Kyle-Back equilibrium model with insider trading and credit risk

- Based on joint works with U. Cetin & A. Danilova -

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Contents

Modelling of default risk I

In order to price firm-specific defaultable bonds, one needs to calculate the default probabilities associated with the firm. There are essentially two approaches in default modelling:

- *The firm value* or *structural approach:* The default occur when the firm value hits below a barrier.
  Ref. Black & Scholes (1973), Merton (1974), later extended by Black & Cox (1976) and others.
  Drawback: When the firm value is continuous in time and known to the market, the default event can be anticipated.

Modelling of default risk II

- *Hazard rate* or *reduced form approach:* This is the case when default time is a random time with an intensity. In mathematical terms, the default time is a totally inaccessible stopping time in the market's filtration so that the default comes as a surprise.
  Ref. Jarrow and Turnbull (1992), Artzner & Delbaen (1995), Duffie, Schroder & Skiadas (1996), Duffie & Singleton (1999) and many papers by M. Jeanblanc & coauthors.

Modelling of default risk III

- Although these two approaches may seem unrelated, one may pass from a structural model to a reduced form model by adding noise to the firm's value as observed in the market (Duffie & Lando (2001)) or restricting the market's information set (Çetin, et al. (2004)).
- Here we propose an equilibrium model to link those two approaches. Our study is based on the insider trading model of Back (1992).

Back's model of insider trading

Inspired by Kyle (1985), Back (1992) studies a market for a bond and a risky asset with three types of participants:

1. *Noise traders:* The noise traders can only observe their own cumulative demands modelled by a standard BM B.
2. *Informed trader:* She knows the value of the risky asset at time 1, given by a r.v. $V \sim \mathcal{N}(0,1) \perp B$. Being risk-neutral, her objective is to maximize her expected profit.
3. *Market maker:* The market maker observes the total order Y and sets the price $S_t = H(t, Y_t)$ of the risky asset to exclude arbitrages and he clears the market.

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The pricing mechanism of the market

- The market maker decides the price looking at the total order, Y^θ, which is given by

$$Y_t^\theta = B_t + \theta_t,$$

where θ_t is the position of the insider in the risky asset at time t.
- Thus, the filtration of the market maker is generated by Y, i.e. $\mathcal{F}_t^Y = \sigma(Y_s : s \leq t)$: The insider's trade is not observed directly by the market maker.
- The market maker has a *pricing rule*, $H \in \mathcal{C}^{1,2}([0,1] \times \mathbb{R})$, to assign the price in the following form:

$$S_t = H(t, Y_t),$$

where S_t is the market price of the risky asset at time t.

The insider's objective

- For each $\mathcal{F}^{\mathrm{I}}$-semimartingale strategy, θ, the insider's wealth at time 1 is defined by

$$W_1^\theta = (V - S_1)\theta_1 + \int_0^1 \theta_{t-} dS_t = \int_0^1 (V - S_{t-}) d\theta_t - [S, \theta]_1,$$

  where $[S, \theta]$ is the quadratic variation of S and θ.
- The admissible trading strategies are those satisfying some mild integrability conditions and such that Y^θ is a Markov process given V. The set of such strategies is denoted with $\mathcal{A}$.
- The insider is risk-neutral with the following objective:

$$\mathbb{E}[W_1^\theta | \mathcal{F}_0^{\mathrm{I}}] \quad \rightarrow \quad \text{maximize!} \qquad (2.1)$$

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Equilibrium

Definition

(H^*, θ^*) is said to form an equilibrium if the following conditions are satisfied:

1. *Market efficiency:* Given θ^*, H^* is a rational pricing rule.
2. *Insider optimality:* Given H^*, θ^* solves insider's optimization problem :

$$W_1^{\theta^*} = \max_{\theta \in \mathcal{A}} \mathbb{E}[W_1^{\theta} | \mathcal{F}_0^{\mathrm{I}}].$$

Equilibrium in the original Back's model

Theorem

Define H^* and α^* by

$$H^*(t,y) = y \qquad \text{and} \qquad \alpha^*_t = \frac{V - Y_t}{1-t}.$$

Let $\theta^*_t = \int_0^t \alpha^*_s ds$. Then, (H^*, θ^*) is an equilibrium.

Note that if Y^* denotes the equilibrium level of the total order, then

$$dY^*_t = dB_t + \frac{V - Y^*_t}{1-t} dt,$$

so that Y^* is a Brownian bridge, in the insider's view, from 0 to V of length 1.

Generalizations: Wu (1999), Cho (2003), Lasserre (2004), ...

A Back's model for defaultable bond. (based on a joint paper with U. Cetin (LSE), F&S 2007)

A company issues a bond that pays 1 at time 1 unless it defaults before that time. The recovery rate is 0 for simplicity.

1. *Noise traders:* As in Back's model, they observe only their own cumulative demand, which is modelled by a standard BM B.
2. *Informed trader:* The insider observes the total order Y and knows τ. Being *risk-neutral*, her objective is to maximize her expected profit.
3. *Market maker:* The market maker observes only the total order Y and whether the default has happened or not. He cannot observe the firm's value Z. He sets the price of the defaultable bond and clears the market.
4. *Default time:* $\tau = \inf\{t : Z_t = -1\}$ where Z is a BM starting from 0, independent of B.

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Pricing mechanism of the market I : $\theta \equiv 0$

- In this case, the total demand is $Y = B$ and the market maker's filtration, $\mathcal{F}^M$, is given by $\mathcal{F}_t^M = \mathcal{F}_t^B \vee \sigma(\tau \wedge t)$, for each t.
- In this case, B and Z independent implies τ is a totally inaccessible stopping time for the market maker **as in reduced form models**, i.e. $\tau \neq \lim_n \tau_n$ for any increasing sequence of stopping times τ_n such that $\tau_n < \tau$ on $\{\tau > 0\}$

The pricing mechanism of the market II

- When the insider trades, the total order at time t is given by

$$Y_t = B_t^\tau + \theta_t^\tau,$$

where θ_t is the position of the insider in the defaultable bond at time t. The market maker also observes whether the default has happened: $\mathcal{F}_t^M := \mathcal{F}_t^Y \vee \sigma(\tau \wedge t)$.

- Let $D_t = \mathbf{1}_{[\tau > t]}$. Thus, if S_t denotes the market price of the bond at time t, we expect

$$S_t = D_t H(t, Y_t),$$

where H is the pricing rule of the market maker.

The pricing mechanism of the market III

- The pricing rule H is assumed to be sufficiently smooth to apply Itô's formula, i.e. $\mathcal{C}^{1,2}$ for $t \in [0,1)$ and $H(1,\cdot) \equiv 1$. Moreover, it is strictly increasing, i.e. $\frac{\partial H}{\partial y}(t,\cdot) > 0$ for all $t \in [0,1]$.
- As in Back, a pricing rule, H, is rational if

$$D_t H(t, Y_t) = \mathbb{E}\left[\mathbf{1}_{[\tau>1]} \middle| \mathcal{F}_t^M \right]. \tag{3.2}$$

The insider's objective I

- For each $\mathcal{F}^{\mathrm{I}}$-semimartingale strategy, θ, the insider's wealth at time 1 is defined by

$$W_1^\theta = \int_0^1 (\mathbf{1}_{[\tau>1]} - S_{t-}) d\theta_t - [S, \theta]_1,$$

  where $[S, \theta]$ is the quadratic variation of S and θ.
- We suppose the insider's strategy is absolutely continuous so that $d\theta_t = \alpha_t dt$. Thus, the insider's wealth at time 1 is

$$W_1^\theta = \int_0^1 (\mathbf{1}_{[\tau>1]} - S_t) \alpha_t dt.$$

The insider's objective II

- The admissible trading strategies θ are absolutely continuous and such that Y^θ is a Markov process given τ. The set of admissible strategies is denoted with $\mathcal{A}$.
- The insider is risk-neutral with the following objective:

$$\mathbb{E}[W_1^\theta | \mathcal{F}_0^I] \quad \rightarrow \quad \text{maximize!} \tag{3.3}$$

Equilibrium

Definition

A pair (H^*, θ^*) is said to form an equilibrium if the following conditions are satisfied:

1. *Market efficiency:* Given θ^*, H^* is a rational pricing rule.
2. *Insider optimality:* Given H^*, θ^* solves the optimization problem in (3.3):

$$W_1^{\theta^*} = \max_{\theta \in \mathcal{A}} \mathbb{E}[W_1^{\theta} | \mathcal{F}_0^{\mathrm{I}}].$$

Implications of insider's optimality I

Recall that insider's problem is

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^1 \left\{ \mathbf{1}_{[\tau>1]} - D_t H(t, Y_t) \right\} \alpha_t dt \middle| \mathcal{F}_0^{\mathrm{I}} \right] \tag{3.4}$$

Let's define for a given $\alpha \in \mathcal{A}$

$$J(t, Y_t) = \sup_{\alpha^{(t)} \in \mathcal{A}(t,\alpha)} \mathbb{E}\left[\int_{t\wedge\tau}^{1\wedge\tau} \left\{ \mathbf{1}_{[\tau>1]} - H(u, Y_u) \right\} \alpha_u^{(t)} du \middle| \mathcal{F}_t^{\mathrm{I}} \right].$$

We are in a similar situation as in Back's model. So, we already know the condition for the optimality.

Implications of insider's optimality II

The solution to (3.4) exists if the following system has a solution:

$$\begin{aligned}\frac{\partial}{\partial y}J(t,y)+\mathbf{1}_{[\tau>1]}-H(t,y) &= 0,\\ \frac{\partial}{\partial t}J(t,y)+\frac{1}{2}\frac{\partial^2}{\partial y^2}J(t,y) &= 0,\end{aligned}$$

which implies

$$\frac{\partial}{\partial t}H(t,y)+\frac{1}{2}\frac{\partial^2}{\partial y^2}H(t,y)=0, \quad t\in[0,\tau]\cap[0,1) \qquad (3.5)$$

i.e. H is space-time harmonic.

Implications of market maker's rationality I

Recall that if H is rational then $D_t H(t, Y_t)$ is $\mathcal{F}^M$-martingale.

- For $t \in [0,1)$, Itô's formula gives

$$
\begin{aligned}
dD_t H(t, Y_t) &= D_{t-} \left\{ \frac{\partial}{\partial t} H_-(t, Y_t) + \frac{1}{2} \frac{\partial^2}{\partial y^2} H_-(t, Y_t) \right\} dt \\
&+ D_{t-} \frac{\partial}{\partial y} H_-(t, Y_t) dY_t + H_-(t, Y_t) dD_t \\
&+ D_1 (1 - H_-(1, Y_1))
\end{aligned}
$$

- Standard filtering arguments imply $dY_t = dB_t^M + \widehat{\alpha}_t dt$ where $\widehat{\alpha}_t = \mathbb{E}[\alpha_t | \mathcal{F}_t^M]$ and B^M is an $\mathcal{F}^M$-Brownian motion.
- However, in an equilibrium, (3.5) must hold and the purely disc. part must vanish.
- Therefore, Y is a BM stopped at τ for the market maker in an equilibrium and, equivalently, $\widehat{\alpha}_t = 0$ on $[t \leq \tau]$.

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Implications of market maker's rationality II

Lemma

Given a rational pricing rule H, $\theta^* \in \mathcal{A}$ is an optimal insider strategy if and only if it satisfies the following two properties:

1. *Inconspicuous insider trade theorem:*

$$\mathbb{E}[\alpha_t^* | \mathcal{F}_t^M] = 0 \text{ on } [t \leq \tau]$$

   where $\alpha_t^* = \frac{d\theta^*}{dt}$;

2. *the corresponding optimal total order Y^* satisfies*

$$\lim_{t \uparrow 1} H(\tau \wedge t, Y^*_{\tau \wedge t}) = \mathbf{1}_{[\tau > 1]}.$$

Equilibrium I

Lemma

The couple (H^*, θ^*) *is an equilibrium if and only if the following two conditions hold:*

1. H^* *solves (3.5) for* $t \in [0, \tau] \cap [0, 1)$,
2. Y^* *is an* $\mathcal{F}^M$*-Brownian motion stopped at* τ *such that*

$$\lim_{t \uparrow 1} H^*(\tau \wedge t, Y^*_{\tau \wedge t}) = \mathbf{1}_{[\tau > 1]}.$$

Rmk: Recall that $\mathbb{P}[\tau > 1 | Z_t] = \mathbf{1}_{[\tau > t]} F(t, Z_t)$ where

$$F(t, y) := \int_{1-t}^{\infty} \frac{y+1}{\sqrt{2\pi x^3}} e^{-\frac{(y+1)^2}{2x}} dx.$$

Equilibrium II

Theorem

There exists an equilibrium (H^*, θ^*) *such that*

$$H^*(t,y) = F(t,y) \quad \textit{for } t < 1,$$

and $H^*(1,\cdot) \equiv 1$*. The equilibrium total order,* Y^**, associated with* θ^* *is the unique strong solution (under* $\mathcal{F}^I$*) to*

$$dY_t = dB_t + \left\{ \frac{1}{1+Y_t} - \frac{1+Y_t}{\tau - t} \right\} \mathbf{1}_{[t \le \tau]} dt, \qquad Y_0 = 0. \quad (3.6)$$

Moreover, $\tau = \inf\{t : Y^*_t = -1\}$ ***predictable under*** $\mathcal{F}^M$.

Given $\tau = l$, Y is a 3-dimensional *Bessel bridge* of length l starting at 0 at time 0 and ending at -1 at time l.

Equilibrium III : Elements of proof

- Existence and uniqueness of a $\mathcal{F}^I$-strong solution Y^* for SDE (3.6) follows from standard results on Bessel bridges (Revuz-Yor)
- Exhibit a weak solution by Jeulin's decomposition of the BM Z under $\mathcal{G} := \mathcal{F}^Z \wedge \sigma(\tau)$:

$$dZ_t = d\beta_t + \left\{ \frac{1}{1+Z_t} - \frac{1+Z_t}{\tau - t} \right\} \mathbf{1}_{[t \le \tau]} dt$$

where β is a $\mathcal{G}$-BM and so independent of τ. It follows that $\tau = \inf\{t : Y^*_t = -1\}$.

Interpretation of the insider's optimal strategy I

- An easy consequence of the previous result is

$$\frac{1}{1+Y_t^*} = \mathbb{E}\left[\frac{1+Y_t^*}{\tau - t}\,\middle|\,\mathcal{F}_t^M\right] \tag{3.7}$$

  i.e. conditional on $\mathcal{F}_t^M$, $(1+Y_t^*)^{-1}$ is the *best approximation in market's view* to the value $\frac{Y_t^*+1}{\tau-t}$.
- Note that the default will happen when Y^* hits -1. Thus, $\frac{Y_t^*+1}{\tau-t}$ is the true *speed of default* and $\frac{1}{Y_t^*+1}$ is the market's expectation.

Interpretation of the insider's optimal strategy II

- Looking at

$$\alpha_t^* = \left\{ \mathbb{E}\left[\frac{1+Y_t^*}{\tau - t} \middle| \mathcal{F}_t^M \right] - \frac{1+Y_t^*}{\tau - t} \right\} \mathbf{1}_{[t \leq \tau]}$$

  we see that the insider sells when the market's expectation of the speed of default is lower than the true speed and buys otherwise.
- This is quite intuitive. Indeed, when the market's expectation for the imminence of the default is low, the bond is relatively overpriced, so the insider sells to increase her profits.

A more realistic case : dynamic insider's information (with U. Cetin & A. Danilova)

Same model except default barrier 0 and ...

1. the insider observes Z_1 continuously on time: $\mathcal{F}_t^I = \mathcal{F}_t^{X,Z}$, $t \geq 0$, τ is a stopping time for him.
2. $Z_t = Z_0 + \int_0^t \sigma(u) dB_u^Z$, $\sigma \geq \epsilon > 0$, B^Z is a standard BM independent of B.
3. Moreover, let $F(t, z)$ be given by $F(t, Z_t)\mathbf{1}_{[\tau > t]} = \mathbb{P}[\tau > 1 | \mathcal{F}_t^Z]$ (same as before).

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Looking for an equilibrium

In this case, a necessary and sufficient condition for (H^*, θ^*) to be an equilibrium is that:

- H^* solves $H_t + \frac{1}{2} H_{yy} = 0$ on $[0, \tau \wedge 1[$ and ...
- Y^* is an $\mathcal{F}^M$-BM stopped at τ such that $\lim_{t \uparrow 1} H^*(\tau \wedge t, Y_{\tau \wedge t}) = \mathbf{1}_{[\tau > 1]}$.

Problem: finding a process Y

- hitting 0 for the first time at τ and ...
- being a BM in its own filtration (progr. enlarged with τ).

Equilibrium I : Statement of the main result

Theorem

There exists an equilibrium (H^*, θ^*) *such that*

$$H^*(t,y) = F(t,y), \quad t < 1$$

and the equilibrium total order Y^* *solves*

$$dY_t = dB_t + \frac{\rho_y(t, Z_t; Y_t)}{\rho(t, Z_t; Y_t)} \mathbf{1}_{[t \le \tau]} dt,$$

where $\rho(t,z;y) = \mathbb{P}[Z_t = z | Y_t = x]$ *and* $\rho_y = \frac{\partial}{\partial y}\rho$*. Moreover, one has* $\tau = \inf\{t > 0 : Y_t^* = 0\}$*. As a consequence,* τ *is a predictable stopping time under the market maker's filtration* $\mathcal{F}^M$*.*

Equilibrium II : Elements of the proof.

Main difficulty : We cannot use standard filtration enlargement results, the dependence between Y and Z can be very complicated.

- Conditional on $\mathcal{F}_t^Z$ and $[\tau > t]$, we have

$$\tau - t = F_t^{-1}\left(\frac{Z_t^2}{N_1^2}\right) \quad \text{in distribution,}$$

  where $N_1 \sim \mathcal{N}(0,1)$ independent of $\mathcal{F}_t^Z$ and $F_t(v) = \int_t^{t+v} \sigma^2(s) ds$.
- We want Y to be a BM hitting -1 for the first time at τ, so that

$$\tau - t = \frac{Y_t^2}{N_2^2} \quad \text{in distribution,}$$

  where $N_2 \sim \mathcal{N}(0,1) \perp \mathcal{F}_t^Y$. So Laplace transforms are equal.

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$$\tau - t = F_t^{-1}\left(\frac{Z_t^2}{N_1^2}\right) \quad \text{in distribution},$$
where $N_1 \sim \mathcal{N}(0,1)$ independent of $\mathcal{F}_t^Z$ and $F_t(v) = \int_t^{t+v} \sigma^2(s)ds$.
- We want Y to be a BM hitting -1 for the first time at τ, so that
$$\tau - t = \frac{Y_t^2}{N_2^2} \quad \text{in distribution},$$
where $N_2 \sim \mathcal{N}(0,1) \perp \mathcal{F}_t^Y$. So Laplace transforms are equal.

Equilibrium III : Elements of the proof (cont'd)

- Let $I(t,z;y)$ be the Laplace transform of $z \mapsto \rho(t,z;y)$ cond. density of $Z_t|Y_t$, which can be chosen $C^{1,2,2}$. Thus

$$I(t,z;y) = \int_0^\infty \exp\left(-\frac{z^2}{2}F_t(y^2x)\right)\eta(x)dx$$

where η is the density of an inverse χ^2-r.v.

- We cannot inverse explicitely I to get ρ, still we can show that

$$\rho_t(t,z;y) - \frac{1}{2}\sigma^2(t)\rho_{yy}(t,z;y) + \frac{1}{2}\rho_{zz}(t,z;y) = 0$$

with boundary conditions $\rho(t,0+;y) = \rho(t,z;0+) = 0$. Moreover,

$$\lim_{y\downarrow 0}\frac{\rho_y(t,z;y)}{\rho(t,z;y)} = -\lim_{z\downarrow 0}\frac{\rho_y(t,z;y)}{\rho(t,z;y)} = +\infty \quad (t,z,y) \in \mathbb{R}^3_{++}$$

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Elements of the proof (cont'd)

- On the other hand, let $S_t^I = g(t, Z_t, Y_t)$ insider's price of defaultable bond, Itô's formula applied to $\rho(t, z; Y_t)$, previous PDE and some filtering get

$$dY_t = dB_t + (Y_t^{-1} - Y_t g(t, Z_t, Y_t))dt = dB_t + \frac{\rho_y}{\rho}(t, Z_t; Y_t)dt$$

  which is a BM in its own filtration $\mathcal{F}^Y$

- Limiting properties of ρ_y/ρ give that Y^* solution of previous SDE hits default barrier 0 for the first time at τ, so that τ is a stopping time wrt $\mathcal{F}^{Y^*}$ and $\mathcal{F}^M = \mathcal{F}^{Y^*}$.
- Y^* is equilibrium total demand and ρ_y/ρ the optimal insider's strategy.

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Concluding remarks and questions

1. In the framework set by Back, we've solved for the equilibrium pricing rule and the equilibrium level of demand.
2. *No expected trade theorem* holds true in the equilibrium.
3. However, the insider reveals part of his information so that τ becomes predictable to the market as in structural models. Nevertheless, Z is still not observable by the market.
4. We've extend these results to the case when insider observes the firm's value continuously on time.
5. Which relationships with dynamic filtrations enlargement? Compare to Föllmer-Wu-Yor, SPA99

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