Systematic Equity-Based Credit Risk: A CEV Model with Jump to Default

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OUTLINE (EQUITY-BASED CREDIT RISK)

- Motivations and results in brief

- Equity price dynamics, equity risk premia, and Equivalent Martingale Measures (EMMs)

- A sensible EMM for hedgers and its implications

- Expected returns on credit-sensitive contracts

- Conclusions

MOTIVATION I

- Equity returns are explained (Vassalou and Xing (JF 2004)) by a default-risk factor from Corporate Bonds (CBs)

- Equity Default Swaps are equity-based Credit Default Swaps (CDSs)

- Assets-value default barriers are based on accounting information and may be ad hoc

OUR MODEL

- Equity is the primitive and default occurs when the stock price falls to zero (no accounting information and no capricious default barrier)

- Equity value either CEV-goes to zero or jumps to zero (Linetsky (MF 2006) and Carr and Linetsky (FS 2006))

- We get equity-based analytical values for CDSs and CBs (Campi and Sbuelz (RL 2005))

MOTIVATION II

- Linetsky (2006) and Carr and Linetsky (2006) do not treat risk premia. Delbaen and Shirakawa (APFM 2001) treat premia for equity volatility elasticities between -1 and 0

- Default risk is systematic (Vassalou and Xing (2004))

- While the objective default intensity $\lambda^{\mathbb{P}}$ is stable, the risk-neutral intensity $\lambda^{\mathbb{Q}}$ peaks at times of bear equity markets

From Berndt, Douglas, Duffie, Ferguson, and Schranz (2005)

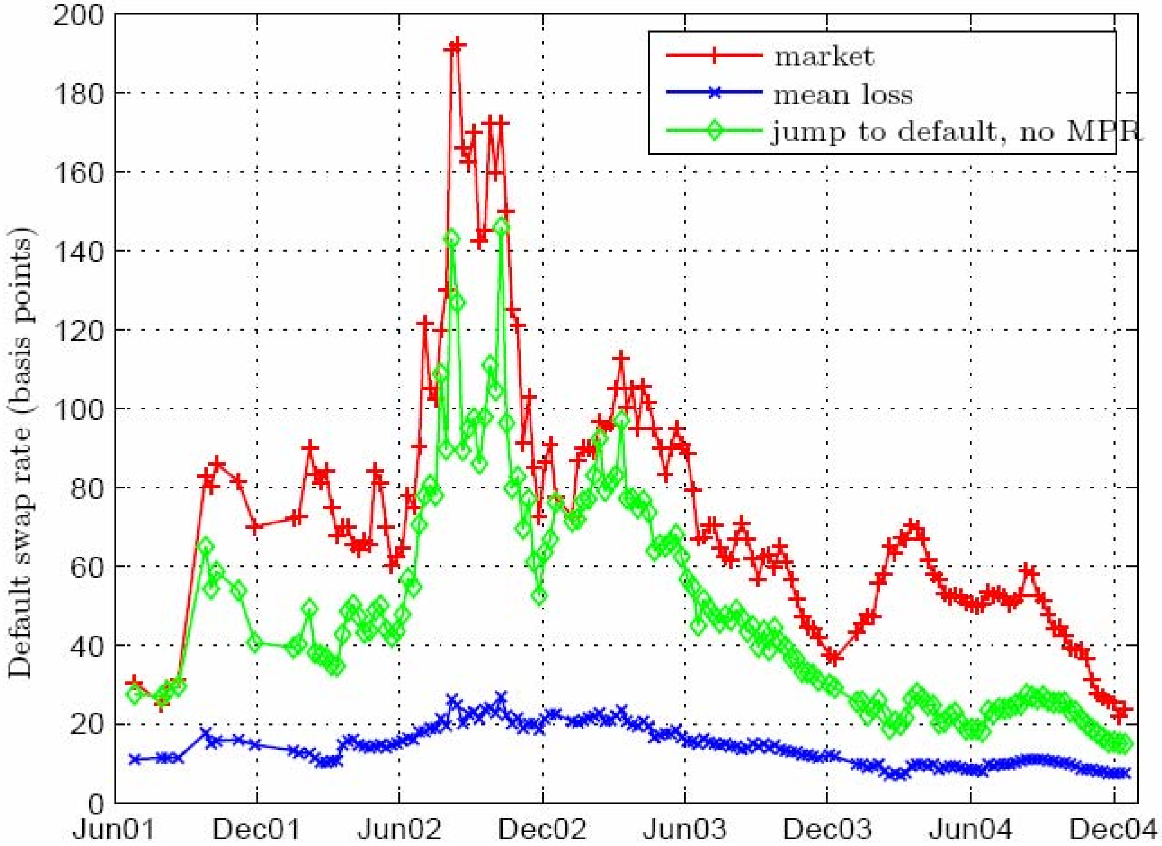

Figure 11: Disney: 5-year actual (market) CDS rates, and modeled CDS rates in the absence of any risk premia (mean loss), and in the absence of non-default mark-to-market risk premia.

OUR MODEL

- Risk premia are treated to show, for any negative elasticity of volatility, the existence of an EMM $\mathbb{Q}$

- Sensible EMMs for hedgers can imply that $\lambda^{\mathbb{Q}}$ rises when equity value dives, even if $\lambda^{\mathbb{P}}$ is constant

- The variance-optimal EMM $\mathbb{Q}^*$ coincides with the minimal EMM (Henderson and Hobson (S:IJPSP 2003), and Schweizer ('GTtQHA' 2001))

$\mathbb{P}$-DYNAMICS OF THE STOCK PRICE PROCESS $\{S\}$

$$\frac{\mathrm{d}S_t}{S_{t-}} + q\mathrm{d}t = \left(r + \theta_t\sigma + \left(E^{\mathbb{P}}\left[e^{\zeta}\right]F_t - 1\right)\lambda_t^{\mathbb{P}}\right)\mathrm{d}t +$$

$$v_t\mathrm{d}z_t^{\mathbb{P}} - \left(\mathrm{d}N_t^{\mathbb{P}} - \lambda_t^{\mathbb{P}}\mathrm{d}t\right),$$

$$v_t = \sigma S_{t-}^{\rho-1}, \qquad \rho - 1 < 0.$$

EQUIVALENT MARTINGALE MEASURES $\mathbb{Q}$ EXIST

- For any $\rho - 1 \in (-\infty, 0)$, the local $\mathbb{P}$-martingale $\{\Pi\}$ with dynamics

$$\frac{\mathrm{d}\Pi_t}{\Pi_{t-}} = -\theta_t \sigma \cdot v_t^{-1} \cdot \mathrm{d}z_t^{\mathbb{P}} + \left(\left(e^{\zeta} F_t - 1\right) \cdot \mathrm{d}N_t^{\mathbb{P}} - \left(E^{\mathbb{P}}\left[e^{\zeta}\right] F_t - 1\right) \lambda_t^{\mathbb{P}} \mathrm{d}t\right)$$

is also a $\mathbb{P}$-martingale and represents the Radon-Nikodym derivative of $\mathbb{Q}$ w.r.t. $\mathbb{P}$, where

$$\left(E^{\mathbb{P}}\left[e^{\zeta}\right] F_t - 1\right) \lambda_t^{\mathbb{P}} = \lambda_t^{\mathbb{Q}} - \lambda_t^{\mathbb{P}}$$

THE VARIANCE-OPTIMAL RISK PRICES

- Up to default, the state-price density corresponding to Π_t is $\pi_t = \exp(-rt)\,\Pi_t$

- The variance-optimal state-price-density process $\{\pi^*\}$ is such that

$$
\begin{aligned}
\theta_t^* \sigma &= \left(\frac{1}{\mathrm{d}t} E_t^{\mathbb{P}} \left[\frac{\mathrm{d}S_t}{S_{t-}} \right] + q - r \right) v_t^2 / \left(v_t^2 + \lambda_t^{\mathbb{P}} \right), \\
e^{\zeta^*} F_t^* &= 1 + \theta_t^* \sigma \cdot \left(v_t^2 \right)^{-1}, \\
\zeta^* &= 0 \quad \mathbb{P}\text{-almost surely.}
\end{aligned}
$$

$\lambda_t^{\mathbb{Q}^*}/\lambda_t^{\mathbb{P}}$ CAN INCREASE WHEN THE EQUITY MARKET GOES DOWN

- Take a constant $\lambda^{\mathbb{P}}$ and a plausibly countercyclical risk premium on equity,

$$\frac{1}{\mathrm{d}t}E_t^{\mathbb{P}}\left[\frac{\mathrm{d}S_t}{S_{t-}}\right]+q-r=\kappa\cdot v_t^2,\quad 0<\kappa<1.$$

- Then, $\lambda_t^{\mathbb{Q}^*}$ is increasing in the diffusive local variance of equity returns:

$$\lambda_t^{\mathbb{Q}^*}=\left(1+\frac{\kappa\cdot v_t^2}{v_t^2+\lambda^{\mathbb{P}}}\right)\lambda^{\mathbb{P}}>\lambda^{\mathbb{P}}.$$

CB'S EXPECTED RETURNS

- We assume that irrespectively of residual maturity, a fraction R of CB's face value Z is recovered at default (Guha and Sbuelz (2005))

- Given semiannual coupons at the rate c, CB's price process is $\{B\}$:

$$E_t^{\mathbb{P}}\left[\frac{\mathrm{d}B_t}{B_{t-}}\right] + \frac{c}{2}\frac{Z}{B_{t-}}\mathbf{1}_{\{t = \text{ COUPON DATE}\}}$$

$$= \left(r + \boldsymbol{\Delta}_t^B \frac{S_{t-}}{B_{t-}}\theta_t\sigma - \left(\frac{RZ - B_{t-}}{B_{t-}}\right)\left(\lambda_t^{\mathbb{Q}} - \lambda_t^{\mathbb{P}}\right)\right)\mathrm{d}t.$$

PUT'S EXPECTED RETURNS

- Consider a European put option on the stock with strike K and maturity T

- The put value process is $\{p\}$:

$$E_t^{\mathbb{P}}\left[\frac{\mathsf{d}p_t}{p_{t-}}\right] \;=\; \left(r + \Delta_t^p \frac{S_{t-}}{p_{t-}}\theta_t\sigma - \left(\frac{e^{-r(T-t)}K - p_{t-}}{p_{t-}}\right)\left(\lambda_t^{\mathbb{Q}} - \lambda_t^{\mathbb{P}}\right)\right)\mathsf{d}t.$$

CONCLUSIONS (EQUITY-BASED CREDIT RISK)

- We use equity as the traded primitive for a detailed analysis of systematic default risk

- Reasonable choices of the pricing kernel can imply that the jump-to-default risk price increases when equity markets dive

- Conjecturally, keenness to be hedged against default risk might be boosted by bear equity markets

EQUITY PRICING WITH CEV-LIKE CASH FLOWS (I)

- The dynamics of the state-price density process $\{\xi\}$ is

$$\frac{d\xi_t}{\xi_t} = -r\mathrm{d}t - \lambda \mathrm{d}Z_t^{\mathbb{P}}.$$

- The dynamics of the cash flow level process $\{D\}$ is

$$\frac{dD_t}{D_t} = \mu \mathrm{d}t + \sigma D_t^{\eta} \mathrm{d}W_t^{\mathbb{P}}, \qquad E\left[dZ_t^{\mathbb{P}} dW_t^{\mathbb{P}}\right] = \rho \mathrm{d}t, \qquad r + \sigma\lambda\rho - \mu > 0.$$

EQUITY PRICING WITH CEV-LIKE CASH FLOWS (II)

- The stock value $S(D)$ is the solution to the following problem:

$$S_D D\mu + \frac{1}{2} S_{DD}\sigma^2 D^{2\eta+2} + D = Sr + S_D \sigma D^{\eta+1}\rho\lambda,$$

$$S(0) = 0, \qquad \lim_{D\to\infty} \frac{S(D)}{D} \leq \frac{1}{r-\mu} < \infty.$$

- Alternatively, one has

$$S(D) = E_0^{\mathbb{P}}\left[\int_0^\infty \frac{\xi_s}{\xi_0} D_s \mathbf{1}_{\{D_s>0\}} \mathrm{d}s\right].$$

ENDOGENOUS DEFAULT WITH CONSTANT RISK

- The perpetual interest payment per unit time to bondholders is c

- A version of the Black & Cox (JF 1976) model can be retrieved

by setting $\eta = 0$:

$$S(D) = \begin{cases} 0 & \text{if } D \leq \underline{D} \\ \frac{D}{r+\sigma\lambda\rho-\mu} - \frac{c}{r} + \left(\frac{c}{r} - \frac{\underline{D}}{r+\sigma\lambda\rho-\mu}\right)\left(\frac{D}{\underline{D}}\right)^{\beta_2} & \text{if } D > \underline{D} \end{cases}$$

$$\underline{D} = \frac{c}{r}\frac{\beta_2}{\beta_2 - 1}(r + \sigma\lambda\rho - \mu),$$

$$\beta_2 = \frac{-\left(\mu - \sigma\lambda\rho - \frac{1}{2}\sigma^2\right) - \left(\left(\mu - \sigma\lambda\rho - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r\right)^{\frac{1}{2}}}{\sigma^2} < 0.$$

CONCLUSION (CEV-LIKE CASH FLOWS)

- The optimal default state equals the business cemetery state if cash flow risk is countercyclical