Dynamic Hedging of Synthetic CDO Tranches with Spread Risk and Default Contagion

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Abstract

We study the hedging of synthetic CDO tranches in a dynamic portfolio credit risk model which incorporates spread risk and default contagion. The model is constructed and studied via Markov-chain techniques. We discuss the immunization of a CDO tranche against spread- and event risk in the Markov-chain model and compare the results with hedge ratios obtained in the standard Gauss copula model. Moreover, we derive model-based dynamic hedging strategies using the concept of risk minimization. Numerical experiments are used to illustrate some of the properties of the risk-minimizing hedging strategies.

Keywords: Dynamic hedging, portfolio credit risk, credit derivatives, incomplete markets, default contagion.

1 Introduction

The risk management for books of synthetic CDO tranches has become an issue of high concern for many investors on credit markets. Typically an investor has taken a protection-seller position in one or several CDO tranches and tries to offset the ensuing risk by taking an opposite (protection-buyer) position in the single-name credit default swaps (CDSs) or in the CDS-index underlying the tranche. In practice, the size of the hedging positions is determined by a pragmatic approach, akin to the use of duration in interest rate risk management: in order to protect a CDO tranche against fluctuations in credit spreads (spread risk) the tranche is first priced via the Gauss copula model, using observed CDS (index) spreads and implied-correlation methodology to determine the model parameters. Next one varies the swap spread of one of the underlying names (name k, say) and defines the so-called *spread delta* of that name as the ratio of the change in the market value of the CDO tranche and of a CDS on name k. The hedge ratios immunizing the tranche against a change in the index spread are determined in a similar way. Sometimes investors also seek to protect their position against defaults in the underlying reference pool (hedging of event risk). The hedge ratio immunizing a CDO tranche against the default of, say, name k (the *default delta*) is taken to be the ratio of the loss in the tranche and in a CDS on name k given that firm k defaults. In computing these losses it is assumed that the credit spreads of the surviving firms are not affected by the default event. Further details on market-standard hedging practice can for instance be found in Neugebauer (2006).

This pragmatic approach has a number of problems: first, the standard way to compute default deltas neglects contagion effects (the fact that on real markets spreads of surviving firms often jump in reaction to default events); second, there is no theoretically consistent

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dynamic-hedging methodology supporting the definition of spread- and default deltas. These deficiencies are not just of a theoretical nature. To begin with, there is widespread evidence for default contagion on credit markets (see for instance Collin-Dufresne, Goldstein & Helwege (2003)) and, as shown below, neglecting contagion effects may lead to inappropriate hedge ratios. Moreover, it is well-known from markets for other types of derivatives that ad-hoc hedging strategies frequently lead to unaccounted drift- and time-decay effects (see for instance El Karoui, Jeanblanc-Picqué & Shreve (1998)). The lack of a sound hedging methodology for portfolio credit derivatives is of course closely related to the fact that the market-standard copula models are static, so that theoretically consistent dynamic hedging strategies cannot be derived in the copula framework.[2]

In this paper we make a first attempt to address these issues. In Section 2, we propose a dynamic credit risk model which allows for the explicit modelling of default contagion and spread risk, and which is therefore an ideal workbench for analyzing the hedging of CDO tranches. The model belongs to the class of models with interacting default intensities such as Jarrow & Yu (2001), Davis & Lo (2001) or Giesecke & Weber (2006); it is particularly close to the Markov-chain models proposed by Frey & Backhaus (2006) or by Arnsdorf & Halperin (2007). In Section 3 we give a formal description of the gains process (the cash-flow dynamics) of CDSs and CDOs; this is a necessary prerequisite for the ensuing analysis. In Section 4 we compute default deltas and spread deltas for the Markov-chain model and compare the results with the values obtained in a Gauss copula model via the standard approach. It turns out that in many cases the hedge ratios differ substantially; as we will see below, this directly linked to contagion effects. In Section 5 we study the dynamic replication of CDO tranches using martingale representation results for marked point processes. With both, spread risk and event risk, markets are typically incomplete, so that we resort to the concept of risk-minimization introduced by Föllmer & Sondermann (1986). Numerical experiments illustrate some of the properties of risk-minimizing hedging strategies. In particular, it turns out that risk-minimizing hedging strategies interpolate between the hedging of spread- and default risk and that deviations from the popular assumption of a homogeneous portfolio can have a sizeable impact on the form and on the performance of hedging strategies.

The dynamic hedging of credit risky securities is studied among others by Bielecki, Jeanblanc & Rutkowski (2004), Elouerkhaoui (2006) and Laurent, Cousin & Fermanian (2007). The latter paper - which was written independently - is closely related to our contribution: similar to us, Laurent et al. (2007) study the hedging of CDO tranches via dynamic hedging with CDS indices in the Markov-chain model of Frey & Backhaus (2006). However, they concentrate on the case without spread risk and hence on complete markets. Since the hedging against random fluctuations of credit spreads is an issue of high concern for many practitioners, we believe that the inclusion of spread risk and the application of incomplete-market methodology is an important extension over their paper.

2 The model

Notation. Fix some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$. The σ-field $\mathcal{F}_t$ represents the information available to investors at time t; all processes introduced below will be $(\mathcal{F}_t)$-adapted. We consider a fixed portfolio of m firms, indexed by $i \in \{1, \dots, m\}$. The $(\mathcal{F}_t)$-stopping time τ_i with values in $(0, \infty)$ represents the default time of firm i. The default state of the portfolio

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[2] This deficienccy is inherent in the copula framework; it cannot be rectified by using more sophisticated copulas than the Gauss copula.

is thus described by the *default indicator process* $Y = (Y_{t,1}, \dots, Y_{t,m})_{t \geq 0}$ where $Y_{t,i} = 1_{\{\tau_i \leq t\}}$; note that $Y_t \in \{0,1\}^m$. Since we consider only models without simultaneous defaults, we can define the *ordered default times* $T_0 < T_1 < \ldots < T_m$ recursively by $T_0 = 0$ and, for $1 \leq n \leq m$, $T_n = \min\{\tau_i : 1 \leq i \leq m, \tau_i > T_{n-1}\}$. By $\xi_n \in \{1, \dots, m\}$ we denote the identity of the firm defaulting at time T_n, i.e. $\xi_n = i$ if $T_n = \tau_i$. We use the following notation for flipping the ith coordinate of a default state: given $y \in \{0,1\}^m$ we define $y^i \in \{0,1\}^m$ by $y^i_i := 1 - y_i$ and $y^i_j := y_j$, $j \in \{1, \dots, m\} \setminus \{i\}$.

The model. We assume throughout the paper that the default-free interest rate is deterministic and equal to $r(t) \geq 0$; $p_0(t,T) = e^{-\int_t^T r(s)ds}$ denotes the price of the default-free zero-coupon bond with maturity T.[3] Moreover, we assume that the measure Q represents a risk-neutral measure used for pricing so that the price of any $\mathcal{F}_T$-measurable claim H is given by

$$H_t := p_0(t,T) E^Q(H \mid \mathcal{F}_t), \quad t \leq T. \tag{1}$$

It is standard practice in the literature to construct portfolio credit risk models directly under a risk-neutral measure Q, because pricing and calibration is done under Q anyhow.

Next we turn to modelling the Q-dynamics of the default indicator process. For this we introduce an $(\mathcal{F}_t)$-adapted state variable process Ψ representing for instance the macroeconomic environment. For tractability reasons Ψ is modelled as a finite-state Markov chain with state-space $S^\Psi = \{\psi_1, \dots, \psi_K\}$. We assume that the default intensity of firm i is given by some nonnegative function $\lambda_i(t, \Psi_t, Y_t)$. In this way fluctuations in Ψ will lead to random fluctuations in credit spreads (spread risk). Moreover, since λ_i depends on the current portfolio state Y_t, the default intensity of a firm may change if there is a change in the default state of other firms in the portfolio, so that default contagion can be modelled explicitly.

The overall state of the economic system is described by the process $\Gamma := (Y_t, \Psi_t)_{t \geq 0}$, modelled as a finite-state Markov chain with state space $S^\Gamma := \{0,1\}^m \times S^\Psi$. The next assumption summarizes the mathematical properties of Γ.

Assumption 2.1 (Dynamics of Γ). Consider bounded functions $\lambda_i : [0,\infty) \times S^\Psi \times \{0,1\}^m \to (0,\infty)$, $1 \leq i \leq m$ and a generator matrix $\boldsymbol{q}^\Psi = \big(q^\Psi(\psi_i, \psi_j)\big)_{1 \leq i,j \leq K}$. We assume that the process Γ is a Markov chain on $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$ with state space S^Γ and transition intensities given by

$$q_t^\Gamma\big((y,\psi),(\tilde{y},\tilde{\psi})\big) := \begin{cases} q^\Psi(\psi,\tilde{\psi}), & \text{if } \tilde{y} = y \text{ and } \tilde{\psi} \neq \psi\ , \\ (1-y_i)\,\lambda_i(t,\psi,y), & \text{if } \tilde{\psi} = \psi \text{ and } \tilde{y} = y^i \text{ for some } 1 \leq i \leq m, \\ 0, & \text{else.} \end{cases} \tag{2}$$

We discuss some of the implications of Assumption 2.1. First, simultaneous defaults of several firms are excluded by assumption. Second, suppose that firm i is in a non-default state at time t ($Y_{t,i} = 0$). The probability that the firm defaults in the small time interval $[t, t+h)$ corresponds to the probability that $\Gamma_t = (Y_t, \Psi_t)$ jumps to the new state $\big(Y_t^i, \Psi_t\big)$ in this time period. Since such a transition occurs with rate $\lambda_i(t, \Psi_t, Y_t)$, it is intuitively clear that under Assumption 2.1 $\lambda_i(t, \Psi_t, Y_t)$ is the default intensity of firm i at time t; a formal argument involving the generator of Γ is given in Frey & Backhaus (2006). Third, the form of the transition intensities in (2) implies that the process Ψ is individually Markov with generator matrix $\boldsymbol{q}^\Psi$ and that there are

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[3] The assumption of deterministic interest rates is routinely made in the literature on portfolio credit derivatives, essentially since the additional complexity of stochastic interest rates is not warranted given the high degree of uncertainty in the calibration of the model parameters related to default dependence.

no joint jumps of Y and Ψ. From a mathematical point of view this assumption could be relaxed without major difficulties; it reflects the fact that we interpret Ψ as an exogenous factor process whose dynamics are not affected by defaults in the portfolio.

Homogeneous-portfolio model. In numerical examples we restrict our attention mostly to homogeneous models where the default-indicator processes of all firms are assumed exchangeable. As discussed in Frey & Backhaus (2006), in that case the default intensities necessarily take the form

$$\lambda_i(t, \Psi_t, Y_t) = h\big(t, \Psi_t, M_t\big)\,, \quad \text{with } M_t := \sum_{i=1}^{m} Y_{t,i}\,, \tag{3}$$

where h is a function from $[0,\infty)\times S^{\Psi}\times\{0,\dots,m\}$ to $(0,\infty)$. With default intensities of the form (3), the process $\tilde{\Gamma} = (M_t, \Psi_t)_{t\geq 0}$ is a Markov chain with state space $S^{\tilde{\Gamma}} := \{0,\dots,m\}\times S^{\Psi}$; see Lemma 2.4 of Frey & Backhaus (2006) for a formal proof. This has two important implications: first, for m large, $\big|S^{\tilde{\Gamma}}\big|$ is much smaller than $\big|S^{\Gamma}\big|$, which facilitates the numerical treatment of the model. Second, with deterministic and identical loss given default δ the cumulative portfolio loss $L_t = \sum_{i=1}^{m} \delta Y_{t,i}$ is proportional to the number of defaults M_t. Hence in the homogeneous-portfolio setting the Markov chain $\tilde{\Gamma}$ can be viewed as a self-contained model for the evolution of the portfolio loss L_t; it is not necessary to model the evolution of the default state of the individual firms. This modelling philosophy is adopted in the so-called *top-down-approach* to credit portfolio modelling; see for instance Giesecke & Goldberg (2007), Schönbucher (2006) or Arnsdorf & Halperin (2007). In fact, the homogenous-portfolio version of our model is mathematically equivalent to the Markov-chain model of Arnsdorf & Halperin (2007). Hence the sophisticated calibration methodology developed by Arnsdorf and Halperin applies to the homogeneous-portfolio version of our model in a straightforward way.[4] From a practical viewpoint the homogeneous-portfolio setup is thus the most useful variant of the Markov-chain model. Nonetheless the theoretical analysis will be carried out for the general model introduced in Assumption 2.1. This permits us in particular to study the impact of portfolio-heterogeneity on the form of the 'optimal' hedge ratios in Section 5.4.

In our analysis we will typically use the following parametric form for the function h, labelled *convex counterparty risk model* (Frey & Backhaus (2006)),

$$h(t,\psi,l) = \lambda_0\psi + \tfrac{\lambda_1}{\lambda_2}\Big(\exp\big(\lambda_2 \tfrac{[l-\mu(t)]^+}{m}\big) - 1\Big)\,, \quad \lambda_0 > 0,\ \lambda_1, \lambda_2 \geq 0\,. \tag{4}$$

Here $\mu(t)$ is some deterministic threshold measuring the expected number of defaults up to time t.[5] The first summand of the intensity function, $\lambda_0\psi$, gives the linear dependency on the factor process; the parameter λ_0 mainly determines the credit quality of firms in the portfolio. The second term models contagion effects: for $l > \mu(t)$ the default intensity $h(t,\psi,l)$ is larger than $\lambda_0\psi$. The parameter λ_1 gives the slope of function $l \mapsto h(t,\psi,l)$ for $l \downarrow \mu(t)$ so that λ_1 models the strength of the default contagion for a 'normal' number of defaults. Note finally that h is convex in l; this implies that a large number of defaults leads to very high values of default intensities, thus triggering a cascade of further defaults. The parameter λ_2 controls the degree of convexity of h and hence the tendency of the model to generate default cascades; as shown in Frey & Backhaus (2006), the calibration of the model to observed CDO tranche spreads typically leads to relatively large values of λ_2.

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[4] Of course homogeneous (top-down) models can be cannot be calibrated to single-name CDS spreads as the model-implied spreads are by definition identical for all firms. Hence these models are usually calibrated to observed index- and tranche spreads.

[5] Typically $\mu(t)$ is obtained by calibration to an observed CDS index spread.

Example 2.2. In the numerical experiments we mostly work with the following parameterization of the homogeneous-portfolio model. Default intensities are given by the convex counterparty risk model (4). We have chosen four different grids of different coarseness for S^{Ψ}, generating different levels of spread-volatility: S_0^{Ψ} corresponds to a constant factor process, S_1^{Ψ} corresponds a low spread volatility, S_2^{Ψ} to a medium volatility and S_3^{Ψ} to a relatively high volatility of credit spreads. Throughout we take $\left|S_i^{\Psi}\right| = 11$ and $\Psi_0 = 0.005$; moreover, it is assumed that Ψ can only jump to neighboring states with transition intensity $\nu^{\Psi} = 5.0$, i.e. the generator matrix $\boldsymbol{q}^{\Psi}$ of Ψ is tridiagonal with off-diagonal elements equal to ν^{Ψ}. In order to obtain reasonable parameter values the model was calibrated to the index-level and to observed tranche spreads of the iTraxx with a maturity of 5 years.[6] The sets $S_0^{\Psi}, \dots, S_3^{\Psi}$ and the corresponding parameters of the convex counterparty risk model (4) are given in Table 1. Note that with increasing spread volatility the contagion parameter λ_1 is reduced in the calibration procedure, as a large part of the dependence between defaults is generated by fluctuations in the common factor Ψ.

S^{Ψ}	λ_0	λ_1	λ_2
$S_0^{\Psi} = \{0.0050, 0.0050, \dots, 0.0050, \dots, 0.0050, 0.0050\}$	0.85910	0.18803	22.125
$S_1^{\Psi} = \{0.0045, 0.0046, \dots, 0.0050, \dots, 0, 0054, 0.0055\}$	0.85827	0.18789	22.126
$S_2^{\Psi} = \{0.0025, 0.0030, \dots, 0.0050, \dots, 0.0070, 0.0075\}$	0.87052	0.16363	24.372
$S_3^{\Psi} = \{0.0005, 0.0014, \dots, 0.0050, \dots, 0.0086, 0.0095\}$	0.89696	0.11306	30.538

Table 1: State space of Ψ and corresponding model parameters.

Remark 2.3. Semi-analytic and numeric methods for computing prices of credit derivatives in Markov-chain models similar to the model introduced in Assumption 2.1 are discussed among others in Frey & Backhaus (2006) or in Herbertsson (2007). It turns out that for homogeneous portfolios analytic approaches based on the Kolmogorov equations or on matrix exponentials work quite well even in large portfolios; for large inhomogeneous portfolios one has to resort to simulation approaches, essentially for combinatoric reasons.

3 Credit Derivatives

In this section we discuss the payments, the market value and the gains process of CDSs and CDOs; this serves to set up the notation and is moreover a necessary prerequisite for studying the dynamic hedging of portfolio credit derivatives. The gains process of a security/position is the sum of the current market value and of the cumulative cash-flows associated with the position (spread-, interest- and default payments). Since we are mainly interested in the case where an investor tries to hedge a protection-seller position in a CDO tranche by taking a protection-buyer position in the CDSs or in the CDS index underlying the transaction, we model the cash flows of the CDO tranche from the viewpoint of a protection seller and the cash flows of CDSs from the viewpoint of a protection buyer.

In the following we denote by T the maturity of all credit derivatives considered. For simplicity we normalize the nominal of each CDS to one; moreover, we assume an identical

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[6] The market-observed spreads used in the calibration were as follows: the index spread is 36 bp; the upfront spread of the equity tranche equals 26% (+500bp running spread); the (running) spread of the [3,6]-tranche is 84bp; the spread of the [6,9]-tranche is 25bp; the spread of the [9,12]-tranche is 12bp; the spread of the [12-22]-tranche is 6 bp.

deterministic default payment of size $\delta \in (0,1)$ for each contract.[7] The spread payments of all credit derivatives are scheduled at N payment dates $0 < z_1 < \cdots < z_N = T$. We define $z_0 := 0$, $\Delta z_n := z_n - z_{n-1}$, and, for $t \geq 0$,

$$n_t^z := \big|\{i = 1, \ldots, N \colon z_i \leq t\}\big|; \tag{5}$$

note that n_t^z is the number of the spread payment dates up to time t.

Single-name CDSs. The market value $V_{t,k}^{\mathrm{CDS}}$ of a protection-buyer position in a CDS on firm k with fixed spread s^k is given by the difference between the value of the default payment and the value of the future premium payments (regular and accrued). Hence we get

$$\begin{aligned} V_{t,k}^{\mathrm{CDS}} = & 1_{\{\tau_k > t\}} E^Q \Bigg(\delta p_0(t, \tau_k) 1_{\{\tau_k \leq T\}} \\ & - s^k \sum_{n=n_t^z+1}^{N} \Big((\Delta z_n) p_0(t, z_n) 1_{\{\tau_k > z_n\}} + (\tau_k - z_{n-1}) p_0(t, \tau_k) 1_{\{z_{n-1} < \tau_k \leq z_n\}} \Big) \Big| \mathcal{F}_t \Bigg). \end{aligned} \tag{6}$$

In the Markov-chain model the Markov-property of Γ implies that $V_{t,k}^{\mathrm{CDS}} = v_k^{\mathrm{CDS}}(t, Y_t, \Psi_t)$ for some function $\tilde{v}_k^{\mathrm{CDS}} \colon [0,T] \times \{0,1\}^m \times S^{\Psi} \to \mathbb{R}$. Note that at a spread payment date $z_n < \tau_k$ there is a jump of size $\Delta s^k \Delta z_n > 0$ in the market value and that $V_{t,k}^{\mathrm{CDS}} \equiv 0$ for $t \geq \tau_k$. The gains process G_k^{CDS} has dynamics

$$dG_{t,k}^{\mathrm{CDS}} = -s^k \{(\Delta z_{n_t^z})(1 - Y_{t,k})\}\, dn_t^z + \Big(\delta - s^k (t - z_{n_t^z})\Big) dY_{t,k} + dV_{t,k}^{\mathrm{CDS}}, \quad t \leq T\,. \tag{7}$$

For convenience premium payments are sometimes modelled by an absolutely-continuous payment stream with rate s^k. In that case (7) simplifies to

$$dG_{t,k}^{\mathrm{CDS}} = -s^k (1 - Y_{t,k}) dt + \delta dY_{t,k} + dV_{t,k}^{\mathrm{CDS}}, \quad t \leq T. \tag{8}$$

CDS indices. The payoff of a CDS-index with fixed spread s^{Ind} on the reference pool equals the payoff of a portfolio consisting of one single-name CDS per name with *identical spreads* $s^k = s^{\mathrm{Ind}}$, $k = 1, \ldots, m$. Denote the cumulative portfolio loss up to time t by $L_t := \sum_{i=1}^m \delta Y_{t,i}$ and define the remaining notional of the index at time t as $N_t^{\mathrm{Ind}} := m - M_t$ (the number of surviving firms at time t). The future cash-flow stream of the default-payment leg is then given by $\int_t^T dL_s$; the future regular premium payments can be written in the form $s^{\mathrm{Ind}} \sum_{n=n_t^z+1}^N (\Delta z_n) N_{z_n}^{\mathrm{Ind}}$ and the accrued premium payments are given by $s^{\mathrm{Ind}} \sum_{k=1}^m (T_k - z_{n-1}) 1_{\{z_{n-1} < T_k \leq z_n\}}$.

The market-value and the gains process can be computed as in (6), (7) or (8). It will turn out that in a homogeneous portfolio the hedge ratios of a CDO tranche with respect to the individual CDSs are identical, $\theta_{t,k} \equiv \theta_t$ for all k. In that case a hedging strategy can be implemented by taking a protection-buyer position of size θ_t directly in the CDS index, which is much easier than running a dynamic portfolio strategy in, say, $m = 125$ single-name CDSs.

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[7] The extension of our analysis to firm-specific but deterministic loss given default δ_i, $1 \leq i \leq m$ involves only notational changes; random recovery rates on the other hand would lead to significant complications. Note that the recovery rates used in the definition of the payoff of CDO tranches and CDS indices are deterministic and identical across firms by market convention.

CDO Tranches. A synthetic CDO tranche on the reference portfolio is characterized by fixed lower and upper attachment points $0 \le l < u \le 1$. The tranche consists of a default payment leg and a premium payment leg. Define the cumulative tranche loss $L_t^{[l,u]}$ by

$$L_t^{[l,u]} := (L_t - ml)^+ - (L_t - mu)^+ \,, \tag{9}$$

i.e. the part of L_t falling in the layer $[l,u]$, and denote the remaining notional of the tranche by $N_t^{[l,u]} := m(u-l) - L_t^{[l,u]}$. At a default time $T_k \le T$ there is a default payment of size

$$\Delta L_{T_k}^{[l,u]} := L_{T_k}^{[l,u]} - L_{T_k-}^{[l,u]} \,.$$

The premium payment leg consists of regular and accrued premium payments. The regular premium payment at date z_n is given by $s^{[l,u]}(\Delta z_n) N_t^{[l,u]}$, $s^{[l,u]}$ the annualized tranche spread. The accrued payment at a default time $\tau \in (z_n, z_{n+1}]$ equals $s^{[l,u]}(\tau - z_n)\Delta L_\tau^{[l,u]}$. For the equity tranche there is moreover an upfront payment at $t = 0$, quoted in the form $s^{\mathrm{upf}} N_0^{[l,u]}$, s^{upf} the so-called upfront spread. The market value of a protection-seller position equals

$$\begin{aligned} V_t^{[l,u]} =& E^Q\Bigg(- \int_t^T p_0(t,s)\, dL_s^{[l,u]} + s^{[l,u]} \sum_{n=n_t^z+1}^{N} \Big\{ p_0(t,z_n)(\Delta z_n) N_{z_n}^{[l,u]} \\ & + \sum_{k=1}^{m} p_0(t,T_k)(T_k - z_{n-1}) \Delta L_{T_k}^{[l,u]} 1_{\{z_{n-1} < T_k \le z_n\}} \Big\} \mid \mathcal{F}_t \Bigg) \,. \end{aligned}$$

Note that in the Markov-chain model $V^{[l,u]} = \tilde{v}^{[l,u]}(t, Y_t, \Psi_t)$ for some function $\tilde{v}[l,u] \colon [0,T] \times \{0,1\}^m \times S^\Psi \to \mathbb{R}$. The gains process $G_t^{[l,u]}$ has dynamics

$$dG_t^{[l,u]} = s^{[l,u]}(\Delta z_{n_t^z})\, N_t^{[l,u]}\, dn_t^z + s^{[l,u]}(t - z_{n_t^z})\, dL_t^{[l,u]} - dL_t^{[l,u]} + dV_t^{[l,u]}. \tag{10}$$

If spread payments are modeled as an absolutely continuous payment stream, (10) becomes

$$dG_t^{[l,u]} = s^{[l,u]} N_t^{[l,u]} dt - dL_t^{[l,u]} + dV_t^{[l,u]}. \tag{11}$$

Market values at a default time. At a default time T_k the market value of a CDO tranche changes for a number of reasons: first, the increase in L_t at T_k makes it more likely that a tranche with $ml > L_t$ will be hit in the future; second, if $\Delta L_{T_k}^{[l,u]} > 0$, there is a change in the remaining nominal of the tranche affecting the size of future premium payments (direct effects); third, with default contagion, the default event impacts the default intensities of the surviving firms and thereby the market value of the tranche (indirect contagion effect). The contagion effect also has an impact on the market value of a non-defaulted CDS. Table 2 shows that the indirect contagion effect can be quite substantial. In this table we compare the change in market value of a non-defaulted CDS and of various CDO tranches at a default time a) in a Gauss copula model and b) in the homogeneous Markov-chain model described in Example 2.2. In case a) we follow standard market practice and assume that the default event has no impact on the default probability of surviving firms. It turns out that the change in market value is much larger for the Markov-chain model with contagion effects than for the Gauss copula model. Moreover, the change in the market value decreases with increasing spread-volatility and hence decreasing interaction-parameter λ_1. It will turn out below that this has a substantial impact on the form of the ensuing hedge ratios.

Product	CDS	[0,3]	[3,6]	[6,9]	[9,12]	[12,22]
Gauss Copula	0.0000	-0.001	-0.103	-0.014	-0.0049	-0.0062
Markov chain, S_0^Ψ	0.0179	-0.369	-0.388	-0.163	-0.1101	-0.3018
Markov chain, S_3^Ψ	0.0114	-0.233	-0.252	-0.091	-0.0645	-0.1939

Table 2: Change in market value $\Delta V_t^{\mathrm{CDS}}$ and $\Delta V_t^{[l,u]}$ at $t = T_1$. Model parameters are given in Example 2.2. The opposite signs in $\Delta V_t^{\mathrm{CDS}}$ and $\Delta V_t^{[l,u]}$ are due to the fact that we consider a protection-buyer position in the CDS and a protection-seller position in the CDO tranches.

4 Sensitivity-based hedging with default contagion

In this section we consider the sensitivity-based hedging strategies used in practice (see for instance Neugebauer (2006)) in the context of the Markov-chain model with spread risk and default contagion. These strategies are not based on any kind of model dynamics; rather one simply attempts to immunize the portfolio against (small) changes in key input parameters.

Immunization against default risk. The *default delta* of a CDO tranche with respect to name i ($\Delta_{t,i}^{\mathrm{def}}$) gives the number of CDSs of firm i one has to hold at time t in order to immunize the portfolio against the change-in-value occurring in the hypothetical scenario where name i defaults at time t. As just explained, in the presence of default contagion the default of firm i impacts the market value $V_{t,j}^{\mathrm{CDS}}$, $j \neq i$. Assume for simplicity that $T_1 > t$ (no defaults in the portfolio up to time t). In that case the vector $(\Delta_{t,1}^{\mathrm{def}}, \dots, \Delta_{t,m}^{\mathrm{def}})$ thus has to solve the following system of m linear equations, indexed with $i \in \{1, \dots, m\}$

$$\begin{aligned} 0 &\overset{!}{=} \Delta G_t^{[l,u]}|_{\tau_i=t} + \sum_{k=1}^{m} \Delta_{t,k}^{\mathrm{def}}\, \Delta G_{t,k}^{\mathrm{CDS}}|_{\tau_i=t} \\ &= \Delta G_t^{[l,u]}|_{\tau_i=t} + \Delta_{t,i}^{\mathrm{def}}\big(\delta - s_i^{\mathrm{CDS}}(t - z_{n_t^z}) - V_{t,i}^{\mathrm{CDS}}\big) + \sum_{k\neq i} \Delta_{t,k}^{\mathrm{def}}\, \Delta V_{t,k}^{\mathrm{CDS}}|_{\tau_i=t}\,. \end{aligned} \tag{12}$$

If firm k has already defaulted we set $\Delta_{t,k}^{\mathrm{def}} = 0$ and omit the kth equation in the system (12). Note that $\Delta G_t^{[l,u]}|_{\tau_i=t}$ consists of the change in the market value

$$\Delta V^{[l,u]}|_{\tau_i=t} = v^{[l,u]}(t, Y_{t-}^i, \Psi_t) - v^{[l,u]}(t, Y_{t-}, \Psi_t)\,,$$

and of the default- and the accrued premium payment due to the default event.

In a homogeneous portfolio Δ^{def} is identical for all firms; the system (12) reduces to

$$\Delta_t^{\mathrm{def}} = -\frac{\Delta G_t^{[l,u]}|_{\tau_i=t}}{(m - M_t - 1)\Delta V_t^{\mathrm{CDS}}|_{\tau_i=t} - V_t^{\mathrm{CDS}} + \delta - s^{\mathrm{CDS}}(t - z_{n_t^z})} = -\frac{\Delta G_t^{[l,u]}|_{\tau_i=t}}{\Delta G_t^{\mathrm{Ind}}|_{\tau_i=t}}\,, \tag{13}$$

and the portfolio can be immunized by taking a position of size Δ_t^{def} directly in the index.

Numerical results. Next we compare Δ^{def} as obtained using the Markov-chain model from Example 2.2 (for deterministic Ψ) with Δ^{def} as obtained from a homogeneous Gauss copula model with tranche correlations and default probabilities calibrated to the same iTraxx data. In the Gauss copula model Δ_i^{def} is given by the ratio $-\Delta G_t^{[l,u]}|_{\tau_i=t} / \Delta G_{t,i}^{\mathrm{CDS}}|_{\tau_i=t}$ where $\Delta G_t^{[l,u]}|_{\tau_i=t}$ is computed under the assumption that the fair swap spread of all non-defaulted firms remains unchanged. Table 3 gives the Δ^{def}-values for both models.[8] The differences between the models

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[8] Here, as in all our numerical experiments, wehave set $\delta = 60\%$ (the standard on the market for synthetic CDO tranches); moreover, we have set $r \equiv 0$ for simplicity.

can be explained by the indirect contagion effect discussed above. Consider first the equity tranche. In the Markov-chain model a protection-buyer position of roughly one CDS per name generates a substantially higher profit than in the copula model, overcompensating the change in the market value of the tranche. As a result the Δ^{def} in the Markov-chain model is lower than in the copula model. For the senior mezzanine and the senior tranches on the other hand, $\Delta G^{[l,u]}$ is significantly higher in the Markov-chain model than in the Gauss copula model, leading to a higher Δ^{def} for the former model class than for the latter.

In interpreting the results from Table 3 one should keep in mind that the numbers have been computed for the case where the factor process Ψ is deterministic. In that case the default contagion in the model is quite strong as the default correlation necessary to explain the observed CDO spreads is generated entirely by contagion effects. With weaker contagion effects the differences in the Δ^{def} generated by the two models is somewhat smaller albeit still substantial.

Product	[0,3]	[3,6]	[6,9]	[9,12]	[12,22]
Gauss Copula	1.002	0.171	0.023	0.008	0.010
Markov chain	0.344	0.138	0.058	0.039	0.107

Table 3: Comparison of Δ^{def} as obtained by the Markov-chain model and the Gauss copula model.

Immunization against spread risk. Market practitioners frequently immunize a protection-seller position in a CDO tranche against fluctuations in the spread of the underlying CDS index. Following market practice (Neugebauer (2006)), we define the index spread delta of a CDO tranche in a homogeneous Gauss copula model at a given spread-level s as

$$\Delta_t^{\text{spread, Gauss}} := -\frac{V_t^{[l,u]}|_{s^{\text{Ind}}=s+1\text{bp}} - V_t^{[l,u]}|_{s^{\text{Ind}}=s}}{V_t^{\text{Ind}}|_{s^{\text{Ind}}=s+1\text{bp}} - V_t^{\text{Ind}}|_{s^{\text{Ind}}=s}}\,. \tag{14}$$

In the homogeneous Markov-chain model of Example 2.2 we have two possibilities for mimicking this definition, both of course somewhat ad hoc. First, we can define the spread delta as in (14), i.e. by recalibrating the Markov-model; in that case $V_t^{[l,u]}|_{s^{\text{Ind}}=s+1\text{bp}}$ is computed by calibrating the level-parameter λ_0 to the spread $s^{\text{Ind}} = s + 1\text{bp}$, leaving all other parameters unchanged. Second, and perhaps more naturally, the spread delta can be defined as

$$\Delta_t^{\text{spread, Markov}} := -\frac{V_t^{[l,u]}|_{\Psi_t=\psi_{i+1}} - V_t^{[l,u]}|_{\Psi_t=\psi_{i-1}}}{V_t^{\text{Ind}}|_{\Psi_t=\psi_{i+1}} - V_t^{\text{Ind}}|_{\Psi_t=\psi_{i-1}}}. \tag{15}$$

In Table 4 we compare the spread deltas for the Markov-chain model of Example 2.2 and the homogeneous Gauss copula model. We observe that the two different definitions of Δ^{spread} in the Markov-model give very similar results. The Gauss copula model leads to larger values for Δ^{spread} than the Markov-model, but the differences are less striking than for Δ^{def}.

5 Dynamic risk-minimizing hedging strategies

5.1 Risk-minimizing hedging strategies

In this section we study the hedging of credit derivatives for the Markov model introduced in Assumption 2.1, using model-based dynamic hedging strategies. We use one CDS per under-

Tranche	[0,3]	[3,6]	[6,9]	[9,12]	[12,22]
Gauss Copula	0.564	0.295	0.082	0.042	0.081
Markov model, recalibrated parameters	0.535	0.141	0.044	0.027	0.070
Markov model, varied Ψ_t (for S_3^Ψ)	0.526	0.128	0.038	0.025	0.074

Table 4: Comparison of Δ^{spread} in the Markov-chain model and in the Gauss copula model.

lying name in the portfolio as hedging instrument.[9] With default- and spread risk, that is for stochastic Ψ, we expect the market to be incomplete, so that a typical CDO tranche cannot be replicated perfectly by dynamic trading in the hedging instruments. In order to deal with this problem we use the concept of *risk minimization* as introduced by Föllmer & Sondermann (1986). Denote by $\tilde{G}_t^{[l,u]}$ and $\tilde{G}_{t,k}^{\text{CDS}}$ the discounted gains processes of the CDO tranche and of the kth CDS. A dynamic hedging strategy $\boldsymbol{\theta} = (\theta_{t,1}, \dots, \theta_{t,m})_{0 \le t \le T}$ is an $(\mathcal{F}_t)$-predictable process so that $\theta_{t,k}$ gives the number of CDS on name k in the portfolio at time t; the description of the strategy is completed by specifying the cash position given by some $(\mathcal{F}_t)$-adapted process θ_0. For any choice of a dynamic hedging strategy we have a representation of the form

$$0 = \tilde{G}_t^{[l,u]} - \tilde{G}_0^{[l,u]} + \sum_{k=1}^m \int_0^t \theta_{s,k} d\tilde{G}_{s,k}^{\text{CDS}} + G_t^\perp, \quad 0 \le t \le T. \tag{16}$$

Here the process $G^\perp$ - which obviously depends on the hedging strategy $\boldsymbol{\theta}$ - represents the hedging error of the strategy. According to Föllmer & Sondermann (1986), a strategy $\boldsymbol{\theta}$ is called risk-minimizing if the the so-called *remaining risk* (the conditional variance of the hedging error) given by

$$E^Q\left((G_T^\perp(\boldsymbol{\theta}) - G_t^\perp(\boldsymbol{\theta}))^2 \;\middle|\; \mathcal{F}_t \right) \tag{17}$$

is minimized over all suitable strategies $\boldsymbol{\theta}$ simultaneously for all $0 \le t \le T$. It is well-known, that a risk-minimizing strategy exists and that it can be computed from the Kunita-Watanabe decomposition of the Q-martingale $\tilde{G}^{[l,u]}$ with respect to the Q-martingales $\tilde{G}_k^{\text{CDS}}$, $1 \le k \le m$; see Föllmer & Sondermann (1986) for details. In particular, the process $G^\perp$ must be orthogonal to the hedging instruments, i.e. $\langle G^\perp, \tilde{G}_k^{\text{CDS}} \rangle_t \equiv 0, \quad k = 1, \dots, m,$. Using this result we can derive from (16) the system of equations

$$d\langle \tilde{G}^{[l,u]}, \tilde{G}_j^{\text{CDS}} \rangle_t = -\sum_{k=1}^m \theta_{t,k} d\langle \tilde{G}_k^{\text{CDS}}, \tilde{G}_j^{\text{CDS}} \rangle_t, \quad j = 1, \dots, m, \tag{18}$$

from which the processes $\theta_1, \dots, \theta_m$ can be determined. The cash-position θ_0 of the strategy is finally determined by the requirement that the discounted market value of the overall portfolio has to be equal to zero,

$$\theta_{t,0} + \sum_{k=1}^m \theta_{t,k} \tilde{V}_{t,k}^{\text{CDS}} + \tilde{V}_t^{[l,u]} \overset{!}{=} 0, \quad 0 \le t \le T\,.$$

In the remainder of the paper we show how to compute the risk-minimizing strategy $\boldsymbol{\theta}$ and study some of its properties. We begin with the case where Ψ is deterministic; in Section 5.3 we discuss the general case where Ψ follows a (non-deterministic) Markov chain.

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[9] This is a starting point; the methodology applies to other sets of hedging instruments with identical computations.

Remark 5.1. There are a number of alternative concepts for hedging in incomplete markets including *local risk-minimization* (Föllmer & Schweizer (1991)); variance-minimizing hedging (see for instance Schweizer (2001)); *superhedging* (see for instance El Karoui & Quenez (1995)); and finally *utility-based* hedging (see for instance Delbaen, Grandits, Rheinländer, Sampieri, Schweizer & Stricker (2002) and Becherer (2004)). We have chosen the concept of risk-minimization both because of its intuitive appeal and for tractability reasons.

5.2 Constant factor process

Computation of the hedging strategy. If Ψ is constant, $\Psi_t \equiv \psi$, the underlying filtration can be taken to be the *default history* $(\mathcal{F}_t^Y)$ with $\mathcal{F}_t^Y = \sigma(Y_s\colon s \le t)$ (as the gains process of all securities involved are $(\mathcal{F}_t^Y)$-adapted in that case). Moreover, we expect the market to be complete, as the number of driving risk factors (the default indicator processes $Y_1, \ldots, Y_m$) is equal to the number of hedging instruments. In this case there is a direct way for computing the hedging strategy $\boldsymbol{\theta}$ which is not based on the system (18): Define the compensated default indicator processes by

$$N_{t,i} = Y_{t,i} - \int_0^{\tau_i \wedge t} \lambda_i(s, \psi, Y_s)\,ds\,, \quad 1 \le i \le m\,. \tag{19}$$

Since there are no joint defaults in our model, the default history $(\mathcal{F}_t^Y)$ is generated by the marked point process $(T_n, \xi_n)_{1 \le n \le m}$ with mark space $\{1, \ldots, m\}$. By standard results from stochastic calculus - see for instance Brémaud (1981), Chapter VIII, Theorem T8 - every $(\mathcal{F}_t^Y)$-martingale can therefore be represented as stochastic integral with respect to the m martingales $N_{t,1}, \ldots, N_{t,m}$, i.e. there are predictable processes $\phi_{t,1}^{[l,u]}, \ldots, \phi_{t,m}^{[l,u]}$ and $\phi_{t,1}^k, \ldots, \phi_{t,m}^k$, $1 \le k \le m$, such that

$$d\tilde{G}_t^{[l,u]} = \sum_{i=1}^m \phi_{t,i}^{[l,u]} dN_{t,i}, \qquad d\tilde{G}_{t,k}^{\mathrm{CDS}} = \sum_{i=1}^m \phi_{t,i}^k dN_{t,i}, \quad k = 1, \ldots, m\,. \tag{20}$$

In order to determine the hedging strategy $\boldsymbol{\theta}$ we argue as follows: From (16) we get, as $G^{\perp} \equiv 0$,

$$d\tilde{G}_t^{[l,u]} = -\sum_{k=1}^m \theta_{t,k}\, d\tilde{G}_{t,k}^{\mathrm{CDS}} = -\sum_{k=1}^m \theta_{t,k}\, d\Big(\sum_{i=1}^m \phi_{t,i}^k dN_{t,i}\Big) = -\sum_{i=1}^m \Big(\sum_{k=1}^m \theta_{t,k} \phi_{t,i}^k\Big)\, dN_{t,i}\,. \tag{21}$$

Denote by $A_t := \{i = 1, \ldots, m : \ Y_{t-,i} = 0\}$ the set of non-defaulted firms immediately prior to time t. Comparing (21) with the first equation in (20), it is immediately seen that a hedging strategy $\boldsymbol{\theta}$ exists if and only if the following system of equations has a solution:

$$\sum_{k \in A_t} \theta_{t,k} \phi_{t,i}^k = -\phi_{t,i}^{[l,u]}, \quad i \in A_t\,, 0 \le t \le T\,; \tag{22}$$

for $k \notin A_t$ we let $\theta_{t,k} = 0$. Note that (22) is a linear system of $|A_t|$ equations for $|A_t|$ unknowns with coefficient matrix $\boldsymbol{\Phi}_t := \big(\phi_{t,i}^j\big)_{i,j \in A_t}$.

It remains to determine the integrands $\phi_{t,i}^k$ and $\phi_{t,i}^{[l,u]}$ in the martingale representation (20). If (20) holds, we have $\Delta\tilde{G}_t^{[l,u]} = \sum_{i=1}^m \phi_{t,i}^{[l,u]} \Delta Y_{t,i}$ and $\Delta\tilde{G}_{t,k}^{\mathrm{CDS}} = \sum_{i=1}^m \phi_{t,i}^k \Delta Y_{t,i}$. Hence

$$\phi_{t,i}^{[l,u]} = 1_{A_t}(i) \Delta\tilde{G}_t^{[l,u]}|_{\tau_i = t} \ \text{ and } \phi_{t,i}^k = 1_{A_t}(i) \Delta\tilde{G}_{t,k}^{\mathrm{CDS}}|_{\tau_i = t}. \tag{23}$$

Summing up, we have the following result.

Proposition 5.2. *If the matrix $\boldsymbol{\Phi}_t$ has full rank for all $t \in [0,T]$, the gains process $G^{[l,u]}$ (and in fact every $\mathcal{F}_T^Y$-measurable claim H) can be replicated by dynamic trading in the savings account and the m CDSs. The trading strategy $\boldsymbol{\theta}$ is given as solution to the linear system* (22) *with coefficients given in* (23)*; the cash-position θ_0 is determined by the equation $\theta_{t,0} + \sum_{k=1}^m \theta_{t,k}\tilde{V}_{t,k}^{CDS} + \tilde{V}_t^{[l,u]} = 0$.*

If we plug the expressions (23) into the linear system (22), it is immediately seen that this system reduces to the system (12) for the spread delta (Δ^{def}) in the Markov-chain model. Hence in the absence of spread risk the dynamic hedging strategy is given by $\boldsymbol{\theta} = (\Delta_{t,1}^{\text{def}}, \ldots, \Delta_{t,m}^{\text{def}})_{0\le t\le T}$. This is quite intuitive as in that case the portfolio is only exposed to default risk.

In a homogeneous portfolio we obviously have $\theta_{t,j} \equiv \theta_t$, $\forall j = 1,\ldots m$ and $\phi_{t,j}^k = \tilde{G}_{t,1}^{\text{CDS}}|_{\tau_2=t} - \tilde{G}_{t-,1}^{\text{CDS}}$, $\forall k \neq j$, $\phi_{t,k}^k = \tilde{G}_{t,1}^{\text{CDS}}|_{\tau_1=t} - \tilde{G}_{t-,1}^{\text{CDS}}$, $\phi_{t,j}^{[l,u]} = \tilde{G}_t^{[l,u]}|_{\tau_1=t} - \tilde{G}_{t-}^{[l,u]}$, $j = 1,\ldots,m$, and we obtain[10]

$$\theta_t = -\frac{\Delta\tilde{G}_t^{[l,u]}|_{\tau_1=t} - \tilde{G}_{t-}^{[l,u]}}{(m-1)(\tilde{G}_{t,1}^{\text{CDS}}|_{\tau_2=t} - \tilde{G}_{t-,1}^{\text{CDS}}) + (\tilde{G}_{t,1}^{\text{CDS}}|_{\tau_1=t} - \tilde{G}_{t-,1}^{\text{CDS}})}\,. \tag{24}$$

Note that the discount factor cancels, so that (24) is in fact equivalent to (13).

Remark 5.3 (The full-rank condition). Conditions ensuring that the full rank condition on $\boldsymbol{\Phi}_t$ holds (and hence market completeness) are discussed in Frey & Backhaus (2006). In particular, it is shown that $\boldsymbol{\Phi}_t$ is complete if the contagion effects are not too strong or if the time to maturity $T-t$ is not too large. A risk-minimizing strategy can of course be computed even if $\boldsymbol{\Phi}_t$ does not have full rank. The necessary computations are a special case of the arguments used in Section 5.3 below, and we omit the details.

5.3 Ψ as Markov chain

If Ψ is random, the gains processes of all securities involved are adapted to $(\mathcal{F}_t^\Gamma)$ (the filtration generated by Y and Ψ) so that we may choose $(\mathcal{F}_t^\Gamma)$ as underlying filtration. In that case we moreover expect the market to be incomplete. The derivation of the risk-minimizing hedging strategy is hence based on the system (18). The main task is to compute the quadratic covariations $\langle \tilde{G}^{[l,u]}, \tilde{G}_k^{\text{CDS}}\rangle_t$ and $\langle \tilde{G}_j^{\text{CDS}}, \tilde{G}_k^{\text{CDS}}\rangle_t$. This is done in two steps.

Step 1: martingale representation. In this step we represent all gains processes as stochastic integrals with respect to the *compensated jump measure* of the Markov chain $\Gamma = (Y_t, \Psi_t)_{t\ge0}$. Recall that $S^\Gamma = \{0,1\}^m \times S^\Psi$, and denote by

$$E^\Gamma := \{(\gamma_1,\gamma_2) \in S^\Gamma \times S^\Gamma : \gamma_1 \neq \gamma_2\}$$

the set of possible transitions of Γ; elements of E^Γ are written in the form $\boldsymbol{e} = (\gamma_1,\gamma_2)$. The *counting measure* μ^Γ associated with the Markov-chain Γ is a measure on $[0,\infty) \times E^\Gamma$; see Section VIII.1 of Brémaud (1981) for the general definition. According to standard measure theory, μ^Γ is uniquely defined by its values on sets of the form $[0,t] \times \{(\gamma_1,\gamma_2)\}$, $(\gamma_1,\gamma_2) \in E^\Gamma$, $t > 0$; here we have

$$\mu^\Gamma\big([0,t] \times \{(\gamma_1,\gamma_2)\}\big) = \sum_{s\le t} 1_{\{\Gamma_{s-}=\gamma_1, \Gamma_s=\gamma_2\}}\,. \tag{25}$$

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[10] For notational simplicity we restrict ourselves to the case $t < T_1$.

The *predictable compensator* of μ^Γ is a measure ν^Γ on $[0,\infty)\times E^\Gamma$ such that for any bounded predictable function $Z\colon \Omega\times[0,\infty)\times E^\Gamma\to\mathbb{R}$ the process

$$M_t^Z = \int_0^t \int_{E^\Gamma} Z(\omega; s, \boldsymbol{e})(\mu^\Gamma - \nu^\Gamma)(ds\times d\boldsymbol{e})$$

is a martingale. In our case ν^Γ is given by

$$\nu^\Gamma\big([0,t]\times\{(\gamma_1,\gamma_2)\}\big) = \int_0^t 1_{\{\Gamma_{s-}=\gamma_1\}} q_s^\Gamma(\gamma_1,\gamma_2)ds\,,\quad (\gamma_1,\gamma_2)\in E^\Gamma, t\geq 0\,, \tag{26}$$

$q_s^\Gamma(\gamma_1,\gamma_2)$ the transition rates of Γ as introduced in Assumption 2.1. It is well-known that every $(\mathcal{F}_t^\Gamma)$-adapted martingale M has a representation of the form

$$M_t = M_0 + \int_0^t \int_{E^\Gamma} Z^M(\omega; s, \boldsymbol{e})(\mu^\Gamma - \nu^\Gamma)(ds\times d\boldsymbol{e})$$

for some predictable random function Z^M; see Brémaud (1981), Chapter VIII, Theorem T8 for details. Applying this result to the discounted gains processes $\tilde{G}^{[l,u]}$, $\tilde{G}_k^{\text{CDS}}$, we get the existence of predictable functions $Z^{[l,u]}$, Z_k^{CDS} such that

$$\tilde{G}_t^{[l,u]} = \tilde{G}_0^{[l,u]} + \int_0^t \int_{E^\Gamma} Z^{[l,u]}(\omega; s, \boldsymbol{e})(\mu^\Gamma - \nu^\Gamma)(ds\times d\boldsymbol{e}) \tag{27}$$

$$\tilde{G}_{t,k}^{\text{CDS}} = \tilde{G}_{0,k}^{\text{CDS}} + \int_0^t \int_{E^\Gamma} Z_k^{\text{CDS}}(\omega; s, \boldsymbol{e})(\mu^\Gamma - \nu^\Gamma)(ds\times d\boldsymbol{e})\,; \tag{28}$$

the computation of $Z^{[l,u]}$ and Z_k^{CDS} is discussed below.

Step 2: computation of the quadratic covariations. We concentrate on computing $\langle \tilde{G}^{[l,u]}, \tilde{G}_k^{\text{CDS}}\rangle_t$; the predictable covariation between the discounted gains processes of the CDSs can be computed analogously. As all processes involved have trajectories of finite variation, we get that the pathwise covariation $[\tilde{G}^{[l,u]}, \tilde{G}_k^{\text{CDS}}]$ is given by

$$[\tilde{G}^{[l,u]}, \tilde{G}_k^{\text{CDS}}]_t = \int_0^t \int_{E^\Gamma} Z^{[l,u]}(s,\boldsymbol{e}) Z_k^{\text{CDS}}(s,\boldsymbol{e})\,\mu^\Gamma(ds\times d\boldsymbol{e})\,.$$

Since $[\tilde{G}^{[l,u]}, \tilde{G}_k^{\text{CDS}}]_t - \langle \tilde{G}^{[l,u]}, \tilde{G}_k^{\text{CDS}}\rangle_t$ is a martingale we thus have

$$\langle \tilde{G}^{[l,u]}, \tilde{G}_k^{\text{CDS}}\rangle_t = \int_0^t \int_{E^\Gamma} Z^{[l,u]}(s,\boldsymbol{e}) Z_k^{\text{CDS}}(s,\boldsymbol{e})\,\nu^\Gamma(ds\times d\boldsymbol{e}) \tag{29}$$

$$= \int_0^t \sum_{\gamma\in S^\Gamma, \gamma\neq\Gamma_{s-}} Z^{[l,u]}\big(s,(\Gamma_{s-},\gamma)\big) Z_k^{\text{CDS}}\big(s,(\Gamma_{s-},\gamma)\big) q_s^\Gamma(\Gamma_{s-},\gamma)\,ds\,. \tag{30}$$

In differential notation we therefore have $d\langle \tilde{G}^{[l,u]}, \tilde{G}_k^{\text{CDS}}\rangle_t = \xi_t^{[l,u],k} dt$ where the predictable process $\xi^{[l,u],k}$ is given by

$$\xi_t^{[l,u],k} = \sum_{\gamma\in S^\Gamma, \gamma\neq\Gamma_{t-}} Z^{[l,u]}\big(t,(\Gamma_{t-},\gamma)\big) Z_k^{\text{CDS}}\big(t,(\Gamma_{t-},\gamma)\big) q_t^\Gamma(\Gamma_{t-},\gamma).$$

Using the special form of the transition intensities of Γ we get

$$\begin{aligned}\xi_t^{[l,u],k} =& \sum_{i=1}^{m}(1-Y_{t-,i})\lambda_i(t,\Psi_{t-},Y_{t-})(Z^{[l,u]}\cdot Z_k^{\mathrm{CDS}})\big(t,\big((Y_{t-},\Psi_{t-}),(Y_{t-}^i,\Psi_{t-})\big)\big)\\ &+\sum_{\psi\in S^\Psi,\psi\neq\Psi_{t-}} q^\Psi(\Psi_{t-},\psi)(Z^{[l,u]}\cdot Z_k^{\mathrm{CDS}})\big(t,\big((Y_{t-},\Psi_{t-}),(Y_{t-},\psi)\big)\big)\,. \end{aligned}\tag{31}$$

Similarly we get for the gains processes of two CDSs $d\langle \tilde G_j^{\mathrm{CDS}},\tilde G_k^{\mathrm{CDS}}\rangle_t=\xi_t^{j,k}dt$, where $\xi_t^{j,k}$ is given by (31) with $Z^{[l,u]}$ replaced by Z_j^{CDS}.

Computation of $Z^{[l,u]}$ and Z_k^{CDS}. It is immediate from (27) and (28) that $Z^{[l,u]}\big(\omega;t,(\gamma_1,\gamma_2)\big)$, and $Z_k^{\mathrm{CDS}}\big(\omega;t,(\gamma_1,\gamma_2)\big)$ are given by the jumps $\Delta\tilde G_t^{[l,u]}$ and $\Delta\tilde G_{t,k}^{\mathrm{CDS}}$ given that $(\Gamma_{t-},\Gamma_t)=(\gamma_1,\gamma_2)$, i.e. given that Γ jumps from γ_1 to γ_2 at time t. recall that by the Markov property of Γ we have for the discounted market value

$$\tilde V_t^{[l,u]}=\tilde v^{[l,u]}(t,Y_t,\Psi_t) \text{ and } \tilde V_{t,k}^{\mathrm{CDS}}=\tilde v_k^{\mathrm{CDS}}(t,Y_t,\Psi_t)\,,$$

for functions $\tilde v^{[l,u]}$ and $\tilde v_k^{\mathrm{CDS}}\colon [0,T]\times\{0,1\}^m\times S^\Psi\to\mathbb{R}$. For notational simplicity we model spread payments by an absolutely continuous payment stream and work with the gains processes (8) and (11). Hence we get for $(\gamma_1,\gamma_2)=\big((y,\psi),(y^i,\psi)\big)$,

$$Z^{[l,u]}\big(\omega;t,(\gamma_1,\gamma_2)\big)=p(0,t)\Delta L_t^{[l,u]}|_{\tau_i=t}+\tilde v^{[l,u]}(t,y^i,\psi)-\tilde v^{[l,u]}(t,y,\psi)\,,\tag{32}$$
$$Z_k^{\mathrm{CDS}}\big(\omega;t,(\gamma_1,\gamma_2)\big)=1_{\{i=k\}}p(0,t)\delta+\tilde v_k^{\mathrm{CDS}}(t,y^i,\psi)-\tilde v_k^{\mathrm{CDS}}(t,y,\psi)\,.\tag{33}$$

Similarly we obtain for $(\gamma_1,\gamma_2)=\big((y,\psi_1),(y,\psi_2)\big)$

$$Z^{[l,u]}\big(\omega;t,(\gamma_1,\gamma_2)\big)=\tilde v^{[l,u]}(t,y,\psi_2)-\tilde v^{[l,u]}(t,y,\psi_1)\,,\tag{34}$$
$$Z_k^{\mathrm{CDS}}\big(\omega;t,(\gamma_1,\gamma_2)\big)=\tilde v_k^{\mathrm{CDS}}(t,y,\psi_2)-\tilde v_k^{\mathrm{CDS}}(t,y,\psi_1)\,;\tag{35}$$

note that in all four cases the right-hand-side is independent of ω.

Summarizing we have

Proposition 5.4. *The risk-minimizing hedging strategy $\boldsymbol\theta=(\theta_{t,1},\dots,\theta_{t,m})_{0\le t\le T}$ is given as solution of the system*

$$\xi_t^{[l,u],j}=-\sum_{k=1}^{m}\theta_{t,k}\xi_t^{k,j}\,,\quad j=1,\dots,m,\ 0\le t\le T\,,\tag{36}$$

with coefficients $\xi^{[l,u],j}$ and $\xi^{k,j}$ defined in (31) *and* (32) *to* (35).

Proof. It is well-known that a risk-minimizing strategy $\boldsymbol\theta$ exists and that it is a predictable process solving the system (18); see for instance Föllmer & Sondermann (1986). Since all quadratic variations involved are absolutely continuous with respect to Lebesgue-measure, the system (18) reduces to (36) and the claim follows. □

Note that we have determined all ingredients necessary to set up the system (36), so that $\boldsymbol\theta$ is easily computed. In the homogeneous-portfolio case things simplify further since θ_t solves the one-dimensional equation $\xi_t^{[l,u]}=-\theta_t\big((m-1)\xi_t^{1,2}+\xi_t^{1,1}\big)$.

5.4 Numerical experiments.

We conclude this section with a small numerical study. Here we focus on two issues: first we compute risk-minimizing hedging strategies for the case when Ψ is a non-deterministic Markov-chain and study the impact of the spread-volatility on the form of the hedging strategy; second we look at the impact of portfolio-heterogeneity on the form and the performance of hedging strategies.

Risk minimization when Ψ is a Markov chain. In order to illustrate the results of Section 5.3 we compute risk-minimizing hedging strategies for the homogeneous models introduced in Example 2.2. Table 5 gives the value θ of the risk-minimizing hedging strategy and compares with Δ^{def} (the immunization-strategy against default risk) and with Δ^{spread} (the immunization strategy against spread risk). As expected, with low spread volatility (small range of the factor state space) the risk-minimizing strategy is close to Δ^{def}; with increasing spread volatility θ comes closer to Δ^{spread}. Hence the risk-minimizing strategy provides a model-based endogenous interpolation between the hedging against spread-risk and the hedging against default risk.

Hedging in heterogenous portfolios. In practice, hedge ratios are frequently computed under the homogeneous-portfolio assumption. This holds true in particular for all hedging studies in top-down-models such as Laurent et al. (2007). It is therefore interesting to study the impact of heterogeneity in the portfolio on the form of the ensuing hedge ratios. For simplicity, we assume that Ψ is deterministic, so that dynamic hedging strategies coincide with the Δ^{def} as computed in Section 4. We consider a portfolio with two (industry) groups of varying credit quality. The default intensity of firms from group κ is modelled by a function $h_\kappa(t, l_\kappa, l)$ which depends on time t, on the number of defaults in group κ, denoted l_κ, and on the overall number of defaults l. Denote by m_κ the number of firms in group κ and by $\mu_\kappa(t)$ the expected number of defaults in group κ. In our simulations we take

$$h_\kappa(t, l_\kappa, l) = \lambda_{\kappa,0} + \frac{\lambda_1}{\lambda_2}\Big\{ \exp\Big(\lambda_2\gamma_\kappa \frac{(l_\kappa - \mu_\kappa(t))^+}{m_\kappa} + \lambda_2(1-\gamma_\kappa)\frac{(l-\mu(t))^+}{m}\Big) - 1\Big\}, \tag{37}$$

with parameters $\lambda_{\kappa,0} > 0$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\gamma_\kappa \in [0,1]$. The first term in the argument of the exponential function in (37) reflects the interaction between firms from the same group; the second term captures the global interaction between defaults in the entire portfolio. The relative strength of these effects is governed by the parameter γ_κ: for γ_κ close to one, firms from group κ are mainly impacted by defaults within group κ; for γ_κ close to zero, the global (portfolio-wide) interaction dominates. In our analysis we compare three different portfolios/parameterizations.

A) Here we consider a homogeneous portfolio of 125 firms with fair CDS spread equal to $s^k = 73$bp for all k.

B) Here we consider a portfolio consisting of 100 ‘good’ names (Group 1, $s^k = 20$bp, $k = 1, \dots, 100$) and 25 ‘bad’ names (Group 2, $s^k = 317$bp, $k = 101, \dots, 125$); the spread of a CDS index on the whole portfolio is $s^{\text{Ind}} = 73$bp so that the average credit quality is the same as in Portfolio A. Moreover, we put $\gamma_1 = \gamma_2 = 0$ so that there is only global interaction between defaults. Intuitively, this parameterization corresponds to a portfolio with one-factor structure.

C) Here the portfolio consists of 25 good names (Group 1) such as financial firms with $s^k = 17.5$bp and 100 medium-quality names (Group 2) with $s^k = 88$bp; again $s^{\text{Ind}} = 73$bp.

> We assume that there is a strong interaction within the group of good firms and put $\gamma_1 = 0.7$; γ_2 is set to 0.2 so that the medium-quality firms are mainly affected by the global portfolio state. Intuitively, this parameterization corresponds to a model with two industry factors, one global factor, and a very strong inter-industry correlation for Group 1 (the good names).

The remaining parameters in (37) have been chosen so that we get (roughly) the same CDO spreads in all three parameterizations.

Recall that for a deterministic factor process the hedging strategy $\boldsymbol{\theta}$ solves the linear system (12) and that the coefficients of this system are given by the change in the gains process of the tranche to be hedged and of the CDSs used as hedging instrument. Hence these changes are largely responsible for the form of the hedge ratios. Numerical values are reported in Table 6 in the appendix: we see that the results for Parameterizations A and B are roughly similar. Note however, that under Parameterization B the default of a bad name (a firm from Group 2) always leads to a smaller absolute change in the gains process than a default of a good name (a firm from Group 1). The reason for this is that in the former case the quality of the remaining portfolio is higher than in the latter case. The results for Parameterization C on the other hand differ widely from the homogeneous-portfolio case. In particular, for the given parameters the default of a good name leads to a substantial deterioration of the credit quality of the overall portfolio, as can be seen from the huge change in the gains process of the index. The hedge ratios for all three parameterizations are given in Table 7 (for $t = 0$). As with the changes in the gains processes, for Portfolio A and B the ensuing hedge ratios are qualitatively similar. However, there are substantial quantitative differences. For Portfolio C on the other hand the strong and asymmetric contagion effects lead to qualitatively different hedge ratios. In particular, in order to hedge the equity- and mezzanine tranches one has to take a protection-seller position in the CDS issued by the good names in Group 1.

An alternative way to assess the robustness of hedging strategies with respect to the assumption of a homogeneous portfolio is to look at the performance at a default-time T_1 of the homogeneous-portfolio strategy, assuming that the actual change in the gains processes corresponds to an inhomogeneous situation. More precisely, we consider a portfolio consisting of a protection-seller position in a CDO tranche or in the CDS-index and of an offsetting protection-buyer position in the CDSs; the size of this protection-buyer position was computed using Parameterization A (see the Portfolio A row of Table 7). Next we compute the hedging error (the change in the gains process of this portfolio) at the first default time T_1, assuming that $\Delta G_{T_1}^{[l,u]}$ and $\Delta G_{k,T_1}^{\mathrm{CDS}}$ are generated by Parameterizations B or C (see the Portfolio B and the Portfolio C row of Table 6). The results of this exercise are contained in Table 8. We report the *relative hedging error* for each tranche (the hedging error normalized by the overall notional); in this way results for different tranches can be compared. We note the following: in case where the actual gains processes are generated by Parameterization B, the homogeneous-portfolio strategy performs quite well. On the other hand, if the actual gains processes are generated by Parameterization C the performance of the homogeneous-portfolio strategy is poor, at least at a default of a firm in Group 1. In fact, the hedging error for the equity- and the junior mezzanine tranche is in the order of 50% of total notional. These findings suggest that hedging strategies based on the assumption of a homogeneous portfolio - and in particular all hedges computed within the top-down approach - might perform well if the real portfolio is heterogeneous with respect to credit quality but relatively homogeneous with respect to the interaction between firms; on the other hand, if the real portfolio exhibits strongly asymmetric contagion effects the homogeneous-portfolio assumption might lead to poorly performing strategies.

6 Conclusion

This paper has studied the (dynamic) hedging of CDO tranches in a portfolio credit risk model with default contagion and random fluctuations in credit spreads. This model was constructed and analyzed with Markov-chain techniques. From our analysis the following findings emerged. First, we studied the impact of default contagion on the market-standard sensitivity-based hedging strategies. It turned out that even a small amount of default contagion has a substantial impact on the form of the ensuing hedge ratios, essentially because of the impact of the default event on the quality of the remaining portfolio. Second, we showed how to compute theoretically consistent dynamic hedging strategies using incomplete-market theory, more specifically the concept of risk-minimization. The main tool in the derivation of these strategies is stochastic calculus for marked point processes. Third, we carried out numerical experiments to study the properties of these strategies. It turned out that risk-minimizing hedging strategies interpolate between the hedging of spread- and default risk in an endogenous fashion. Moreover, we showed that deviations from the popular assumption of a homogeneous portfolio can have a substantial impact on the form and on the performance of hedging strategies.

These are important results on dynamic hedging in credit markets. In particular, the sizeable differences between the market-standard sensitivity-based hedging strategies computed in the copula framework and the dynamic hedging strategies derived in our setup with spread risk and default contagion show that the current hedging practice is subject to a substantial amount of model risk. A systematic study (both simulation-based and empirical) of the model risk associated with the hedging of credit derivatives, perhaps building on the insights from Cont (2006), is therefore a logical next step. Due to its versatility the Markov-chain model proposed in the first part of the present paper could be a useful tool in this analysis. However, such a study is a major undertaking and is therefore deferred to further research.

A Tables

A1. Risk-minimization when Ψ is a Markov chain

	[0,3]-tranche			[3,6]-tranche			[6,9]-tranche		
	θ	Δ^{def}	Δ^{spread}	θ	Δ^{def}	Δ^{spread}	θ	Δ^{def}	Δ^{spread}
S_0^Ψ	0.344	0.344	-	0.138	0.138	-	0.058	0.058	-
S_1^Ψ	0.348	0.345	0.476	0.138	0.138	0.143	0.057	0.058	0.049
S_2^Ψ	0.414	0.366	0.491	0.136	0.134	0.138	0.050	0.053	0.045
S_3^Ψ	0.469	0.414	0.526	0.127	0.126	0.128	0.041	0.045	0.038

	[9,12]-tranche			[12,22]-tranche		
	θ	Δ^{def}	Δ^{spread}	θ	Δ^{def}	Δ^{spread}
S_0^Ψ	0.039	0.039	-	0.107	0.107	-
S_1^Ψ	0.039	0.039	0.031	0.106	0.107	0.082
S_2^Ψ	0.033	0.037	0.029	0.093	0.103	0.079
S_3^Ψ	0.028	0.032	0.025	0.084	0.096	0.074

Table 5: Comparison of the risk-minimizing hedging strategy θ with Δ^{def} and Δ^{spread} for different choices of the state spaces S^Ψ; Δ^{spread} has been computed using(15)

A2. Hedging in heterogenous portfolios

	ΔG^{CDS_1}	ΔG^{CDS_2}	ΔG^{Ind}	$\Delta G^{[0,3]}$	$\Delta G^{[3,6]}$	$\Delta G^{[6,9]}$	$\Delta G^{[9,12]}$	$\Delta G^{[12,22]}$
Portfolio A	0.0161	0.0161	-2.600	-0.471	-0.582	-0.236	-0.134	-0.315
Portfolio B								
default in Group 1	0.0177	0.0234	-2.752	-0.525	-0.772	-0.387	-0.233	-0.474
default in Group 2	0.0149	0.0197	-2.201	-0.467	-0.645	-0.322	-0.193	-0.392
Portfolio C								
default in Group 1	0.1386	0.1019	-13.451	-0.660	-1.295	-0.929	-0.807	-2.355
default in Group 2	0.0083	0.0146	-1.617	-0.466	-0.558	-0.211	-0.113	-0.258

Table 6: Changes due to a default in the gains process of the non-defaulted CDSs, of the index and of various CDO tranches for Portfolios A, B and C. The numbers were computed for $t = 0$.

Product		Index	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$
Portfolio A	Δ^{def}	1.000	0.181	0.224	0.091	0.051	0.121
Portfolio B	Δ_1^{def} (good firms)	1.029	0.155	0.237	0.119	0.072	0.146
	Δ_2^{def} (bad firms)	0.802	0.168	0.185	0.090	0.053	0.108
Portfolio C	Δ_1^{def} (good firms)	1.039	-0.575	-0.514	-0.043	0.084	0.370
	Δ_2^{def} (bad firms)	0.979	0.286	0.325	0.108	0.049	0.088

Table 7: Hedge-ratio (or equivalently Δ^{def}) for the CDS index and for various CDO tranches at $t = 0$, assuming that $T_1 > 0$.

Product	Index	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$
ΔG as in Parameterization B						
Rel. hedging error at default in Group 1	0.1%	0.2%	-3.0%	-3.2%	-2.2%	-0.9%
Rel. hedging error at default in Group 2	0.3%	-0.1%	-1.9%	-2.4%	-1.7%	-0.7%
ΔG as in Parameterization C						
Rel. hedging error at default in Group 1	0.5%	50.5%	49.8%	9.5%	-2.3%	-5.2%
Rel. hedging error at default in Group 2	0.5%	-1.6%	-1.4%	-0.2%	0.1%	0.1%

Table 8: Performance at $t = T_1$ of the hedging strategy computed for Parameterization A, assuming that the actual change in the gains processes is generated by Parameterizations B or C. The hedging error is represented as percentage of the total notional of the tranche. The latter equals 125 for the index, 3.75 for the $[0,3]$ up to the $[9,12]$ tranche and 12.5 for the $[12,22]$ tranche

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