

Insider trading in credit markets with dynamic information asymmetry

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Back's model of insider trading

Inspired by Kyle (1985), Back (1992) studies a market for a bond and a stock with three types of participants:

1. *Noise traders*: The noise traders have no information about the future value of the risky asset and can only observe their own cumulative demands. The cumulative demand is modeled by a standard Brownian motion denoted with B .
2. *Informed trader*: The insider knows the value of the risky asset at time 1, which is given by a standard normal random variable, V , independent of B . Being risk-neutral, her objective is to maximize her expected profit.
3. *Market maker*: The market maker observes the total order and sets the price of the risky asset to clear the market.

The pricing mechanism of the market

- ▶ The market maker decides the price looking at the total order, Y^θ , which is given by

$$X_t^\theta = B_t + \theta_t,$$

where θ_t is the position of the insider in the risky asset at time t .

- ▶ Thus, the filtration of the market maker is the one generated by X and is denoted with \mathcal{F}^X . Note that θ is not necessarily adapted to \mathcal{F}^X , i.e. the insider's trade is not observed directly by the market maker.
- ▶ The market maker has a *pricing rule*, $H : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$, to assign the price in the following form:

$$S_t = H(t, X_t),$$

where S_t is the market price of the risky asset at time t .

Objectives of the insider and the market maker

- ▶ The market maker chooses a *rational* pricing rule, i.e. he chooses a pricing rule so that

$$H(t, X_t) = \mathbb{E}[V | \mathcal{F}_t^X],$$

for every $t \in [0, 1]$.

- ▶ The insider aims to maximize her expected profit out of trading.

Equilibrium

Definition 1

A pair (H^*, θ^*) is said to form an equilibrium if H^* is a pricing rule, $\theta^* \in \mathcal{A}$, and the following conditions are satisfied:

1. *Market efficiency condition:* Given θ^* , H^* is a rational pricing rule.
2. *The optimality condition:* Given H^* , θ^* maximizes the expected profit of the insider.

Equilibrium

Theorem 2

Define H^* and α^* by

$$H^*(t, x) = x \quad \text{and} \quad \alpha_t^* = \frac{V - X_t}{1 - t}.$$

Let $\theta^* = \int_0^t \alpha_s^* ds$. Then, (H^*, θ^*) is an equilibrium.

Note that if X^* denotes the equilibrium level of the total order, then

$$dX_t^* = dB_t + \frac{V - X_t^*}{1 - t} dt,$$

so that X^* is a Brownian bridge, in the insider's view, from 0 to V of length 1. This means that the insider drives the total demand to the true value of the asset while not changing its distribution at the same time.

An equilibrium model for a defaultable bond

A company issues a bond that pays \$1 at time 1 unless it defaults before that time. We further suppose

$$\tau := \inf\{t > 0 : Z_t = -1\},$$

where Z is a martingale representing the firm value. In Campi & Çetin (2007), it is assumed that

- ▶ Z is a Brownian motion, and
- ▶ The insider knows τ from the beginning.
- ▶ The objectives for the market maker and the insider and the structure of the noise traders are as in Back.

Equilibrium in the setting of Campi & Çetin

Theorem 3

There exists an equilibrium (H^*, θ^*) such that

$$H^*(t, x) := \int_{1-t}^{\infty} \frac{x+1}{\sqrt{2\pi y^3}} e^{-\frac{(x+1)^2}{2y}} dy,$$

and the equilibrium total order, X^* , associated with θ^* solves on $[t < \tau]$

$$dX_t = dB_t + \left\{ \frac{1}{1+X_t} - \frac{1+X_t}{\tau-t} \right\} dt. \quad (1)$$

Moreover, X^* is an \mathcal{F}^M -Brownian motion stopped at τ .

Note that, this time, $1+X^*$ is a 3-dimensional Bessel bridge of length τ in insider's view. Moreover, τ is an \mathcal{F}^{X^*} -stopping time. Indeed,

$$\tau = \inf\{t > 0 : X_t^* = -1\}.$$

This result shows in particular that the pricing methodology turns from reduced-form into structural.

Dynamic information asymmetry

Keeping the definition of the default time and the defaultable bond the same, we now assume that the firm value is given by

$$Z_t = Z_0 + \int_0^t \sigma(u) dB_u^Z$$

where

- ▶ Z_0 is a random variable strictly greater than -1 , a.s.,
- ▶ σ is a deterministic function such that

$$\int_0^t \sigma^2(s) ds < \infty, \quad \forall t > 0,$$

- ▶ B^Z is a standard Brownian motion.

Assumptions on the market participants

- ▶ The market maker chooses a rational Markovian pricing rule, H , as in Back by looking at the total demand and whether the default has occurred or not, i.e.

$\mathcal{F}^M = \mathcal{F}^X \vee \sigma(\tau \wedge t)$, where

$$X_t = B_t^\tau + \theta_{t \wedge \tau}.$$

- ▶ The noise traders are modeled by a standard Brownian motion, B , independent of Z .
- ▶ The insider observes the process Z continuously in time. Since the pricing rule is monotone, she also observes the total demand; thus, her filtration, \mathcal{F}^I , is generated by B and Z .
- ▶ Insider's objective is to maximize expected profits.
- ▶ The notion of equilibrium is as in Back.

The characterization of the equilibrium

Let $F(t, z)$ be given by:

$$F(t, Z_t) \mathbf{1}_{\{\tau > t\}} = \mathbb{E}[\mathbf{1}_{\{\tau > 1\}} | \mathcal{F}_t^Z]$$

Lemma 4

If a rational pricing rule H satisfies

$$H_t + \frac{1}{2} H_{xx} = 0 \quad (2)$$

then there exists a solution of

$$V_t + \frac{1}{2} V_{xx} + \frac{1}{2} \sigma^2(t) V_{zz} = 0 \quad (3)$$

$$V_x + F(t, z) - H(t, x) = 0 \quad (4)$$

with $V(1, x, z) \geq V(1, x^*, z)$ for any $x \neq x^*$ and z and with $V(\tau, x, -1) \geq V(\tau, x^*, -1)$ for any $x \neq x^*$ and $\tau \leq 1$, where x^* is the solution of $H(x, \tau \wedge 1) = \mathbf{1}_{\{\tau > 1\}}$

As one can guess, this V will be closely related to the value function for the insider's optimization problem. This V can be used to verify the statement of the following theorem.

Theorem 5

The couple (H^, θ^*) is an equilibrium if and only if the following two conditions hold:*

1. H^* solves (2),
2. X^* is an \mathcal{F}^M -Brownian motion stopped at $\tau \wedge 1$ such that X^* hits -1 for the first time at τ (on $[\tau \leq 1]$).

Thus, our aim is to find a process X hitting -1 for the first time at τ and is a solution to

$$dX_t = dB_t + g(t, X_t, Z_t)dt, \quad \text{on } [t < \tau], \quad (5)$$

for some appropriate function g .

Some initial thoughts

- ▶ First of all, the function g in (5) has to be somewhat irregular. If, e.g., $\mathcal{E}\left(-\int_0^\cdot g(s, X_s, Z_s)dB_s\right)$ is a uniformly integrable martingale, one can show by a Girsanov transformation that τ is a totally inaccessible stopping time with respect to \mathcal{F}^X , therefore, it cannot be the first hitting time of a Brownian motion.
- ▶ In case $Z_0 \equiv 0$ and $\sigma \equiv 1$ one may want to try a projection of the Bessel bridge defined in (1), e.g. the solution to

$$dX_t = dB_t + \left\{ \frac{1}{1+X_t} - \frac{1+X_t}{(1+Z_t)^2} \right\} dt,$$

exploiting the fact that $\mathbb{E}\left[\frac{1}{\tau-t} \mid \mathcal{F}_t^Z\right] = (1+Z_t)^{-2}$. However, this X is not a Brownian motion in its own filtration.

Indeed, in order for X to be a Brownian motion we need $\mathbb{E}[(Z_t + 1)^{-2} | \mathcal{F}_t^X] = (1 + X_t)^{-2}$. Standard nonlinear filtering theory yields

$$d\mathbb{E}[(Z_t + 1)^{-2} | \mathcal{F}_t^X] = dM_t + 3\mathbb{E}[(Z_t + 1)^{-4} | \mathcal{F}_t^X] dt,$$

where M is a martingale. Moreover, Itô's formula gives

$$d(1 + X_t)^{-2} = dN_t + 3(X_t + 1)^{-4} dt,$$

where N is another martingale. But this means

$$\mathbb{E}[(Z_t + 1)^{-4} | \mathcal{F}_t^X] = (\mathbb{E}[(Z_t + 1)^{-2} | \mathcal{F}_t^X])^2,$$

which implies $Z_t \in \mathcal{F}_t^X$ and, thus, $Z = B$!

Some related work

- ▶ A dynamic version of Back's model (i.e. insider trading with a stock) has been studied by Back and Pedersen (1998) and Föllmer, Wu and Yor (1999).
- ▶ The idea is to construct a Brownian motion that hits W_1 at time 1 using a continuous observation of a Brownian motion W . Here W_1 is the time 1 value of W .
- ▶ A solution cannot be obtained unless the Brownian motion W has a time varying variance.
- ▶ Föllmer, et al. study a slightly more general problem of canonical decomposition of linear transformations of two independent Brownian motions using Volterra representation of Gaussian processes. They obtain conditions under which the linear transformation is a Brownian motion in its own filtration and hits a standard normal random variable at time 1.

Some necessary conditions

We now turn to our problem and obtain some necessary conditions for the existence of a solution. For this purpose, we need the distribution of τ .

Proposition 1

Let Z and τ be as defined earlier. Then, for each $t \geq 0$, the conditional density of $\tau - t$ on the event $[\tau > t]$ given \mathcal{F}_t^Z is given by

$$p(t; z, v) := \mathbb{P}[\tau - t \in dv | \mathcal{F}_t^Z] = \frac{Z_t + 1}{\sqrt{2\pi F_t^3(v)}} e^{-\frac{(Z_t + 1)^2}{2F_t(v)}} \sigma^2(t+v), \quad v \geq 0, \quad (6)$$

where $F_t(v) := \int_t^{v+t} \sigma^2(s) ds$.

This shows, in particular, that conditional on \mathcal{F}_t^Z and $[\tau > t]$ the following identity in distribution holds:

$$\tau - t = F_t^{-1} \left(\frac{(Z_t + 1)^2}{N_1^2} \right), \quad (7)$$

where N_1 is a standard normal random variable independent of \mathcal{F}_t^Z . In fact the identity in distribution holds even if we condition on $\mathcal{F}_t^{X,Z}$ since $\mathcal{F}_t^{X,Z} = \mathcal{F}_t^{B,Z}$ and B and Z are independent. By similar reasoning,

$$\tau - t = \frac{(X_t + 1)^2}{N_2^2}, \quad (8)$$

where N_2 is a standard normal r.v. independent of \mathcal{F}_t^X , if τ is the first hitting time for X as well.

On the conditional distribution of Z

Now, define $\tilde{X} := X + 1$ and $\tilde{Z} := Z + 1$ and let $R(t, \cdot)$ denote the conditional cumulative distribution function of \tilde{Z}_t given \mathcal{F}_t^X on the event $[\tau > t]$. The expressions (7) and (8) along with the independence of N_1 from $\mathcal{F}_t^{Z, X}$ imply that for every $t \geq 0$ and $0 \leq y < 1 - t$

$$\int_0^\infty R\left(t, \sqrt{\frac{F_t(y)}{u}}\right) \eta(u) du = \bar{\eta}\left(\frac{y}{\tilde{X}_t^2}\right), \quad (9)$$

where η is the probability density function of the inverse chi-square given by

$$\eta(x) = \frac{1}{\sqrt{2\pi x^3}} \exp\left\{-\frac{1}{2x}\right\}, \quad x \in \mathbb{R}_+,$$

and $\bar{\eta}$ is the associated cumulative distribution function. From above it is easy to see that the conditional distribution of \tilde{Z}_t given \mathcal{F}_t^X must depend only on the value of \tilde{X}_t .

A condition on the initial distribution of Z

Since Z_0 is independent from X_0 the above expression effectively puts a restriction that should be met jointly by the function F_0 and the initial distribution of Z .

Proposition 2

Suppose $Z_0 = a - 1$, a.s. for some $a > 0$. Then, in order for Z and X hit -1 at the same time it is necessary that $\sigma \equiv a$.

The following gives the condition on the function F_0 when Z_0 is a conditioned normal random variable.

Proposition 3

Suppose that $1 - R(0, z) = P(\sqrt{s}N > z | N > 0)$ where N is a standard normal random variable. Then, Z and X hit -1 at the same time, if they ever hit before time 1, if

$$F_0(t) = s \tan^2 \left(\frac{\pi}{2} \bar{\eta}(t) \right). \quad (10)$$

On the function σ

- ▶ By differentiating above F_0 we get

$$\sigma^2(t) = \pi s \frac{\sin(\frac{\pi}{2}\bar{\eta}(t))}{\cos^3(\frac{\pi}{2}\bar{\eta}(t))} \eta(t).$$

- ▶ σ is bounded since

$$\lim_{t \rightarrow 0} \sigma(t) = 0 \text{ and } \lim_{t \rightarrow \infty} \sigma(t) = \sqrt{\frac{2}{\pi}} s.$$

The conditional density of \tilde{Z}

Let

$$q(t, z, v) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(z-v)^2}{2t}\right) - \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(z+v)^2}{2t}\right).$$

The above is the transition density of a Brownian motion killed at 0. Later, we shall show that the function

$$\rho(t, x, z) := \int_0^\infty \int_0^\infty q(1-t, u, x) q(F_0(t), z, v) f(u, v) du dv,$$

is the conditional density of \tilde{Z}_t given $\tilde{X}_t = x$ upto time 1 for an appropriate positive function f .

The PDE associated with ρ

For a sufficiently well-behaved f we should have

1. $\lim_{x \rightarrow 0} \rho(t, x, z) = \lim_{z \rightarrow 0} \rho(t, x, z) = 0$ and
- 2.

$$\rho_t(t, z, x) - \frac{1}{2}\sigma^2(t)\rho_{zz}(t, z, x) + \frac{1}{2}\rho_{xx}(t, z, x) = 0, \quad (11)$$

on \mathbb{R}_{++}^3 .

Why is ρ the conditional density?

- ▶ If \tilde{X} is a Brownian motion that hits 0 for the first time at τ , then the joint density of \tilde{X} and \tilde{Z} equals

$$q(t, 1, x)\rho(t, x, z)$$

by Bayes' theorem.

- ▶ On the other hand, this joint density *should* satisfy the Kolmogorov's forward equation defined by the vector Markov process (\tilde{X}, \tilde{Z}) .
- ▶ We shall see that for a specific drift term for \tilde{X} , $q(t, 1, x)\rho(t, x, z)$ satisfies the associated Kolmogorov equation and \tilde{X} is a Brownian motion hitting 0 for the first time at τ .

Further properties of ρ

- ▶ Observe that as $x \rightarrow 0$, $\rho(t, x, \cdot)$ approaches to the delta function at 0, which means that as the process \tilde{X} gets close to 0 \tilde{Z} is forced to be near 0.
- ▶ Moreover,

$$\lim_{x \rightarrow 0} \frac{\rho_x(t, x, z)}{\rho(t, x, z)} = +\infty,$$

for every $z > 0$. Thus, if we consider the solution, if exists, to the SDE

$$d\tilde{X}_t = dB_t + \frac{\rho_x(t, \tilde{X}_t, \tilde{Z}_t)}{\rho(t, \tilde{X}_t, \tilde{Z}_t)} dt$$

drift term is going to get so large to kill the Brownian term if X gets very close to 0 when \tilde{Z} is away from zero.

How to determine f ?

In order to determine the function f , consider the limit of ρ as $t \rightarrow 0$, which equals

$$\int_0^\infty q(1, u, x) f(u, z) du.$$

Note that the above expression should give the distribution of \tilde{Z}_0 when $x = 1$.

Proposition 4

Suppose that the conditional density of \tilde{Z}_t given \mathcal{F}_t^X on the event $[\tau > t]$ is given by ρ . Suppose further that $1 - R(0, z) = P(\sqrt{s}N > z | N > 0)$ for some $s > 0$ and let μ denote the associated density. Then,

$$f(u, v) = \frac{\partial}{\partial v} \mathbb{P} \left(\tau > \frac{s}{v^2} | Z_0 = u \right) \frac{\mu(u)}{q(1, u, 1)}, \quad (12)$$

where

$$F_0(t) = s \tan^2 \left(\frac{\pi}{2} \bar{\eta}(t) \right).$$

More on f

Suppose $s \leq 1$. It can be shown by ordinary dominated convergence theorems that if f is defined by (12), then

1. $\lim_{x \rightarrow 0} \rho(t, x, z) = \lim_{z \rightarrow 0} \rho(t, x, z) = 0$ and

2.

$$\rho_t(t, z, x) - \frac{1}{2}\sigma^2(t)\rho_{zz}(t, z, x) + \frac{1}{2}\rho_{xx}(t, z, x) = 0,$$

on \mathbb{R}_{++}^3 .

Construction of \tilde{X} that hits 0 for the first time at τ

From now on we *assume* that ρ is determined by the function f , which is defined by (12), and that the initial distribution of \tilde{Z}_0 is given by $1 - R(0, z) = P(\sqrt{s}N > z | N > 0)$ for some fixed $s \in (0, 1]$. The following follows from the standard results on the existence of strong solutions for SDEs.

Proposition 5

There exists a unique continuous strong solution to

$$d\tilde{X}_t = dB_t + \frac{\rho_X(t, \tilde{X}_t, \tilde{Z}_t)}{\rho(t, \tilde{X}_t, \tilde{Z}_t)} dt,$$

up to L_n for every $n > 0$ where

$$L_n := \inf\{t > 0 : \tilde{X}_t \notin (n^{-1}, \infty) \text{ or } \tilde{Z}_t \notin (n^{-1}, \infty)\} \wedge 1$$

Extending the definition of \tilde{X}

Let $L := \sup_n L^n$. Note that $L \leq \tau \wedge 1$, a.s..

- ▶ It can be shown that $\lim_{n \rightarrow \infty} \tilde{X}_{L^n}$ exists. Thus, we can define \tilde{X} by

$$d\tilde{X}_t = dB_t + \frac{\rho_x(t, \tilde{X}_t, \tilde{Z}_t)}{\rho(t, \tilde{X}_t, \tilde{Z}_t)} dt,$$

for $t < L$, and $\tilde{X}_L = \lim_{n \rightarrow \infty} \tilde{X}_{L^n}$. This way \tilde{X} is continuous and nonnegative over $[0, L]$.

Making \tilde{X} hit 0 at τ

Define

$$\mathcal{L}_t = \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \sigma^2(t) \frac{\partial^2}{\partial z^2} + \frac{\rho_x(t, \cdot)}{\rho(t, \cdot)} \frac{\partial}{\partial x}.$$

Lemma 6

For any $m \in \mathbb{N}$ there exist constants $\alpha > 0$, $c > 0$ and $\lambda > 0$ such that the function

$$\varphi(t, x, z) = -\frac{\alpha \rho(t, x, z)}{x} \ln(\alpha \rho(t, x, z)) + c \quad (13)$$

is positive and satisfies

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_t \right) \varphi \leq \lambda \varphi \quad (14)$$

for any $(t, x, z) \in [0, 1] \times [0, \infty) \times [m^{-1}, m]$ and

$$\lim_{x \downarrow 0} \varphi(t, x, z) = +\infty, \quad z > 0 \quad (15)$$

Lemma continued

For any $m \in \mathbb{N}$ there exist constants $\alpha > 0$, $c > 0$ and $\lambda > 0$ such that the function

$$\psi(t, x, z) = \frac{\alpha \rho(t, x, z)}{z} \ln \alpha \rho(t, x, z) - c \quad (16)$$

is negative and satisfies

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_t \right) \psi \geq \lambda \psi \quad (17)$$

for any $(t, x, z) \in [0, 1] \times [m^{-1}, m] \times [0, \infty)$ and

$$\lim_{z \downarrow 0} \psi(t, x, z) = -\infty, \quad x > 0 \quad (18)$$

\tilde{X} hits 0 for the first time at τ

Take $z_0 > 0$ and m such that $z_0 \in (m^{-1}, m)$. An easy application of Itô's formula gives that $e^{-\lambda t} \varphi(t, X_t, Z_t)$ is a supermartingale up to $\tau_m \wedge 1$ where

$\tau_m := \inf\{t > 0 : Z_t \notin (m^{-1}, m)\}$, so that Fatou's lemma yields

$$\begin{aligned}\varphi(0, 1, z_0) &\geq \mathbb{E}[e^{-\lambda L \wedge \tau_m} \varphi(\tau_m \wedge L, X_{\tau_m \wedge L}, Z_{\tau_m \wedge L})] \\ &\geq \mathbb{E}[e^{-\lambda L} \varphi(L, X_L, Z_L) 1_{\{L < \tau_m \wedge 1\}}]\end{aligned}$$

Suppose that $\mathbb{P}[L < \tau_m \wedge 1] > 0$. On the set $\{L < \tau_m \wedge 1\}$, $X_L = 0$ and $Z_L > 0$, therefore, due to (15),

$\varphi(L, X_L, Z_L) = \varphi(L, 0, Z_L) = +\infty$, which leads to a contradiction.

Hence $\mathbb{P}[L < \tau_m \wedge 1] = 0$ for any m , so $\mathbb{P}[L = \tau \wedge 1] = 1$.

Next, using ψ one can similarly show that \tilde{Z} cannot hit 0 before \tilde{X} hits m^{-1} for any $m \in \mathbb{N}$. This in turn yields $\tilde{X}_L = 0$ on $[\tau \leq 1]$, hence the claim.

Transition densities for \tilde{X} and \tilde{Z}

Let Y denote the vector process (\tilde{X}, \tilde{Z}) with the generator

$$\mathcal{L}_t = \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \sigma^2(t) \frac{\partial^2}{\partial z^2} + \frac{\rho_x(t, \cdot)}{\rho(t, \cdot)} \frac{\partial}{\partial x}. \quad (19)$$

There exists a unique positive function $p(s, y; t, w)$, $0 \leq s < t$ and $y, w \in \mathbb{R}_+^2$, which is continuous jointly with respect to all its variables and has the property that if $\varphi \in C_0^\infty(\mathbb{R}_+^2)$ and $g \in C_0^\infty([0, \infty) \times \mathbb{R}_+^2)$, then

for each $t > 0$ the function

$$h(s, y) = \int p(s, y; t, w) \varphi(w) dw + \int_s^t du \int p(s, y; u, w) g(u, w) dw \quad (20)$$

is in $C_b^{1,2}([0, \infty) \times \mathbb{R}_+^2)$ and satisfies

$$\frac{\partial h}{\partial s} + \mathcal{L}_s h = -g, \quad 0 \leq s < t,$$

with $h(t, \cdot) = \varphi$. Moreover, the function P defined by

$$P(s, y; t, \Gamma) = \int_{\Gamma} p(s, y; t, w) dw, \quad \Gamma \in \mathcal{B}_{\mathbb{R}_+^2},$$

is the transition probability function for Y .

\tilde{X} is a stopped Brownian motion

- ▶ ρ as defined above is also the fundamental solution to the forward equation

$$\rho_t = -\frac{\partial}{\partial x} \left(\frac{\rho_x(t, y)}{\rho(t, y)} \rho \right) + \frac{1}{2} \rho_{xx} + \frac{1}{2} \sigma^2(t) \rho_{zz}.$$

It is straightforward to check that $\rho q(\cdot, 1, \cdot)$ satisfies the forward equation with the given initial distribution for Y .

- ▶ Thus, the conditional density of \tilde{Z} equals ρk where k is a function of x and t only.
- ▶ If k is not identically equal to 1, then \tilde{X} , in its own filtration, is a Brownian motion with a nonzero drift. However, the distribution of the first hitting time for a Brownian motion with drift cannot be the same as the distribution defining the first hitting time of a Brownian motion.
- ▶ Thus, $k \equiv 1$ and the density for \tilde{X} is given $q(\cdot, 1, \cdot)$, hence \tilde{X} is a Brownian motion starting at 1 and stopped at 0.