VALUATION OF CREDIT DEFAULT SWAPS AND SWAPTIONS

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Abstract. This paper presents a conceptual framework for valuation of single-name credit derivatives, and recuperates, in some cases generalizing, a few of known results in credit risk theory. Valuation is viewed with respect to a given state price and relative to a general numeraire. Survival probabilities and default recoveries are considered as processes adapted to a subfiltration, following Jeanblanc and Rutosksy [JR], or, in the special case of Cox processes, Lando [L]. A result of Duffie and Singleton [DS] on pricing bonds with recovery in terms of loss ratio is reproduced. The notion of coadapted change of numeraire is introduced, and its invariants identified and studied. The concept of a credit claim is formalized by introducing notions of T-claims, τ-claims, and $\mathcal{T}$-streams. Application is made to credit default swaps and swaption, and a known Black-Scholes approximation for the latter is derived.

1. Introduction

There is a simple Black-Scholes formula European for credit default swaptions, also known as credit spread options (see [S] and [AG]). The only data needed for pricing is the default-free discount curve, the par CDS curve, the recovery ratio, and a single "forward spread volatility" parameter. Should this market become liquid, the implied Black-Scholes volatility from this formula could provide a standard for market quotation of these products.

A pedagogical purpose of this paper is to examine more closely the underlying assumptions and structure that give rise to this formula. Another aim is to use this simple application as a platform to illustrate, hopefully in a clear and conceptual exposition, a few of ideas and results of the well-developed and growing theory of credit risk valuation, and add some observations of our own.

The probabilistic results of this paper are for the most part well known in one form or another. What distinguishes our approach is

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Key words and phrases. Credit default swap, credit spread option, state-price density, numeraire martingale measure, stopping time, subfiltration, conditional independence, coadapted numeraires.

the financial model framework within which they are developed from first principles. The prevalent approach takes as numeraire the continuous money market price β_t of the form $\exp(\int_0^t r_s ds)$. Likewise, it assumes an absolutely continuous (conditional) survival probability of the form $\alpha_t = \exp(-\int_0^t \lambda_s ds)$, where λ_t is the hazard rate. We lift these restrictions, and allow both to be general semimartingales.

The advantage of working with a general numeraire β_t is that the framework is numeraire-invariant upfront, enabling seamless change of numeraire later. Change of numeraire has proved beneficial in many financial applications, especially in valuation of options and swaptions. As we change numeraire, the survival probability with respect to the new numeraire measure will in general change too. As such, we should allow general survival probabilities α_t. However, the main significance of working with a general α_t probably lies in connection with multi-name credit risk theory. This paper is confined to a single credit event.

Credit derivative structures, even single-name ones, are complex enough to warrant probabilistic formalization of their terms. We attempt this through an extensive nomenclature of terms and notation for various structures, culminating in the definition of a credit default swaption. Along the way we introduce notions as $\mathcal{H}_T$-claims, τ-claims, and survival and recovery $\mathcal{T}$-streams, and encounter such arcane symbolism as $A^{\delta^\alpha_{\mathcal{T}}}$, $B^{T,L^\alpha_{\mathcal{T}}}$, and $(B^{T,L_{\mathcal{T}}}_{t^*} - KA^{\delta_{\mathcal{T}}}_{t^*})^+\beta/\beta_{t^*}$. All this may seem to put undue burden on presentation. Nevertheless, we hope that in balance it is a worthwhile contribution to mathematical formalization of credit derivative contracts, which, if not legally binding, at least enriches the financial content and market applicability of the abstract theory.

These and other lengthy terminology like ”state price density, numeraire-relative, default valuation model $(\mathcal{M}, \beta, \tau, \mathcal{H}_t)$” or ”$\mathcal{H}_t$-coadapted change of numeraire” are exercises in building financial context around probabilistic constructs with an eye toward application to CDS valuation. They give the paper an applied orientation towards a CDS swaption pricing formula and a Black-Scholes approximation.

Credit derivatives pricing is theoretically most interesting when there is a correlation between default and the numeraire. The case where the two are independent under the numeraire measure is quite easy by comparison. The difference is essentially one between assuming a deterministic versus a stochastic survival probability α_t. The latter notion is elaborated in the literature in different ways.

Duffie and Singleton [DS] concentrate on the intensity of the default-time τ, whose integral is the compensator of $1_{\tau \leq t}$, a quantity intrinsic to τ and the given filtration $\mathcal{F}_t$. As this $1_{\tau \leq t}$-martingale intensity drops to zero after default, they take as given an essentially arbitrary positive extension h_t of it past default, with respect to which stochastic survival probability α_t may be defined as $\exp(-\int_0^t h_s ds)$.

Other approaches focus attention on an auxiliary subfiltration $\mathcal{H}_t$ of the original filtration $\mathcal{F}_t$ which explains default but does not predict it exactly . One is based on Cox processes, as in Lando [L]. There, as in [DS], a stochastic hazard rate (extended positive intensity) is given, and α_t defined as before. Further, the hazard rate (hence α_t) is assumed $\mathcal{H}_t$-adapted, and default time τ is constructed to occur at the first time α_t reaches a draw from an $\mathcal{H}$-independent uniform sample.

Jeanblanc and Rutkowski [JR], which we follow closely, take τ as given like [DS], and $\mathcal{H}_t$ as given like [L]. But, unlike them, they do not take α_t as given. Rather, they construct α_t as the conditional probability $P[\tau > t \,|\, \mathcal{H}_t]$ of survival given $\mathcal{H}_t$. This generalizes the Cox setting of [L]. Notions of complementary and conditionally independent subfiltrations are employed to further analyze credit risk. These conditions are satisfied in the setting of [L].

All three approaches provide an extension of credit derivative prices past default. An advantage of the subfiltration $\mathcal{H}_t$ approach is that it provides explicit expressions for these extensions as conditional expectations with respect to $\mathcal{H}_t$. As all such extensions are $\mathcal{H}_t$-adapted, $\mathcal{H}_t$ is best chosen as a "large" subfiltration.

A numeraire independent framework is properly built upon the notion of the state price density ξ_t. It is an intuitively appealing concept and provides a rich financial context. Here, a contingent claim C is characterized by its deflated price $\xi_t C_t$ being a martingale. This enables valuation by taking expectation: $C_t = E_t[\xi_T C_T]/\xi_t$, for $t \le T$.

A numeraire β is in principle any positive claim. In practice, its price β_t is deemed observable, whereas ξ_t may not be so. As such, it makes sense to price claims relative to a numeraire. For this purpose, using ξ_t and β_t, a numeraire measure is constructed under which relative claim prices C_t/β_t are martingale. The numeraire measure may be unobservable, but still, some observability is incorporated by the numeraire itself.

From purely mathematical point of view, the theory collapses to the special case $\xi_t = \beta_t = 1$. This special case practically strips the theory of its financial content and reduces it raw-bone to probability theory. While interesting on its own right, for financial applications such as ours, it is inappropriate to assume $\xi_t = \beta_t = 1$, as we would then be ignoring market trends relevant to derivatives valuation.

Change of numeraire in presence of default has also been studied in Schönbucher[S]. That approach is different than ours, as the numeraire there drops to zero after default. Accordingly, the "default swap measure" is not an equivalent measure, albeit absolutely continuous. Relative prices are only defined on paths where default has not occurred. As such, the forward credit spread S_t is not defined globally. Here, we allow only strictly positive numeraires. This results in a spread S_t extended globally to paths where default has occurred. In fact S_t will be a $\mathcal{H}_t$-adapted martingale under a numeraire measure P^A, and the time 0 swaption price will be $A_0 E^A(S_t - K)^+$.

The next section introduces and discusses the proposed framework. Section 3 contains basic valuation results from [L] and their generalizations in [JR], adapted to the present setup. It also derives and generalizes two versions of a result in [DS], pricing a bond with recovery as a bond without recovery but enhanced survival probability, in terms of recovery and loss ratios. Section 4 is devoted to change of numeraire. It introduces the notion of $\mathcal{H}_t$-coadapted numeraires, studies the invariants of this equivalence relation, and points out its connection with conditional independence.

The final section applies the results to credit default swap and swaptions, establishing pricing formulae in a general numeraire and a simplified option pricing formula in a swap numeraire. The latter can be approximated by a Black-Scholes formula, which is also derived in [S] and reported in the recent book of Arvanitis and Gregory [AG].

For the sake of clarity and continuity, all proofs are provided, even when they may be standard, occasionally in footnotes as reminders.

2. A RELATIVE VALUATION FRAMEWORK FOR CREDIT DERIVATIVES

This section introduces notation and terminology and sets up the model structure. It contains no results. Instead, there is a discussion of the reason for the chosen framework and its difference with more common approaches.

For the purposes of this paper, we introduce definitions throughout in italic *Words* with first letters capitalized.

2.1. **State price density valuation model.**

A *State Price Density Valuation Model* $\mathcal{M}$ is a 6-tuple $\mathcal{M} = (\overline{T}, \Omega, \mathcal{F}, \mathcal{F}_t, P, \xi_t)$, where

$0 < \overline{T} < \infty$ is the terminal date;

$(\Omega, \mathcal{F}, \mathcal{F}_t, P)_{(0 \leq t \leq \overline{T})}$ is a filtered probability space satisfying the usual conditions, with $\mathcal{F} = \mathcal{F}_{\overline{T}}$ and $\mathcal{F}_0$ equal the trivial subalgebra of events of probability 0 or 1;

ξ_t is a a.s. positive, integrable semimartingale with $\xi_0 = 1$ a.e.

In what follows t, s, and T will denote times in $[0, \overline{T}]$.

We denote $E_t[\cdot] := E[\cdot \,|\, \mathcal{F}_t]$.

In this model, P represents the *actual* probability measure, variously also referred to as the subjective, later, objective, historical, empirical, or physical probability measure.

$\mathcal{F}_t$ represents all information up to and including time t.

The process ξ_t is the so-called *state price density.* For each time t and state ω we can think of $\xi_t(\omega)$ as the price at time 0 of a claim that pays, at time t, $1/P(d\omega)$ at state $\omega \in d\omega$ and 0 elsewhere ($d\omega \in \mathcal{F}_t$). More precisely, the price at time 0 of a claim that pays 1 at time-t if an event $E \in \mathcal{F}_t$ occurs is thought to equal $\int_E \xi_t dP = E[\xi_t 1_E]$. As such, $E[\xi_t]$ is none other than the price at time 0 of a t-maturity zero-coupon bond. More generally, the price at time 0 of a time-t payoff C_t is considered to be $E[C_t\xi_t]$.

The state price density is also used to price claims at time $t > 0$. Namely, it postulates that for $t < T$, $\xi_T(\omega)/\xi_t(\omega)$ equals the price at time t of a claim that at time T pays $1/P(d\omega)$ at state $\omega \in d\omega$ and 0 elsewhere ($d\omega \in \mathcal{F}_T$). More precisely, the price at time t of a claim that pays 1 at time T if an event $E \in \mathcal{F}_T$ occurs equals $E_t[\xi_T 1_E]/\xi_t$. More generally, the price at time t of a T-payoff C_T is defined to be $E_t[C_T\xi_T/\xi_t]$.

We summarize the above definition of claim's price as follows:

A process C_t is the price of a claim if and only if $\xi_t C_t$ is a martingale.

The state price density ξ_t is also viewed as a *deflator*. The condition that ξ_t be a supermartingale (on the average decreasing) is viewed as being equivalent to positive interest rates. In fact, the price of a T-maturity zero-coupon is by definition $E_t[\xi_T/\xi_t]$, which is less than or equal to one if and only if ξ_t is supermartingale. This assumption is however not necessary and not made in what follows. Nevertheless, it is useful to think of ξ_t as a deflator, and of $\xi_t C_t$ as the deflated price of the claim. The basic tenet of valuation therefore postulates that *deflated claim prices are martingales.*

2.2. Money market numeraire and the risk-neutral measure.

Instead of using the state-price density, an equivalent, but different and more common approach to modelling is in terms of a P-equivalent measure Q called the *risk-neutral measure* and a finite variation, $\mathcal{F}_t$-adapted process $\beta_t > 0$ called the *(continuous) money market* numeraire. Further β_t is often assumed to be absolutely continuous, hence of the form $\beta_t = \exp(\int_0^t r_s ds)$, for some process r_t representing the *instantaneous interest rate.*

In this approach, claim prices C_t are defined by: *A process C_t is the price of a claim if and only if C_t/β_t is a Q-martingale.*

Such an approach can be mapped to the state-price-density approach and vice versa. Specifically, we can associate with such a pair (Q, β_t) the state price density $\xi_t^{Q,\beta} := E_t[dQ/dP]/\beta_t$. Conversely, given ξ_t, we can always find a finite variation process $\beta_t^\xi > 0$ such that $\xi_t\beta_t^\xi$ is a martingale. Further, if ξ_t is a special semimartingale with $\xi_{t-} > 0$, there exists a unique predictable, finite variation process such that $\xi_t\beta_t^\xi$ is a positive martingale. This $\beta_t^\xi > 0$ is the money market numeraire associated with ξ_t, and the associated risk-neutral measure Q^ξ is defined by $dQ^\xi/dP = \xi_{\overline{T}}\beta_{\overline{T}}$. These associations are inverses of each other.

We **conjecture** that in this case, a right-continuous version of the limit below exists and equals β_t^ξ:

$$\beta_t^\xi = \lim_{N\to\infty} \prod_{n=1}^{N} \frac{\xi_{t\frac{n-1}{N}}}{E[\xi_{t\frac{n}{N}} \,|\, \mathcal{F}_{t\frac{n-1}{N}}]}.$$

The financial interpretation of this conjecture is that the discrete, simple compounded money-market numeraire (also referred to as the "spot-Libor numeraire"), which in general has infinite variation (and positive quadratic variation), converges to the finite-variation continuous money market numeraire.

The significance of β_t^ξ (or r_t) is that it is deemed an observable process. The unobservability of ξ_t is absorbed into the risk-neutral measure Q^ξ. Both β_t^ξ and Q^ξ are part of ξ_t: β_t is an observable part of ξ_t and Q^ξ contains any unobservable part. So, in effect by switching to valuation relative to β_t^ξ, one incorporates some observability into valuation.

2.3. **Numeraire relative valuation model.**

Relative to the chosen state price density ξ_t, a *numeraire* can be viewed as the price of a positive claim, that is, as any process $\beta_t > 0$ such that deflated price $\xi_t\beta_t$ is a martingale. However, we find it more convenient here to define:

A *Numeraire* $\beta \in \mathcal{F}$ is a $\mathcal{F}$-measurable random variable such that $\beta > 0$ a.e. and $\xi\beta$ is integrable, where $\xi := \xi_{\overline{T}}$.

The *Price* of numeraire β is the unique process $\beta_t > 0$ such that $\xi_t\beta_t$ is a martingale, namely, $\beta_t := E_t[\xi\beta]/\xi_t$. Note, $\beta_{\overline{T}} = \beta$. Conversely, given such process $\beta_t > 0$, its terminal value $\beta_{\overline{T}}$ defines a numeraire whose price is β_t.

A *(State Price Density) Numeraire Relative Valuation Model* is a pair $\mathcal{N} = (\mathcal{M}, \beta)$, where $\mathcal{M}$ is a valuation model and β is numeraire.

The reason for introducing a numeraire is that its price β_t may be both observable and relevant to the claim at hand. If so, we can decompose $\xi_t = (\xi_t\beta_t)/\beta_t$ into the observable inverse price $1/\beta_t$ and the martingale $\xi_t\beta_t$ which contains any unobservable parts of ξ_t. In this way, a numeraire reduces the unobservability of ξ_t.

The *Numeraire Measure* associated to relative valuation model $(\mathcal{M}, \beta)$ is the equivalent measure $Q^{\xi,\beta}$ whose density $dQ^{\xi,\beta}/dP := \xi\beta/\beta_0$ is the normalized terminal value $\xi\beta$ (and therefore, whose martingale density is $\xi_t\beta_t/\beta_0$.) As is well-known, for any claim C, its *numeraire-relative price* C_t/β_t is a $Q^{\xi,\beta}$-martingale.

The simplest example of a numeraire is $1/\xi$. Note, its price process is the inflator $1/\xi_t$. Also note, $Q^{\xi,1/\xi} = P$.

Given a sufficiently well-behaved ξ_t (a special semimartingale), we have already visited another example, namely, the money market numeraire $\beta^\xi := \beta^\xi_{\overline{T}}$. Its price process is the the money market price β_t^ξ, because, by its definition, $\beta_t^\xi \xi_t$ is the martingale. In this case, clearly, $Q^{\xi,\beta^\xi} = Q^\xi$.

But, the money market numeraire β_t^ξ is not the only market observable price. There are other observable numeraires, such as bonds, that may be more suitable for the valuation at hand. In our view, a general purpose relative valuation model should admit an arbitrary numeraire for relative valuation. In other words, it should be numeraire independent. This is not the case in most approaches, as they regard relative valuation only with respect to the continuous money market numeraire.

A different but equivalent approach to ours here, which is also less common and does not sacrifice generality, would be to develop relative valuation in terms of a given pair (Q, β_t), where Q is an equivalent measure and $\beta_t > 0$ is a general semimartingale viewed as the numeraire. This is similar but more general than the money market approach, as β_t is not required to have finite variation. Then, if desired, a consistent state price density ξ_t can be defined as $\xi_t = E_t[dQ/dP]/\beta_t$. However, we prefer to take as given ξ_t and β_t, and derive the equivalent measure Q as the numeraire measure $Q^{\xi,\beta}$, because the concept of state-price density appears a more intuitive financial first principle than that of an equivalent measure Q.

2.4. **Numeraire-relative default valuation model.** A *(Single-Name) Default Valuation Model* is a triple $(\mathcal{M}, \tau, \mathcal{H}_t)$, where

$\mathcal{M} = (\overline{T}, \Omega, \mathcal{F}, \mathcal{F}_t, P, \xi_t)$ is a valuation model,

τ is an $\mathcal{F}_t$ stopping time (*default time*),

$\mathcal{H}_t$ is a sub-filtration of $\mathcal{F}_t$, satisfying the usual conditions ($\mathcal{H}_0 = \mathcal{F}_0$).

One thinks of the subfiltration $\mathcal{H}_t$ as an auxiliary information set of variables that explain the likelihood (intensity of) default, but cannot exactly predict it.

Set $\mathcal{H} := \mathcal{H}_{\overline{T}}$.

The *conditional survival probabilities* α_t and $\overline{\alpha}_t$ of default time τ are

$$\alpha_t := P[\tau > t \,|\, \mathcal{H}_t];$$

$$\overline{\alpha}_t := P[\tau > t \,|\, \mathcal{H}].$$

Note, $0 \le \alpha_t \le 1$, and α_t is $\mathcal{H}_t$-adapted.

Note, $0 \le \overline{\alpha}_t \le 1$, and $\overline{\alpha}_t$ decreasing.

Note, if $\overline{\alpha}_t$ is deterministic, then $\alpha_t = \overline{\alpha}_t$ and τ is independent of $\mathcal{H}$.[1]

Note, if α_t is deterministic and strictly deceasing, then, for $x \in [\alpha_{\overline{T}}, 1]$, $P[\alpha_\tau < x] = x$.[2]

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[1] For bounded $H \in \mathcal{H}$, $E[\tau > t \,|\, H] = E[E[\tau > t \,|\, \mathcal{H}]H] = E[\tau > t \,|\, \mathcal{H}]E[H \,|\, \mathcal{H}]$.

[2] $P[\alpha_\tau < x] = P[\tau > \alpha^{-1}(x)] = E[P[\tau > \alpha^{-1}(x) \,|\, \mathcal{H}_{\alpha^{-1}(x)}]] = E[\alpha(\alpha^{-1}(x)) \,|\, \mathcal{H}_{\alpha^{-1}(x)}] = 1$.

A *(Numeraire)-Relative Default Valuation Model* is a 4-tuple $(\mathcal{M}, \beta, \tau, \mathcal{H}_t)$, where $(\mathcal{M}, \beta)$ is a numeraire-relative valuation model, and $(\mathcal{M}, \tau, \mathcal{H}_t)$ is a default valuation model.

3. Valuation of credit derivatives

Fix a default valuation model $(\mathcal{M}, \tau, \mathcal{H}_t)$.

For any stochastic process X_t, set $X := X_{\overline{T}}$.

3.1. **Claims, T-claims, their prices and $\mathcal{H}_t$-prices.**

A *Claim* C is a $\mathcal{F}$-measurable random variable $C \in \mathcal{F}$ such that ξC is integrable. We write $C \in \mathcal{C}$.

Define the *Price* and *$\mathcal{H}_t$-Price* of a claim C by (with $\xi := \xi_{\overline{T}}$)

$$C_t := \mathcal{V}_t^C := E_t[\xi C]/\xi_t,$$

$$V_t^C := E[\xi C \,|\, \mathcal{H}_t]/\xi_t.$$

Note, $\xi_t C_t$ is a $\mathcal{F}_t$-martingale, and $\xi_t V_t^C$ is a $\mathcal{H}_t$-martingale.

Note, if ξ_t is $\mathcal{H}_t$-adapted, then $V_t^C = E[C_t \,|\, \mathcal{H}_t]$.

We can alternatively, but equivalently, view a claim as a $\mathcal{F}_t$-adapted process C_t such that $\xi_t C_t$ is a $\mathcal{F}_t$-martingale. The corresponding claim is then the terminal value $C_{\overline{T}}$. Note, then $\mathcal{V}_t^{C_{\overline{T}}} = C_t$.[3].

A claim C *Does not Price non-$\mathcal{H}_t$-Risk* if $\xi_t C_t$ is $\mathcal{H}_t$-adapted.

Proposition 3.1. *A claim C does not price non-$\mathcal{H}_t$-risk if and only if $V_t^C = C_t$ for all t.*

Proof. If $\xi_t C_t$ is $\mathcal{H}_t$ adapted, then, iterating expectation, $\xi_t C_t$ is an $\mathcal{H}_t$ martingale. But, so is $\xi_t V_t^C$, and both martingales have the same terminal value ξC. So, they are equal. Therefore $C_t = V_t^C$. The converse is obvious because $\xi_t V_t^C$ is $\mathcal{H}_t$-adapted by definition.

□

A *T-Claim* is a claim C such that $C\xi \in \mathcal{F}_T$. We write $C \in \mathcal{C}_T$.

Note then, $C_T = C\xi/\xi_T$. We can think of C_T as the payoff at time T of the T-claim C. The claim C itself is then viewed as the *inflated* value of this payoff at the terminal date.

Let C be a T-claim. Then, note, C is a s-claim for $s > T$, $C_s = C\xi/\xi_s$ for all $s \geq T$, and $C_t = E_t[\xi_T C_T]/\xi_t$ for all t (not just for $t \leq T$).

T-claims are constructed as follows. For any $F_T \in \mathcal{F}$ such that $F_T \xi_T$ is integrable, $C^{F_T} := F_T \xi_T/\xi$ is a T-claim. Note, $C_T^{F_T} = F_T$ and $C_t^{F_T} = E_t[F_T \xi_T]/\xi_t$.

T-maturity zero-coupon bond is the T-claim $D^T := \xi_T/\xi$. Note, $D_T^T = 1$ and $D_t^T := E_t[\xi_T]/\xi_t$.

T-maturity survival zero-coupon bond is the T-claim $D^{1_{\tau>T}} := 1_{\tau>T}\xi_T/\xi$. Note, $D_T^{1_{\tau>T}} = 1_{\tau>T}$, and $D_t^{1_{\tau>T}} := E_t[1_{\tau>T}\xi_T]/\xi_t$.

We will later present more complex examples of claims, such as survival streams, default protection streams, and credit swaptions.

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[3] Because the two martingales have same terminal value

3.2. **$\mathcal{H}$-claims, $\mathcal{H}_T$-claims, and τ-claims.** A *$\mathcal{H}$-Claim* is a claim C such that $C\xi \in \mathcal{H}$.[4] We write $C \in \mathcal{C}_{\mathcal{H}}$.

Note, for any claim C, $V^C = E[\xi C \,|\, \mathcal{H}]/\xi$. So, C is a $\mathcal{H}$-claim if and only if $V^C = C$.

Any claim C that does not price not $\mathcal{H}_t$ risk is obviously an $\mathcal{H}$-claim. We later study special case models where the converse is also true.

A *$\mathcal{H}_T$-Claim* is a claim C such that $C\xi \in \mathcal{H}_T$. We write $C \in \mathcal{C}_{\mathcal{H}_T}$.

Note, if C is an $\mathcal{H}_T$ claim then $V_T^C = C\xi/\xi_T = C_T$. In fact, $V_s^C = C_s$ for all $s \geq T$.

Proposition 3.2. *For any $\mathcal{H}$-claim C , $V_t^{1_{\tau>T}C} = V_t^{\overline{\alpha}_T C}$ for all t, T.*

For any $\mathcal{H}_T$-claim C, in addition, $V_t^{1_{\tau>T}C} = V_t^{\alpha_T C}$ for all t, T.

Proof. For $C \in \mathcal{C}_{\mathcal{H}}$,

$$\xi_t V_t^{1_{\tau>T}C} = E[1_{\tau>T}\xi C \,|\mathcal{H}_t] = E[1_{\tau>T}\xi C \,|\, \mathcal{H} \,|\mathcal{H}_t] = E[E[1_{\tau>T} \,|\, \mathcal{H}]\xi C \,|\mathcal{H}_t] = \xi_t V_t^{\overline{\alpha}_T C}.$$

Similarly, for $C \in \mathcal{C}_{\mathcal{H}_T}$,

$$\xi_t V_t^{1_{\tau>T}C} = E[1_{\tau>T}\xi C \,|\mathcal{H}_t] = E[1_{\tau>T}\xi C \,|\, \mathcal{H}_T \,|\mathcal{H}_t] = E[E[1_{\tau>T} \,|\, \mathcal{H}_T]\xi C \,|\mathcal{H}_t] = \xi_t V_t^{\alpha_T C}.$$

□

A τ-Claim is a claim C such that ξC is $\mathcal{F}_\tau$-measurable, where, as usual, $\mathcal{F}_\tau := \{E \in \mathcal{F} : E \cap \{\tau \leq t\} \in \mathcal{F}_t, \; \forall\, t \leq \overline{T}\}$.

Equivalently, a τ-claim is a claim such that $C_{\tau\wedge\overline{T}} = \xi C/\xi_{\tau\wedge\overline{T}}$.

So, a τ-claim C satisfies $C_{\tau\wedge\overline{T}} \in \mathcal{F}_\tau$.

For $\tau < \overline{T}$, we can think of C_τ as the *recovery value* of the τ-claim at the time of default. When $C_{\tau_-} > 0$, the random variable $C_\tau/C_{\tau_-} \in \mathcal{F}_\tau$ is the *pre-default recovery ratio.*

For a τ-claim, the optional stopping theorem implies $C_{\tau\wedge t} = \xi_t C_t/\xi_{\tau\wedge t}$.

In other words, for $\tau \leq t$, $\xi_\tau C_\tau = \xi_t C_t$.

τ-claims are constructed as follows. Let R_t (the *recovery process*) be a bounded $\mathcal{F}_t$-adapted optional process (e.g., left or right continuous.) Then, $C = R_{\tau\wedge\overline{T}}\xi_{\tau\wedge\overline{T}}/\xi$ is a τ-claim, if it is a claim. Note, $C_{\tau\wedge\overline{T}} = R_{\tau\wedge\overline{T}}$.

3.3. **Valuation of survival T-claims and recovery τ-claims.** Following [JR], we say $\mathcal{H}_t$ *is a complementary subfiltration of* $\mathcal{F}_t$ *with respect to* τ (or $\mathcal{H}_t$ is $(\tau, \mathcal{F}_t)$ complementary) if

(i) $\alpha_t > 0$ a.s., all t.

(ii) $\mathcal{F}_t = \sigma(1_{\tau>s}H_t, s \leq t, H_t \in \mathcal{H}_t)$, i.e., $\mathcal{F}_t$ is generated by the variates $1_{\tau>s}H_t$, or, equivalently, $\mathcal{F}_t = \mathcal{H}_t \vee \mathcal{F}_t^\tau$ for all t, where $\mathcal{F}_t^\tau$ is the filtration generated by τ (i.e., by events $\{\tau > s\}$, $s \leq t$).[5]

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[4] We emphasize that an $\mathcal{H}$-claim is one whose *deflated* value is $\mathcal{H}$-measurable, not (necessarily) one which is $\mathcal{H}$ measurable itself.

[5] In the results that follow, condition (ii) can be weakened to $\mathcal{F} = \mathcal{H} \vee \mathcal{F}^\tau \vee \mathcal{G}$, and $\mathcal{F}_t \subset \mathcal{H}_t \vee \mathcal{F}_t^\tau \vee \mathcal{G}$, for some σ-algebra $\mathcal{G}$ independent of $\mathcal{H} \vee \mathcal{F}^\tau$ (or more generally, conditionally independent of $\mathcal{F}^\tau$ given $\mathcal{H}$).

Note that, iterating expectation, $P[\tau > t] = E[\alpha_t] > 0$. Hence (i) implies $P[\tau > t] > 0$ for all t. Similarly, (i) implies , $\overline{\alpha}_t = E[\alpha_t \,|\, \mathcal{H}] > 0$ a.s. Since $1_{\tau>0} = \alpha_0$, (i) also implies $\tau > 0$ a.e.

Condition (i) also implies that τ is *not* an $\mathcal{H}_t$ stopping time, unless $\tau > \overline{T}$ a.e. Indeed, if $1_{\tau>t}$ is $\mathcal{H}_t$ measurable for some $t > 0$, then $\alpha_t = 1_{\tau>t}$. So, if (i) holds, it follows $\tau > t$ a.e.

So, condition (i) in effect says that $\mathcal{H}_t$ does not predict τ.

One of the main results of single-name credit risk theory can be phrased as stating the following and its corollary.

Theorem 3.3. *Suppose $\mathcal{H}_t$ is $(\tau, \mathcal{F}_t)$-complementary. Then, for $t < T$ and any claim C*

$$\mathcal{V}_t^{1_{\tau>T}C} = \frac{1_{\tau>t}}{\alpha_t} V_t^{1_{\tau>T}C}. \qquad (t < T,\ C \in \mathcal{C}) \tag{3.1}$$

Proof. This is an immediate consequence of the fact that under assumptions (i)-(ii), for any integrable $F \in \mathcal{F}$, we have

$$E[1_{\tau>T}F \,|\, \mathcal{F}_t] = \frac{1_{\tau>t}}{\alpha_t}\, E[1_{\tau>T}F \,|\, \mathcal{H}_t].$$

This is shown in [JR]. A different way to see it is as follows.

Let X denote the RHS. By the definition of conditional expectation, we need to show that $E[XF_t] = E[1_{\tau>T}FF_t]$ for all bounded $F_t \in \mathcal{F}_t$. Because of the complementary assumption, it suffices to check this when $F_t = 1_{\tau>s}H_t$, $s \leq t$, $\mathcal{H}_t \in \mathcal{H}_t$. Since $1_{\tau>T}1_{\tau>s} = 1_{\tau>T}$, we have, $E[1_{\tau>T}FF_t] = E[1_{\tau>T}FH_t]$. But, since $1_{\tau>s}1_{\tau>t} = 1_{\tau>t}$ and $E[X \,|\, \mathcal{H}_t] = E[1_{\tau>T}F \,|\, \mathcal{H}_t]$, iterating expectation, we also have

$$E[XF_t] = E[XH_t] = E[E[X\,|\mathcal{H}_t]H_t] = E[E[1_{\tau>T}F\,|\mathcal{H}_t]H_t] = E[1_{\tau>T}FH_t].$$

□

Remark 3.4. From this result it follows that $1_{\tau>t}/\alpha_t$ is a martingale.[6]

Corollary 3.5. *Suppose $\mathcal{H}_t$ is $(\tau, \mathcal{F}_t)$-complementary. Then,*

(i) For any $\mathcal{H}$-claim C, and $t < T$,

$$\mathcal{V}_t^{1_{\tau>T}C} = \frac{1_{\tau>t}}{\alpha_t} V_t^{\overline{\alpha}_T C}. \qquad (t < T,\ C \in \mathcal{C}_{\mathcal{H}}) \tag{3.2}$$

(ii) For any $\mathcal{H}_T$-claim C ,

$$\mathcal{V}_t^{1_{\tau>T}C} = \frac{1_{\tau>t}}{\alpha_t} V_t^{\alpha_T C}. \qquad (\forall t,\ C \in \mathcal{C}_{\mathcal{H}_T}) \tag{3.3}$$

(iii) For any $\mathcal{H}_s$-claim C and $s < T$

$$\mathcal{V}_t^{1_{s<\tau\leq T}C} = \frac{1_{\tau>t}}{\alpha_t} V_t^{(\alpha_s-\alpha_T)C}. \qquad (\forall t,\ s < T,\ C \in \mathcal{C}_{\mathcal{H}_s}) \tag{3.4}$$

---

[6] Indeed, by the formula in the proof, $E_t[1_{\tau>T}/\alpha_T] = 1_{\tau>t}E[1_{\tau>T}/\alpha_T \,|\, \mathcal{H}_t]/\alpha_t$. But, iterating expectation given $\mathcal{H}_T$, we have $E[1_{\tau>T}/\alpha_T \,|\, \mathcal{H}_t] = 1$.

Proof. (i) and (ii) are a direct consequence of the theorem and Proposition 3.2. (iii) follows from (ii), using, $1_{s<\tau\leq T} = 1_{\tau>s} - 1_{\tau>T}$. $\square$

This result effectively reduces valuation of $\mathcal{V}_t^{1_{\tau>T}C}$ to valuation of $V_t^{\alpha_T C}$ when $C \in \mathcal{C}_{\mathcal{H}_T}$. But, this is simply the valuation of a claim without explicit dependence on default time, because, $\alpha_T \in \mathcal{H}_T$.

When $\xi_t = \exp(-\int_0^t r_s ds)$ and $\alpha_t = \exp(-\int_0^t \lambda_s ds)$, Eq. (3.3) is none other than a formula in [L] stating

$$E_t[1_{\tau>T} e^{-\int_t^T r_s ds} C_T] = 1_{\tau>t} E[e^{-\int_t^T (r_s+\lambda_s)ds} C_T \mid \mathcal{H}_t].$$

Another main result of credit risk theory is valuation of credit claims with recovery. Eq. (3.4) for s-claims can be used to derive a pricing formula for τ-claims, in a manner much resembling that in [JR].

Theorem 3.6. *Suppose $\mathcal{H}_t$ is $(\tau, \mathcal{F}_t)$-complementary and α_t is a $\mathcal{H}_t$-semimartingale. Let R_t be a bounded predictable process such that $R_t\xi_{t-}$ is $\mathcal{H}_t$-adapted and $\int_s^T R_t \xi_{t-} d\alpha(t)$ is integrable. Then,*

$$\mathcal{V}_t^{1_{s<\tau\leq T} R_\tau \xi_\tau/\xi} = \frac{1_{\tau>t}}{\alpha_t} V_t^{-\int_s^T R_u \xi_{u-} d\alpha(u)/\xi}. \qquad (t \leq s < T) \tag{3.5}$$

Proof. Set $t_n = s + n(T-s)/N$, $n = 0, \dots, N$.

Since $R_{t_n}\xi_{t_n}/\xi$ is a t_n-claim, using Eq.(4.4) and linearity, we have

$$\mathcal{V}_t^{\sum_{n=0}^{N-1} 1_{t_n<\tau\leq t_{n+1}} R_{t_n}\xi_{t_n}/\xi} = \frac{1_{\tau>t}}{\alpha_t} V_t^{\sum_{n=0}^{N-1}(\alpha_{t_n}-\alpha_{t_{n+1}}) R_{t_n}\xi_{t_n}/\xi}.$$

But, as N gets large, the LHS converges to the LHS of Eq. (3.5) and the RHS converges to the RHS of Eq. (3.5) (uniformly in probability). $\square$

When $\xi_t = \exp(-\int_0^t r_s ds)$ and $\alpha_t = \exp(-\int_0^t \lambda_s ds)$, the above equation for a τ-claim is none other than a formula in [L] stating

$$E_t[1_{s<\tau\leq T} e^{-\int_t^\tau r_s ds} R_\tau] = 1_{\tau>t} E[\int_s^T e^{-\int_t^s (r_u+\lambda_u)du} \lambda_s R_s ds \mid \mathcal{H}_t].$$

Combining the previous results, we obtain pricing for claims, such as defaultable bonds, that have a promised survival payoff as well as a recovery in case of default, as discussed in [DS].

Corollary 3.7. *Assume the previous theorem setup. Then, for any $\mathcal{H}_T$-claim C, and all $t \leq T$*

$$\mathcal{V}_t^{1_{\tau>T}C + 1_{\tau\leq T} R_\tau \xi_\tau/\xi} = 1_{\tau\leq t} R_\tau \xi_\tau/\xi_t + \frac{1_{\tau>t}}{\alpha_t} V_t^{\alpha_T C - \int_t^T R_s \xi_{s-} d\alpha_s/\xi}. \tag{3.6}$$

Moreover, on the stochastic interval $\tau \leq T$, we have

$$\mathcal{V}_{\tau-}^{1_{\tau>T}C + 1_{\tau\leq T} R_\tau \xi_\tau/\xi} = \frac{1}{\alpha_{\tau-}} V_{\tau-}^{\alpha_T C - \int_t^T R_s \xi_{s-} d\alpha(s)/\xi}, \qquad \tau \leq T. \tag{3.7}$$

Proof. Decomposing, $1 = 1_{\tau\le t} + 1_{\tau>t}$, we have

$$\mathcal{V}_t^{1_{\tau>T}C+1_{\tau\le T}\,R_\tau\xi_\tau/\xi} = \mathcal{V}_t^{1_{\tau\le t}\,R_\tau\xi_\tau/\xi} + \mathcal{V}_t^{1_{\tau>T}C+1_{t<\tau\le T}\,R_\tau\xi_\tau/\xi}.$$

But, $1_{\tau\le t}\,R_\tau\xi_\tau$ is $\mathcal{F}_t$-measurable. Hence $\mathcal{V}_t^{1_{\tau\le t}\,R_\tau\xi_\tau/\xi} = 1_{\tau\le t}R_\tau\xi_\tau/\xi_t$. The second terms are also equal by the previous theorem, applied with $s = t$. Eq. (3.7) now follows too, because at τ^- default has not occurred, and therefore the first term does not contribute.

□

3.4. **Recovery and loss ratio representations.** The following is an adaption to the present framework of a result in [DS].

Theorem 3.8. *Assume the previous corollary setup, and further that, $C > 0$ a.e., and $R_t \ge 0$, $\alpha_{t\_} > 0$, and α_t has finite variation. Then*

$$\mathcal{V}_t^{1_{\tau>T}C+1_{t<\tau\le T}\,R_\tau\xi_\tau/\xi} = \frac{1_{\tau>t}}{\alpha'_t}V_t^{\alpha'_T C}, \qquad (t \le T)$$

where

$$\alpha'_t := \alpha_t/\mathcal{E}\Big(\int_0^t \frac{R_s d\alpha_s}{V_s^{\alpha_T C-\int_s^T R_u\xi_{u\_}\,d\alpha(u)/\xi}}\Big).$$

Proof. In view of the previous corollary it suffices to show

$$V_t^{\alpha_T C-\int_t^T R_u\xi_{u\_}\,d\alpha(u)/\xi} = \frac{\alpha_t}{\alpha'_t}V_t^{\alpha'_T C}. \qquad (t \le T)$$

As both sides are equal at T, and $\xi_t V_t^{\alpha'_T C}$ is a $\mathcal{H}_t$-martingale, it is enough to show that

$$M_t := \xi_t\frac{\alpha'_t}{\alpha_t}V_t^{\alpha_T C-\int_t^T R_u\xi_{u\_}\,d\alpha(u)/\xi}$$

is a $\mathcal{H}_t$-martingale. Set

$$N_t := \xi_t V_t^{\alpha_T C-\int_0^T R_u\xi_{u\_}\,d\alpha_u/\xi}.$$

Note

$$N_t + \int_0^t R_s\xi_{s\_}\,d\alpha_s = \frac{\alpha_t}{\alpha'_t}M_t.$$

Further, since α_t has finite variation, so does α_t/α'_t. Therefore, by the product rule

$$dN_t + R_t\xi_{t\_}\,d\alpha_t = M_t d\frac{\alpha_t}{\alpha'_t} + \frac{\alpha_{t\_}}{\alpha'_{t\_}}dM_t.$$

But, by the definition of α'_t and M_t,

$$d(\frac{\alpha_t}{\alpha'_t}) = \xi_{t\_}R_t d\alpha_t/M_t.$$

Hence, $dN_t = \alpha_{t\_}dM_t/\alpha'_{t\_}$. So, $dM_t = \alpha'_{t\_}dN_t/\alpha_{t\_}$. Since N_t is a $\mathcal{H}_t$-martingale, it follows that M_t is a $\mathcal{H}_t$ local martingale on $[0, T]$.

If additionally $E[\int_0^T \frac{\alpha'^2_{t_-}}{\alpha^2_{t_-}} d[N_t, N_t]] < \infty$ (a regularity assumption that should be added to the assumptions of the theorem), then M_t will be a $\mathcal{H}_t$ (square integrable) martingale.

□

Remark 3.9. Note, if α_t is continuous (in addition to having finite variation), then, in the definition of α'_t above, we can replace $V_s^{\alpha_T C - \int_s^T R_u \xi_{u_-} d\alpha(u)/\xi}$ by $V_{s_-}^{\alpha_T C - \int_s^T R_u \xi_{u_-} d\alpha(u)/\xi}$, and the stochastic exponential by ordinary exponential. That is, then

$$\alpha'_t = \alpha_t / \mathcal{E}(\int_0^t \frac{R_s d\alpha_s}{V_{s_-}^{\alpha_T C - \int_s^T R_u \xi_{u_-} d\alpha(u)/\xi}}) = \alpha_t \exp(\int_0^t \frac{-R_s d\alpha_s}{V_{s_-}^{\alpha_T C - \int_s^T R_u \xi_{u_-} d\alpha(u)/\xi}}).$$

This result has an intuitively appealing interpretation. In view of Corollary 3.5, we can think of $V_t^{\alpha'_T}/\alpha'_t$ as what the price of the claim $1_{\tau > T} C$ would have been, if the conditional survival probability of τ were α'_t instead of α_t. So, the result is basically saying that the predefault price of a claim with recovery is the same as that of the same claim with no recovery, provided the latter is valued with respect to the increased conditional survival probability α'_t.

The following is a different version of essentially the same result. It is somewhat less satisfactory, as it avoids an inherent recursion in the previous result. But, it does not require that α_t have finite variation (or C be positive), although it does require a continuity assumption (which is satisfied if $\mathcal{H}_t$ is a Brownian filtration).

Theorem 3.10. *Suppose $\mathcal{H}_t$ is $(\tau, \mathcal{F}_t)$-complementary, $\alpha_{t_-} > 0$, and α_t is a $\mathcal{H}_t$-semimartingale. Let ϕ_t be a positive, locally bounded $\mathcal{H}_t$-predictable process. Set*

$$\alpha'_t := \alpha_t / \mathcal{E}(\int_0^t \phi_s d\alpha_s / \alpha_{s_-});$$

$$R_t := \phi_t V_{t_-}^{\alpha'_T C} / \alpha'_{t_-}.$$

Assume R_t is bounded and $\int_0^T R_t \xi_{t_-} d\alpha(t)$ is integrable. Let C be a $\mathcal{H}_T$ claim. Assume $\xi_t V_t^{\alpha'_T C}$ is continuous. Then,

$$\mathcal{V}_t^{1_{\tau > T} C + 1_{t < \tau \le T} R_\tau \xi_\tau / \xi} = \frac{1_{\tau > t}}{\alpha'_t} V_t^{\alpha'_T C}. \qquad (t \le T)$$

Proof. In view of the previous corollary it suffices to show

$$V_t^{\alpha_T C - \int_t^T R_u \xi_{u_-} d\alpha(u)/\xi} = \frac{\alpha_t}{\alpha'_t} V_t^{\alpha'_T C}. \qquad (t \le T)$$

Using the definition of α'_t, we calculate

$$d(\frac{\alpha_t}{\alpha'_t}) = d\mathcal{E}_t(\int_0^\cdot \phi_s d\alpha_s / \alpha_{s_-}) = \mathcal{E}_{t_-}(\int_0^\cdot \phi_s d\alpha_s / \alpha_{s_-}) \phi_t d\alpha_t / \alpha_{t_-} = \phi_t \frac{d\alpha_t}{\alpha'_{t_-}}.$$

Hence, integrating from t to T,

$$\frac{\alpha_T}{\alpha'_T} = \frac{\alpha_t}{\alpha'_t} + \int_t^T \phi_s \frac{d\alpha_s}{\alpha'_{s_-}}.$$

Hence,

$$\alpha'_T \frac{\alpha_t}{\alpha'_t} C = \alpha_T C - \alpha'_T \int_t^T \phi_s \frac{d\alpha_s}{\alpha'_{s_-}} C.$$

Therefore it remains to show

$$V_t^{\int_t^T R_s \xi_{s_-} d\alpha(s)/\xi} = V_t^{\alpha'_T \int_t^T \phi_s \frac{d\alpha_s}{\alpha'_{s_-}} C}.$$

But, using the definition R_t and the continuity assumption, we have

$$E[\int_t^T R_s \xi_{s_-} d\alpha(s) \,|\, \mathcal{H}_t] = E[\int_t^T \phi_s E[\alpha'_T C \,|\, \mathcal{H}_s] d\alpha(s)/\alpha'_{s_-} \,|\, \mathcal{H}_t]$$

$$= \int_t^T E[E[\phi_s \alpha'_T C d\alpha(s)/\alpha'_{s_-} \,|\, \mathcal{H}_s] \,|\, \mathcal{H}_t] = E[\int_t^T \phi_s \alpha'_T C d\alpha(s)/\alpha'_{s_-} \,|\, \mathcal{H}_t].$$

□

When α_t is continuous and of finite variation, we can, as in [DS], express α'_t in terms of the *loss ratio* $1 - \phi_t$:

$$\alpha'_t = \alpha_t \exp(-\int_0^t \phi_s d\alpha_s/\alpha_s) = \exp(\int_0^t (1 - \phi_s) d\alpha_s/\alpha_s).$$

In the absolutely continuous case, the *hazard rate* $-d\alpha'_t/\alpha'_t$ of α'_t is therefore the loss ratio $1 - \phi_t$ times the hazard rate $-d\alpha_t/\alpha_t$ of α_t.

Using Ito's lemma, it is easy to see that in general [7]

$$\alpha'_t = \mathcal{E}(\int_0^t (1 - \phi_s)(d\alpha_s/\alpha_{s_-} - \phi_s d[\alpha_s, \alpha_s]/\alpha_{s_-}^2)).$$

The following two corollaries are immediate consequence of the above two theorems and the previous section.

Corollary 3.11. *Under the assumptions either of above two theorems,*

$$\mathcal{V}_{\tau_-}^{1_{\tau > T} C + 1_{\tau \le T} R_\tau \xi_\tau/\xi} = \frac{V_{\tau_-}^{\alpha'_T C}}{\alpha'_{\tau_-}}, \qquad \tau \le T.$$

Corollary 3.12. *Under the assumptions either of above two theorems,*

$$\mathcal{V}_t^{1_{\tau > T} C + 1_{t < \tau \le T} R_\tau \xi_\tau/\xi} = \mathcal{V}_t^{1_{\tau > T} \frac{\alpha'_T \alpha_t}{\alpha'_t \alpha_T} C}. \qquad (t \le T)$$

---

[7] When α_t is continuous, we further have

$$\alpha'_t = \exp(\int_0^t (1 - \phi_s) d\alpha_s/\alpha_s - (1 - \phi_s^2) d[\alpha_s, \alpha_s]/2\alpha_s^2).$$

The last result can be interpreted as saying that the pre-default price of a claim with recovery is the same as that of a claim with no recovery but a certain higher payoff.

4. NUMERAIRE-RELATIVE VALUATION OF CREDIT DERIVATIVES

A Numeraire is an *a.e.* positive claim β. We write $\beta \in \mathcal{C}^+$.

We do not require that β_t have finite variation, or be $\mathcal{H}_t$-adapted.

For the rest of the paper, $\beta > 0$ denotes a numeraire.

4.1. **Valuation relative to a numeraire and its associated measure.**

For any claim A with $A_0 \neq 0$, define its associated measure P^A by[8]

$$\frac{dP^A}{dP} := \xi \frac{A}{A_0}.$$

Note, P^A is an equivalent measure if and only if A is a numeraire.

We refer to P^β as the *(β-Associated)Numeraire Measure.*

Note, if B is any claim with $B_0 = \beta_0$, then $dP^B/dP^\beta = B/\beta$.

Note, $C \in \mathcal{F}$ is a claim if and only if C/β is P^β-integrable.

As already mentioned, a numeraire incorporates observability into valuation. The previous results can be viewed as valuation relative to the numeraire $1/\xi_t$, which may be deemed unobservable. Here, we wish to extend those results relative to any numeraire. But, this is only a matter of change of notation, as explained below.

A well known property of change of measure is that for any σ-subalgebra $\mathcal{G}$ of $\mathcal{F}$, and any P^β-integrable $F \in \mathcal{F}$, we have

$$E^\beta[F \,|\, \mathcal{G}] = \frac{E[F\xi\beta \,|\, \mathcal{G}]}{E[\xi\beta \,|\, \mathcal{G}]}.$$

This implies that if C is a claim then C_t/β_t is a P^β-martingale:

$$E^\beta[C/\beta \,|\, \mathcal{H}_t] = E[\xi C \,|\, \mathcal{H}_t]/E[\xi\beta \,|\, \mathcal{H}_t] = C_t/\beta_t.$$

For this reason the numeraire measure P^β is also referred to as the *martingale measure* associated to numeraire β.

We think of C_t/β_t as the claim's numeraire-relative price. If we view the numeraire price β_t as an inflator, then we can also think of the relative price C_t/β_t as a deflated price. But, of course it is different than the ξ-deflated price $C_t\xi_t$ (unless $\beta_t = 1/\xi_t$). But, as C_t/β_t is a P^β-martingale, we can view $1/\beta_t$ as a state price density, not with respect to P, but with respect to the numeraire martingale measure P^β.

For the numeraire relative valuation model $(\mathcal{M}, \beta)$, let $\mathcal{M}^\beta$ denote the valuation model $\mathcal{M}^\beta := (\overline{T}, \Omega, \mathcal{F}, P^\beta, \mathcal{F}_t, 1/\beta_t)$. Note, $\mathcal{M}^{\frac{1}{\xi}} = \mathcal{M}$.

We re-denote the previous symbols $\mathcal{S}$ associated with the original $\mathcal{M}$ as symbols $\mathcal{S}^\beta$ for the model $\mathcal{M}^\beta$, by replacing ξ_t with $1/\beta_t$ and P with P^β. Specifically, for any claim C, we set

---

[8] This defines a *probability* measure, because A is a claim, hence $E[\xi A] = A_0$.

$$\mathcal{V}_t^{\beta C} := \beta_t E_t^\beta[C/\beta] = C_t,$$

$$V_t^{\beta C} := \beta_t E^\beta[C/\beta \,|\, \mathcal{H}_t].$$

Actually, $\mathcal{V}_t^{\beta C} = C_t$. In fact, $\mathcal{V}_t^{\beta C}/\beta_t$ is by definition a P^β-martingale with the same terminal value as the P^β-martingale C_t/β_t. So the two martingales are equal.

So, the notation $\mathcal{V}_t^{\beta C}$ is superfluous. But, the notation $V_t^{\beta C}$ is not so. In general, if B is another numeraire, then[9]

$$V_t^{B C} = \frac{B_t}{V_t^{\beta B}} V_t^{\beta C}.$$

Note also, $V_t^{\beta\beta} = \beta_t$.

For any claim A with $A_0 \neq 0$, we denote its *Numeraire-Relative Conditional Survival Probabilities*

$$\alpha_t^A := P^A[\tau > t \,|\, \mathcal{H}_t];$$
$$\overline{\alpha}_t^A := P^A[\tau > t \,|\, \mathcal{H}].$$

We also redefine various types of claims relative to the numeraire.

A *T^β-Claim* is a claim C such that $C/\beta \in \mathcal{F}_T$.

A T^β-claim can be thought of a time T payoff $C_T \in \mathcal{F}_T$ that is invested in the numeraire β and held till the terminal date. The associated claim C is identified as the inflated terminal value $C = \beta C_T/\beta_T$.

There is not "much" difference between T-claims and T^β-claims. Let $C_T \in \mathcal{F}_T$ be such that $C_T\xi_T$ is P-integrable. Then $C_T\xi_T/\xi$ is a T-claim and $C_T\beta/\beta_T$ is a T^β-claim. Although, they are in general *different* claims, yet, for $t \leq T$ we have $\mathcal{V}_t^{C_T\xi_T/\xi} = \mathcal{V}_t^{C_T\beta/\beta_T}$.[10]

Note, if β is a T-claim, then C is a T^β-claim if and only if C is a T-claim.

We define other types of claims relative to β similarly, e.g.:

A *$\mathcal{H}^\beta$-Claim* is a claim C such that $C/\beta \in \mathcal{H}$.

A *$\mathcal{H}_T^\beta$-Claim* is a claim C such that $C/\beta \in \mathcal{H}_T$.

A claim C is a *τ^β-Claim* if $C/\beta \in \mathcal{F}_\tau$.

We may now apply Theorem 3.3 to the default valuation model $(\mathcal{M}^\beta, \tau, \mathcal{H}_t)$. All we have to do is change notation from P to P^β and from ξ_t to $1/\beta_t$, to immediately conclude

Theorem 4.1. *Suppose $\mathcal{H}_t$ is $(\tau, \mathcal{F}_t)$-complementary. Then, for $t < T$ and any claim C*

$$\mathcal{V}_t^{1_{\tau>T}C} = \frac{1_{\tau>t}}{\alpha_t^\beta} V_t^{\beta 1_{\tau>T}C}. \qquad (t < T,\ C \in \mathcal{C}) \tag{4.1}$$

---

[9] $V_t^{BC} := B_t E^B[C/B \,|\, \mathcal{H}_t] = B_t E^\beta[C/\beta \,|\, \mathcal{H}_t]/E^\beta[B/\beta \,|\, \mathcal{H}_t] = B_t V_t^{\beta C}/V_t^{\beta B}$.

[10] Indeed, both sides are equal at $t = T$ (they equal C_T), and ξ_t times either is a martingale.

Evidently, the other results of previous section translate similarly. For the sake of brevity, we will write down only one more.

Corollary 4.2. *Suppose $\mathcal{H}_t$ is $(\tau, \mathcal{F}_t)$-complementary. Then,*

(i) For any $\mathcal{H}^\beta$-claim C, and $t < T$

$$\mathcal{V}_t^{1_{\tau>T}C} = \frac{1_{\tau>t}}{\alpha_t^\beta} V_t^{\beta\overline{\alpha}_T^\beta C}. \qquad (t < T,\ C \in \mathcal{C}_{\mathcal{H}^\beta}) \tag{4.2}$$

(ii) For any $\mathcal{H}_T^\beta$-claim C ,

$$\mathcal{V}_t^{1_{\tau>T}C} = \frac{1_{\tau>t}}{\alpha_t^\beta} V_t^{\beta\alpha_T^\beta C}. \qquad (\forall t,\ C \in \mathcal{C}_{\mathcal{H}_T^\beta}) \tag{4.3}$$

(iii) For any $\mathcal{H}_s^\beta$-claim C and $s < T$

$$\mathcal{V}_t^{1_{s<\tau\leq T}C} = \frac{1_{\tau>t}}{\alpha_t^\beta} V_t^{(\alpha_s^\beta - \alpha_T^\beta)C}. \qquad (\forall t,\ s < T,\ C \in \mathcal{C}_{\mathcal{H}_s^\beta}) \tag{4.4}$$

An important concept for further development is "coadaptedness." A claim C is β-*Coadapted* if C_t/β_t is $\mathcal{H}_t$-adapted.

Proposition 4.3. *A claim C is β-coadapted if and only if $V_t^{\beta^C} = C_t$.*

Proof. If C is β-coadapted, then the P^β-martingale C_t/β_t is also a $(P^\beta, \mathcal{H}_t)$-martingale (by iterating expectation). But, so is $V_t^{\beta^C}$. Both martingales same the same terminal value, so they are equal. The converse is obvious because $V_t^{\beta^C}/\beta_t$ is $\mathcal{H}_t$-adapted. □

A β-coadapted claim is priced by the operator $\mathcal{V}_t^C$ in the same way that it is β-relative valued by the operator $V_t^{\beta^C}$. By definition, it satisfies $C\xi \in \mathcal{H}$, and its β-relative value $E[C/\beta \,|\, \mathcal{H}_t]$ equals $E_t[C/\beta]$. This interchange of $E[\cdot \,|\, \mathcal{H}_t]$ with $E_t[\cdot]$ is shown in [L] to hold automatically in the Cox setting. In [JR], equivalent conditions are studied that make such an interchange possible. The expectation that appear in [DS] and [S] are all $\mathcal{F}_t$-expectations as well. In our context, the property we are after is that any $\mathcal{H}^\beta$-claim C be automatically β-coadapted. We will return to this after studying numeraires B which are β-coadapted.

Note, if β does not price non-$\mathcal{H}_t$-risk, then C does not price non-$\mathcal{H}_t$-risk if and only if C is β-coadapted.

4.2. **Coadapted change of numeraire.** Two numeraires β and B are $\mathcal{H}_t$-*Coadapted Numeraires* if B_t/β_t is $\mathcal{H}_t$-adapted.

Clearly, coadaptedness is an equivalence relation on $\mathcal{C}^+$.

Proposition 4.4. *Let β and B to numeraires. The following conditions are equivalent*

(i) β and B are $\mathcal{H}_t$-coadapted.

(ii) B_t/β_t is a $(P^\beta, \mathcal{H}_t)$-martingale.

(iii) $dP^B{}_{|\mathcal{H}_t}/dP^\beta{}_{|\mathcal{H}_t} = B_t/\beta_t$.

Proof. That (ii) implies (i) is obvious. Conversely, we already know B_t/β_t is a $(P^\beta, \mathcal{F}_t)$ martingale. If it is also $\mathcal{H}_t$-adapted, it must also be a $(P^\beta, \mathcal{H}_t)$ martingale. That (iii) implies (i) is obvious. Finally, $dP^B{}_{|\mathcal{H}_t}/dP^\beta{}_{|\mathcal{H}_t} = E^\beta[B/\beta \,|\, \mathcal{H}_t]$ is always a $(P^\beta, \mathcal{H}_t)$-martingale with terminal value B/β. But if (ii) holds, so is B_t/β_t. Hence they are a.s. equal. □

We next show that various quantities of interest are invariant under coadapted change of numeraire.

Proposition 4.5. *Suppose β and B are $\mathcal{H}_t$-coadapted numeraires, then for all claims C,*[11]

$$V_t^{B^C} = V_t^{\beta^C}.$$

Proof. By its definition $V_t^{B^C}/B_t$ is a $(P^B, \mathcal{H}_t)$-martingale. From previous proposition, part (iii), it follows $V_t^{B^C}/\beta_t$ is a $(P^\beta, \mathcal{H}_t)$-martingale. But, so is $V_t^{\beta^C}/\beta$. Further both martingales have the same terminal value C. Hence they are equal. □

The following elementary result appears new, at least in this form.

Proposition 4.6. *Suppose β and B are $\mathcal{H}_t$-coadapted numeraires, then for all integrable, $\mathcal{F}_t$-adapted processes F_t, $E^B[F_t \,|\, \mathcal{H}] = E^\beta[F_t \,|\, \mathcal{H}]$ and $E^B[F_t \,|\, \mathcal{H}_t] = E^\beta[F_t \,|\, \mathcal{H}_t]$.*

Proof.

$$E^B[F_t \,|\, \mathcal{H}] = \frac{E^\beta[F_t B/\beta \,|\, \mathcal{H}]}{E^\beta[B/\beta \,|\, \mathcal{H}]} = \frac{E^\beta[F_t \,|\, \mathcal{H}] B/\beta}{B/\beta} = E^\beta[F_t \,|\, \mathcal{H}].$$

$$\begin{aligned} E^B[F_t \,|\, \mathcal{H}_t] &= \frac{E^\beta[F_t B/\beta \,|\, \mathcal{H}_t]}{E^\beta[B/\beta \,|\, \mathcal{H}_t]} = \frac{E^\beta[E_t^\beta[F_t B/\beta] \,|\, \mathcal{H}_t]}{E^\beta[E_t^\beta[B/\beta] \,|\, \mathcal{H}_t]} = \frac{E^\beta[F_t E_t[B/\beta] \,|\, \mathcal{H}_t]}{E^\beta[B_t/\beta_t \,|\, \mathcal{H}_t]} \\ &= \frac{E^\beta[F_t B_t/\beta_t \,|\, \mathcal{H}_t]}{B_t/\beta_t} = \frac{E^\beta[F_t \,|\, \mathcal{H}_t] B_t/\beta_t}{B_t/\beta_t} = E^\beta[F_t \,|\, \mathcal{H}_t]. \end{aligned}$$

□

Corollary 4.7. *Suppose β and B are $\mathcal{H}_t$-coadapted numeraires. Then $\alpha_t^B = \alpha_t^\beta$ and $\overline{\alpha}_t^B = \overline{\alpha}_t^\beta$.*

Note, β and $1/\xi$ are coadapted and if only if β does not price non-$\mathcal{H}_t$ risk. In this the case, the application of above proposition to $A = 1/\xi$ provides that $\alpha_t^\beta = \alpha_t$, $\overline{\alpha}_t^\beta = \overline{\alpha}_t$, and, for any claim C, $V_t^{\beta^C} = V_t^C$.

Note, if β and B are coadapted numeraires, then a claim C is β-coadapted if and only if C is B-coadapted. In this case, as we saw before $V_t^{\beta^C} = C_t = V_t^{B^C}$.

We next examine conditions under which coadaptedness holds.

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[11] Conversely, if this equation holds for $C = \beta$ and $C = B$ alone, then β and B are $\mathcal{H}_t$-coadapted numeraires.

4.3. **Conditionally independent σ-algebras.**

Temporarily changing notation in this subsection, let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be three σ-algebras of Ω with $\mathcal{H} \subset \mathcal{F} \cap \mathcal{G}$. Assume a probability measure is given on $\mathcal{F} \vee \mathcal{G}$. Then, as is well known, the following three conditions are equivalent.[12]

(i) $\forall$ bounded $F \in \mathcal{F}$ and $G \in \mathcal{G}$, $E[FG \,|\, \mathcal{H}] = E[F \,|\, \mathcal{H}] E[G \,|\, \mathcal{H}]$.
(ii) $\forall$ bounded $F \in \mathcal{F}$, $E[F \,|\, \mathcal{G}] = E[F \,|\, \mathcal{H}]$.
(iii) $\forall$ bounded $G \in \mathcal{G}$, $E[G \,|\, \mathcal{F}] = E[G \,|\, \mathcal{H}]$.

If any (and hence all) these conditions are satisfied, then $\mathcal{F}$, $\mathcal{G}$ are said to be *conditionally independent given* $\mathcal{H}$.[13]

We point out that if these conditions hold, then in fact (i) holds for all *square-integrable* $F \in \mathcal{F}$ and $G \in \mathcal{G}$, and (ii) and (iii) hold for all *integrable* $F \in \mathcal{F}$ and $G \in \mathcal{G}$ (not just for bounded ones).

4.4. **Conditionally independent subfiltration $\mathcal{H}_t$ - martingale invariance property.**

Returning to our main setup, we follow [JR] closely in this subsection.

Let Q be any P-equivalent measure. (Think of Q as P^β).

We say the subfiltration $\mathcal{H}_t$ of $\mathcal{F}_t$ is a *Q-conditionally independent subfiltration* if for all t, $\mathcal{F}_t$ and $\mathcal{H}$ are Q-conditionally independent given $\mathcal{H}_t$.[14] By the previous section, this is the case if and only if any (hence all) of the following three conditions are satisfied.

(i) $\forall t$, $\forall$ bounded $F_t \in \mathcal{F}_t$, $H \in \mathcal{H}$, $E^Q[F_t H \,|\, \mathcal{H}_t] = E^Q[F_t \,|\, \mathcal{H}_t] E^Q[H \,|\, \mathcal{H}_t]$.
(ii) $\forall t$, $\forall$ bounded $F_t \in \mathcal{F}_t$, $E^Q[F_t \,|\, \mathcal{H}] = E^Q[F_t \,|\, \mathcal{H}_t]$.
(iii) $\forall t$, $\forall$ bounded $H \in \mathcal{H}$, $E^Q[H \,|\, \mathcal{F}_t] = E^Q[H \,|\, \mathcal{H}_t]$.
(Again, in (i) "bounded" can be replaced by "square-integrable" and in (ii) and (iii) by "integrable").

This condition is equivalent to *Q-Martingale Invariance Property*:
(iv) *Every $(\mathcal{H}_t, Q)$ martingale is also an $(\mathcal{F}_t, Q)$ martingale.*[15]

---

[12] By symmetry, it suffices to show equivalence of (i) and (ii). Assume (ii). Let $F \in \mathcal{F}$ and $G \in \mathcal{G}$ be bounded. Then, $E[FG \,|\, \mathcal{H}] = E[FG \,|\, \mathcal{G} \,|\, \mathcal{H}] = E[E[F \,|\, \mathcal{G}] G \,|\, \mathcal{H}]] = E[E[F \,|\, \mathcal{H}] G \,|\, \mathcal{H}]] = E[F \,|\, \mathcal{H}] E[G \,|\, \mathcal{H}]]$. Conversely, assume (i). Let $F \in \mathcal{F}$ be bounded. Set $G := E[F \,|\, \mathcal{G}]$. Note, $E^2[G \,|\, \mathcal{H}] = E[F \,|\, \mathcal{H}] E[G \,|\, \mathcal{H}]] = E[FG \,|\, \mathcal{H}] = E[FE[F \,|\, \mathcal{G}] \,|\, \mathcal{H}] = E[FE[F \,|\, \mathcal{G}] \,|\, \mathcal{G} \,|\, \mathcal{H}] = E[E^2[F \,|\, \mathcal{G}] \,|\, \mathcal{H}] = E[G^2 \,|\, \mathcal{H}]$. Thus, G is $\mathcal{H}$-measurable. So, $E[F \,|\, \mathcal{H}] = E[F \,|\, \mathcal{G} \,|\, \mathcal{H}] = E[G \,|\, \mathcal{H}] = G = E[F \,|\, \mathcal{G}]$.

[13] This will imply $\mathcal{F} \cap \mathcal{G} = \mathcal{H}$. (The converse is usually false.)

[14] This implies $\mathcal{H}_t = \mathcal{F}_t \cap \mathcal{H}$. The converse of course is not true.

[15] This condition is stated for square integrable martingales in [JR], where the equivalence with (i)-(iii) is noted, and several references provided. In our setting with $\overline{T} < \infty$, it holds for all martingales. For $\overline{T} = \infty$, it holds for closed martingales, i.e., those of the form $E_t^Q[F]$ for some Q-integrable $F \in \mathcal{F}$.

The equivalence (iv) with (iii) is quite clear.[16]

4.5. β-conditionally-independent subfiltration $\mathcal{H}_t$.

Given a numeraire β, the subfiltration $\mathcal{H}_t$ is a *β-Conditionally-Independent Subfiltration of $\mathcal{F}_t$* if $\mathcal{H}_t$ is a P^β-conditionally independent subfiltration.

Proposition 4.8. *$\mathcal{H}_t$ is a β-conditionally-independent subfiltration of $\mathcal{F}_t$ if and only if all $\mathcal{H}^\beta$-claims are β-coadapted.*

Proof. Assume $\mathcal{H}_t$ is a β-conditionally independent. Then, for any claim C, $V_t^{\beta^C}$ will also be a $\mathcal{F}_t$ martingale under P^β. But, if C is a $\mathcal{H}^\beta$-claim, then $V^{\beta^C} = C$. So, the two $\mathcal{F}_t$-martingales $V_t^{\beta^C}$ and C_t have the same terminal values and are equal. Conversely, if M_t is $(\mathcal{H}_t, P^\beta)$ martingale, then $C := M\beta$ is an $\mathcal{H}^\beta$-claim, hence, by assumption, β-coadapted. Therefore, $E[M \,|\, \mathcal{F}_t] = C_t/\beta_t = V_t^C/\beta_t = E[M \,|\, \mathcal{H}_t] = M_t$. ☐

Corollary 4.9. *If $\mathcal{H}_t$ is β-conditionally independent, then a claim C is β-coadapted if and only if it is a $\mathcal{H}^\beta$-claim, i.e., $C/\beta \in \mathcal{H}$.*

Thrice application of the above proposition yields

Proposition 4.10. *Let $\mathcal{H}_t$ be a β-conditionally-independent subfiltration of $\mathcal{F}_t$. Let B be another numeraire. Then, $\mathcal{H}_t$ is a B-conditionally-independent subfiltration of $\mathcal{F}_t$ if and only if B and β are $\mathcal{H}_t$-coadapted.*

Proof. Assume $\mathcal{H}_t$ is B-conditionally independent. The above proposition applied to the numeraire B, and claim $C = \beta$ implies B and β are coadapted. Conversely, let C be any claim such that $C/B \in \mathcal{H}$. Then, assuming B and β are coadapted, we have $C/\beta \in \mathcal{H}$. Hence by the above proposition, C is β-coadapted. But, this implies C is B-coadapted, as B and β are assumed coadapted. By the above proposition applied to the numeraire B, this in turn implies $\mathcal{H}_t$ is B-conditionally-independent.

☐

Bringing default time τ into the picture, we have the following characterization from [JR].

Proposition 4.11. *$\mathcal{H}_t$ is a β-conditionally-independent subfiltration of $\mathcal{F}_t^\tau \vee \mathcal{H}_t$ if and only if $\alpha_t^\beta = \overline{\alpha}_t^\beta$, a.s., all t (or equivalently $\overline{\alpha}_t^\beta$ is $\mathcal{H}_t$-adapted).*

Proof. The "only if" part follows at once from Sec. 4.4, condition (ii), applied to $F_t = 1_{\tau>t}$. Conversely, it suffices to show (ii) holds for bounded random variables F_t of the form $1_{\tau>s}H_t$, $s \le t$, $H_t \in \mathcal{H}_t$,

---

[16] Indeed, suppose (iii) holds. Let M_t be a $(\mathcal{H}_t, Q)$ martingale. Then, for $t < T$, we have $E^Q[M_T \,|\, \mathcal{F}_t] = E^Q[M_T \,|\, \mathcal{H}_t] = M_t$. Hence, (iv) holds. Conversely, suppose (iv) holds. Let $H \in \mathcal{H}$ be bounded. Set $M_t := E^Q[H \,|\, \mathcal{H}_t]$. By (iv) M_t is also a $\mathcal{F}_t$ martingale. Hence $M_t = E^Q[M_{\overline{T}} \,|\, \mathcal{F}_t] = E^Q[H \,|\, \mathcal{F}_t]$. So, (iii) holds.

because $\mathcal{F}_t^\tau \vee \mathcal{H}_t$ is generated by such F_t. Clearly, $E^\beta[1_{\tau>s}H_t \,|\, \mathcal{H}] = \overline{\alpha}_s H_t$. But, if $\overline{\alpha}_t^\beta$ is $\mathcal{H}_t$-adapted, then, iterating expectation a few times,

$$E^\beta[1_{\tau>s}H_t \,|\, \mathcal{H}_t] = E^\beta[1_{\tau>s} \,|\, \mathcal{H}_t]H_t = E^\beta[\overline{\alpha}_s^\beta \,|\, \mathcal{H}_t]H_t = \overline{\alpha}_s^\beta H_t = E^\beta[1_{\tau>s}H_t \,|\, \mathcal{H}].$$

□

In particular, if $\mathcal{H}_t$ is a β-conditionally-independent subfiltration of $\mathcal{F}_t^\tau \vee \mathcal{H}_t$, then α_t is decreasing (and hence of finite variation).

A general example of a complementary, conditionally independent subfiltration $\mathcal{H}_t$ is the Cox setting of [L]. It is amenable to Monte-Carlo simulation and also discussed as a standard general construction of default models in [JR] and elsewhere. Its interesting feature is that α_τ^β is uniformly distributed and is P^β-independent of $\mathcal{H}$.[17]

Combining the above and part (i) of Corollary 4.2 yields

Corollary 4.12. *Suppose $\mathcal{H}_t$ is $(\tau, \mathcal{F}_t)$-complementary and β-conditionally independent. Then for all $\mathcal{H}^\beta$-claims C (not just $\mathcal{H}_T^\beta$-claims)*

$$\mathcal{V}_t^{1_{\tau>T}C} = \frac{1_{\tau>t}}{\alpha_t^\beta}\mathcal{V}_t^{\alpha_T^\beta C}. \qquad (t < T,\ C \in \mathcal{C}_{\mathcal{H}^\beta})$$

5. Application to credit streams and swaptions

Fix a relative valuation default model $(\mathcal{M}, \beta, \tau, \mathcal{H}_t)$.

For simplicity of notation, we enforce V_t^C and α_t now stand for $V_t^{\beta C}$ and α_t^β, as opposed to previously that they stood for the special case $\beta = 1/\xi$.

5.1. **Survival and default protection $\mathcal{T}$-streams and swaptions.**

In finance, we often encounter a "stream" of claims. To formalize this, we define a *Tenor* $\mathcal{T}$ to be a sequence $\mathcal{T} = (T_1, \ldots, T_N)$ such that $0 < T_1 < \ldots < T_N \leq \overline{T}$.

Fix a tenor $\mathcal{T}$.

A claim C is a $\mathcal{T}$-*Stream*, if $C = \sum_1^N C_n$, for some T_n^β-claims C_n. We write $C \in \mathcal{C}_\mathcal{T}$.[18]

If C_n are $\mathcal{H}_{T_n}^\beta$ claims, i.e., $C_n/\beta \in \mathcal{H}_{T_n}$, we call C a $\mathcal{H}_\mathcal{T}$-*Stream*, and write $C \in \mathcal{C}_{\mathcal{H}_\mathcal{T}}$.

We now list some examples of $\mathcal{T}$-streams as definitions used later.

$(\mathcal{T}, \delta_\mathcal{T})$-*Default-Free Stream*: $C = \beta \sum_1^N \delta_n/\beta_{T_n}$, , where $\delta_n \in \mathcal{H}_{T_n}^\beta$ is bounded. (Normally $\delta_n \simeq T_n - T_{n-1}$.) It is a $\mathcal{H}_\mathcal{T}$-stream, and often used as numeraire to price vanilla interest-rate swaptions.

$(\mathcal{T}, \delta_\mathcal{T})$-*Survival Stream*: $C = \beta \sum_1^N 1_{\tau>T_n}\delta_n/\beta_{T_n}$. Multiplied by spread K, it becomes the spread premium leg of a T-forward CDS.

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[17] This may be true for all complementary, β-conditionally independent $\mathcal{H}_t$.

[18] We should really call it a $\mathcal{T}^\beta$-*Stream* and write $C \in \mathcal{C}_{\mathcal{T}^\beta}$. But, we have already noted that price at time $t < T$ of a time T-payoff C_T does not depend on which numeraire it is invested it until the terminal date.

Its associated (absolutely continuous but not equivalent) measure is called the "default swap measure" in [S]. We denote this claim $A^{\delta_\mathcal{T}}$.

$(\mathcal{T}, \delta_\mathcal{T})$-*$\alpha$-Survival Stream*: $C = \beta \sum_1^N \alpha_{T_n}\delta_n/\beta_{T_n}$. It is a $\mathcal{H}_\mathcal{T}$-stream. When positive, we will use it as the numeraire for valuation of an option to swap the stream $A^{\delta_\mathcal{T}}$ with another claim. We denote this claim $A^{\delta^\alpha_\mathcal{T}}$.

$(T, \mathcal{T}, L_\mathcal{T})$-*(Default) Protection Stream*: $C = \beta \sum_1^N 1_{T_{n-1}<\tau\le T_n} L_n/\beta_{T_n}$, where $L_n \in \mathcal{H}^\beta_{T_n}$ is bounded, and $T_0 := T$. It represents the protection leg of a T-forward CDS. We denote this claim $B^{T,L_\mathcal{T}}$. If actually, $L_n \in \mathcal{H}^\beta_{T_{n-1}}$, we say this claim is a *Previsible* protection stream.

$(T, \mathcal{T}, L_\mathcal{T})$-*$\alpha$-Default Protection Stream*: $C = \beta \sum_1^N (\alpha_{T_{n-1}} - \alpha_{T_n})L_n/\beta_{T_n}$, where $L_n \in \mathcal{H}_{T_n}$ is bounded, and $T_0 := T$. We denote this claim $B^{T,L_\mathcal{T}\alpha}$.

A swap claim is the difference of two other claims, often two streams. A *payers T-forward CDS with coupon K* is the claim

$$B^{T,L_\mathcal{T}} - KA^{\delta_\mathcal{T}} = \beta \sum_{n=1}^N 1_{T_{n-1}\tau\le T_n} L_n/\beta_{T_n} - K\beta \sum_{n=1}^N 1_{\tau>T_n}\delta_n/\beta_{T_n} \in \mathcal{C}_\mathcal{T}.$$

A t^*-expiry *Credit Spread Option* with strike spread K is the claim

$$C^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}} := \frac{\beta}{\beta_{t^*}}(B^{T,L_\mathcal{T}}_{t^*} - KA^{\delta_\mathcal{T}}_{t^*})^+ \in \mathcal{C}^\beta_{t^*}.$$

5.2. **Pricing CDS and credit swaptions.** Let $t \le t^* \le T$, and $\mathcal{T}$ be a tenor with $T_1 > T$.

Let $A^{\delta_\mathcal{T}} := \beta \sum_1^N 1_{\tau>T_n}\delta_n/\beta_{T_n}$ be a $\delta_\mathcal{T}$-survival stream.

Let $B^{T,L_\mathcal{T}} := \beta \sum_1^N 1_{T_{n-1}<\tau\le T_n} L_n/\beta_{T_n}$ be a $(T, L_\mathcal{T})$-previsible protection stream.

Let $C^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}} := \beta(B^{T,L_\mathcal{T}}_{t^*} - KA^{\delta_\mathcal{T}}_{t^*})^+/\beta_{t^*}$ be a spread option.

Let $A^{\delta^\alpha_\mathcal{T}} := \beta \sum_1^N \alpha_{T_n}\delta_n/\beta_{T_n}$ be the $\delta_\mathcal{T}$-α-survival stream associated to $A^{\delta_\mathcal{T}}$, and $B^{T,L^\alpha_\mathcal{T}} := \beta \sum_1^N (\alpha_{T_{n-1}} - \alpha_{T_n})L_n/\beta_{T_n}$ be the $(T, L_\mathcal{T})$-α-protection stream associated to $B^{T,L_\mathcal{T}}$.

The following results constitute a principal objective of this paper.

Theorem 5.1. *Suppose $\mathcal{H}_t$ is $(\tau, \mathcal{F}_t)$-complementary. Then, the following valuation formulae hold for $t \le t^* \le T$*

$$A^{\delta_\mathcal{T}}_t = \frac{1_{\tau>t}}{\alpha_t} V^{A^{\delta^\alpha_\mathcal{T}}}_t = 1_{\tau>t}\frac{\beta_t}{\alpha_t}\sum_{n=1}^N E^\beta[\delta_n \frac{\alpha_{T_n}}{\beta_{T_n}} \mid \mathcal{H}_t].$$

$$B^{T,L_\mathcal{T}}_t = \frac{1_{\tau>t}}{\alpha_t} V^{B^{T,L^\alpha_\mathcal{T}}}_t = 1_{\tau>t}\frac{\beta_t}{\alpha_t}\sum_{n=1}^N E^\beta[L_n \frac{\alpha_{T_{n-1}} - \alpha_{T_n}}{\beta_{T_n}} \mid \mathcal{H}_t].$$

$$C^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}}_t = \frac{1_{\tau>t}}{\alpha_t} V_t^{\frac{\beta}{\beta_{t^*}}(V^{B^{T,L^\alpha_\mathcal{T}}}_{t^*} - KV^{A^{\delta^\alpha_\mathcal{T}}}_{t^*})^+} = 1_{\tau>t}\frac{\beta_t}{\alpha_t} E^\beta[(V^{B^{T,L^\alpha_\mathcal{T}}}_{t^*} - KV^{A^{\delta^\alpha_\mathcal{T}}}_{t^*})^+/\beta_{t^*} \,|\mathcal{H}_t].$$

Moreover, if $A^{\delta^\alpha_\mathcal{T}} > 0$ a.e. is a numeraire, then, changing numeraire

$$C_t^{K,t^*,T,L_{\mathcal{T}},\delta_{\mathcal{T}}} = 1_{\tau>t}\frac{A_t^{\delta^\alpha_{\mathcal{T}}}}{\alpha_t^{A^{\delta^\alpha_{\mathcal{T}}}}}E^{A^{\delta^\alpha_{\mathcal{T}}}}[(\frac{V_{t^*}^{B^{T,L^\alpha_{\mathcal{T}}}}}{V_{t^*}^{A^{\delta^\alpha_{\mathcal{T}}}}} - K)^+ \mid \mathcal{H}_t].$$

Furthermore, if $\mathcal{H}_t$ is a β-conditionally independent subfiltration, the above hold even when $B^{T,L_{\mathcal{T}}}$ is not previsible.

Proof. The first two formulae follow from linearity and the previous section. By the first equality of the first two formulae applied at $t = t^*$,

$$C^{K,t^*,T,L_{\mathcal{T}},\delta_{\mathcal{T}}} = \frac{1_{\tau>t^*}}{\alpha_t^*}\frac{\beta}{\beta_{t^*}}(V_{t^*}^{B^{T,L^\alpha_{\mathcal{T}}}} - KV_{t^*}^{A^{\delta^\alpha_{\mathcal{T}}}})^+.$$

So again by the previous section (noting a cancellation of α_{t^*})

$$C_t^{K,t^*,T,L_{\mathcal{T}},\delta_{\mathcal{T}}} = \frac{1_{\tau>t}}{\alpha_t}V_t^{\frac{\beta}{\beta_{t^*}}(V_{t^*}^{B^{T,L^\alpha_{\mathcal{T}}}} - KV_{t^*}^{A^{\delta^\alpha_{\mathcal{T}}}})^+},$$

establishing the first equality in the third formula, from which the second equality follows by definition. Moreover, if $A^{\delta^\alpha_{\mathcal{T}}}$ is a numeraire, then, applying this same result to $A^{\delta^\alpha_{\mathcal{T}}}$ instead of β, we have

$$C_t^{K,t^*,T,L_{\mathcal{T}},\delta_{\mathcal{T}}} = 1_{\tau>t}\frac{A_t^{\delta^\alpha_{\mathcal{T}}}}{\alpha_t^{A^{\delta^\alpha_{\mathcal{T}}}}}E^{A^{\delta^\alpha_{\mathcal{T}}}}[(\frac{V_{t^*}^{A^{\delta^\alpha_{\mathcal{T}}}B^{T,L^\alpha_{\mathcal{T}}}}}{A_{t^*}^{\delta^\alpha_{\mathcal{T}}}} - K)^+ \mid \mathcal{H}_t].$$

(Here, we used $V_{t^*}^{A^{\delta^\alpha_{\mathcal{T}}}A^{\delta^\alpha_{\mathcal{T}}}} = A_{t^*}^{\delta^\alpha_{\mathcal{T}}}$.) But, $V_{t^*}^{A^{\delta^\alpha_{\mathcal{T}}}B^{T,L^\alpha_{\mathcal{T}}}}/A_{t^*}^{\delta^\alpha_{\mathcal{T}}} = V_{t^*}^{B^{T,L^\alpha_{\mathcal{T}}}}/V_{t^*}^{A^{\delta^\alpha_{\mathcal{T}}}}$. Hence, the fourth formula follows. The last statement is a consequence of Corollary 4.12.

□

The above valuation formulae in the arbitrary numeraire β for survival and protection streams, and the credit spread option to swap these streams, enable, in principle, to evaluate their prices by Monte-Carlo simulation of $\mathcal{H}_t$-adapted processes. Note, the values at time $t = 0$ simplify, e.g.,

$$C_0^{K,t^*,T,L_{\mathcal{T}},\delta_{\mathcal{T}}} = \beta_0 E^\beta[(V_{t^*}^{B^{T,L^\alpha_{\mathcal{T}}}} - KV_{t^*}^{A^{\delta^\alpha_{\mathcal{T}}}})^+/\beta_{t^*}] = A_0^{\delta^\alpha_{\mathcal{T}}}E^{A^{\delta^\alpha_{\mathcal{T}}}}[(\frac{V_{t^*}^{B^{T,L^\alpha_{\mathcal{T}}}}}{V_{t^*}^{A^{\delta^\alpha_{\mathcal{T}}}}} - K)^+].$$

As effective as the above formulae may be for valuation purposes, they involve the β-value operator V_t, rather than the numeraire-independent price operator $\mathcal{V}_t$. Further, in the change of numeraire formula, $\alpha_t^{A^{\delta^\alpha_{\mathcal{T}}}}$ appears instead of $\alpha_t^\beta =: \alpha_t$. Also, there is no guarantee that various $E[\cdot \mid \mathcal{H}_t]$ that appear in these formula can be replaced by $E_t[\cdot]$. Fortunately, as we have already seen, these nuisances are removed by the requirement of coadaptedness, a condition that is implied by conditional independence.

As an immediate consequence of the above theorem and the previous section, we have the following two results.

Theorem 5.2. *Suppose $\mathcal{H}_t$ is $(\tau, \mathcal{F}_t)$-complementary. If the two claims $A^{\delta^\alpha_\mathcal{T}}$ and $B^{T,L^\alpha_\mathcal{T}}$ are β-coadapted, we have the following pricing formulae.*

$$A_t^{\delta_\mathcal{T}} = \frac{1_{\tau>t}}{\alpha_t} A_t^{\delta^\alpha_\mathcal{T}} = 1_{\tau>t} \frac{\beta_t}{\alpha_t} \sum_{n=1}^{N} E_t^\beta [\delta_n \frac{\alpha_{T_n}}{\beta_{T_n}}].$$

$$B_t^{T,L_\mathcal{T}} = \frac{1_{\tau>t}}{\alpha_t} B_t^{T,L^\alpha_\mathcal{T}} = 1_{\tau>t} \frac{\beta_t}{\alpha_t} \sum_{n=1}^{N} E_t^\beta [L_n \frac{\alpha_{T_{n-1}} - \alpha_{T_n}}{\beta_{T_n}}].$$

$$C_t^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}} = \frac{1_{\tau>t}}{\alpha_t} \mathcal{V}_t^{\frac{\beta}{\beta_{t^*}}(B_{t^*}^{T,L^\alpha_\mathcal{T}} - K A_{t^*}^{\delta^\alpha_\mathcal{T}})^+} = \frac{1_{\tau>t}}{\alpha_t} E_t^\beta [(B_{t^*}^{T,L^\alpha_\mathcal{T}} - K A_{t^*}^{\delta^\alpha_\mathcal{T}})^+ \frac{\beta_t}{\beta_{t^*}}].$$

Moreover, if $A^{\delta^\alpha_\mathcal{T}} > 0$ a.e. is a numeraire, then, changing numeraire measure (and noting by coadaptedness, $\alpha_t^{A^{\delta^\alpha_\mathcal{T}}} = \alpha_t^\beta =: \alpha_t$),

$$C_t^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}} = 1_{\tau>t} \frac{A_t^{\delta^\alpha_\mathcal{T}}}{\alpha_t} E_t^{A^{\delta^\alpha_\mathcal{T}}} (\frac{B_{t^*}^{T,L^\alpha_\mathcal{T}}}{A_{t^*}^{\delta^\alpha_\mathcal{T}}} - K)^+.$$

Furthermore, the ratio

$$S_t := \frac{B_t^{T,L^\alpha_\mathcal{T}}}{A_t^{\delta^\alpha_\mathcal{T}}} \qquad (t \leq T)$$

is a $\mathcal{H}_t$-adapted $(P^{A^{\delta^\alpha_\mathcal{T}}}, \mathcal{F}_t)$-martingale.[19]

Corollary 5.3. *If $A^{\delta^\alpha_\mathcal{T}}/\beta \in \mathcal{H}$, $B^{T,L^\alpha_\mathcal{T}}/\beta \in \mathcal{H}$, and $\mathcal{H}_t$ is $(\tau, \mathcal{F}_t)$-complementary and also a β-conditionally independent subfiltration of $\mathcal{F}_t$, then, as before, $A^{\delta^\alpha_\mathcal{T}}$ and $B^{T,L^\alpha_\mathcal{T}}$ are β-coadapted, and consequently the above pricing formulae and statement about the spread S_t hold.*

The process S_t above is the *Forward Par Credit Spread.*

Note, when $\tau(\omega) > t$, dividing the price above of stream $B^{T,L_\mathcal{T}}$ by the price of stream $A^{\delta_\mathcal{T}}$, the terms α_t cancel, and we have

$$S_t(\omega) = \frac{B_t^{T,L_\mathcal{T}}(\omega)}{A_t^{\delta_\mathcal{T}}(\omega)}. \qquad (\tau(\omega) > t)$$

For $\tau(\omega) \leq t$, both prices are zero, and we cannot perform this division. Nevertheless, $S_t(\omega)$ still exists globally, as defined.

The spread option pricing representation above in the numeraire $A^{\delta^\alpha_\mathcal{T}}$ for $C_t^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}}$ can be viewed as a *call* option price on the credit spread S_t struck at K. Note, at $t = 0$ it simplifies nicely to

$$C_0^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}} = A_0^{\delta^\alpha_\mathcal{T}} E^{A^{\delta^\alpha_\mathcal{T}}} (S_{t^*} - K)^+.$$

Moreover, $E^{A^{\delta^\alpha_\mathcal{T}}} S_{t^*} = S_0$.

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[19] Of course in all these formula $E_t[\cdot]$ can also be replaced by $E[\cdot \mid \mathcal{H}_t]$. And, S_t is also a $(P^{A^{\delta^\alpha_\mathcal{T}}}, \mathcal{H}_t)$-martingale.

5.3. **Black-Scholes approximation for the credit swaption price.** Assume $\mathcal{H}_t$ is a Brownian filtration. This implies that all $(P^{A^{\delta^\alpha_\mathcal{T}}}, \mathcal{H}_t)$-martingales are continuous. In particular, S_t is continuous.

Assume also $B^{T,L^\alpha_\mathcal{T}} > 0$ a.e., so that $S_t > 0$ a.s.

If $\log S_t$ has deterministic quadratic variation, then, because S_t is a continuous $P^{A^{\delta^\alpha_\mathcal{T}}}$-martingale, $\log S_t$ will follow a $P^{A^{\delta^\alpha_\mathcal{T}}}$-Gaussian process, and hence S_{t^*} will be $P^{A^{\delta^\alpha_\mathcal{T}}}$-lognormally distributed. Then, clearly, the above theorem provides a Black-Scholes formula for the credit default swaption.

In general, $\log S_t$ will not have deterministic quadratic variation. Nevertheless, we can define an approximation to S_t with this property.

Consider the Hilbert space of all square-integrable $(P^{A^{\delta^\alpha_\mathcal{T}}}, \mathcal{H}_t)$-martingales M_t with $M_0 = 0$, equipped with the inner product

$$\langle M_t, N_t \rangle = E^{A^{\delta^\alpha_\mathcal{T}}}[M_{t^*} N_{t^*}] = E^{A^{\delta^\alpha_\mathcal{T}}}[M_{t^*}, N_{t^*}].$$

(Note, for an Ito process $\| \int_0^\cdot \sigma_s dz_s \|^2 = E^{A^{\delta^\alpha_\mathcal{T}}}[\int_0^{t^*} \sigma_s^2 ds]$.) Let π denote the orthogonal projection onto the closed subspace of all $(P^{A^{\delta^\alpha_\mathcal{T}}}, \mathcal{H}_t)$-square-integrable (Gaussian) martingales with deterministic quadratic variation. (Note, for an Ito process, $\pi \int_0^t \sigma_s dz_s = \int_0^t E^{A^{\delta^\alpha_\mathcal{T}}}[\sigma_s] dz_s$.)

Assume $S_{t_-} > 0$. Then $\mathcal{L}(S_t) := \int_0^t dS_s / S_{s_-}$ is a $(P^{A^{\delta^\alpha_\mathcal{T}}}, \mathcal{H}_t)$ local martingale. Assume it is a square integrable martingale. Set

$$v_t^2 := E^{A^{\delta^\alpha_\mathcal{T}}}[\pi \mathcal{L}(S_t), \pi \mathcal{L}(S_t)] = \mathrm{var}^{A^{\delta^\alpha_\mathcal{T}}}[\pi \mathcal{L}(S_t)], \qquad v_{t,t^*} := \sqrt{v_{t^*}^2 - v_t^2}.$$

(Note, if $\mathcal{L}(S_t) = \int_0^t \sigma_s dz_s$ is an Ito process, then, $v_t^2 = \int_0^t (E^{A^{\delta^\alpha_\mathcal{T}}}[\sigma_s])^2 ds$.)

The *Approximate Swaption Price Process* $\widehat{C}_t^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}}$ is

$$\widehat{C}_t^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}} := 1_{\tau > t} \frac{A_t^{\delta^\alpha_\mathcal{T}}}{\alpha_t} (S_t N(\frac{\log S_t/K}{v_{t,t^*}} + \frac{v_{t,t^*}}{2}) - K N(\frac{\log S_t/K}{v_{t,t^*}} - \frac{v_{t,t^*}}{2})).$$

This Gaussian (deterministic volatility) approximation $C_t^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}} \approx \widehat{C}_t^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}}$ is exact if (and only if) $\mathcal{L}(S_t)$, or equivalently $\log S_t$, has deterministic quadratic variation. In general, this approximation effectively replaces $E_t^{A^{\delta^\alpha_\mathcal{T}}}(S_{t^*} - K)^+$ by $E_t^{A^{\delta^\alpha_\mathcal{T}}}(S_t \mathcal{E}(\pi \mathcal{L}(S_{t^*})) - K)^+$, or what is the same, the conditional $P^{A^{\delta^\alpha_\mathcal{T}}}$-distribution of S_{t^*} given $\mathcal{F}_t$ is approximated by that of $S_t \mathcal{E}(\pi \mathcal{L}(S_{t^*}))$.

The approximate swaption price at time 0 satisfies

$$\widehat{C}_0^{K,t^*,T,L_\mathcal{T},\delta_\mathcal{T}} = A_0^{\delta^\alpha_\mathcal{T}} (S_0 N(\frac{\log S_0/K}{v_{t^*}} + \frac{v_{t^*}}{2}) - K N(\frac{\log S_0/K}{v_{t^*}} - \frac{v_{t^*}}{2})).$$

A similar spread option pricing formula at $t = 0$ appears in [S] and [AG].

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