Dynamic Hedging of Credit Derivatives with Spread- and Contagion Risk

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1. Introduction

Most protection-sellers on CDO tranches try to offset some of the risk by hedging, i.e. by taking protection-buyer positions in CDSs in the underlying names.

Current practice is based on

- Gauss-copula model for pricing (with tranche-dependent correlations)
- Sensitivities for hedging

Deficiencies (at least from a methodological viewpoint)

- Sensitivity-based hedging neglects contagion effects
- No model-based dynamic hedging strategies (Gauss copula model essentially static ⇒ dynamic trading strategies difficult to derive)

Our contribution

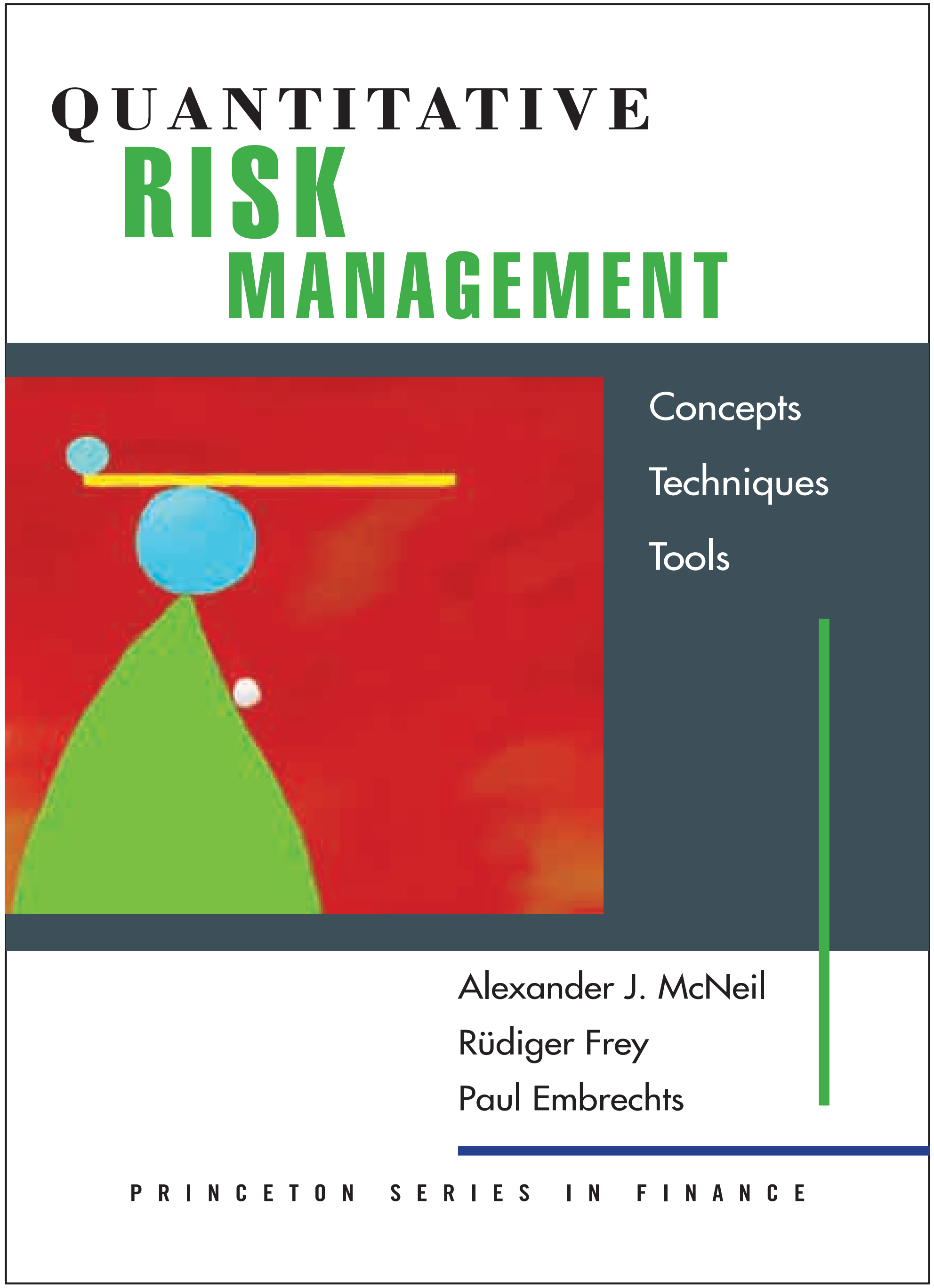
In this talk we try to address these issues. We

- introduce a dynamic Markov model for default intensities with spread- and contagion risk as 'work-bench' for our analysis
- study sensitivity-based hedging in this model and compare to sensitivities in Gauss copula model
- derive model-based dynamic hedging strategies using concept of risk minimization and compare to sensitivity-based hedging.

Outline

- Introduction
- Gains process of CDS- and CDO positions.
- A Markov model with spread- and contagion risk
- Sensitivity-based hedging
- Risk-minimizing dynamic hedging strategies

For background information on portfolio credit risk models see e.g. [McNeil et al., 2005]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2. Gains process of CDS- and CDO positions

The gains process of a financial position/security measures overall change in value (change in market value and intermediate cash-flows such as spread-, interest- default- and dividend payments)

Basic Notation. Consider m firms with default times τ_i, $1 \leq i \leq m$. We introduce

- default indicator process $\mathbf{Y}_t = (Y_{t,1}, \ldots, Y_{t,m})$ with $Y_{t,i} = 1_{\{\tau_i \leq t\}}$.
- Ordered default times denoted by $T_0 < T_1 < \ldots < T_m$; $\xi_n \in \{1, \ldots, m\}$ gives identity of the firm defaulting at time T_n.
- for given LGD δ_i, the cumulative loss $L_t = \sum_{i=1}^m \delta_i Y_{t,i}$.
- Risk-free interest rate $r(t)$; let $p_0(t_1, t_2) = \exp(-\int_{t_1}^{t_2} r(u)du$ (default-free discount factor)

Market value and gains process for CDSs

Consider CDS on name k with spread s, default payment δ and spread payment dates $0 < z_1 < \cdots < z_N = T$. Denote pricing measure by Q. Then the market value in t of a protection-buyer position equals (for $\tau_k > t$)

$$V_t^{\mathsf{CDS}} = E^Q\Big(\delta p_0(t,\tau_k) 1_{\{\tau_k \le T\}} - s \sum_{n=n(t)}^{N} \big\{(z_n - z_{n-1}) p_0(t, z_n) 1_{\{\tau_k > z_n\}} + (\tau_k - z_{n-1}) p_0(t,\tau_k) 1_{\{z_{n-1} < \tau_k \le z_n\}}\big\} \mid \mathcal{F}_t\Big).$$

Corresponding gains process G^{CDS} satisfies $G_0^{\mathsf{CDS}} = 0$,

$$dG_t^{\mathsf{CDS}} = -s(1 - Y_{t,k})dt + \delta dY_{t,k} + dV_t^{\mathsf{CDS}}$$

(modelling spread payments as continuous payment stream).

Market value and gains process for CDOs

Payment Description. Consider CDO-tranche with attachment points $l < u$, notional given by $N_t^{[l,u]} = (u - L_t)^+ - (l - L_t)^+$; define cumulative loss of tranche $[l, u]$ as $L_t^{[l,u]} := N_0^{[l,u]} - N_t^{[l,u]}$.

- Default payments. At kth default time $T_k < T$ protection-seller makes default payment $\Delta L_{T_k}^{[l,u]}$ (the part of the portfolio loss which falls in the layer $[l, u]$).

- Premium payments. Protection-seller receives periodic premium payments at $0 < z_1 < \cdots < z_N = T$ of size $s^{[l,u]}(z_n - z_{n-1})N_{z_n}^{[l,u]}$, $s^{[l,u]}$ the tranche spread. At default-date T_k he receives an accrued premium of size $s^{[l,u]}(T_k - z_{n-1})\Delta L_{T_k}^{[l,u]}$.

Market value and gains process for CDOs

Market value of protection-seller position given by

$$V_t^{[l,u]} = E^Q\Big(- \int_t^T dL_s^{[l,u]} + s^{[l,u]} \sum_{n=n(t)}^{N} \Big\{ p_0(t, z_n)(z_n - z_{n-1}) N_{z_n}^{[l,u]}$$

$$+ \sum_{k=1}^{m} p_0(t, T_k)(T_k - z_{n-1}) L_{T_k}^{[l,u]} 1_{\{z_{n-1} < T_k \le z_n\}} \Big\} \mid \mathcal{F}_t \Big)$$

Corresponding gains process $G^{[l,u]}$ satisfy $G_0^{[l,u]} = s^{\mathsf{upf}}(u - l)$,

$$dG_t^{[l,u]} = s^{[l,u]} N_t^{[l,u]} dt - dL_t^{[l,u]} + dV_t^{[l,u]},$$

(again for spread payments as continuous payment stream).

3. A Model with Spread- and Contagion Risk

Basic idea. Default intensities are model primitives (other than in copula models). Default intensity of firm i is modelled as function $\lambda_i(\Psi_t, \mathbf{Y}_t)$ of some Markovian factor process Ψ and of default-state $\mathbf{Y}_t$ of portfolio at time t. In this way default intensities are model primitives, and default contagion can be modelled explicitly.

Advantages. Intuitive and explicit parameterization of dependence between defaults; spread risk (via Ψ) and contagion (via dependence of λ on $\mathbf{Y}_t$) can be modelled explicitly $\Rightarrow$ useful tool for studying dynamic hedging of credit derivatives; Markov process techniques available for analysis and simulation of the model.

Disadvantage. Calibration to term structure of defaultable bonds or CDSs more difficult than with copula models.

Construction via Markov Chains.

a) Deterministic Ψ. Model is conveniently defined as Markov chain with state space $S = \{0,1\}^m$ and transition rates (from $\mathbf{y}$ to $\mathbf{x}$)

$$\lambda(\mathbf{y}, \mathbf{x}) = \begin{cases} 1_{\{y_i = 0\}} \lambda_i(\mathbf{y}), & \text{if } \mathbf{x} = \mathbf{y}^i \text{ for some } i \in \{1 \dots, m\}, \\ 0 & \text{else,} \end{cases}$$

where $\mathbf{y}^i \in S$ is obtained from $\mathbf{y} \in S$ by flipping ith coordinate.
Intuition. Chain can jump only to neighbouring states which differ from $\mathbf{Y}_t$ by exactly one default; if $Y_{t,i} = 0$, the probability of a jump in $[t, t+h)$ to state $\mathbf{Y}_t^i$ (default of firm i) is $\approx h\lambda_i(\mathbf{Y}_t)$.

b) Stochastic Ψ. We assume that Ψ is an autonomous Markov process (mostly a finite state Markov chain); $\mathbf{Y}$ is conditionally Markov and $\Gamma_t = (\Psi_t, \mathbf{Y}_t)$ is Markov [Frey and Backhaus, 2004].

Modelling Default Intensities

Default intensities $\lambda_i(\psi, \mathbf{y})$ are essential ingredient of the model. For computational simplicity we use a homogenous model where

$$\lambda_i(\Psi_t, \mathbf{Y}_t) = h\left(\Psi_t, M(\mathbf{Y}_t)\right) \text{ for } M(\mathbf{y}) = \sum_{i=1}^{m} y_i \,. \qquad (1)$$

Example: Convex counterparty risk model.

$$h(\psi, l) = \lambda_0 \psi + \frac{\lambda_1}{\lambda_2}\left(e^{\lambda_2 \left(l/m - \bar{\mu}(t)\right)^+} - 1\right), \; \lambda_0 > 0, \lambda_1 \geq 0, \lambda_2 \geq 0.$$

Here $\bar{\mu}(t)$ measures expected proportion of defaulted firms until t; convexity parameter λ_2 controls tendency of the model to generate default cascades.

[Frey and Backhaus, 2007]: Model can explain correlation skew of CDOs for λ_2 reasonably large.

Modelling Ψ

Ψ is a finite-state Markov chain with state space S^{Ψ}, $\Psi_0 = 0.005$ and transition-rate between neighboring states equal to 5.0. Three different parameterizations, all calibrated to the same set of CDS-index and CDO-tranche spreads.

- Low spread volatility: $S_1^{\Psi} = \{0.0045, 0.0046, \ldots, 0.005, \ldots, 0.0055\}$; $\lambda_0 = 0.86$, $\lambda_1 = 0.188$, $\lambda_2 = 22.1$
- Medium spread volatility: $S_2^{\Psi} = \{0.0025, 0.003, \ldots, 0.005, \ldots, 0.0075\}$ $\lambda_0 = 0.87$, $\lambda_1 = 0.163$, $\lambda_2 = 24.3$;
- High spread volatility: $S_3^{\Psi} = \{0, 0.001, \ldots, 0.005, \ldots, 0.01\}$; $\lambda_0 = 0.91$, $\lambda_1 = 0.097$, $\lambda_2 = 33.1$;

Note that with increasing spread volatility the contagion-parameter λ_1 is being reduced.

4. Sensitivity-based hedging

Hedging spread risk. Market standard [Neugebauer, 2006]: The 'Delta' Δ_i measures (percentage) amount of CDS-protection that needs to be bought to hedge a protection seller position in a CDO tranche against small fluctuations in the spread of credit i.

- In Gauss copula model one has $\Delta_i = \dfrac{\partial V_t^{[l,u]} / \partial s_i^{\text{CDS}}}{\partial V_{t,i}^{\text{CDS}} / \partial s_i^{\text{CDS}}}$.
- In Markov model with discrete state space of Ψ we use the approximation

$$\Delta^+(\psi_i) = \frac{V_t^{[l,u]}|_{\Psi_t=\psi_{i+1}} - V_t^{[l,u]}|_{\Psi_t=\psi_i}}{V_t^{\text{CDS}}|_{\Psi_t=\psi_{i+1}} - V_t^{\text{CDS}}|_{\Psi_t=\psi_i}}.$$

Hedging spread risk: numerical results

Tranche		[0,3]	[3,6]	[6,9]	[9,12]	[12,22]
Spread		26 %	84 bp	24 bp	14 bp	11 bp
Tranche Correlation		17.30 %	3.22 %	9.93 %	15.81 %	27.46 %
Gauss Cop.	Δ	0.61	0.23	0.06	0.03	0.07
S_1^{Ψ}	Δ^+	0.472	0.143	0.049	0.031	0.082
S_5^{Ψ}	Δ^+	0.473	0.139	0.046	0.030	0.081
S_{10}^{Ψ}	Δ^+	0.509	0.129	0.038	0.026	0.078

Note that in Markov model Deltas are typically lower than in Gauss copula model

Hedging event (default) risk

Default-risk sensitivity (Value on Default VOD_i). in Gauss-copula model measures change in P&L of a position assuming that name i defaults and that all other spreads remain unchanged.

We define VOD_i, $1 \leq i \leq m$, in the Markov-model by requiring that for all i the change in the gains process of the hedged portfolio at τ_i is zero. Default contagion $\Rightarrow$ default of firm i impacts $V_{t,j}^{\mathsf{CDS}}$ for $j \neq i$. Hence $(\mathrm{VOD}_1, \ldots, \mathrm{VOD}_m)$ solves the following system of equations

$$0 \overset{!}{=} \Delta G_t^{[l,u]}|_{t=\tau_i} + \sum_{j=1}^{m} \mathrm{VOD}_j \, \Delta G_{t,j}^{\mathsf{CDS}}|_{t=\tau_i}$$

$$= \Delta G_t^{[l,u]}|_{t=\tau_i} + \mathrm{VOD}_i(\delta - V_{t-,i}^{\mathsf{CDS}}) + \sum_{j \neq i} \mathrm{VOD}_j \, \Delta V_{t,j}^{\mathsf{CDS}}|_{t=\tau_i}$$

Value on Default: Numerical Results

	VOD in the Markov model	VOD in the Copula model
$[0, 3]$	0.344	1.002
$[3, 6]$	0.138	0.171
$[6, 9]$	0.058	0.023
$[9, 12]$	0.039	0.008
$[12, 22]$	0.107	0.010

Comparison of the Value on Default obtained by the Markov and the Gauss Copula model for deterministic spreads (for a homogeneous portfolio VOD is identical for all firms).

Difference due to contagion-effect: for non-defaulted firm j $\Delta V_{t,j}^{\text{CDS}}|_{t=\tau_i} > 0$, i.e. we make a profit on the protection-buyer positions in the non-defaulted CDS.

5. Risk-minimizing dynamic hedging strategies

Overview. We determine dynamic hedging strategies using one CDS per underlying name as hedging instrument. With spread- and event risk market is incomplete (unless more than one hedging instrument per name available) $\Rightarrow$ use concept of risk minimization.

Risk-minimizing strategies [Föllmer and Sondermann, 1986]. W.l.o.g. let $r \equiv 0$. We seek a representation of the form

$$G_t^{[l,u]} - G_0^{[l,u]} = \sum_{j=1}^{m} \int_0^t \theta_{s,i} dG_{s,i}^{\text{CDS}} + L_t, \;\; 0 \le t \le T, \qquad (2)$$

- L represents the hedge error - such that for each t the remaining risk (conditional error variance) $E_Q\big((L_T - L_t)^2 \mid \mathcal{F}_t\big)$ is minimized.

Risk-minimizing strategies

It is well-known that L in (2) must be orthogonal to the hedging instruments, $\langle L, G^{\mathsf{CDS}}_{\cdot,i}\rangle_t \equiv 0$, $1 \le i \le m$. Hence $\boldsymbol{\theta}_t = (\theta_{t,1}, \dots, \theta_{t,m})$ can be determined from the equations

$$d\langle G^{[l,u]}, G^{\mathsf{CDS}}_j\rangle_t = \sum_{i=1}^{m} \theta_{t,i} d\langle G^{\mathsf{CDS}}_i, G^{\mathsf{CDS}}_j\rangle_t, \quad j = 1, \dots, m. \tag{3}$$

Strategy $\boldsymbol{\theta}_t$ is computed from (3) in a two-step procedure.

- represent $G^{[l,u]}$ and G^{CDS}_j as stochastic integrals with respect to $N_{t,i} := Y_{t,i} - \int_0^{t\wedge\tau_i} \lambda_i(\Psi_s, \mathbf{Y}_s) ds$ and the compensated jump-measure of Ψ and compute the quadratic covariations in (3)
- Solve the linear system (3).

Qualitative Results

- If Ψ is constant (no spread risk) the market is typically complete $L \equiv 0$ and the hedging strategy corresponds to the VOD.
- With both spread- and event risk (stochastic Ψ) market is incomplete.
- For homogeneous portfolios one can also use the underlying index as single hedging instrument; with inhomogeneous models this is no longer true.
- For low spread volatility (state space S_1^Ψ) $\boldsymbol{\theta}$ is close to the VOD; with high spread volatility (state space S_3^Ψ) spread risk becomes more important and $\boldsymbol{\theta}$ is relatively close to the 'Delta'.

Numerical Results

	[0,3]-tranche			[3,6]-tranche			[6,9]-tranche		
	θ	VOD	Δ^+	θ	VOD	Δ^+	θ	VOD	Δ^+
S_0^Ψ	0.344	0.344	-	0.138	0.138	-	0.058	0.058	-
S_1^Ψ	0.348	0.345	0.472	0.138	0.138	0.143	0.057	0.058	0.049
S_5^Ψ	0.414	0.366	0.473	0.136	0.134	0.139	0.050	0.053	0.046
S_{10}^Ψ	0.483	0.432	0.507	0.126	0.123	0.129	0.039	0.043	0.038

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