r_i}(V_\tau - F_\tau) \mid \mathcal{F}_n] \tag{6}$$

with $\operatorname{ess}\sup$ taken over all the $(\mathcal{F}_n)_n$-stopping times τ taking integer values between n and N, and $V - F$ is given by

$$V_n - F_n = A_N \sum_{i=n+1}^{N} B\left(n\frac{T}{N}, i\frac{T}{N}\right) - \frac{A_N}{\rho_N}\left(1 - \frac{1}{(1+\rho_N)^{N-n}}\right)$$

with the zero-coupon values B now given by

$$B\left(n\frac{T}{N}, i\frac{T}{N}\right) = \mathbb{E}_{\mathbb{Q}}[e^{-\frac{1}{N}\sum_{k=n}^{i-1} r_k} \mid \mathcal{F}_n]$$

Numerical solution (2)

If the short rate r is a Markov chain, then the zero-coupon values $B(\frac{n}{N}T, \frac{i}{N}T)$ are deterministic functions of n and r_n, as well as $V_n - F_n$, thus in (6) the conditional expectation is a function of n and r_n.

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Thus (6) can be solved by building the Snell envelope and using the Dynamic Programming principle: define recursively the functions W_n, $n = 0, \dots, N$, as

$$\begin{aligned} W_N(r) &:= V_N(r) - F_N(r) \equiv 0, \qquad (7) \\ W_n(r) &:= \max\left((V_n - F_n)(r), \mathbb{E}[e^{-\frac{1}{N}r_{n+1}} W_{n+1}(r_{n+1}) \mid r_n = r]\right) \end{aligned}$$

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Given r_0, the computational cost of calculating $(W_n)_n$ depends on the model that we choose for $(r_n)_n$.

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We choose to use a recombining binomial tree dynamics for r. If the number of states that r_n can assume is $O(n)$ (in the common binomial case, $= n+1$), then one can use backward induction to calculate the functions $(W_n)_n$, having to calculate them in every one of the $O(\sum_{n=1}^{N} n) = O(\frac{N(N+1)}{2})$ nodes of the tree.

Computationally simple trees

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As noticed before, in this case the total number of nodes up to time N is at most equal to $K\frac{N(N+1)}{2}$.
If we use a binomial tree, typically $K = 1$, and in order to calculate the functions (7) one has to calculate at each node of the tree two expectations (of $V_n(r_n)$ and of $W_{n+1}(r_{n+1})$) over two possible future outcomes and then a maximum, ending up with a bounded number of operations at each node of the tree.

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Divide the interval $[0, T]$ in N equal subintervals with length $h := T/N$, and consider a process $(r_t^h)_{t\in[0,T]}$, which is constant on the subintervals and, at each time hk, $k = 1, \dots, N$, jumps upwards or downwards with probability q and $1 - q$, respectively.

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More precisely, take $q_h, R^+_h, R^-_h : \mathbb{R} \times [0, +\infty)$ such that

$$q_h(r, hk) \in [0, 1], \qquad -\infty < R^-_h(r, hk) \le R^+_h(r, hk) < \infty,$$

and define the process r^h such that

$$\mathbb{P}(r^h_{(k+1)h} = R^+_h(r, hk) | r^h_{hk} = r) = q_h(r, hk), \tag{8}$$

$$\mathbb{P}(r^h_{(k+1)h} = R^-_h(r, hk) | r^h_{hk} = r) = 1 - q_h(r, hk), \tag{9}$$

The process r^h is a Markov chain.

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The process r^h is a Markov chain.
Under suitable assumptions on q_h, R_h^+, R_h^-, the processes $(r^h)_h$ converge weakly to r.

Computationally simple trees: the idea

In order to obtain a recombining binomial tree (thus computational simplicity), we must require that the total displacement of a up-movement followed by a down-movement is equal to the analogous displacement when the movements have reverse order.

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$$\begin{aligned} R^+(r,t) - r + R^-\left(R^+(r,t), t+h\right) - R^+(r,t) = \\ = \quad R^-(r,t) - r + R^+\left(R^-(r,t), t+h\right) - R^-(r,t) \end{aligned}$$

i.e. $R^-(R^+(r,t), t+h) = R^+(R^-(r,t), t+h)$, must be true for all $t = nh$, $r \in \mathbb{R}$.

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i.e. $R^{-}(R^{+}(r,t), t+h) = R^{+}(R^{-}(r,t), t+h)$, must be true for all $t = nh$, $r \in \mathbb{R}$.
We now follow Nelson and Rawaswany and present computationally simple trees for two classical models of r.

Weak convergence to the Vasicek model

Assume that r follows the Vasicek model

$$\mathrm{d}r_t = \beta(\alpha - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{10}$$

with $\beta > 0$,

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$$R_h^+(r,t) = r + \sigma\sqrt{h}, \qquad R_h^-(r,t) = r - \sigma\sqrt{h},$$

$$q_h(r,t) = \begin{cases} \frac{1}{2} + \sqrt{h}\frac{\beta}{2\sigma}(\alpha - r) & \text{if } 0 \leq \frac{1}{2} + \sqrt{h}\frac{\beta}{2\sigma}(\alpha - r) \leq 1, \\ 0 & \text{if } \frac{1}{2} + \sqrt{h}\frac{\beta}{2\sigma}(\alpha - r) < 0, \\ 1 & \text{otherwise} \end{cases}$$

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With this choice, one can prove that the sequence $(r^h)_h$ converges weakly to r (drift $\leftrightarrow q_h$, diffusion $\leftrightarrow R_h^\pm$),

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With this choice, one can prove that the sequence $(r^h)_h$ converges weakly to r (drift $\leftrightarrow q_h$, diffusion $\leftrightarrow R_h^{\pm}$), and we have a computationally simple tree.

Numerical results for the Vasicek model

We now present a numerical solution of the prepayment problem. We set as model parameters $r_0 = 0.03$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$. For each node, we indicate with a red cross an optimal decision to prepay (stop) and with a white dot an optimal decision to continue.

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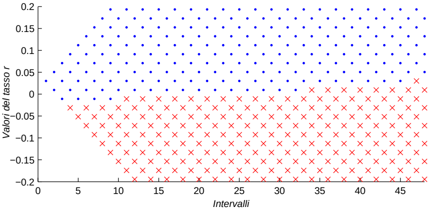
The first graphic has $\rho = 0.04$, $T = 2$ years and $N = 48$ subintervals (corresponding to prepayment decisions taken every 15 days).



Numerical results for the Vasicek model

The second graphic has $\rho = 0.04$, $T = 20$ years and $N = 240$ (prepayment decisions every month).

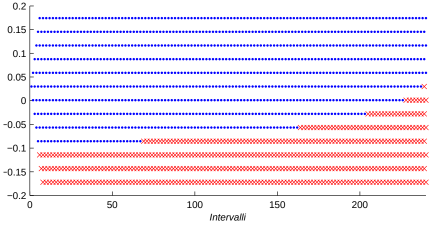


Figure: Optimal prepayment with $N = 240$ (other parameters: $r_0 = 0.03$, $T = 20$, $\rho = 0.04$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$)

Weak convergence to non-gaussian models

In order to build a computationally simple tree for the CIR model, we first make a transform of the state variable $X_t := X(r_t, t)$ with $X \in C^{1,2}$. If r satisfies Equation (4), then

$$\mathrm{d}X = \left(\frac{\partial X}{\partial t} + \mu \frac{\partial X}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 X}{\partial r^2} \right) \mathrm{d}t + \sigma \frac{\partial X}{\partial r}\, \mathrm{d}W_t.$$

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Now choose X such that

$$X(r, t) = \int^r \frac{\mathrm{d}z}{\sigma(z, t)}, \tag{11}$$

With this choice, the diffusion term of $X(r_t, t)$ is constant and we can again build a computationally simple tree in a similar way as we did for the Vasicek model.

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If X is invertible, then we can come back to the process r by applying the inverse transformation.

Weak convergence to the CIR model

Assume that r follows the CIR model

$$\mathrm{d}r = \beta(\alpha - r)\mathrm{d}t + \sigma\sqrt{r}\mathrm{d}W_t,$$

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$$X(r) := \int^r \frac{\mathrm{d}Z}{\sigma\sqrt{Z}} = \frac{2\sqrt{r}}{\sigma}, \text{ with inverse } R(x) := \frac{\sigma^2 x^2}{4}\mathbf{1}_{x>0}$$

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In this case it is required, when r is near 0, to make an upward jump J_h^+ which is a multiple of the corresponding downward jump J_h^-. Then, by defining

$$\begin{aligned} R_h^{\pm}(x,t) &:= R(x \pm J_h^{\pm}(x,t)\sqrt{h}), \\ q_h(x,t) &:= \frac{h\beta(\alpha - R(x)) + R(x) - R_h^-(x)}{R_h^+(x) - R_h^-(x)}\mathbf{1}_{R_h^+(x)>0}, \end{aligned}$$

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Numerical results for the CIR model

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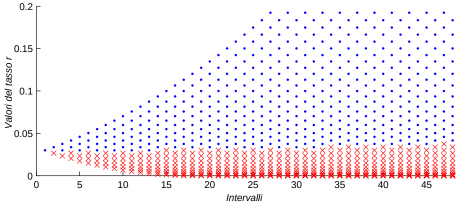

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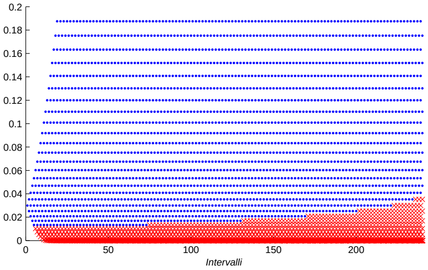

Figure: Optimal prepayment with $N = 24$ (other parameters: $r_0 = 0.03$, $T = 2$, $\rho = 0.04$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$)

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For example, assume that a mortgage has a so-called entrance rate ρ_1 up to the time T_1 and $\rho_2 > \rho_1$ for the remaining time $[T_1, T]$.
Then it is well known that the repayment intensities in the two periods (in continuous time) are

$$
\begin{aligned}
A(t) &= A_1 = \frac{P\rho_1}{1-(\frac{1}{1+\rho_1})^N}, \quad t < T_1, \\
A(t) &= A_2 = \frac{P\rho_2}{1-(\frac{1}{1+\rho_2})^{N-N_1}} \frac{1-(\frac{1}{1+\rho_1})^{N-N_1}}{1-(\frac{1}{1+\rho_1})^N}, \quad t > T_1
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\end{aligned}
$$

In this case, the prepayment option is written on

$$
F_t = \int_0^T e^{-\int_t^u \rho \, ds} A(u)\, du, \qquad V_t = \int_0^T B(t,u)A(u)\, du
$$

Numerical results for an "entrance rate" mortgage (CIR)

Take as model parameters $r_0 = 0.03$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$. The following graphic has $\rho_1 = 0.03$ for a period $T_1 = 3$ years and $\rho_2 = 0.04$ for a total of $T = 20$ years and $N = 120$ subintervals (corresponding to prepayment decisions taken every 2 months).