

# MINIMAL BLOW-UP SOLUTIONS TO THE MASS-CRITICAL INHOMOGENEOUS NLS EQUATION

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ABSTRACT. We consider the mass-critical focusing nonlinear Schrödinger equation in the presence of an external potential, when the nonlinearity is inhomogeneous. We show that if the inhomogeneous factor in front of the nonlinearity is sufficiently flat at a critical point, then there exists a solution which blows up in finite time with the maximal (unstable) rate at this point. In the case where the critical point is a maximum, this solution has minimal mass among the blow-up solutions. As a corollary, we also obtain unstable blow-up solutions of the mass-critical Schrödinger equation on some surfaces. The proof is based on properties of the linearized operator around the ground state, and on a full use of the invariances of the equation with an homogeneous nonlinearity and no potential, *via* time-dependent modulations.

## 1. INTRODUCTION

1.1. **Setting of the problem and main result.** We consider the equation

$$(1.1) \quad \begin{aligned} i\partial_t u + \Delta u - V(x)u + g(x)|u|^{4/d}u &= 0, \quad x \in \mathbf{R}^d, \\ u|_{t=0} &= u_0 \in H^1(\mathbf{R}^d) \end{aligned}$$

where  $d = 1$  or  $d = 2$ ,  $g$  and  $V$  are real smooth functions on  $\mathbf{R}^d$ , bounded as well as their derivatives, and  $g$  is positive at least in an open subset of  $\mathbf{R}^d$ . We investigate blowing up solutions to (1.1). One of the application that we have in mind is the study of finite time blow-up for solutions to the nonlinear Schrödinger equation on surfaces. The link between these two problems is detailed in §1.3 below.

First, let us recall some classical arguments (see e.g. [15]). The nonlinearity is energy-subcritical, so for any initial condition  $u_0 \in H^1$ , there exists a maximal interval of existence  $]T_-(u_0), T_+(u_0)[$ , and a solution  $u$  of (1.1) such that

$$u \in C(]T_-, T_+[ , H^1).$$

Furthermore if  $T_+ < +\infty$ , then  $\lim_{t \rightarrow T_+} \|\nabla u(t)\|_{L^2} = +\infty$ . The mass  $M = \|u(t)\|_{L^2}^2$  and the energy

$$E = \int \left( \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{2} V(x) |u(t, x)|^2 - \frac{1}{\frac{4}{d} + 2} g(x) |u(t, x)|^{\frac{4}{d} + 2} \right) dx$$

are independent of  $t \in ]T_-, T_+[$ .

We consider the ground state  $Q$ , which is (up to translations and sign change) the minimal mass real-valued solution of the equation

$$\Delta Q + Q^{1+4/d} = Q, \quad x \in \mathbf{R}^d.$$

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Recall that  $Q$  is  $C^\infty$ , radial, and exponentially decreasing at infinity. Furthermore,  $Q$  is the critical point for the Gagliardo–Nirenberg inequality ([33])

$$(1.2) \quad \|\psi\|_{L^{2+4/d}}^{2+4/d} \leq C \|\nabla\psi\|_{L^2}^2 \|\psi\|_{L^2}^{4/d}, \quad \forall \psi \in H^1(\mathbf{R}^d).$$

In the homogeneous case  $V = 0$ ,  $g = 1$ , the equation

$$(1.3) \quad i\partial_t u + \Delta u + |u|^{4/d}u = 0$$

is stable by the pseudo-conformal transformation: if  $u$  is a solution of (1.3), so is  $\tilde{v}(t, x)$  defined by

$$(1.4) \quad \tilde{v}(t, x) = (\mathcal{T}u)(t, x) := \frac{e^{i\frac{|x|^2}{4t}}}{t^{d/2}} \bar{u}\left(\frac{1}{t}, \frac{x}{t}\right).$$

Applying this transformation to the stationary solution  $e^{it}Q$ , one gets a solution of Equation (1.3)

$$S(t, x) = e^{-i/t} \frac{e^{i\frac{|x|^2}{4t}}}{t^{d/2}} Q\left(\frac{x}{t}\right),$$

that blows up at time  $t = 0$ , and such that  $\|\nabla u(t)\|_{L^2} \approx \frac{1}{t}$  as  $t \rightarrow 0$ . A classical argument shows, as a consequence of (1.2) that this is the minimal mass solution blowing-up in finite time (see [33]).

When equation (1.3) is perturbed in such a way that the pseudo-conformal transformation is no longer valid, there are only few known examples of blow-up solutions with the same growth rate. Consider the same equation (1.3) posed on an open subset of  $\mathbf{R}^d$  (with Dirichlet or Neumann boundary conditions) or on a flat torus. Then one can construct blow-up solutions as perturbation of  $S(t, x)$  with an exponentially small error when  $t \rightarrow 0$ : see [26] for  $d = 1$  and [12] for  $d = 2$ . The proof relies on a fixed point argument around a truncation of  $S(t, x)$ . The linear term is considered as a source term, and is controlled in spaces of functions decaying exponentially in time.

The recent work [20] is devoted to a 4 dimensional inhomogeneous mass critical Hartree equation. In the corresponding homogeneous case, a pseudo-conformal transformation similar to (1.4) is known, yielding a blow-up solution analogous to  $S(t, x)$ . Under the assumption that the perturbation vanishes at some large order at the blow-up point, a pseudo-conformal, minimal mass blow-up solution of the perturbed equation is constructed. In this case, the solution is only a polynomial perturbation of the explicit ground state pseudo-conformal blow-up solution, and the proof of [26] and [12] is no longer valid. The construction of [20] relies on an adaptation of an argument of Bourgain and Wang [10].

In the general setting of (1.1), the strategy of [26] and [12] does not work neither unless both  $g$  and  $V$  are constant around the blow-up point. The argument of Bourgain and Wang is easy to adapt and gives, as in [20], a minimal mass solution under strong flatness conditions on  $g$  and  $V$  at the blow-up point (see Remark 1.5 and Section 2). Our goal is to weaken as much as possible these conditions: we construct blow-up solutions for any potential  $V$ , assuming only a vanishing condition to the order 2 on  $g - g(x_0)$  at the blow-up point  $x_0$ . We assume for simplicity that  $x_0 = 0$  and that  $g(0) = 1$ , the general case  $x_0 \in \mathbf{R}^d$ ,  $g(x_0) > 0$  follows by space translation and scaling. For  $s \geq 0$ , we denote

$$\Sigma^s = \left\{ \psi \in H^s(\mathbf{R}^d) \mid |x|^s \psi \in L^2(\mathbf{R}^d) \right\} = H^s(\mathbf{R}^d) \cap \mathcal{F}(H^s(\mathbf{R}^d)),$$

and we shall drop the index for  $\Sigma^1$ . Our main result is the following:

**Theorem 1.1.** *Let  $d = 1$  or  $d = 2$  and  $V, g \in C^\infty(\mathbf{R}^d; \mathbf{R})$ . Assume that for all  $\alpha$ ,  $\partial^\alpha g, \partial^\alpha V \in L^\infty$  and*

$$(1.5) \quad g(0) = 1 \quad ; \quad \frac{\partial g}{\partial x_j}(0) = \frac{\partial^2 g}{\partial x_j \partial x_k}(0) = 0, \quad 1 \leq j, k \leq d.$$

*Then there exists  $T > 0$ ,  $u \in C([0, T[, \Sigma)$  solution of (1.1) on  $]0, T[$  such that*

$$(1.6) \quad \left\| u(t) - \tilde{S}(t) \right\|_{\Sigma} \xrightarrow{t \rightarrow 0^+} 0, \quad \text{where } \tilde{S}(t, x) = e^{-itV(0)} \frac{e^{i\frac{|x|^2}{4t} - i\theta(\frac{1}{t})}}{t^{d/2}} Q\left(\frac{x}{t}\right),$$

*where  $\theta$  is a continuous real-valued function such that  $\theta(\tau) = \tau + o(\tau)$  as  $\tau \rightarrow +\infty$ .*

**Remark 1.2.** As can be easily guessed, we do not really need the boundedness of  $\partial^\alpha g, \partial^\alpha V$  for all  $\alpha$ 's, but for a finite number of multiindices, which we do not wish to make explicit.

**Remark 1.3.** The pseudo-conformal blow-up regime of Theorem 1.1, where the blow-up rate  $\|\nabla u(t)\|_{L^2}$  is of order  $1/t$  around  $t = 0$ , is unstable and non-generic, as opposed to the blow-up regime at a rate  $\left(\frac{\log(|\log |t||)}{t}\right)^{1/2}$  highlighted (in space dimension 1) by G. Perelman [27] (see also [25]). This log-log regime was shown to be generic in all dimension, under a spectral assumption if  $d \geq 2$ , in a series of papers of F. Merle and P. Raphaël (see e.g [24, 30]). This assumption was checked in the case  $d \leq 4$ , and the main properties of the log-log regime persist for  $d = 5$  (see [17]). Theorem 1.1 may also be seen as a structural stability property for the pseudo-conformal blow-up regime: this regime persists under some perturbation of the equation.

**Remark 1.4.** Note that (1.6) implies that  $\|u(t)\|_2^2 = \|Q\|_2^2$ . If we assume furthermore that  $|g| \leq 1$ , the solution constructed in Theorem 1.1 has minimal mass for blow-up. This is consistent with the conjecture that the non-generic blow-up occurs at the boundary of the manifold of all blowing-up solutions. Note also that  $g$  may not remain everywhere positive: we consider a localized phenomenon.

**Remark 1.5.** Establishing Theorem 1.1 is much easier if we assume that  $V - V(0)$  and  $g - g(0)$  vanish to high order at  $x = 0$ . This is the analogue of Theorem 1 of [20] in the context of Hartree equation. In Section 2, we give, in this less general setting, a quick proof of (1.6) which is an adaptation of [10] and simplifies the argument of [20]. In this case we can assume that  $\theta(\tau) = \tau$ , but we do not know if this holds in the general context of Theorem 1.1. The main difficulty of the proof of Theorem 1.1 under the general assumption is to combine the strategy of [10] with modulation theory to relax the high order flatness assumption to the weaker assumption (1.5).

We next discuss two particular cases. If  $g \equiv 1$ , our theorem shows that for any real-valued smooth potential  $V$  which is bounded on  $\mathbf{R}^d$  as well as all its derivatives, for any point  $x_0 \in \mathbf{R}^d$ , there exists a solution of

$$i\partial_t u + \Delta u - V(x)u + |u|^{4/d}u = 0$$

blowing-up at  $x_0$  at a pseudo-conformal rate. Little is known about blow-up solutions for this equation, except in some particular cases (where  $V$  is unbounded) where algebraic miracles provide a good understanding: if  $V$  is linear in  $x$ , Avron-Herbst formula shows that  $V$  does not change the blow-up rate ([14]). If  $V(x) = \pm\omega^2|x|^2$ ,  $V$  changes the blow-up time, but not the blow-up rate ([13]). Our result shows that the  $S(t)$  blow-up rate remains for any bounded potential (e.g. obtained after truncating the above potential).

Equation (1.1) in the case  $V \equiv 0$  was studied by F. Merle in [23]. Assume for the sake of simplicity that

$$g(0) = 1 \quad \text{and} \quad \forall x \neq 0, |g(x)| < 1.$$

In this case,  $g$  attains its maximum at 0. In [23], it is shown, assuming  $g \in C^1$ ,  $V = 0$ , and an additional bound on  $g$  and its derivative, that for any mass  $M > \|Q\|_{L^2}^2$  and close to  $\|Q\|_{L^2}^2$  there exists a blow-up solution  $u$  of (1.1) such that  $\|u_0\|_{L^2}^2 = M$ . It is also shown that a critical mass blow-up solution must concentrate at the critical point 0. Furthermore, if there exists  $\alpha \in ]0, 1[$  such that  $g$  satisfies

$$(1.7) \quad \nabla g(x) \cdot x \leq -|x|^{1+\alpha}$$

for small  $x$ , then there is no critical mass solution. Note that this assumption implies that  $g$  is not  $C^2$ . The existence of minimal mass blow-up solutions for  $g$  which do not satisfy (1.7) is left open in [23]. Theorem 1.1 answers positively to this question for smooth  $g$ , except in the critical case  $\nabla g(0) = 0$  and  $\nabla^2 g(0) \neq 0$ , which includes the case  $\alpha = 1$  in (1.7). We conjecture that non-generic minimal mass blow-up solutions of (1.1) may exist in some cases, but with a distorted blow-up profile  $\tilde{S}$ . This distortion is suggested by the fact that the modulations used to construct the blow-up solution become larger when  $g - g(0)$  and  $V - V(0)$  vanish to lower order at 0.

We do not address here the question of the existence of non-generic blow-up solutions of (1.1) with supercritical mass. This existence was proven for equation (1.3) in [10] in space dimensions 1 and 2. See also the refinement of J. Krieger and W. Schlag [21] in space dimension 1, and [20] for the case of Hartree nonlinearity in space dimension 4. Let us also mention the conjecture, stated in [27], that there is a codimension one submanifold of initial data of equation (1.3) in  $H^1$  leading to pseudo-conformal blow-up. Another interesting but difficult issue that is not treated here is the uniqueness (in some sense to be precised) of the minimal-mass solution constructed in Theorem 1.1, in the spirit of the work of F. Merle [22] on equation (1.3) (see also [18], and [2] for partial results in the case of a plane domain).

**1.2. Strategy of the proof.** The key ingredient of the proof is a result of M. Weinstein [34] on the properties of the linearized NLS operator around the ground state, which implies that the instability of the linearized equation is only polynomial, not exponential.

We first consider, as in [10] the pseudo-conformal transformation (1.4). Thus  $u$  is a solution to (1.1) on  $]0, T[$  if and only if  $\tilde{v}$  is solution to the following equation on  $] \frac{1}{T}, +\infty[$ :

$$(1.8) \quad i\partial_t \tilde{v} + \Delta \tilde{v} - \frac{1}{t^2} V\left(\frac{x}{t}\right) \tilde{v} + g\left(\frac{x}{t}\right) |\tilde{v}|^{4/d} \tilde{v} = 0.$$

Intuitively, for large time, the potential term is negligible (it belongs to  $L_t^1 L_x^\infty$ , hence it is short range in the sense of [16]), and the inhomogeneity can be approximated by its value at the origin. Therefore, a good asymptotic model for (1.8) should be given by the solution (with the same behavior as  $t \rightarrow +\infty$ ) to the “standard” mass-critical focusing nonlinear Schrödinger equation (1.3). As in [33], we want to construct a blow-up solution to (1.1) by constructing a solution  $\tilde{v}$  to (1.8) which behaves like the solitary wave  $e^{it}Q(x)$  (which solves (1.3)) as  $t \rightarrow +\infty$ . In the case  $g = 1$ , there is a huge literature concerning the existence and stability of solitary waves associated to (1.8) when the potential  $1/t^2 V(x/t)$  is replaced by a *time independent* potential: therefore, these results seem of no help to study the blow-up phenomenon.

In a first approximation, we look for a solution of the form

$$(1.9) \quad \tilde{v} = e^{it}(Q + h).$$

Therefore  $\tilde{v}$  is a solution of (1.8) if and only if

$$(1.10) \quad i\partial_t h + \Delta h - h - \frac{1}{t^2}V\left(\frac{x}{t}\right)(Q + h) + g\left(\frac{x}{t}\right)|Q + h|^{4/d}(Q + h) - Q^{1+4/d} = 0.$$

Consider the linearized operator near  $Q$

$$(1.11) \quad Lf := -\Delta f + f - \left(\frac{2}{d} + 1\right)Q^{4/d}f - \frac{2}{d}Q^{4/d}\bar{f}.$$

In [34], it is shown that the semi-group  $e^{itL}$  is bounded in the orthogonal space of a  $2d+4$  dimensional space  $S$ , the space of secular modes, where it grows polynomially. This allows us to construct the solution  $h$  of (1.10) as a fixed point in a space of functions that decay polynomially as  $t \rightarrow +\infty$ . Namely, we can write (1.10) as

$$(1.12) \quad i\partial_t h - Lh = R(h),$$

where  $R(h)$  is, roughly speaking, the sum of a source term involving  $Q$ ,  $V$  and  $g$ , of a similar linear term where  $Q$  is replaced by  $h$ , and of a term which is nonlinear in  $h$ . The latter is essentially harmless, since we expect  $h$  to be small. The first two terms can be proved small provided that we require a sufficient vanishing for  $V$  and  $g - 1$  at the origin to balance the polynomial growth of the semi-group  $e^{itL}$  on  $S$ . This approach is sketched in §2 below. Note that even though intuitively, it is natural to expect  $1/t^2V(x/t)$  and  $g(x/t) - 1$  to be negligible for large time, proving this requires fine spectral properties of  $L$ , since the  $S(t)$  behavior is unstable. In the case where  $V$  and  $g - 1$  are not too flat at the origin, we need even more information.

In order to loosen the assumptions on the local behavior of  $V$  and  $g$  near the origin, we use all the invariances associated to (1.3) to neutralize as many secular modes as possible. There is a  $2d + 3$  dimensional family of modulations, given by the scaling, space-translation, gauge, Galilean, and conformal invariances. By modulating the function  $\tilde{v}$  thanks to these transformations, we can eliminate all secular modes but one, limiting the growth of the operator  $e^{itL}$ . This allows us to decrease the order to which  $V$  and  $g - 1$  vanish at the origin, so as to infer Theorem 1.1. This approach is quite similar in spirit to [21] for  $L^2$ -critical Schrödinger equation, and to [7, 8], where an  $L^2$ -supercritical Schrödinger equation is considered, for which the invariances are therefore fewer.

One of the difficulties of our proof is to include the choice of the modulation parameters in the definition of the operator defining the fixed point. In this context, the contraction property seems hard to check and we bypass this difficulty by using the Schauder fixed point theorem. A key step is to obtain energy estimates on an evolution equation with a time-dependent operator, which is the sum of the linearized operator  $L$  and a time-dependent perturbative term which is given by the modulation.

**1.3. Application to NLS on surfaces.** Let us first recall that the other known blow-up regime, the log-log regime, is not only more stable on  $\mathbf{R}^d$ : it is structurally stable, in the sense that it persists in other geometries. The case of a domain was settled by F. Planchon and P. Raphaël [29], and the one of a general Riemannian manifold by N. Burq, P. Gérard and P. Raphaël [11].

As a consequence of Theorem 1.1, we are able to construct blow-up solutions — with  $1/t$  blow-up speed, and with profile related to  $Q$  — on surfaces flat enough at

the blow-up point. To this purpose we consider rotationally symmetric manifolds. Such a manifold  $M$  is a Riemannian manifold of dimension 2, given by the metric

$$ds^2 = dr^2 + \phi^2(r) d\omega^2,$$

where  $d\omega^2$  is the metric on the sphere  $\mathbb{S}^1$ , and  $\phi$  is a smooth function  $C^\infty([0, \infty[)$ , positive on  $]0, \infty[$ , such that  $\phi^{(\text{even})}(0) = 0$  and  $\phi'(0) = 1$ . These conditions on  $\phi$  yield a smooth manifold (see e.g. [28]). For example,  $\mathbf{R}^2$  and the hyperbolic space  $\mathbf{H}^2$  are such manifolds, with  $\phi(r) = r$  and  $\phi(r) = \sinh r$ , respectively. The volume element is  $\phi(r)$ , and the distance to the origin from a point of coordinates  $(r, \omega)$  is  $r$ . Finally, the Laplace–Beltrami operator on  $M$  is

$$\Delta_M = \partial_r^2 + \frac{\phi'(r)}{\phi(r)} \partial_r + \frac{1}{\phi^2(r)} \Delta_{\mathbf{S}^1}.$$

Now, if we consider  $\tilde{u}$  a radial solution of NLS on  $M$  (recall that  $d = 2$ )

$$(1.13) \quad i\partial_t \tilde{u} + \Delta_M \tilde{u} + |\tilde{u}|^2 \tilde{u} = 0,$$

then the radial function  $u$  defined by

$$\tilde{u}(t, r) = u(t, r) \left( \frac{r}{\phi(r)} \right)^{1/2}$$

satisfies Equation (1.1) with

$$V(r) = \frac{1}{2} \frac{\phi''(r)}{\phi(r)} - \frac{1}{4} \left( \left( \frac{\phi'(r)}{\phi(r)} \right)^2 - \frac{1}{r^2} \right), \text{ and } g(r) = \frac{r}{\phi(r)}.$$

Therefore we are in the framework of Theorem 1.1, up to conditions of flatness of the metrics at the blow-up point and of boundedness of  $V$  and  $g$  at infinity. These boundedness conditions corresponds to conditions on the growth of the unit ball volume of the manifold at infinity.

This proves the existence of a blow-up solution of speed  $1/t$  and critical mass for such surfaces. Notice that the hyperbolic space  $\phi(r) = \sinh r$  correspond to the borderline case  $\partial_r^2 g(0) \neq 0$ , which we do not reach with our method. The motivation for this case would be to complete the available information: the virial identity yields a sufficient blow-up condition which is weaker than in the Euclidean case ([3]), and for defocusing nonlinearities (or focusing nonlinearities with small data), the geometry of the hyperbolic space strongly alters scattering theory, since long range effects which are inevitable in the Euclidean case, vanish there (see [5, 19, 1]; see also [6, 4]).

We conclude this subsection by giving explicit examples of surfaces satisfying the above assumptions.

**Example 1.6** (Compact perturbations of the hyperbolic and Euclidian planes). Let  $c_0, d_0 \in \mathbf{R}$  and consider  $\phi \in C^\infty([0, +\infty[)$  such that  $\phi(r) = r + c_0 r^5 + \mathcal{O}(r^7)$  as  $r \rightarrow 0$ , and  $\phi(r) = \sinh(r) + d_0$  or  $\phi(r) = r + d_0$  for large  $r$ . Then there exists a solution  $\tilde{u}$  of (1.13) that blows up at time  $t = 0$  at the origin  $r = 0$ , and such that  $\|\nabla \tilde{u}(t)\|_{L^2} \approx 1/t$  as  $t \rightarrow 0$ . An example of such a surface in the case  $\phi(r) = r + d_0$  for large  $r$  is given by the surface  $M$  of  $\mathbf{R}^3$  equipped with the induced Euclidean metric and defined by the equation  $x = f(y^2 + z^2)$ , where  $f : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is a smooth nondecreasing function such that  $f(0) = f'(0) = 0$  and  $f(s) = x_0 > 0$  for large  $s$ .

**Remark 1.7.** Many simple manifolds do not enter in our framework, as they do not satisfy the boundedness conditions on  $V$  and  $g$  at infinity. Examples are given by the surfaces of  $\mathbf{R}^3$  defined by the equation  $x = (y^2 + z^2)^k$ ,  $k \geq 2$ , with the induced Euclidean metric, which are spherically symmetric manifolds such that  $g = r/\phi(r)$  satisfies assumption (1.5), but grows polynomially at infinity. We do not know if

this is only a technical point and it would be interesting, in view of these examples, to relax the boundedness conditions on  $V$  and  $g$  at infinity to a polynomial growth. The case of non-flat compact surfaces, even with strong symmetry assumptions, is also completely open.

**1.4. Structure of the paper.** In §2, we sketch the proof of Theorem 1.1 under strong flatness assumptions on  $V$  and  $g$  near the origin. The result then follows in a rather straightforward fashion from a standard fixed point argument, relying on estimates on the linearized operator  $L$  due to M. Weinstein. In §3, we introduce the full family of modulations, in order to reduce the proof of Theorem 1.1. In §4, we recall some more precise properties on the linearized operator  $L$ , which are crucial for tuning the modulation, as presented in §5. Once the modulation is settled, we study the non-secular part of the remainder in §6. The proof of Theorem 1.1 is then completed in §7, thanks to compactness arguments. Minor technical results are detailed in two appendices, for the sake of completeness.

## 2. PROOF OF A WEAKER RESULT

In this section, we sketch the proof of Theorem 1.1 under the

**Assumption 2.1.** *Let  $d = 1$  or  $2$ , and  $V, g \in C^\infty(\mathbf{R}^d; \mathbf{R})$ . Assume that for all  $\alpha$ ,  $\partial^\alpha g, \partial^\alpha V \in L^\infty$ , and that there exist  $m_V \geq 7$  and  $m_g \geq 9$  such that:*

$$\begin{aligned} \forall |\beta| \leq m_V, \quad |\partial^\beta V(x)| &\leq C_\beta |x|^{m_V - |\beta|} \text{ if } |x| \leq 1, \\ \forall |\beta| \leq m_g, \quad |\partial^\beta (g(x) - 1)| &\leq C_\beta |x|^{m_g - |\beta|} \text{ if } |x| \leq 1. \end{aligned}$$

We will need the following property of  $L$ , which is a consequence of [34] (see also [9, Proposition 1.38]). Recall that the linearized operator  $L$  is defined by

$$Lf := -\Delta f + f - \left(\frac{2}{d} + 1\right) Q^{4/d} f - \frac{2}{d} Q^{4/d} \bar{f}.$$

**Proposition 2.2.** *One can decompose  $H^1(\mathbf{R}^d)$  as  $H^1 = S \oplus M$ , with  $S$  (of finite dimension) and  $M$  stable by  $e^{itL}$  and such that, if  $P_M$  and  $P_S$  denote the projections on  $M$  and  $S$ , respectively, the following holds. If  $s \geq 1$  and  $\psi \in H^s$ , then for all  $t \geq 1$ ,*

$$\begin{aligned} \|e^{itL} P_S(\psi)\|_{H^s} &\leq C(1+t^3) \int |\psi(x)| e^{-c|x|} dx, \\ \|e^{itL} P_M(\psi)\|_{H^s} &\leq C\|\psi\|_{H^s}. \end{aligned}$$

Also, if  $s' \geq 1$  and  $\psi \in \Sigma^{s'}$ , then for all  $t \geq 1$ ,

$$\begin{aligned} \||x|^{s'} e^{itL} P_S(\psi)\|_{L^2} &\leq C(1+t^3) \int |\psi(x)| e^{-c|x|} dx, \\ \||x|^{s'} e^{itL} P_M(\psi)\|_{L^2} &\leq C\||x|^{s'} \psi\|_{L^2} + C(1+t^{s'}) \|\psi\|_{H^{s'}}. \end{aligned}$$

In particular, we have for all  $s \geq 1$  and  $\psi \in H^s(\mathbf{R}^d)$ ,

$$(2.1) \quad \|e^{itL} \psi\|_{H^s} \leq C(1+|t|^3) \|\psi\|_{H^s},$$

and for all  $\psi \in \Sigma$ ,

$$(2.2) \quad \||x| e^{itL} \psi\|_{L^2} \leq C\||x| \psi\|_{L^2} + C(1+|t|^3) \|\psi\|_{H^1}.$$

In order to prove Theorem 1.1, we need to find a solution of

$$i\partial_t h - Lh = R(h) \quad ; \quad \|h(t)\|_\Sigma \xrightarrow{t \rightarrow +\infty} 0.$$

We now give the expression of  $R(h)$ :  $R(h) = R_{NL}(h) + R_L(h) + R_0$ , with

$$\begin{aligned} R_{NL}(h) &= -g\left(\frac{x}{t}\right) \left[ |Q + h|^{4/d}(Q + h) - Q^{1+4/d} - \left(\frac{2}{d} + 1\right) Q^{4/d}h - \frac{2}{d} Q^{4/d}\bar{h} \right], \\ R_L(h) &= \left[ 1 - g\left(\frac{x}{t}\right) \right] \left[ \left(\frac{2}{d} + 1\right) Q^{4/d}h + \frac{2}{d} Q^{4/d}\bar{h} \right] + \frac{1}{t^2} V\left(\frac{x}{t}\right) h, \\ R_0(t, x) &= \left[ 1 - g\left(\frac{x}{t}\right) \right] Q(x)^{1+4/d} + \frac{1}{t^2} V\left(\frac{x}{t}\right) Q(x). \end{aligned}$$

We construct a fixed point for the functional

$$(2.3) \quad \mathcal{M}(h)(t, x) = \int_t^{+\infty} e^{i(\tau-t)L} iR(h)(\tau, x) d\tau,$$

that we decompose as  $\mathcal{M}(h) = \mathcal{M}_{NL}(h) + \mathcal{M}_L(h) + \mathcal{M}_0$ , in accordance with the decomposition of  $R$ . Let  $s > d/2$  with  $s \geq 1$ ,  $T > 1$ , and  $4 < b < a$  real numbers to be chosen later. We can prove that  $\mathcal{M}$  is a contraction on the ball of radius one  $B_{a,b,T}$  of the space

$$E_{a,b,T} = \{ \psi \in C([T, +\infty[; H^s \cap \Sigma) \mid \|\psi\|_E < \infty \},$$

where

$$\|\psi\|_E := \sup_{t \geq T} (t^a \|\psi(t)\|_{H^s} + t^b \| |x| \psi(t) \|_{L^2}).$$

In the sequel we will denote by  $C$  a positive constant, that may change from line to line and depend on  $a$ ,  $b$ , and  $s$  but **not on  $T$** .

Since the assumptions made in this paragraph are not as general as in Theorem 1.1, we shall only sketch the main steps of the arguments which lead to the conclusion of Theorem 1.1.

*Bound on the nonlinear terms.* There exists  $C > 0$  such that

$$\forall h, f \in B_{a,b,T}, \quad \|\mathcal{M}_{NL}(h) - \mathcal{M}_{NL}(f)\|_E \leq \frac{C}{T^{a-4}} \|h - f\|_E.$$

This estimate follows from (2.1), (2.2) and the definition of  $E_{a,b,T}$ , which is an algebra embedded in  $L^\infty(\mathbf{R}^d)$ . Note also that  $4/d \geq 1$ , so  $R_{NL}$  contains nonlinear terms which are at least quadratic in  $h$ .

*Bound on the first linear term.* There exists  $C > 0$  such that

$$\forall h, f \in B_{a,b,t_0}, \quad \|\mathcal{M}_L^1(h) - \mathcal{M}_L^1(f)\|_E \leq \frac{C}{T^{m_g-4}} \|h - f\|_E,$$

where

$$\mathcal{M}_L^1(h)(t, x) = \int_t^{+\infty} e^{i(\tau-t)L} i \left[ g\left(\frac{x}{\tau}\right) - 1 \right] \left[ \left(\frac{2}{d} + 1\right) Q^{4/d}h + \frac{2}{d} Q^{4/d}\bar{h} \right] d\tau.$$

The key remark is that  $Q$  decays exponentially. If  $|x| \leq \tau$ , by assumption on  $g$ ,

$$\left| \left[ g\left(\frac{x}{\tau}\right) - 1 \right] Q(x)^{4/d} \right| \leq C \left| g\left(\frac{x}{\tau}\right) - 1 \right| e^{-c|x|} \leq \frac{C}{\tau^{m_g}} |x|^{m_g} e^{-c|x|} \leq \frac{C}{\tau^{m_g}} e^{-\frac{c}{2}|x|}.$$

If  $|x| \geq \tau$ , in view of the boundedness of  $g$  and the exponential decay of  $Q$ ,

$$\left| \left[ g\left(\frac{x}{\tau}\right) - 1 \right] Q(x)^{4/d} \right| \leq C e^{-c|x|} \leq C e^{-\frac{c}{2}\tau} e^{-\frac{c}{2}|x|}.$$

Hence the bound

$$\forall x \in \mathbf{R}^d, \forall \tau \geq 1, \quad \left| \left[ g\left(\frac{x}{\tau}\right) - 1 \right] Q(x)^{4/d} \right| \leq \frac{C}{\tau^{m_g}} e^{-\frac{c}{2}|x|}.$$

Proceeding along the same lines, we infer

$$(2.4) \quad \left\| \left[ g\left(\frac{\cdot}{\tau}\right) - 1 \right] Q^{4/d} \right\|_{H^s} \leq \frac{C}{\tau^{m_g}},$$

and we get by (2.1) and (2.2), the bound on the first linear term.

*Bound on the second linear term.* There exists  $C > 0$  such that for all  $h, f \in B_{a,b,T}$ ,

$$\|\mathcal{M}_L^2(h) - \mathcal{M}_L^2(f)\|_E \leq C \left( \frac{1}{T^{m_V-2}} + \frac{1}{T} + \frac{1}{T^{a-b}} \right) \|h - f\|_E,$$

where

$$\mathcal{M}_L^2(h)(t, x) = \int_t^{+\infty} e^{i(\tau-t)L} \left( \frac{i}{\tau^2} V\left(\frac{x}{\tau}\right) h(\tau, x) \right) d\tau.$$

We have, for  $\tau \geq 1$ ,

$$\left\| V\left(\frac{\cdot}{\tau}\right) \right\|_{W^{s,\infty}} \leq C, \quad \text{hence} \quad \left\| V\left(\frac{\cdot}{\tau}\right) h \right\|_{H^s} \leq C \|h\|_{H^s}.$$

Like above, we also have

$$(2.5) \quad \left\| V\left(\frac{x}{\tau}\right) e^{-c|x|} \right\|_{H^s} \leq \frac{C}{\tau^{m_V}}.$$

By decomposing  $\mathcal{M}_L^2(h)$  on its  $M$  and  $S$  components, we can use the estimates of Proposition 2.2 to get the desired bound on the second linear term.

*Bound on the source term.* There exists  $C > 0$  such that

$$\|\mathcal{M}_0\|_E \leq C \left( \frac{1}{T^{m_g-a-4}} + \frac{1}{T^{m_V-a-2}} \right).$$

This follows easily from (2.4) and (2.5).

*Conclusion.* Gathering all the previous estimates together, we have:

$$(2.6) \quad \forall f, h \in B_{a,b,T}, \quad \|\mathcal{M}(h) - \mathcal{M}(f)\|_E \leq C \mathfrak{w}(T) \|h - f\|_E, \quad \text{where}$$

$$\mathfrak{w}(T) = \frac{1}{T^{a-4}} + \frac{1}{T^{m_g-4}} + \frac{1}{T^{m_V-2}} + \frac{1}{T} + \frac{1}{T^{a-b}} + \frac{1}{T^{m_g-a-4}} + \frac{1}{T^{m_V-2-a}}.$$

Therefore, for  $m_V > 6$  and  $m_g > 8$  (this corresponds to the assumption made in this paragraph, since  $m_V$  and  $m_g$  are integers, by regularity of  $V$  and  $g$ ), we can choose  $a, b$  with  $4 < b < a$  such that all the powers of  $T$  in (2.6) are positive. Hence we can pick  $T$  large enough such that

$$(2.7) \quad \forall f, h \in B_{a,b,T}, \quad \|\mathcal{M}(h) - \mathcal{M}(f)\|_E \leq \frac{1}{2} \|h - f\|_E.$$

Taking  $f = 0$  in (2.7), we see that  $\mathcal{M}$  maps  $B_{a,b,T}$  into  $B_{a,b,T}$ . Furthermore, (2.7) shows that  $\mathcal{M}$  is a contraction on  $B_{a,b,T}$ , which concludes the proof of Theorem 1.1 under Assumption 2.1.

### 3. INTRODUCING A MODULATION

We now wish to replace the assumption made in the previous section by the assumptions of Theorem 1.1, which we rewrite:

**Assumption 3.1.** *Let  $d = 1$  or  $2$ , and  $V, g \in C^\infty(\mathbf{R}^d; \mathbf{R})$ . Assume that for all  $\alpha$ ,  $\partial^\alpha g, \partial^\alpha V \in L^\infty$ :*

$$\forall |\beta| \leq 1, \quad |\partial^\beta V(x)| \leq C_\beta |x|^{1-|\beta|} \text{ if } |x| \leq 1,$$

$$\forall |\beta| \leq 3, \quad |\partial^\beta (g(x) - 1)| \leq C_\beta |x|^{3-|\beta|} \text{ if } |x| \leq 1.$$

At first sight, the above assumption on  $V$  is stronger than in Theorem 1.1. This difference is irrelevant though, in view of the following remark. For a potential  $V$  as in Theorem 1.1, replacing  $u(t, x)$  by  $u(t, x)e^{itV(0)}$  amounts to changing  $V$  to  $V - V(0)$ , a potential which satisfies the above assumption. This explains the presence of the factor  $e^{-itV(0)}$  in the statement of Theorem 1.1.

**3.1. Modulation and linearization.** As explained in the introduction, we want to obtain a solution to

$$(3.1) \quad i\partial_t \tilde{v} + \Delta \tilde{v} - \frac{1}{t^2} V\left(\frac{x}{t}\right) \tilde{v} + g\left(\frac{x}{t}\right) |\tilde{v}|^{4/d} \tilde{v} = 0 \quad ; \quad \|\tilde{v}(t) - e^{i\theta(t)} Q\|_{\Sigma} \xrightarrow{t \rightarrow +\infty} 0,$$

where  $\theta(t) = t + o(t)$  as  $t \rightarrow +\infty$ . Introduce the following modulations:

$$(3.2) \quad \tilde{v}(t, x) = e^{i(q_1(t) + q_3(t) \cdot x + q_5(t) |x|^2)} \frac{1}{q_4(t)^{d/2}} v\left(\gamma(t), \frac{x}{q_4(t)} - q_2(t)\right),$$

with  $q_1, q_4, q_5, \gamma \in \mathbf{R}$  and  $q_2, q_3 \in \mathbf{R}^d$ . The functions  $v$  and  $\tilde{v}$  have similar properties as  $t \rightarrow +\infty$  if, morally,

$$(3.3) \quad q_1(t), q_2(t), q_3(t), q_5(t) \xrightarrow{t \rightarrow +\infty} 0 \quad ; \quad q_4(t) \xrightarrow{t \rightarrow +\infty} 1 \quad ; \quad \gamma(t) \underset{t \rightarrow +\infty}{\sim} t.$$

We give a rigorous meaning to this line below. Note that the second point implies the last one if we assume

$$\dot{\gamma}(t) = \frac{1}{q_4(t)^2}.$$

From now on, we define  $\gamma$  as

$$(3.4) \quad \gamma(t) = \tau_0 + \int_{\tau_0}^t \frac{1}{q_4(\sigma)^2} d\sigma,$$

where  $\tau_0$  is a large time to be determined later. We introduce the new time and space variables

$$\begin{aligned} (\tau, y) &= \left( \gamma(t), \frac{x}{q_4(t)} - q_2(t) \right), \quad \text{or, equivalently} \\ (t, x) &= \left( \gamma^{-1}(\tau), q_4(\gamma^{-1}(\tau)) (y + q_2(\gamma^{-1}(\tau))) \right). \end{aligned}$$

With our choice for  $\gamma$ , (3.1) is equivalent to

$$(3.5) \quad i\partial_\tau v + \Delta v = V_p v - g_p |v|^{4/d} v + iZ_p(v),$$

where we have denoted, for  $p(\tau) = (p_1, p_2, p_3, p_4, p_5) \in \mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}$ :

$$\begin{aligned} iZ_p(v) &= (p_1 + p_3 \cdot y + p_5 |y|^2) v + ip_2 \cdot \nabla v + ip_4 \left( \frac{d}{2} + y \cdot \nabla \right) v, \\ g_p(\tau, y) &= g\left(\frac{x}{t}\right) = g\left(\frac{q_4(y + q_2)}{\gamma^{-1}(\tau)}\right), \\ V_p(\tau, y) &= \frac{q_4^2}{t^2} V\left(\frac{x}{t}\right) = \frac{q_4^2}{\gamma^{-1}(\tau)^2} V\left(\frac{q_4(y + q_2)}{\gamma^{-1}(\tau)}\right). \end{aligned}$$

The parameters  $q_2$  and  $q_4$  are assessed in  $\gamma^{-1}(\tau)$  and substituting  $x = q_4(y + q_2)$ ,

$$\begin{aligned} p_1 &= q_4^2 \dot{q}_1 + q_4^3 \dot{q}_3 \cdot q_2 + q_4^4 \dot{q}_5 |q_2|^2 + q_4^2 |q_3 + 2q_4 q_5 q_2|^2, \\ p_2 &= q_4^2 \dot{q}_2 + q_4 \dot{q}_4 q_2 - 2q_4 a - 4q_4^2 q_5 q_2, \\ p_3 &= q_4^3 \dot{q}_3 + 2q_4^4 \dot{q}_5 q_2 + 4q_4^3 q_3 q_5 + 8q_4^4 q_5^2 q_2, \\ p_4 &= q_4 \dot{q}_4 - 4q_5 q_4^2, \\ p_5 &= q_4^4 \dot{q}_5 + 4q_4^4 q_5^2. \end{aligned}$$

The following rewriting essentially block diagonalizes the above system:

$$\begin{aligned} p_4 &= q_4 (\dot{q}_4 - 4q_5 q_4) . \\ p_5 &= q_4^4 (\dot{q}_5 + 4q_5^2) . \\ p_2 &= q_4^2 \dot{q}_2 - 2q_4 q_3 + p_4 q_2 . \\ p_3 &= q_4^3 (\dot{q}_3 + 4q_3 q_5) + 2q_2 p_5 . \\ p_1 &= p_3 \cdot q_2 - p_5 |q_2|^2 + q_4^2 (\dot{q}_1 + |q_3|^2) . \end{aligned}$$

Note that we have not examined the asymptotic condition as  $t \rightarrow +\infty$ . We analyze this aspect more precisely below (see §3.3). We write  $v = e^{i\tau}(Q+w)$ : Equation (3.5) is equivalent to

$$(3.6) \quad i\partial_\tau w - Lw - iZ_p(w) = iR_p(w) + iZ_p(Q),$$

where  $L$  is the linearized operator (1.11), and  $R_p(w) = R_{NL}(w) + R_L(w) + R_0$

$$(3.7) \quad \begin{cases} iR_{NL}(w) = -g_p \times (F(Q+w) - F(Q) - \ell(w)), \\ iR_L(w) = (1 - g_p) \times \ell(w) + V_p w, \\ iR_0 = (1 - g_p) \times F(Q) + V_p Q, \end{cases}$$

with

$$F(z) = |z|^{4/d} z \quad ; \quad \ell(w) = \left( \frac{2}{d} + 1 \right) Q^{4/d} w + \frac{2}{d} Q^{4/d} \bar{w}.$$

As we will often write the equation (3.6) as  $\partial_t w + iLw = (\dots)$ , we also forced a multiplication by  $i$  in the definition of  $R_p$ . Note that  $R_{NL}$ ,  $R_L$  and  $R_0$  also depend on the parameter  $p$ , although we will usually not indicate it with an index.

The sequel of this section is as follows. In §3.2, we show that one can recover the modulations  $q_1, \dots, q_5$  and the original variables  $t$  and  $x$  from the parameters  $p_1, \dots, p_5$  and the modulated variables  $\tau$  and  $y$ . In §3.3, we reduce Theorem 1.1 to the proof of an existence theorem in the modulated variables  $\tau$  and  $y$ .

**3.2. From  $p$  to the modulation.** From now on, we will work only in the modulated variables  $\tau$  and  $y$ , and consider, by abuse of notation, the modulations  $q_k$  as functions of  $\tau$ . Denoting by  $'$  the derivative with respect to  $\tau$ , that is  $\dot{f} = \frac{1}{q_4} f'$ , the above system reads:

$$(3.8) \quad \begin{cases} p_4 = \frac{q_4'}{q_4} - 4q_5 q_4^2 . \\ p_5 = q_4^2 q_5' + 4q_4^4 q_5^2 . \\ p_2 = q_4' - 2q_4 q_3 + p_4 q_2 . \\ p_3 = q_4 q_3' + 4q_4^3 q_3 q_5 + 2q_2 p_5 . \\ p_1 = p_3 \cdot q_2 - p_5 |q_2|^2 + q_4' + q_4^2 |q_3|^2 . \end{cases}$$

Recall that we seek  $q_4 = 1 + q_{4r}$ , with

$$q_1, q_2, q_3, q_{4r}, q_5 \xrightarrow[t \rightarrow +\infty]{} 0.$$

Consider these functions as unknowns, to be sought, for  $c > 0$ , in

$$(3.9) \quad W(c, \tau_0) = \{f \in C([\tau_0, \infty[), \quad \|f\|_{c, \tau_0} := \sup_{\tau \geq \tau_0} \tau^c |f(\tau)| < \infty\}.$$

Our main assumption here is  $p_j \in W(c(p_j), \tau_0)$ , for  $1 \leq j \leq 5$ .

**Lemma 3.2.** *Let  $c(p_3) = c(p_5) > 2$ ,  $c(p_1) > 1$ ,  $c(p_2) = c(p_4) > 1$ . Then if  $\tau_0$  is sufficiently large the following holds. Let  $p_j \in W(c(p_j), \tau_0)$ ,  $1 \leq j \leq 5$  such that*

$$\forall j \in \{1, \dots, 5\}, \quad \|p_j\|_{c(p_j), \tau_0} \leq 1.$$

Then there exists a unique family of parameters  $q_1, q_2, q_3, q_{4r}, q_5$ , such that the system (3.8) holds with

- $q_2, q_{4r} \in W(c(q_2), \tau_0)$  with  $c(q_2) = (\min(c(p_5) - 2, c(p_4) - 1))^-$ .
- $q_3, q_5 \in W(c(q_3), \tau_0)$  with  $c(q_3) = c(p_3)/2$ .
- $q_1 \in W(c(q_1), \tau_0)$  with  $c(q_1) = \min(c(p_1) - 1, c(p_3) - 1)$ ,

and

$$\|q_1\|_{c(q_1), \tau_0} + \|q_2\|_{c(q_2), \tau_0} + \|q_3\|_{c(q_3), \tau_0} + \|q_{4r}\|_{c(q_2), \tau_0} + \|q_5\|_{c(q_3), \tau_0} \leq 1.$$

Finally, the variables  $(\tau, y)$  and  $(t, x)$  are uniformly equivalent:

$$\frac{1}{2} \leq \frac{d\tau}{dt} \leq 2 \quad ; \quad \frac{1}{2} \langle x \rangle \leq \langle y \rangle \leq 2 \langle x \rangle.$$

**Remark 3.3.** Under the assumptions of the lemma, we can define implicitly the variable  $t$  from the variable  $\tau$  in view of the formula (3.4).

*Proof.* The first two equations in (3.8) determine  $q_{4r}$  and  $q_5$ . Then the next two yield  $q_2$  and  $q_3$ , while we infer  $q_1$  from the last equation. Thus we first consider

$$(3.10) \quad \begin{cases} q'_{4r} - 4q_5 = p_4(1 + q_{4r}) + 4q_5q_{4r}(3 + 3q_{4r} + q_{4r}^2), \\ q'_5 = \frac{p_5}{(1 + q_{4r})^2} - 4(1 + q_{4r})^2q_5^2. \end{cases}$$

Introduce the corresponding homogeneous system:

$$\frac{d}{d\tau} \begin{pmatrix} q_{4r} \\ q_5 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q_{4r} \\ q_5 \end{pmatrix}.$$

The square of the above matrix is zero, and we infer:

$$\exp \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}.$$

Duhamel's formula for (3.10) thus reads:

$$\begin{aligned} q_{4r}(\tau) &= - \int_{\tau}^{\infty} \left[ p_4(\sigma) (1 + q_{4r}(\sigma)) + 4q_5(\sigma)q_{4r}(\sigma) (3 + 3q_{4r}(\sigma) + q_{4r}^2(\sigma)) \right] d\sigma \\ &\quad - \int_{\tau}^{+\infty} 4(\tau - \sigma) \left( \frac{p_5(\sigma)}{(1 + q_{4r}(\sigma))^2} - 4(1 + q_{4r}(\sigma))^2q_5^2(\sigma) \right) d\sigma, \\ q_5(\tau) &= - \int_{\tau}^{\infty} \left[ \frac{p_5(\sigma)}{(1 + q_{4r}(\sigma))^2} - 4(1 + q_{4r}(\sigma))^2q_5^2(\sigma) \right] d\sigma. \end{aligned}$$

Denoting  $N(k) = \|k\|_{c(k), \tau_0}$ , the first right hand side is controlled by

$$\begin{aligned} \int_{\tau}^{\infty} \left( \sigma^{-c(p_4)} + \sigma^{-c(q_{4r})-c(q_5)} N(q_{4r})N(q_5) + \tau\sigma^{-c(p_5)} + \tau\sigma^{-2c(q_5)} N(q_5)^2 \right) d\sigma \\ \lesssim \tau^{1-c(p_4)} + \tau^{1-c(q_{4r})-c(q_5)} + \tau^{2-\min(c(p_5), 2c(q_5))}. \end{aligned}$$

The second right hand side is controlled by  $\tau^{1-\min(c(p_5), 2c(q_5))}$ . We can solve the above system by a fixed point argument in the class that we consider, provided that  $\tau_0$  is sufficiently large, as soon as

$$\begin{aligned} c(q_{4r}) + 1 < c(p_4) & \quad ; \quad 1 < c(q_5), \\ c(q_{4r}) + 2 < \min(c(p_5), 2c(q_5)) & \quad ; \quad c(q_5) + 1 < \min(c(p_5), 2c(q_5)). \end{aligned}$$

This boils down to

$$c(p_4) > 1 \quad ; \quad c(p_5) > 2,$$

in which case we may take

$$c(q_{4r}) = (\min(c(p_5) - 2, c(p_4) - 1))^- \quad ; \quad c(q_5) = \frac{1}{2}c(p_5).$$

Note also that  $\tau_0$  can be chosen independent of  $p$  such that  $N(p) \leq 1$ .

The system yielding  $(q_2, q_3)$  is similar (the constant 4 becomes a 2):

$$\begin{cases} q_2' - 2q_3 = p_2 + 2q_{4r}q_3 - p_4q_2 \\ q_3' = -4(1 + q_{4r})^2 q_3q_5 + \frac{p_3 - 2q_2p_5}{1 + q_{4r}}. \end{cases}$$

Under the extra assumption  $c(p_2) = c(p_4)$  and  $c(p_3) = c(p_5)$ , we may take

$$c(q_2) = c(q_{4r}) \quad ; \quad c(q_3) = c(q_5).$$

It is clear that we may choose  $c(q_1) = \min(c(p_1) - 1, c(p_3) - 1)$ .

The inequalities:

$$\begin{aligned} \left| \frac{d}{dt}(\tau - t) \right| &= \left| \frac{1}{q_4^2} - 1 \right| = \left| (1 + q_{4r})^{-2} - 1 \right| \lesssim \frac{1}{t^{c(q_2)}}, \\ |y_j - x_j| &= \left| \left( \frac{1}{q_4} - 1 \right) x_j - q_2 \right| \lesssim \frac{1}{t^{c(q_2)}} (|x_j| + 1) \end{aligned}$$

imply the last part of the lemma.  $\square$

The following lemma is a direct consequence of the proof of the previous result:

**Lemma 3.4.** *Let  $p$  and  $\tilde{p}$  satisfy the assumptions of Lemma 3.2. Assume in addition that for all  $k$ ,  $c(p_k) = c(\tilde{p}_k) = c(p) > 2$ . Denote by  $q$  and  $\tilde{q}$  the corresponding modulations provided by Lemma 3.2. We have*

$$|q_4(\tau) - \tilde{q}_4(\tau)| + |q_2(\tau) - \tilde{q}_2(\tau)| \lesssim \frac{1}{\tau^{c(p)-2}} \max_{1 \leq k \leq 5} \|p_k - \tilde{p}_k\|_{c(p), \tau_0}.$$

*Proof.* Subtract the Duhamel's formulations to systems (3.10) associated to  $p$  and  $\tilde{p}$ , respectively. Denoting  $e_q(\tau) = |q_4 - \tilde{q}_4| + \tau|q_5 - \tilde{q}_5|$ , we have immediately

$$\begin{aligned} e_q(\tau) &\lesssim \tau \int_{\tau}^{\infty} \left( \left( \frac{1}{\sigma^{c(p)}} + \frac{1}{\sigma^{1+c(q_5)}} \right) e_q(\sigma) + \frac{1}{\sigma^{c(p)}} \max_{1 \leq k \leq 5} \|p_k - \tilde{p}_k\|_{c(p), \tau_0} \right) d\sigma \\ &\lesssim \tau \int_{\tau}^{\infty} \left( \frac{1}{\sigma^{1+c(q_5)}} e_q(\sigma) + \frac{1}{\sigma^{c(p)}} \max_{1 \leq k \leq 5} \|p_k - \tilde{p}_k\|_{c(p), \tau_0} \right) d\sigma \end{aligned}$$

From Lemma 3.2,  $c(q_5) = c(p)/2 > 1$ . We can then apply Gronwall lemma to  $\tilde{e}_q(\tau) = e_q(\tau)/\tau$ , and the first estimate follows. The estimate for  $q_2$  proceeds along the same lines.  $\square$

**3.3. Reduced problem.** In the rest of this paper, we show the following:

**Theorem 3.5.** *Let Assumption 3.1 be satisfied. There exists  $\tau_0 > 0$ , a modulation  $p$  such that  $p_j \in W(c(p), \tau_0)$  with  $c(p) > 2$ , and a solution  $w \in C([\tau_0, \infty]; \Sigma)$  to*

$$(3.11) \quad i\partial_{\tau}w - Lw - iZ_p(w) = iR_p(w) + iZ_p(Q),$$

such that

$$\|w(\tau)\|_{H^1} \leq \frac{C}{\tau^{2^-}}, \quad \|\langle y \rangle w(\tau)\|_{L^2} \leq \frac{C}{\tau^{1^-}}.$$

*Theorem 3.5 implies Theorem 1.1.* Writing  $v(\tau, y) = e^{i\tau}(Q(y) + w(\tau, y))$ , we first see that Theorem 3.5 implies the existence of  $p = p(\tau)$  like above, and a solution  $v \in C([\tau_0, \infty]; \Sigma)$  to

$$\begin{aligned} i\partial_{\tau}v + \Delta v &= V_p v - g_p |v|^{4/d} v + iZ_p(v), \\ \|v(\tau) - e^{i\tau}Q\|_{H^1} &\leq \frac{C}{\tau^{2^-}}, \quad \|\langle y \rangle (v(\tau) - e^{i\tau}Q)\|_{L^2} \leq \frac{C}{\tau^{1^-}}. \end{aligned}$$

If this holds, then Lemma 3.2 yields a modulation  $q$  such that

$$(3.12) \quad |q_1(t)| + |q_2(t)| + |q_4(t) - 1| \xrightarrow{t \rightarrow +\infty} 0, \quad |q_3(t)| + |q_5(t)| \leq \frac{C}{t^{1+}},$$

and a solution of (3.1),

$$\tilde{v}(t, x) = e^{i(q_1(t)+q_3(t) \cdot x + q_5(t)|x|^2)} \frac{1}{q_4(t)^{d/2}} v \left( \gamma(t), \frac{x}{q_4(t)} - q_2(t) \right).$$

In view of the behavior of the  $H^1$  and  $\mathcal{F}H^1$  norms via the pseudo-conformal transformation, the only point we have to check to get Theorem 1.1 is that indeed,

$$t \|\tilde{v}(t) - e^{i\theta(t)}Q\|_{\dot{H}^1} \xrightarrow{t \rightarrow +\infty} 0 \quad \text{and} \quad \|\tilde{v}(t) - e^{i\theta(t)}Q\|_{\mathcal{F}H^1} \xrightarrow{t \rightarrow +\infty} 0$$

for some  $\theta(t) = t + o(t)$ . We choose  $\theta(t) = \gamma(t)$ : (3.4) and Lemma 3.2 show that indeed,  $\gamma(t) = t + o(t)$  as  $t \rightarrow \infty$ . We readily verify that it suffices to prove

$$t \left\| e^{i(q_1(t)+q_3(t) \cdot x + q_5(t)|x|^2)} \frac{1}{q_4(t)^{d/2}} e^{i\gamma(t)} Q \left( \frac{x}{q_4(t)} - q_2(t) \right) - e^{i\gamma(t)} Q(x) \right\|_{\dot{H}^1} \xrightarrow{t \rightarrow +\infty} 0$$

and

$$\left\| e^{i(q_1(t)+q_3(t) \cdot x + q_5(t)|x|^2)} \frac{1}{q_4(t)^{d/2}} e^{i\gamma(t)} Q \left( \frac{x}{q_4(t)} - q_2(t) \right) - e^{i\gamma(t)} Q(x) \right\|_{\mathcal{F}H^1} \xrightarrow{t \rightarrow +\infty} 0.$$

This in turn can be checked by elementary computations, in view of (3.12), and since  $Q \in \mathcal{S}(\mathbf{R}^d)$ .  $\square$

As suggested by the statement of Theorem 3.5, we construct *simultaneously* the modulation  $p$  and the remainder  $w$ . We will see in Section 5 that these two unknowns are related through a nonlinear process.

#### 4. THE LINEARIZED OPERATOR

To prove Theorem 3.5, we need more precise properties concerning the linearized operator  $L$  than those recalled in Proposition 2.2. We use again refined estimates proved in [34] (see also [10]).

As in [34], we identify  $\mathbf{C}$  with  $\mathbf{R}^2$ , and the space of complex-valued functions  $H^1(\mathbf{R}^d, \mathbf{C})$  with the space  $H^1(\mathbf{R}^d, \mathbf{R}) \times H^1(\mathbf{R}^d, \mathbf{R})$ , considering the operator  $\mathcal{L} = iL$  as an operator on  $L^2 \times L^2$  with domain  $H^2 \times H^2$ :

$$\mathcal{L} = iL = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}, \quad L_+ = -\Delta + 1 - \left( \frac{4}{d} + 1 \right) Q^{4/d}, \quad L_- = -\Delta + 1 - Q^{4/d}.$$

Note that  $\mathcal{L}$  is not self-adjoint. We denote by

$$\langle f, g \rangle = \int_{\mathbf{R}^d} f_1 g_1 + \int_{\mathbf{R}^d} f_2 g_2,$$

the scalar product on  $L^2(\mathbf{R}^d) \times L^2(\mathbf{R}^d)$ . The space of secular modes is defined by

$$S := \bigcup_{\kappa \geq 1} N(\mathcal{L}^\kappa),$$

where  $N(A)$  is the null-space of the operator  $A$ . We next specify the space  $S$  and the dynamics of  $e^{itL}$  on  $S$ . Note that by direct calculation,

$$(4.1) \quad \begin{cases} L_- (|x|^2 Q) = -4 \left( \frac{d}{2} Q + x \cdot \nabla Q \right), & L_- Q = 0, \\ -L_+ \left( \frac{d}{2} Q + x \cdot \nabla Q \right) = 2Q, \end{cases}$$

$$(4.2) \quad L_- (x_\ell Q) = -2\partial_{x_\ell} Q, \quad L_+ (\partial_{x_\ell} Q) = 0.$$

Furthermore, there exists only one radial function  $\tilde{Q}$  such that

$$L_+ \tilde{Q} = -|x|^2 Q.$$

Consider for  $1 \leq \ell \leq d$

$$\begin{aligned} n_1 &= -i\alpha_0^{-1}Q; & n_{2,\ell} &= -\beta_0^{-1}\partial_{x_\ell}Q, & n_{3,\ell} &= i\beta_0^{-1}x_\ell Q \\ n_4 &= \alpha_0^{-1}\left(\frac{d}{2}Q + x \cdot \nabla Q\right), & n_5 &= -i\alpha_0^{-1}\left(\frac{1}{2}|x|^2Q + \gamma_0Q\right), & n_6 &= \alpha_0^{-1}\tilde{Q}, \end{aligned}$$

(where  $\alpha_0, \beta_0, \gamma_0$  are normalization constants,  $\alpha_0, \beta_0 > 0$ ). Then

$$(4.3) \quad \begin{cases} \mathcal{L}n_1 = \mathcal{L}n_{2,\ell} = 0, & \mathcal{L}n_4 = -2n_1, & \mathcal{L}n_{3,\ell} = 2n_{2,\ell}, \\ \mathcal{L}n_5 = 2n_4, & \mathcal{L}n_6 = -2n_5 + 2\gamma_0n_1. \end{cases}$$

This shows that all  $n_j$ 's are in the space  $S$ . By similar computations, the following functions are in the space  $S^* = \bigcup_{\kappa \geq 1} N((\mathcal{L}^*)^\kappa)$ :

$$\begin{aligned} m_1 &= i\tilde{Q}, & m_{2,\ell} &= x_\ell \tilde{Q}, & m_{3,\ell} &= -i\partial_{x_\ell} \tilde{Q}, \\ m_4 &= -\frac{1}{2}|x|^2Q - \gamma_0Q, & m_5 &= i\frac{d}{2}Q + ix \cdot \nabla Q, & m_6 &= -Q. \end{aligned}$$

Moreover,  $M = (S^*)^\perp$ , and  $\langle n_k, m_j \rangle = \delta_{jk}$ , so that

$$P_S h = \sum_{1 \leq j \leq 6} \nu_j n_j, \text{ where } \nu_j = \langle h, m_j \rangle.$$

As a consequence, in view of (4.3), the exact dynamics of  $e^{itL}$  on  $S$  is obtained.

**Proposition 4.1.** *Let  $G \in C(\mathbf{R}; H^1 \times H^1)$ , and  $W$  such that*

$$(4.4) \quad \partial_t W + iLW = G$$

*Denote  $\nu_j = \langle W, m_j \rangle$  and  $d_j = \langle G, m_j \rangle$ . Then,*

$$\begin{aligned} \nu'_1 &= 2\nu_4 - 2\gamma_0\nu_6 + d_1 & \nu'_{2,\ell} &= -2\nu_{3,\ell} + d_{2,\ell} & \nu'_{3,\ell} &= d_{3,\ell} \\ \nu'_4 &= -2\nu_5 + d_4 & \nu'_5 &= 2\nu_6 + d_5 & \nu'_6 &= d_6. \end{aligned}$$

## 5. TUNING THE MODULATION

Our approach consists of a careful examination of (3.11). As we have seen in the previous section, we can write  $H^1 = M \oplus S$ . Recall that  $S$ , the generalized kernel of  $iL$ , is a finite dimensional space, and that the group  $e^{itL}$  is bounded on  $M$ . To construct the wave operator of Theorem 3.5, we have to control the secular part of  $w$  (its  $S$  component). We decompose  $w$  into  $w = w_S + w_M$ . By noticing that

$$(5.1) \quad \begin{aligned} Z_p(Q) &= -i(p_1 + p_3 \cdot y + p_5|y|^2)Q + p_2 \cdot \nabla Q + p_4 \left( \frac{d}{2} + y \cdot \nabla \right) Q \\ &= p_1\alpha_0n_1 - p_3\beta_0n_3 + 2p_5\alpha_0(n_5 - \gamma_0n_1) - p_2\beta_0 \cdot n_2 + \alpha_0n_4 \end{aligned}$$

is in  $S$ , we deduce the projected equations on  $S$  and on  $M$ . Namely, we want to construct a solution to the system

$$(5.2) \quad \partial_\tau w_S + iLw_S = P_S R_p(w) + P_S Z_p(w) + Z_p(Q),$$

$$(5.3) \quad \partial_\tau w_M + iLw_M - P_M Z_p(w_M) = P_M R_p(w) + P_M Z_p(w_S).$$

We introduce

$$(5.4) \quad \begin{aligned} \Phi(w)(\tau) &= \int_\tau^\infty e^{i(\tau-\sigma)L} (P_S R_p(w) + P_S Z_p(w) + Z_p(Q)) d\sigma + \Phi_2(w)(\tau) \\ &= \Phi_1(w)(\tau) + \Phi_2(w)(\tau), \end{aligned}$$

where  $\Phi_2(w) = \phi$  is the solution (in  $M$  for all  $\tau$ ) of the equation

$$\partial_\tau \phi + iL\phi - P_M Z_p(\phi) = P_M R_p(w) + P_M Z_p(w_S).$$

The existence of  $\Phi_2(w)$  will be shown in §6. In the present section, we define the modulation parameter  $p$ , and estimate  $\Phi_1(w)$ . The main point in our approach is that  $p$  depends on  $w$ , and is chosen so that the secular part  $\Phi_1(w)$  of  $\Phi(w)$  belongs to  $\text{span}(n_6)$ . As  $p$  also appears in the definition of  $\Phi$  in (5.4), the dependence of  $\Phi$  upon  $w$  is more implicit (and more nonlinear) than it may seem.

As it is standard, we shall construct in Section 7 a fixed point for  $\Phi$ . However, we shall not use Banach–Picard result (based on contractions), but rather the Schauder fixed point argument (based on compactness).

The modulation parameter  $p$  is defined from  $w$  in Proposition 5.3 of §5.2. To implement this idea, we first prove some *a priori* estimates for arbitrary  $p$ .

### 5.1. General estimates.

**Lemma 5.1.** *Let Assumption 3.1 be satisfied, and*

$$\|p_k\|_{c(p), \tau_0} \leq 1, \quad \|\tilde{p}_k\|_{c(p), \tau_0} \leq 1, \quad k \in \{1, \dots, 5\},$$

where  $c(p) \in ]2, 3[$ . Then, for every fixed  $w$ , we have the pointwise estimates

$$(5.5) \quad |R_{NL}(w)| \lesssim Q|w|^2 + \sum_{3 \leq j \leq 1+4/d} |w|^j,$$

$$(5.6) \quad |R_L(w)| \lesssim \frac{\langle y \rangle^3}{\tau^3} e^{-c\langle y \rangle} |w| + \frac{1}{\tau^2} \min\left(1, \frac{\langle y \rangle}{\tau}\right) |w|,$$

$$(5.7) \quad |R_p(w) - R_{\tilde{p}}(w)| \lesssim \frac{1}{\tau^{c(p)+1}} \|p - \tilde{p}\|_{c(p)} \left( e^{-c\langle y \rangle} + \langle y \rangle^3 |w| \langle w \rangle^{4/d} \right)$$

$$(5.8) \quad |R_{NL,p}(w) - R_{NL,\tilde{p}}(w)| \lesssim \frac{1}{\tau^{c(p)+1}} \|p - \tilde{p}\|_{c(p), \tau_0} \langle y \rangle^3 |w|^2 \langle w \rangle^{4/d-1},$$

$$(5.9) \quad |R_{L,p}(w) - R_{L,\tilde{p}}(w)| \lesssim \frac{1}{\tau^{c(p)+1}} \|p - \tilde{p}\|_{c(p), \tau_0} \langle y \rangle |w|.$$

*Proof.* Estimates (5.5) and (5.6) follow from the definition of  $R_{NL}$  and of  $R_L$  (see (3.7)), and, for (5.6), from Assumption (3.1).

Next we estimate  $|g_p - g_{\tilde{p}}|$ . Notice that

$$\left| \frac{1}{\gamma^{-1}(\tau)} - \frac{1}{\tilde{\gamma}^{-1}(\tau)} \right| = \left| \frac{\tilde{\gamma}^{-1}(\tau) - \gamma^{-1}(\tau)}{\gamma^{-1}(\tau)\tilde{\gamma}^{-1}(\tau)} \right|.$$

We have

$$\frac{d\gamma^{-1}(\tau)}{d\tau} = q_4(\tau)^2 = (1 + q_{4r}(\tau))^2.$$

Therefore, by Lemma 3.4:

$$\left| \frac{d}{d\tau} (\tilde{\gamma}^{-1}(\tau) - \gamma^{-1}(\tau)) \right| \lesssim |q_{4r}(\tau) - \tilde{q}_{4r}(\tau)| \lesssim \frac{1}{\tau^{c(p)-2}} \|p - \tilde{p}\|_{c(p), \tau_0},$$

Integrating between  $\tau_0$  and  $\tau$  and using that  $\gamma(\tau_0) = \tilde{\gamma}(\tau_0) = \tau_0$  we get, since  $c(p) < 3$ ,

$$|\tilde{\gamma}^{-1}(\tau) - \gamma^{-1}(\tau)| \lesssim \frac{1}{\tau^{c(p)-3}} \|p - \tilde{p}\|_{c(p), \tau_0}.$$

This rather poor estimate yields the more interesting one

$$\left| \frac{1}{\gamma^{-1}(\tau)} - \frac{1}{\tilde{\gamma}^{-1}(\tau)} \right| \lesssim \frac{1}{\tau^{c(p)-1}} \|p - \tilde{p}\|_{c(p), \tau_0}.$$

Denote  $\lambda = q_4/\gamma^{-1}$ , and  $\tilde{\lambda}$  its counterpart associated to  $\tilde{p}$ . We can write

$$g_p(\tau, y) - g_{\tilde{p}}(\tau, y) = g(\lambda y + \lambda q_2) - g(\tilde{\lambda} y + \tilde{\lambda} \tilde{q}_2).$$

Note that Assumption 3.1 implies

$$|g(a) - g(b)| \lesssim |a - b| (|a|^2 + |b|^2).$$

Invoking Lemma 3.2 and Lemma 3.4, we deduce

$$\begin{aligned} |g_p(\tau, y) - g_{\tilde{p}}(\tau, y)| &\lesssim \left( |\lambda - \tilde{\lambda}| |y| + |\lambda q_2 - \tilde{\lambda} \tilde{q}_2| \right) \left( \lambda^2 + \tilde{\lambda}^2 \right) \langle y \rangle^2 \\ &\lesssim \frac{1}{\tau^{c(p)-1}} \|p - \tilde{p}\|_{c(p), \tau_0} \langle y \rangle \times \frac{1}{\tau^2} \langle y \rangle^2 \\ &\lesssim \frac{1}{\tau^{c(p)+1}} \|p - \tilde{p}\|_{c(p), \tau_0} \langle y \rangle^3. \end{aligned}$$

We have a similar estimate on  $V_p - V_{\tilde{p}} = \lambda^2 V(\lambda(y + q_2)) - \tilde{\lambda}^2 V(\tilde{\lambda}(y + \tilde{q}_2))$ :

$$|V_p(\tau, y) - V_{\tilde{p}}(\tau, y)| \lesssim \frac{1}{\tau^{c(p)+1}} \|p - \tilde{p}\|_{c(p), \tau_0} \langle y \rangle.$$

By definition, we have (without splitting the terms as in (3.7))

$$-iR_p(w) = g_p \times |Q + w|^{4/d} (Q + w) - V_p \times (Q + w) - F(Q) - \ell(w).$$

We also have

$$|Q + w|^{1+4/d} \lesssim Q^{1+4/d} + |w|^{1+4/d},$$

and the estimate (5.7) follows.

Estimates (5.8) and (5.9) of the lemma are a straightforward consequence of the definitions (3.7) of  $R_{L,p}$  and  $R_{NL,p}$ , and of the above estimates.  $\square$

We introduce the notation, for  $1 \leq j \leq 6$ ,

$$(5.10) \quad D_j(p)(\tau) = \langle P_S R_p(w) + P_S Z_p(w), m_j \rangle.$$

For  $p = (p_1, \dots, p_5)$ ,  $c > 0$ , denote, once and for all,

$$|p(\tau)| = \max_{1 \leq k \leq 5} |p_k(\tau)|, \quad \|p\|_{c, \tau_0} = \max_{1 \leq k \leq 5} \|p_k\|_{c, \tau_0}.$$

**Lemma 5.2.** *Let Assumption 3.1 be satisfied. If*

$$\|p\|_{c(p), \tau_0} \leq 1,$$

where  $c(p) > 2$ , then we have for all  $\tau \geq \tau_0$ ,

$$(5.11) \quad \begin{aligned} |D_j(p)(\tau)| &\lesssim \|w\|_{H^1}^{1+4/d} + \|w\|_{H^1}^2 + \frac{1}{\tau^2} \|w\|_{L^2} + |p(\tau)| \|w\|_{L^2} \\ &+ \begin{cases} 0 & \text{if } j = 2, 4, 6 \\ 1/\tau^{1+c(p)^-} + 1/\tau^4 & \text{if } j = 1, 5 \\ 1/\tau^3 & \text{if } j = 3. \end{cases} \end{aligned}$$

*Proof.* Taking the  $L^2$ -norm in  $y$  in the pointwise estimate (5.5), Sobolev embedding yields:

$$\|R_{NL}(w)(\tau)\|_{L^2} \lesssim \sum_{2 \leq k \leq 1+4/d} \|w(\tau)\|_{H^1}^k.$$

By the pointwise estimate (5.6) we get

$$\|R_L(w)(\tau)\|_{L^2} \lesssim \frac{1}{\tau^2} \|w(\tau)\|_{L^2}.$$

These estimates yield, since  $m_j \in \mathcal{S}(\mathbf{R}^d)$ ,

$$|D_j(p)(\tau)| \lesssim \sum_{2 \leq k \leq 1+4/d} \|w(\tau)\|_{H^1}^k + \frac{1}{\tau^2} \|w(\tau)\|_{L^2} + |\langle R_0, m_j \rangle| + |p(\tau)| \|w(\tau)\|_{L^2},$$

Notice that  $R_0$  is purely imaginary, that  $m_2, m_4$  and  $m_6$  are real, and thus

$$\forall j \in \{2, 4, 6\} \quad \langle R_0, m_j \rangle = 0,$$

which yields the first case in (5.11).

By Assumption 3.1, the Taylor expansion of  $g$  near the origin reads:

$$g(x) = 1 + \sum_{|\alpha|=3} c_\alpha x^\alpha + \mathcal{O}(|x|^4).$$

In view of Lemma 3.2, we infer

$$\begin{aligned} g_p(\tau, y) &= 1 + \frac{q_4^3}{(\gamma^{-1}(\tau))^3} \sum_{|\alpha|=3} c_\alpha (y + q_2)^\alpha + \mathcal{O}\left(\frac{|q_4(y + q_2)|^4}{\tau^4}\right) \\ &= 1 + \frac{q_4^3}{(\gamma^{-1}(\tau))^3} \sum_{|\alpha|=3} c_\alpha y^\alpha + \mathcal{O}\left(\frac{\langle y \rangle^2}{\tau^{3+c(q_2)}}\right) + \mathcal{O}\left(\frac{\langle y \rangle^4}{\tau^4}\right). \end{aligned}$$

Notice that if  $j \in \{1, 5\}$ ,  $m_j$  is a radial function. Thus if  $|\alpha| = 3$ ,

$$\int y^\alpha m_j = 0.$$

Arguing similarly on  $V$ , we infer

$$|\langle P_S R_0, m_j \rangle| = |\langle R_0, m_j \rangle| = |\langle (1 - g_p)F(Q) + V_p Q, m_j \rangle| \lesssim \frac{1}{\tau^{3+c(q_2)}} + \frac{1}{\tau^4}.$$

Lemma 3.2 then yields the second case in (5.11). To prove the third case, we use the pointwise estimate

$$|R_0| \lesssim \left\langle \frac{y}{\tau} \right\rangle^3 Q^{1+4/d} + \frac{1}{\tau^2} \left( \mathbf{1}_{|y|>\tau} + \frac{\langle y \rangle}{\tau} \mathbf{1}_{|y|\leq\tau} \right) Q.$$

Since  $Q$  decays exponentially, this yields

$$\|R_0(\tau)\|_{L^2} \lesssim \frac{1}{\tau^3},$$

and the third case in (5.11) follows.  $\square$

**5.2. Control of the secular modes by projection.** Our approach consists in defining  $\Phi$  so as to limit the role of the secular modes (to the sixth mode).

**Proposition 5.3.** *Let  $\varepsilon \in ]0, 1/3[$ . Then if  $\tau_0 > 0$  is large enough we have the following property. Let  $w \in C([\tau_0, \infty[; H^1)$  with*

$$(5.12) \quad \sup_{\tau \geq \tau_0} \tau^{2-\varepsilon} \|w(\tau)\|_{H^1} \leq 1.$$

*There exists a unique modulation parameter  $p = p(w)$ , such that, for  $\tau \geq \tau_0$ ,*

$$(5.13) \quad |p(\tau)| \leq \frac{1}{\tau^{3-3\varepsilon}},$$

*and*

$$\Phi_1(w)(\tau) = \int_\tau^\infty e^{i(\tau-\sigma)L} (P_S R(w) + P_S Z_p(w) + Z_p(Q)) d\sigma \in \text{span } n_6.$$

*Furthermore, for this choice of  $p$*

$$(5.14) \quad \forall \tau \geq \tau_0, \quad |\langle \Phi_1(w)(\tau), m_6 \rangle| \leq \frac{C}{\tau^{3-2\varepsilon}},$$

*where  $C$  does not depend on  $w$ .*

*Proof.* We introduce, for arbitrary  $p$ ,

$$(5.15) \quad d_j(p)(\tau) = \langle P_S R_p(w) + P_S Z_p(w) + Z_p(Q), m_j \rangle = \langle D_j(w) + Z_p(Q), m_j \rangle.$$

By the explicit expression (5.1) of  $Z_p(Q)$  we get the relations between  $d_j$  and  $D_j$ :

$$\begin{aligned} d_1(p) &= D_1(p) + \alpha_0 p_1 - 2\alpha_0 \gamma_0 p_5, & d_2(p) &= D_2(p) - \beta_0 p_2, & d_3(p) &= D_3(p) - \beta_0 p_3 \\ d_4(p) &= D_4(p) + \alpha_0 p_4, & d_5(p) &= D_5(p) + 2\alpha_0 p_5, & d_6(p) &= D_6(p), \end{aligned}$$

where  $\alpha_0, \beta_0, \gamma_0$  are real constants,  $\alpha_0, \beta_0 > 0$ . From (5.12), we know that  $w$  tends to zero as  $\tau \rightarrow +\infty$ . Recalling that  $Z_p(Q) \in S$  for any parameter  $p$ , the stability of  $S$  by  $e^{itL}$  shows that  $\Phi_1(w) \in S$ . Denote, as in Proposition 4.1,

$$\Phi_1(w)(\tau) = \sum_{j=1}^6 \nu_j(\tau) n_j.$$

By Proposition 4.1,

$$\nu_6(\tau) = - \int_{\tau}^{+\infty} d_6 = - \int_{\tau}^{+\infty} D_6(p),$$

which is well-defined in view of (5.11), (5.12), (5.13). We want  $\nu_j$  to vanish, for  $1 \leq j \leq 5$ , so in view of Proposition 4.1, we would like to impose

$$d_2 = d_3 = d_4 = 0 \quad ; \quad d_5 = -2\nu_6 \quad ; \quad d_1 = 2\gamma_0\nu_6.$$

The proposition follows if we get a fixed point  $p$  in the unit ball of  $(W(3-3\varepsilon, \tau_0))^5$  (the space  $W$  is defined by (3.9)) for the operator  $\Psi(p) = \tilde{p} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5)$ :

$$\begin{aligned} \tilde{p}_5 &= \frac{1}{2\alpha_0} \left( -D_5(p) + 2 \int_{\tau}^{\infty} D_6(p) \right) \quad ; \quad \tilde{p}_4 = -\frac{1}{\alpha_0} D_4(p) \\ \tilde{p}_j &= \frac{1}{\beta_0} D_j(p), \quad j = 2, 3 \quad ; \quad \tilde{p}_1 = -\frac{1}{\alpha_0} D_1(p) - \frac{\gamma_0}{\alpha_0} D_5(p). \end{aligned}$$

Let  $B$  be the closed unit ball in  $(W(3-3\varepsilon, \tau_0))^5$ . We first show that  $B$  is stable by  $\Psi$ . By (5.12), and since  $0 < \varepsilon < 1/2$ , we have, for  $\tau \geq \tau_0 \gg 1$ ,

$$(5.16) \quad \begin{aligned} \|w\|_{H^1}^{1+4/d} + \|w\|_{H^1}^2 + \frac{1}{\tau^2} \|w\|_{L^2} + |p(\tau)| \|w\|_{L^2} \\ \lesssim \frac{1}{\tau^{(1+4/d)(2-\varepsilon)}} + \frac{1}{\tau^{4-2\varepsilon}} + \frac{1}{\tau^{4-\varepsilon}} + \frac{1}{\tau^{5-4\varepsilon}} \leq \frac{1}{\tau^{4-2\varepsilon}}. \end{aligned}$$

By definition of  $\Psi$ , (5.16) and the estimates (5.11) on  $D_j$ , we get, for  $j \in \{2, 3, 4\}$

$$\begin{aligned} |\tilde{p}_j(\tau)| &\lesssim |D_j(p)(\tau)| \lesssim \|w\|_{H^1}^{1+4/d} + \|w\|_{H^1}^2 + \frac{1}{\tau^2} \|w\|_{L^2} + \frac{1}{\tau^3} + |p(\tau)| \|w\|_{L^2} \\ &\leq \frac{C}{\tau^3} \leq \frac{1}{\tau^{3-3\varepsilon}}, \end{aligned}$$

if  $\tau \geq \tau_0$  and  $\tau_0$  is chosen sufficiently large. In view of the estimates (5.11) and (5.16), we have

$$(5.17) \quad \int_{\tau}^{+\infty} |D_6(p)(\sigma)| d\sigma \lesssim \int_{\tau}^{+\infty} \frac{1}{\sigma^{4-2\varepsilon}} d\sigma \lesssim \frac{1}{\tau^{3-2\varepsilon}},$$

provided  $\tau_0 \gg 1$ . By the estimate (5.11) (second case) and (5.16) we get, taking again  $\tau \geq \tau_0 \gg 1$ ,

$$|\tilde{p}_5(\tau)| \lesssim |D_5(p)(\tau)| + \int_{\tau}^{+\infty} |D_6(p)(\sigma)| d\sigma \lesssim \frac{1}{\tau^{4-2\varepsilon}} + \frac{1}{\tau^{4-3\varepsilon}} + \frac{1}{\tau^{3-2\varepsilon}} \leq \frac{1}{\tau^{3-3\varepsilon}},$$

and similarly

$$|\tilde{p}_1(\tau)| \lesssim |D_1(p)(\tau)| + |D_5(p)(\tau)| \leq \frac{1}{\tau^{3-3\varepsilon}}.$$

As a consequence  $\tilde{p} = \Psi(p) \in B$ , and the stability property of  $\Psi$  is settled.

It remains to prove the contraction property of  $\Psi$ ,

$$(5.18) \quad \|\Psi(p) - \Psi(\tilde{p})\|_{3-3\varepsilon, \tau_0} \leq \kappa \|p - \tilde{p}\|_{3-3\varepsilon, \tau_0},$$

for all  $p, \tilde{p} \in B$ , with  $\kappa < 1$ . In view of the definition of  $\Psi$ , it is enough to show that if  $\iota$  is small, and  $\tau_0$  is chosen large enough, we have, for  $\tau \geq \tau_0$ ,

$$(5.19) \quad \|D_j(p) - D_j(\tilde{p})\|_{3-3\varepsilon, \tau_0} \leq \iota \|p - \tilde{p}\|_{3-3\varepsilon, \tau_0},$$

for  $1 \leq j \leq 5$ , and

$$(5.20) \quad \|D_6(p) - D_6(\tilde{p})\|_{4-3\varepsilon, \tau_0} \leq \iota \|p - \tilde{p}\|_{3-3\varepsilon, \tau_0}.$$

Recall that by definition,

$$D_j(p) = \langle P_S R_p(w) + P_S Z_p(w), m_j \rangle.$$

We have

$$\begin{aligned} |\langle P_S Z_p(w) - P_S Z_{\tilde{p}}(w), m_j \rangle| &\lesssim |p_k(\tau) - \tilde{p}_k(\tau)| \|w(\tau)\|_{L^2} \\ &\lesssim \frac{1}{\tau^{2-\varepsilon}} |p_k(\tau) - \tilde{p}_k(\tau)| \lesssim \frac{1}{\tau^{5-4\varepsilon}} \|p - \tilde{p}\|_{3-3\varepsilon, \tau_0}. \end{aligned}$$

By the pointwise estimate (5.7) we get

$$|\langle R_p(w) - R_{\tilde{p}}(w), m_j \rangle| \lesssim \frac{1}{\tau^{4-3\varepsilon}} \|p - \tilde{p}\|_{3-3\varepsilon, \tau_0}.$$

Taking  $\tau_0$  larger if necessary, we deduce the estimate (5.19).

To prove (5.20), we argue as in the proof of Lemma 5.2:

$$D_6(p) = \langle R_{NL}(w) + R_L(w) + P_S Z_p(w), m_6 \rangle,$$

that is, the contribution of  $R_0$  vanishes, since  $R_0$  is purely imaginary and  $m_6$  is real. We then invoke inequalities (5.8) and (5.9) of Lemma 5.1, to infer:

$$\begin{aligned} |\langle R_p(w) - R_{\tilde{p}}(w), m_6 \rangle| &\lesssim \frac{1}{\tau^{4-3\varepsilon}} \|p - \tilde{p}\|_{3-3\varepsilon, \tau_0} \|w(\tau)\|_{H^1} \langle \|w(\tau)\|_{H^1} \rangle^{4/d} \\ &\lesssim \frac{1}{\tau^{4-3\varepsilon}} \times \frac{1}{\tau^{2-\varepsilon}} \|p - \tilde{p}\|_{3-3\varepsilon, \tau_0}, \end{aligned}$$

which gives (5.20) and concludes the proof of the contraction property (5.18).

Therefore there exists a fixed point  $p \in B$  for  $\Psi$ . For this  $p$ , we have  $\nu_j(\tau) = 0$ , for  $1 \leq j \leq 5$ . Moreover, since

$$\Phi_1(w)(\tau) = \nu_6(\tau)n_6,$$

it remains to show (5.14), that is, to check that  $|\nu_6(\tau)| \lesssim 1/\tau^{3-2\varepsilon}$ . This follows immediately from (5.17) and the fact that  $\nu'_6 = D_6$ .  $\square$

## 6. THE NON-SECULAR PART

As announced in the previous paragraph, we now study the  $M$ -component of  $w$ , which has to solve (5.3). For this, we consider the operator  $\Phi_2$ , that is, we study the equation

$$(6.1) \quad \partial_\tau \phi + iL\phi - P_M Z_p(\phi) = F \quad ; \quad \|\phi(\tau)\|_\Sigma \xrightarrow{\tau \rightarrow +\infty} 0,$$

where  $F \in C([\tau_0, \infty[; M)$ .

**6.1. Energy estimates.** Recall the important property, established in [34]: on  $M$ , the  $H^1$  norm  $\|\cdot\|_{H^1}$  is equivalent to  $\|\cdot\|_M$ , where

$$\|\phi\|_M^2 = \operatorname{Re} \langle L\phi, \phi \rangle.$$

**Lemma 6.1.** *Let  $\kappa \in \{0, 1, 2\}$ , and  $F \in L^1([\tau_0, \infty[; \Sigma^{2\kappa+1})$ . Suppose that  $\phi \in C([\tau_0, \infty[; M \cap \Sigma^{2\kappa+1})$  solves (6.1) and tends to 0 in  $\Sigma^{2\kappa+1}$  as  $\tau \rightarrow +\infty$ . There*

exists  $C > 0$  such that for all  $\tau \geq \tau_0$ , the following holds:

$$\begin{aligned} \|\phi(\sigma)\|_{H^{2\kappa+1}} &\leq C \int_{\tau}^{\infty} \left( \|F(\sigma)\|_{H^{2\kappa+1}} \right. \\ &\quad \left. + |p(\sigma)| \left( \|\phi(\sigma)\|_{H^{2\kappa+1}} + \|\langle y \rangle \langle \nabla \rangle^{2\kappa} \phi(\sigma)\|_{L^2} \right) \right) d\sigma, \\ \|\langle y \rangle^{2\kappa+1} \phi(\sigma)\|_{L^2} &\leq C \int_{\tau}^{\infty} \left( \|\langle y \rangle^{2\kappa+1} F(\sigma)\|_{L^2} + \|\langle y \rangle^{2\kappa} \nabla \phi(\sigma)\|_{L^2} + \|\phi(\sigma)\|_{L^2} \right. \\ &\quad \left. + |p(\sigma)| \|\langle y \rangle^{2\kappa+1} \phi(\sigma)\|_{L^2} \right) d\sigma. \end{aligned}$$

*Proof.* We begin with the first inequality in the case  $\kappa = 0$ . Multiply (6.1) by  $L\bar{\phi}$ , integrate with respect to  $y$  and consider the real part:

$$\operatorname{Re} \int_{\mathbf{R}^d} \partial_{\tau} \phi L\bar{\phi} - \operatorname{Re} \int_{\mathbf{R}^d} P_M Z_p(\phi) L\bar{\phi} = \operatorname{Re} \int_{\mathbf{R}^d} F L\bar{\phi}.$$

We readily check the identity

$$\operatorname{Re} \int_{\mathbf{R}^d} \partial_{\tau} \phi L\bar{\phi} = \frac{1}{2} \frac{d}{d\tau} \|\phi\|_M^2.$$

A straightforward integration by parts yields

$$\left| \int_{\mathbf{R}^d} F L\bar{\phi} \right| \lesssim \|F\|_{H^1} \|\phi\|_{H^1}.$$

It remains to estimate

$$(6.2) \quad \operatorname{Re} \int_{\mathbf{R}^d} P_M Z_p(\phi) L\bar{\phi} = \operatorname{Re} \int_{\mathbf{R}^d} Z_p(\phi) L\bar{\phi} - \operatorname{Re} \int_{\mathbf{R}^d} P_S Z_p(\phi) L\bar{\phi}.$$

We start with the first term. Recall that

$$Z_p(\phi) = -i(p_1 + p_3 \cdot y + p_5 |y|^2) \phi + p_2 \cdot \nabla \phi + p_4 \left( \frac{d}{2} + y \cdot \nabla \right) \phi$$

and

$$L\bar{\phi} = -\Delta \bar{\phi} + \bar{\phi} - \left( \frac{2}{d} + 1 \right) Q^{4/d} \bar{\phi} - \frac{2}{d} Q^{4/d} \phi.$$

We have, by elementary integration by parts:

$$\begin{aligned} \operatorname{Re} i \int p_1 \phi \Delta \bar{\phi} &= 0, \\ \left| \operatorname{Re} i \int p_3 \cdot y \phi \Delta \bar{\phi} \right| &= \left| \operatorname{Im} \int \phi p_3 \cdot \nabla \bar{\phi} \right| \leq |p_3| \|\phi\|_{L^2} \|\phi\|_{H^1}, \\ \left| \operatorname{Re} i \int p_5 |y|^2 \phi \Delta \bar{\phi} \right| &= 2 \left| p_5 \operatorname{Im} \int \phi y \cdot \nabla \bar{\phi} \right| \leq 2|p_5| \|\phi\|_{H^1} \|\langle y \rangle \phi\|_{L^2}, \\ \operatorname{Re} \int p_2 \cdot \nabla \phi \Delta \bar{\phi} &= 0, \\ \operatorname{Re} \int p_4 \left( \frac{d}{2} + y \cdot \nabla \right) \phi \Delta \bar{\phi} &= -p_4 \int |\nabla \phi|^2. \end{aligned}$$

We infer:

$$\left| \operatorname{Re} \int Z_p(\phi) \Delta \bar{\phi} \right| \leq C(|p_3| + |p_4|) \|\phi\|_{H^1}^2 + |p_5| \|\phi\|_{H^1} \|\langle y \rangle \phi\|_{L^2}.$$

We easily deduce that the first term in (6.2) is controlled by

$$\left| \operatorname{Re} \int Z_p(\phi) L\bar{\phi} \right| \leq C|p| \|\phi\|_{H^1} (\|\phi\|_{H^1} + \|\langle y \rangle \phi\|_{L^2}).$$

For the remaining second term in (6.2) we use the structure of the space  $S$ ,

$$\left| \operatorname{Re} \int P_S Z_p(\phi) \bar{L}\phi \right| = \left| \sum_{1 \leq j \leq 6} \langle Z_p(\phi), m_j \rangle \operatorname{Re} \int n_j L\bar{\phi} \right|.$$

Integrating by parts both in the scalar product and in the integral, we get

$$\left| \operatorname{Re} \int P_S Z_p(\phi) \bar{L}\phi \right| \leq C|p| \|\phi\|_{L^2}^2.$$

Summarizing, we have obtained

$$\frac{d}{d\tau} \|\phi\|_M^2 \leq C \|F\|_{H^1} \|\phi\|_{H^1} + C|p| \|\phi\|_{H^1} (\|\phi\|_{H^1} + \|\langle y \rangle \phi\|_{L^2}).$$

Since the  $M$ -norm and the  $H^1$ -norm are equivalent on  $M$ , the first inequality of the lemma follows in the case  $\kappa = 0$ .

Let  $\kappa \geq 1$ . We write (6.1) as

$$\partial_\tau \phi + iL\phi - Z_p(\phi) = F - P_S Z_p(\phi).$$

Applying the operator  $(iL)^\kappa$  we get

$$\partial_\tau ((iL)^\kappa \phi) + iL((iL)^\kappa \phi) - (iL)^\kappa Z_p(\phi) = (iL)^\kappa F - (iL)^\kappa P_S Z_p(\phi).$$

Hence

$$\partial_\tau ((iL)^\kappa \phi) + iL((iL)^\kappa \phi) - Z_p(iL)^\kappa \phi = (iL)^\kappa F + [(iL)^\kappa, Z_p] \phi - (iL)^\kappa P_S Z_p(\phi),$$

where  $[(iL)^\kappa, Z_p]$  denotes the commutator of the operators  $(iL)^\kappa$  and  $Z_p$ . By direct computation, the commutator  $[iL, Z_p]$  is an operator of order 2 in  $(\langle y \rangle, \nabla)$ , which is only of order 1 in  $\langle y \rangle$  and whose coefficients are multiples of  $p_1, \dots, p_5$ :

$$\begin{aligned} [iL, Z_p] \phi &= \left[ iL, -i(p_1 + p_3 \cdot y + p_5 |y|^2) + p_2 \cdot \nabla + p_4 \left( \frac{d}{2} + y \cdot \nabla \right) \right] \phi \\ &= -i \left[ \Delta, -i(p_3 \cdot y + p_5 |y|^2) + p_4 y \cdot \nabla \right] \phi \\ &\quad - i \left( \frac{2}{d} + 1 \right) \left[ Q^{4/d}, p_2 \cdot \nabla + p_4 y \cdot \nabla \right] \phi - i \frac{2}{d} \left[ Q^{4/d}, p_2 \cdot \nabla + p_4 y \cdot \nabla \right] \bar{\phi} \\ &= -2(p_3 \cdot \nabla + 2p_5 y \cdot \nabla + dp_5 + ip_4 \Delta) \phi \\ &\quad + i \left( \frac{2}{d} + 1 \right) \left( p_2 \cdot \nabla (Q^{4/d}) + p_4 y \cdot \nabla (Q^{4/d}) \right) \phi \\ &\quad + i \frac{2}{d} \left( p_2 \cdot \nabla (Q^{4/d}) + p_4 y \cdot \nabla (Q^{4/d}) \right) \bar{\phi}. \end{aligned}$$

Furthermore

$$[(iL)^\kappa, Z_p] = \sum_{j=0}^{\kappa-1} (iL)^j [iL, Z_p] (iL)^{\kappa-j-1}.$$

Hence

$$\|[(iL)^\kappa, Z_p] \phi\|_{H^1} \leq |p| \left( \|\phi\|_{H^{2\kappa+1}} + \left\| \langle y \rangle \langle \nabla \rangle^{2\kappa} \phi \right\|_{L^2} \right).$$

Notice also that

$$\|(iL)^\kappa P_S(Z_p(\phi))\|_{H^1} = \left\| \sum_{1 \leq j \leq 6} \langle Z_p(\phi), m_j \rangle (iL)^\kappa n_j \right\|_{H^1} \lesssim |p| \|\phi\|_{L^2}.$$

Denoting  $\phi_\kappa = (iL)^\kappa \phi$ , we see that  $\phi_\kappa(\tau) \in M$  and that it solves

$$\partial_\tau \phi_\kappa + iL\phi_\kappa - P_M Z_p(\phi_\kappa) = (iL)^\kappa F + [(iL)^\kappa, Z_p] \phi - (iL)^\kappa P_S Z_p(\phi) + P_S Z_p(\phi_\kappa).$$

From the case  $\kappa = 0$  and the previous estimates, we get:

$$\begin{aligned} \|(iL)^\kappa \phi(\tau)\|_{H^1} &\lesssim \int_\tau^{+\infty} \left( \|F(\sigma)\|_{H^{2\kappa+1}} \right. \\ &\quad \left. + |p(\sigma)| \left( \|\phi(\sigma)\|_{H^{2\kappa+1}} + \|\langle y \rangle \langle \nabla \rangle^{2\kappa} \phi(\sigma)\|_{L^2} \right) \right) d\sigma. \end{aligned}$$

Noting that for a large constant  $K$ , we have for all  $f \in M$ ,

$$\|(iL)^\kappa f\|_{H^1} + K\|f\|_{H^1} \approx \|f\|_{H^{2\kappa+1}},$$

and using the case  $\kappa = 0$  to bound  $\|\phi(\tau)\|_{H^1}$ , we get the first estimate of the lemma.

To conclude, we estimate the momenta: for  $s \in \mathbf{N}$ , we compute more generally

$$\begin{aligned} \frac{1}{2} \frac{d}{d\tau} \int \langle y \rangle^{2s} |\phi|^2 &= \operatorname{Re} \int \langle y \rangle^{2s} \partial_\tau \phi \bar{\phi} = \operatorname{Re} \int \langle y \rangle^{2s} (-iL\phi + P_M Z_p(\phi) + F) \bar{\phi} \\ &= \operatorname{Im} \int \langle y \rangle^{2s} L\phi \bar{\phi} + \operatorname{Re} \int \langle y \rangle^{2s} P_M Z_p(\phi) \bar{\phi} + \operatorname{Re} \int \langle y \rangle^{2s} F \bar{\phi}. \end{aligned}$$

By a direct integration by parts

$$(6.3) \quad \left| \operatorname{Im} \int \langle y \rangle^{2s} L\phi \bar{\phi} \right| \lesssim \|\langle y \rangle^{s-1} \nabla \phi\|_{L^2} \|\langle y \rangle^s \phi\|_{L^2} + \|\phi\|_{L^2}^2.$$

Furthermore,

$$\operatorname{Re} \int \langle y \rangle^{2s} P_M Z_p(\phi) \bar{\phi} = \operatorname{Re} \int \langle y \rangle^{2s} Z_p(\phi) \bar{\phi} - \operatorname{Re} \int \langle y \rangle^{2s} P_S Z_p(\phi) \bar{\phi}.$$

On the one hand,

$$\begin{aligned} \left| \operatorname{Re} \int \langle y \rangle^{2s} Z_p(\phi) \bar{\phi} \right| &= \left| \operatorname{Re} \int \langle y \rangle^{2s} p_2 \cdot \nabla \phi \bar{\phi} + p_4 \langle y \rangle^{2s} \left( \frac{d}{2} + y \cdot \nabla \right) \phi \bar{\phi} \right| \\ &\lesssim \max_{k=2,4} |p_k| \|\langle y \rangle^s \phi\|_{L^2}^2. \end{aligned}$$

On the other hand,

$$\left| \operatorname{Re} \int \langle y \rangle^{2s} P_S Z_p(\phi) \bar{\phi} \right| = \left| \sum_{1 \leq j \leq 6} \langle Z_p(\phi), m_j \rangle \operatorname{Re} \int n_j \langle y \rangle^{2s} \bar{\phi} \right| \lesssim |p| \|\phi\|_{L^2}^2.$$

Hence

$$(6.4) \quad \left| \operatorname{Re} \int \langle y \rangle^{2s} P_M Z_p(\phi) \bar{\phi} \right| \lesssim |p| \|\langle y \rangle^s \phi\|_{L^2}^2.$$

Combining (6.3), (6.4), we obtain the second estimate of the lemma, concluding this proof.  $\square$

**6.2. Refined a priori estimates.** In the sequel, we consider  $0 < \varepsilon < 1/3$ , and extra smallness assumptions will be precised when needed.

**Lemma 6.2.** *Let  $\tau_0 > 0$  and  $p \in C([\tau_0, \infty])^5$  such that*

$$\forall \tau \geq \tau_0, \quad |p(\tau)| \leq \frac{1}{\tau^{3-3\varepsilon}}.$$

For  $a, b > 0$ . Let

$$\begin{aligned} X(a, b, \delta) &= \left\{ \phi \in C([\tau_0, \infty]; M \cap \Sigma^\delta), \quad \|\phi\|_{X(a, b, \delta)} < \infty \right\}, \text{ where} \\ \|\phi\|_{X(a, b, \delta)} &= \sup_{\tau \geq \tau_0} \tau^a \|\phi(\tau)\|_{H^\delta} + \sup_{\tau \geq \tau_0} \tau^b \left\| \langle y \rangle^\delta \phi(\tau) \right\|_{L^2}. \end{aligned}$$

Assume that  $F \in X(a+1+\eta, b+1+\eta, \delta)$ , with  $\eta > 0$  and

$$(6.5) \quad \delta < a - b < \delta(2 - 3\varepsilon), \quad 1 \leq \delta \leq 5.$$

Let  $\mu > 0$ . If  $\tau_0$  is sufficiently large, every solution  $\phi \in X(a, b, \delta)$  of (6.1) satisfies

$$(6.6) \quad \|\phi\|_{X(a,b,\delta)} \leq \mu \|F\|_{X(a+1+\eta, b+1+\eta, \delta)}.$$

*Proof.* We prove first the lemma in the cases  $\delta = 1$  and  $\delta = 5$ .

**First case:**  $\delta = 1$ . Denote by

$$M_1 = \|F\|_{X(a+1+\eta, b+1+\eta, 1)}.$$

The  $H^1$ -estimate and the momentum estimate with  $\kappa = 0$  of Lemma 6.1 read, along with the assumption on  $p$ :

$$\begin{aligned} \|\phi(\tau)\|_{H^1} &\leq C \int_{\tau}^{\infty} \left( \frac{M_1}{\sigma^{a+1+\eta}} + \frac{1}{\sigma^{3-3\varepsilon}} (\|\phi(\sigma)\|_{H^1} + \|\langle y \rangle \phi(\sigma)\|_{L^2}) \right) d\sigma, \\ \|\langle y \rangle \phi(\tau)\|_{L^2} &\leq C \int_{\tau}^{\infty} \left( \frac{M_1}{\sigma^{b+1+\eta}} + \frac{1}{\sigma^{3-3\varepsilon}} \|\langle y \rangle \phi(\sigma)\|_{L^2} + \|\phi(\sigma)\|_{H^1} \right) d\sigma. \end{aligned}$$

We apply Lemma A.1 with the following data:

$$\alpha_1 = \beta_1 = 0 ; \alpha_2 = \beta_2 = 1 ; a_1 = a_2 = b_1 = 2 - 3\varepsilon ; b_2 = -1.$$

This is possible under the assumptions  $a, b > 0$  and  $1 < a - b < 2 - 3\varepsilon$ , which are fulfilled in the context of Lemma 6.2. We then have

$$\|\phi\|_{X(a,b,1)} \leq \mu \|F\|_{X(a+1+\eta, b+1+\eta, 1)},$$

for  $\tau_0$  sufficiently large.

**Second case:**  $\delta = 5$ . Denote by

$$M_5 = \|F\|_{X(a+1+\eta, b+1+\eta, 5)}.$$

To proceed in a similar way as in the first case, we use interpolation estimates (B.8) and (B.9). By Lemma 6.1 in the case  $\kappa = 2$ , we obtain

$$\begin{aligned} \|\phi(\sigma)\|_{H^5} &\leq C \int_{\tau}^{\infty} \left( \frac{M_5}{\sigma^{a+1+\eta}} \right. \\ &\quad \left. + \frac{1}{\sigma^{3-3\varepsilon}} \left( \|\phi(\sigma)\|_{H^5} + \|\langle y \rangle^5 \phi(\sigma)\|_{L^2}^{1/5} \|\phi(\sigma)\|_{H^5}^{4/5} \right) \right) d\sigma, \\ \|\langle y \rangle^5 \phi(\sigma)\|_{L^2} &\leq C \int_{\tau}^{\infty} \left( \frac{M_5}{\sigma^{b+1+\eta}} + \|\langle y \rangle^5 \phi(\sigma)\|_{L^2}^{4/5} \|\phi(\sigma)\|_{H^5}^{1/5} + \|\phi(\sigma)\|_{L^2} \right. \\ &\quad \left. + \frac{1}{\sigma^{3-3\varepsilon}} \|\langle y \rangle^5 \phi(\sigma)\|_{L^2} \right) d\sigma. \end{aligned}$$

We apply Lemma A.1 with the following data:

$$\alpha_1 = \beta_3 = 0 ; \beta_2 = 1 ; \alpha_2 = \beta_1 = \frac{1}{5} ; a_1 = a_2 = b_3 = 2 - 3\varepsilon ; b_1 = b_2 = -1.$$

This is possible under the assumptions  $a, b > 0$  and  $5 < a - b < 10 - 15\varepsilon$ , which are fulfilled in the context of Lemma 6.2. We then have

$$\|\phi\|_{X(a,b,5)} \leq \mu \|F\|_{X(a+1+\eta, b+1+\eta, 5)},$$

for  $\tau_0$  sufficiently large. Summarizing, we have obtained the lemma in the following cases:

$$\begin{aligned} 1 &< a - b < 2 - 3\varepsilon \text{ and } \delta = 1, \\ 5 &< a - b < 5(2 - 3\varepsilon) \text{ and } \delta = 5. \end{aligned}$$

By complex interpolation, the lemma follows under condition (6.5).  $\square$

### 6.3. The Cauchy problem on $M$ .

**Proposition 6.3.** *Let  $\tau_0 > 0$  and  $p \in C([\tau_0, \infty])^5$  such that*

$$\forall \tau \geq \tau_0, \quad |p(\tau)| \leq \frac{1}{\tau^{3-3\varepsilon}}.$$

*Assume that  $F \in X(a+1+\eta, b+1+\eta, \delta)$ , with  $a, b > 0$ ,  $\eta > 0$  and*

$$\delta < a - b < \delta(2 - 3\varepsilon), \quad 1 \leq \delta \leq 5.$$

*where the space  $X$  was defined in Lemma 6.2. Then (6.1) has a unique solution  $\phi \in X(a, b, \delta)$ . Furthermore, this solution satisfies (6.6).*

*Proof.* The proof is set up in the same spirit as the existence of Møller's wave operators. Let  $\chi(\tau) = 1 - H(\tau)$ , where  $H$  is the Heaviside function, be the function equal to 1 for  $\tau < 0$  and 0 for  $\tau > 0$ . We first consider the case where  $F \in L^1([\tau_0, \infty]; M \cap \Sigma^5)$ . For  $(\tau_n)_n$  a sequence going to  $+\infty$ , consider

$$(6.7) \quad \partial_\tau \phi_n + iL\phi_n - P_M Z_p(\phi_n) = \chi(\tau - \tau_n)F \quad ; \quad \phi_n|_{\tau=1+\tau_n} = 0.$$

To begin with, we remove the projection  $P_M$  from the left hand side, and consider

$$(6.8) \quad \partial_\tau \phi_n + iL\phi_n - Z_p(\phi_n) = \chi(\tau - \tau_n)F \quad ; \quad \phi_n|_{\tau=1+\tau_n} = 0.$$

We show that for every  $n$ , (6.8) has a unique solution  $\phi_n \in C([\tau_0, \infty]; \Sigma^5)$ . To see this, remove the modulation by reversing the approach presented in §3: recalling (3.2), define  $\tilde{\phi}_n$  by

$$\tilde{\phi}_n(t, x) = e^{i(q_1(t)+q_3(t) \cdot x + q_5(t)|x|^2)} \frac{1}{q_4(t)^{d/2}} \phi_n \left( \gamma(t), \frac{x}{q_4(t)} - q_2(t) \right),$$

where  $\gamma$  is given by (3.4) and the  $q_j$ 's are well-defined function of the  $p_k$ 's in view of Lemma 3.2. We check that (6.8) is then equivalent to an equation of the form

$$i\partial_t \tilde{\phi}_n + \Delta \tilde{\phi}_n = W_1 \tilde{\phi}_n + W_2 \overline{\tilde{\phi}_n} + \tilde{F}_n \quad ; \quad \tilde{\phi}_n|_{t=t_n} = 0,$$

where the notation  $\tilde{F}_n$  is obvious,  $t_n = \gamma^{-1}(\tau_n + 1)$ , and the potentials are given by

$$W_1(t, x) = \frac{1}{q_4(t)^2} \left( 1 - \left( \frac{2}{d} + 1 \right) Q \left( \frac{x}{q_4(t)} - q_2(t) \right)^{4/d} \right),$$

$$W_2(t, x) = -\frac{2}{dq_4(t)^2} Q \left( \frac{x}{q_4(t)} - q_2(t) \right)^{4/d} e^{2i(q_1(t)+q_3(t) \cdot x + q_5(t)|x|^2)}.$$

We note that  $W_j \in L^\infty([t_0, \infty]; W^{5, \infty}(\mathbf{R}^d))$ ,  $j = 1, 2$ . We can then construct  $\tilde{\phi}_n$  in  $C([t_0, \infty]; L^2)$ : a fixed point argument yields  $\tilde{\phi}_n$  on small time intervals (with a non-trivial initial data in order to repeat the process), and we can split  $[t_0, t_n]$  into finitely many time intervals on which we can control the  $L_t^\infty L_x^2$ -norm of  $\tilde{\phi}_n$  by the  $L_t^1 L_x^2$ -norm of  $\tilde{F}_n$  on the same time interval. We can proceed along the same line to construct  $\tilde{\phi}_n$  in  $C([t_0, \infty]; H^5)$ , and then infer that  $\tilde{\phi}_n$  is also in  $C([t_0, \infty]; \Sigma^5)$  (with  $\tilde{\phi}_n|_{t \geq t_n} = 0$ ). We skip the easy details.

We deduce that (6.8) has a unique solution  $\phi_n \in C([\tau_0, \infty]; \Sigma^5)$ . The case of (6.7) follows easily, by rewriting it as

$$\partial_\tau \phi_n + iL\phi_n - Z_p(\phi_n) = -P_S Z_p(\phi_n) + \chi(\tau - \tau_n)F \quad ; \quad \phi_n|_{\tau=1+\tau_n} = 0,$$

and by recalling that

$$P_S Z_p(\phi_n) = \sum_{j=1}^6 \langle Z_p(\phi_n), m_j \rangle n_j, \quad \text{hence } \|P_S Z_p(\phi_n)(\tau)\|_{\Sigma^5} \lesssim \frac{1}{\tau^{3-3\varepsilon}} \|\phi_n(\tau)\|_{L^2}.$$

The important point which we must note now is that  $\phi_n \in C([t_0, \infty[; M \cap \Sigma^5)$ , which is compactly supported in time, has no secular part. This is so thanks to Proposition 4.1, and the integral formulation of (6.7), which can be written as:

$$\phi_n(\tau) = \int_{\tau}^{1+\tau_n} e^{i(\sigma-\tau)L} ((\chi(\sigma - \tau_n)F(\sigma) + P_M Z_p(\phi_n)(\sigma)) d\sigma, \quad \tau \geq \tau_0.$$

Since  $\chi(\cdot - \tau_n)F \in L^1([\tau_0, \infty[; M \cap \Sigma^5)$ , Proposition 4.1 shows that the right hand side of the above equation has no non-trivial  $S$ -component. Therefore,  $\phi_n(\tau) \in M$ .

To conclude, we note that under the assumptions of the proposition,  $\chi(\cdot - \tau_n)F$  converges to  $F$  in  $X(a+1+\eta/2, b+1+\eta/2, \delta)$ . Since (6.1) is linear, Lemma 6.2 shows that  $\phi_n$  is a Cauchy sequence in  $X(a, b, \delta)$ , thus it converges in this space to  $\phi$  solution to (6.1) which satisfies (6.6).

By density, we can also assume  $F \in X(a+1+\eta, b+1+\eta, \delta)$ . Uniqueness follows also from Lemma 6.2.  $\square$

## 7. FIXED POINT ARGUMENT

In this section we show Theorem 3.5.

Recall that we have defined the operator  $\Phi$  as follows:

$$\begin{aligned} \Phi(w)(\tau) &= \int_{\tau}^{\infty} e^{i(\tau-\sigma)L} (P_S R_p(w) + P_S Z_p(w) + Z_p(Q)) d\sigma + \Phi_2(w)(\tau) \\ &= \Phi_1(w)(\tau) + \Phi_2(w)(\tau), \end{aligned}$$

where  $\Phi_2(w) = \phi$  is the solution (in  $M$ ) of the equation

$$\partial_{\tau}\phi + iL\phi - P_M Z_p(\phi) = P_M R_p(w) + P_M Z_p(P_S w).$$

The modulation  $p$  is a function of  $w$  itself, defined in §5, Proposition 5.3. To prove Theorem 3.5 (hence Theorem 1.1), we show that  $\Phi$  has a fixed point in a suitable space. Consider for  $0 < \varepsilon < 1/3$  and  $1 < \delta$

$$Y(\delta, \varepsilon, \tau_0) = \left\{ w \in C([\tau_0, \infty[; M \cap \Sigma^{\delta}) + C([\tau_0, \infty[; \text{span } n_6); \|w\|_{\delta, \varepsilon, \tau_0} < \infty \right\}$$

where  $\|w\|_{\delta, \varepsilon, \tau_0}$  is defined as

$$\sup_{\tau \geq \tau_0} \tau^{2-\varepsilon} \|P_M w(\tau)\|_{H^{\delta}} + \sup_{\tau \geq \tau_0} \tau^{2-2\varepsilon-\delta} \|\langle y \rangle^{\delta} P_M w(\tau)\|_{L^2} + \sup_{\tau \geq \tau_0} \tau^{3-3\varepsilon} |\langle w(\tau), m_6 \rangle|.$$

**7.1. Stability.** The main result of this section is the following:

**Proposition 7.1.** *Let  $\delta \in ]1, 2[$ , and  $0 < \varepsilon < 1/4$  so that  $\varepsilon < 1 - \delta/2$ . There exists  $\tau_0 > 0$  such that  $\Phi$  maps the closed unit ball of  $Y(\delta, \varepsilon, \tau_0)$  to itself.*

*Proof.* For  $w \in Y(\delta, \varepsilon, \tau_0)$ , Proposition 5.3 yields a modulation  $p$  such that

$$\sup_{\tau \geq \tau_0} \tau^{3-3\varepsilon} |p(\tau)| \leq 1.$$

By Proposition 6.3,  $P_S \Phi(w) = \Phi_1(w)$ . Since by Proposition 5.3,  $\Phi_1(w) \in \text{span}(n_6)$ , the secular part of  $\Phi(w)$  has the suitable structure for  $Y$ . Moreover, by (5.14), for  $\tau \geq \tau_0$ ,

$$|\langle \Phi_1(w)(\tau), m_6 \rangle| \leq \frac{C}{\tau^{3-2\varepsilon}} \leq \frac{C}{\tau_0^{\varepsilon}} \frac{1}{\tau^{3-3\varepsilon}}.$$

Therefore, increasing  $\tau_0$  if necessary,

$$\sup_{\tau \geq \tau_0} \tau^{3-3\varepsilon} |\langle \Phi(w)(\tau), m_6 \rangle| \leq \frac{1}{2}.$$

Thus  $\Phi_1(w)$  is in the  $1/2$ -ball of  $Y(\delta, \varepsilon, \tau_0)$ .

To control the non-secular part  $P_M\Phi(w) = \Phi_2(w)$ , we apply Proposition 6.3 with

$$F = P_MR_p(w) + P_MZ_p(P_Sw).$$

We look for  $a$  and  $b$  such that  $F \in X(a+1+\eta, b+1+\eta, \delta)$ . We note that

$$P_MZ_p(P_Sw) = \langle w, m_6 \rangle P_MZ_p(n_6),$$

so we have the estimate

$$\|P_MZ_p(P_Sw)(\tau)\|_{\Sigma^\delta} \lesssim |\langle w(\tau), m_6 \rangle| |p(\tau)| \lesssim \frac{1}{\tau^{3-3\varepsilon}} \times \frac{1}{\tau^{3-3\varepsilon}} = \frac{1}{\tau^{6-6\varepsilon}}.$$

The delicate term, which explains the assumption  $\delta < 2$ , is the last one,  $P_MR_p(w)$ . We treat separately the contributions of  $R_0$ ,  $R_L$  and  $R_{NL}$ . Since  $d \leq 2$  and  $\delta > 1$ ,  $H^\delta(\mathbf{R}^d)$  is an algebra, and we infer

$$\begin{aligned} \|P_MR_{NL}(w)\|_{H^\delta} &\leq \|R_{NL}(w)\|_{H^\delta} + \|P_SR_{NL}(w)\|_{H^\delta} \\ &\lesssim \|w\|_{H^\delta}^2 + \|w\|_{H^\delta}^{1+4/d} \lesssim \frac{1}{\tau^{2(2-\varepsilon)}}. \end{aligned}$$

From the pointwise estimate (5.5),

$$\begin{aligned} \|\langle y \rangle^\delta P_MR_{NL}(w)\|_{L^2} &\leq \|\langle y \rangle^\delta R_{NL}(w)\|_{L^2} + \|\langle y \rangle^\delta P_SR_{NL}(w)\|_{L^2} \\ &\lesssim \|w\|_{L^2}^2 + \|\langle y \rangle^\delta w\|_{L^2} \sum_{2 \leq j \leq 4/d} \|w\|_{H^\delta}^j \lesssim \frac{1}{\tau^{2(2-\varepsilon)}}. \end{aligned}$$

We next treat the contribution of  $R_L$ . Using that  $\tau^2 V_p$  is bounded in the Sobolev space  $W^{2,\infty}$ , uniformly for  $\tau \geq 1$ , we get

$$\|P_MR_L(w)\|_{H^\delta} \lesssim \frac{1}{\tau^2} \|w\|_{H^\delta} \lesssim \frac{1}{\tau^{4-\varepsilon}}.$$

Denoting  $\beta = 2 - 2\varepsilon - \delta \in ]0, 1[$  the decay rate of the  $\delta$ -momentum of  $w$  in the definition of  $Y(\delta, \varepsilon, \tau_0)$ , and using simply the boundedness of the external potential  $V$ , we infer

$$\|\langle y \rangle^\delta P_MR_L(w)\|_{L^2} \lesssim \frac{1}{\tau^3} \|w\|_{L^2} + \frac{1}{\tau^2} \|\langle y \rangle^\delta w\|_{L^2} \lesssim \frac{1}{\tau^{2+\beta}}.$$

The term  $P_MR_0$  can be estimated in a similar way, up to the fact that the  $H^\delta$ -norm and the momenta of  $Q$  do not decay in time:

$$\|P_MR_0\|_{H^\delta} \lesssim \frac{1}{\tau^3} \quad ; \quad \|\langle y \rangle^\delta P_MR_0\|_{L^2} \lesssim \frac{1}{\tau^3}.$$

Summarizing, we have obtained

$$\begin{aligned} \|F\|_{H^\delta} &\lesssim \frac{1}{\tau^{6-6\varepsilon}} + \frac{1}{\tau^{4-2\varepsilon}} + \frac{1}{\tau^{4-\varepsilon}} + \frac{1}{\tau^3} \lesssim \frac{1}{\tau^3}, \\ \|\langle y \rangle^\delta F\|_{L^2} &\lesssim \frac{1}{\tau^{6-6\varepsilon}} + \frac{1}{\tau^{4-2\varepsilon}} + \frac{1}{\tau^{2+\beta}} + \frac{1}{\tau^3} \lesssim \frac{1}{\tau^{2+\beta}}, \end{aligned}$$

meaning that  $F \in X(3, 2+\beta, \delta)$ . We can then apply Proposition 6.3 provided that there exists  $a, b, \eta > 0$  with

$$\delta < a - b < \delta(2 - 3\varepsilon),$$

such that  $F \in X(a+1+\eta, b+1+\eta, \delta)$ . We take  $a+1+\eta = 3$  (note that this constraint comes from  $R_0$ ). This requires  $a = 2 - \eta$  ( $\eta > 0$  can be arbitrarily small), and since on the other hand, we must have  $a > \delta$ , this explains why we have assumed  $\delta < 2$ . By taking  $\eta = \frac{\varepsilon}{2}$  and  $b = \beta + \frac{\varepsilon}{2}$ , we get as a constraint

$$\delta < 2 - \varepsilon - \beta < \delta(2 - 3\varepsilon).$$

As  $\varepsilon < 1/4$ , this condition is fulfilled by the choice  $\beta = 2 - 2\varepsilon - \delta$ . Therefore, by Proposition 6.3  $\Phi_2(w) \in X(2 - \frac{\varepsilon}{2}, \beta + \frac{\varepsilon}{2}, \delta)$ . By increasing  $\tau_0$  if necessary,  $\Phi_2(w)$  is also in the  $1/2$ -ball of  $Y(\delta, \varepsilon, \tau_0)$ , and the proposition follows.  $\square$

**7.2. Compactness.** We recall the following compactness result, which is a particular case of [31, Corollary 4].

**Theorem 7.2** (From [31]). *Let  $X \subset B \subset Y$  be Banach spaces such that  $X$  is compactly embedded into  $B$ , and  $B$  is continuously embedded into  $Y$ . Let  $\tau_0 < \tau_1$  and  $F$  be a subset of  $L^\infty([\tau_0, \tau_1]; X)$  such that  $\{\frac{\partial v}{\partial \tau}, v \in F\}$  is bounded in  $L^\infty([\tau_0, \tau_1]; Y)$ . Then  $F$  has compact closure in  $C([\tau_0, \tau_1]; B)$ .*

Fix  $\varepsilon, \delta$  as in Proposition 7.1. Let  $K$  be the closed unit ball of  $Y(\delta, \varepsilon, \tau_0)$ . By Proposition 7.1, the operator  $\Phi$  maps  $K$  into itself. Notice that  $K$  is closed into  $Y(\delta', \varepsilon', \tau_0)$  if  $\delta' < \delta, \varepsilon' > \varepsilon$  and  $2\varepsilon + \delta < 2\varepsilon' + \delta'$ . In this subsection we show the following lemma.

**Lemma 7.3.** *Let  $0 < \varepsilon < \varepsilon'$  and  $1 < \delta' < \delta < 2$  and assume  $2\varepsilon + \delta < 2\varepsilon' + \delta' < 2$  (this implies that  $Y(\delta, \varepsilon, \tau_0)$  is continuously embedded into  $Y(\delta', \varepsilon', \tau_0)$ ). The image  $\Phi(K)$  of  $K$  has compact closure in  $Y(\delta', \varepsilon', \tau_0)$ .*

*Proof.* It is sufficient to show that for all  $r > 0$ , there exists a finite number  $N$  of functions in  $\psi_n \in Y(\delta', \varepsilon', \tau_0)$ , such that

$$(7.1) \quad \psi \in \Phi(K) \implies \exists n \in \{1, \dots, N\}, \|\psi - \psi_n\|_{\delta', \varepsilon', \tau_0} < r.$$

Recall first that  $\Phi(K) \subset K$ . Thus for  $\psi \in \Phi(K)$ ,

$$\begin{aligned} \tau^{2-\varepsilon'} \|P_M \psi(\tau)\|_{H^{\delta'}} + \tau^{3-3\varepsilon'} |\langle \psi(\tau), m_6 \rangle| + \tau^{2-2\varepsilon'-\delta'} \|\langle y \rangle^{\delta'} P_M \psi(\tau)\|_{L^2} \\ \leq \tau^{\varepsilon-\varepsilon'} + \tau^{3\varepsilon-3\varepsilon'} + \tau^{2\varepsilon+\delta-2\varepsilon'-\delta'}. \end{aligned}$$

Let  $\tau_1$  such that  $\tau_0 < \tau_1/2$  and

$$\left(\frac{\tau_1}{2}\right)^{\varepsilon-\varepsilon'} + \left(\frac{\tau_1}{2}\right)^{3\varepsilon-3\varepsilon'} + \left(\frac{\tau_1}{2}\right)^{2\varepsilon+\delta-2\varepsilon'-\delta'} < \frac{r}{2}.$$

From the two preceding inequalities, we get that for  $\psi \in \Phi(K)$ ,

$$(7.2) \quad \tau \geq \frac{\tau_1}{2} \implies \tau^{2-\varepsilon'} \|P_M \psi(\tau)\|_{H^{\delta'}} + \tau^{3-3\varepsilon'} |\langle \psi(\tau), m_6 \rangle| + \tau^{2-2\varepsilon'-\delta'} \|\langle y \rangle^{\delta'} P_M \psi(\tau)\|_{L^2} < \frac{r}{2}.$$

Next, consider the set

$$F = \left\{ \Phi(w) \Big|_{[\tau_0, \tau_1]}, w \in K \right\}.$$

We will show that the assumptions of Theorem 7.2 hold with

$$X = \Sigma^\delta, \quad B = \Sigma^{\delta'}, \quad Y = \Sigma^{\delta-2},$$

where we define (as  $\delta < 2$ )  $\Sigma^{\delta-2} = H^{\delta-2} + \mathcal{F}(H^{\delta-2})$ . Note that  $0 < \delta' < \delta$ , so that  $X$  is compactly embedded in  $B$ . The fact that  $\Phi(K) \subset K$  shows that  $F$  is a bounded subset of  $C([\tau_0, \tau_1]; \Sigma^\delta)$ . Furthermore if  $\phi = \Phi(w) \in K$  then  $\phi = \phi_M + \phi_S$  where

$$\begin{aligned} \partial_\tau \phi_S + iL\phi_S &= P_S R_p(w) + P_S Z_p(w) + Z_p(Q), \\ \partial_\tau \phi_M + iL\phi_M - P_M Z_p(\phi_M) &= P_M R_p(w) + P_M Z_p(w_S). \end{aligned}$$

Using that  $\phi$  and  $w$  are in  $K$ , we get that  $\partial_\tau \phi \in C([\tau_0, +\infty[; \Sigma^{\delta-2})$  and that  $\partial_\tau \phi|_{[\tau_0, \tau_1]}$  is uniformly bounded in  $\Sigma^{\delta-2}$  with a bound which is independent of  $\phi$ . By Theorem 7.2,  $F$  has compact closure in  $C([\tau_0, \tau_1]; \Sigma^\delta)$ . As a consequence, there exist  $\tilde{\psi}_1, \dots, \tilde{\psi}_N$  such that

$$(7.3) \quad \forall \tilde{\psi} \in F, \exists n \in \{1, \dots, N\}, \sup_{\tau_0 \leq \tau \leq \tau_1} \left\| \tilde{\psi}(\tau) - \tilde{\psi}_n(\tau) \right\|_{\Sigma^{\delta'}} < \frac{r}{6\tau_1^\kappa},$$

where  $\kappa = \max\{2 - \varepsilon', 3 - 3\varepsilon', 2 - 2\varepsilon' - \delta'\} > 0$ .

Let  $\chi \in C^\infty([\tau_0, +\infty[)$ , supported in  $[\tau_0, \tau_1]$ , such that  $0 \leq \chi \leq 1$ , and  $\chi = 1$  on  $[\tau_0, \tau_1/2]$ . For  $1 \leq n \leq N$ , let  $\psi_n = \chi \tilde{\psi}_n$ . We show that the  $\psi_n$ 's satisfy (7.1), which will conclude the proof of the lemma. Let  $\psi \in \Phi(K)$ . By (7.3), there exists  $n \in \{1, \dots, N\}$  such that

$$\|\chi\psi - \psi_n\|_{L^\infty(\tau_0, +\infty, \Sigma^{\delta'})} \leq \left\| \psi - \tilde{\psi}_n \right\|_{L^\infty(\tau_0, \tau_1, \Sigma^{\delta'})} < \frac{r}{A\tau_1^\kappa},$$

where  $A$  is a large universal constant to be specified later. And thus, using that  $\chi\psi$  is supported in  $[\tau_0, \tau_1]$ ,

$$\forall \tau \geq \tau_0, \quad \|\chi\psi(\tau) - \psi_n(\tau)\|_{\Sigma^{\delta'}} < \frac{r}{A\tau^\kappa}.$$

This implies, if  $A$  is large enough,

$$\begin{aligned} \tau^{2-\varepsilon'} \|\chi P_M \psi(\tau) - P_M \psi_n(\tau)\|_{H^{\delta'}} + \tau^{3-3\varepsilon'} |\langle \chi\psi(\tau) - \psi_n(\tau), m_6 \rangle| \\ + \tau^{2-2\varepsilon'-\delta'} \|\langle y \rangle^{\delta'} (\chi P_M \psi(\tau) - P_M \psi_n(\tau))\|_{L^2} < \frac{r}{2}. \end{aligned}$$

Furthermore, by (7.2),

$$\begin{aligned} \tau^{2-\varepsilon'} \|(1-\chi)P_M \psi(\tau)\|_{H^{\delta'}} + \tau^{3-3\varepsilon'} |(1-\chi) \langle \psi(\tau), m_6 \rangle| \\ + \tau^{2-2\varepsilon'-\delta'} \|(1-\chi) \langle y \rangle^{\delta'} P_M \psi(\tau)\|_{L^2} < \frac{r}{2}. \end{aligned}$$

Hence (7.1). The proof is complete.  $\square$

**7.3. End of the proof.** The following proposition will allow us to use Schauder's Theorem in order to prove Theorem 3.5.

**Proposition 7.4.** *Let  $\delta, \delta' \in ]1, 2[$ , and  $0 < \varepsilon, \varepsilon' < 1/4$  so that  $2\varepsilon + \delta < 2\varepsilon' + \delta' < 2$ . The closed unit ball  $K$  of  $Y(\delta, \varepsilon, \tau_0)$  is closed in  $Y(\delta', \varepsilon', \tau_0)$ . In addition, the map  $\Phi : K \rightarrow K$  is continuous for the topology of  $Y(\delta', \varepsilon', \tau_0)$ .*

*Proof.* In view of Proposition 7.1, we need only prove the continuity. We start with an estimate of the difference of two parameters  $p, \tilde{p}$  defined from two different functions  $w, \tilde{w} \in Y(\delta', \varepsilon', \tau_0)$ . Recall that the existence of  $p$  was proved in Proposition 5.3 as a fixed point of the operator  $\Psi(p) = \Psi_w(p)$ , and that  $w \in Y(\delta', \varepsilon', \tau_0)$  implies  $\|p\|_{3-3\varepsilon', \tau_0} < \infty$ . We have

$$\begin{aligned} |p(\tau) - \tilde{p}(\tau)| &= |\Psi_w(p)(\tau) - \Psi_{\tilde{w}}(\tilde{p})(\tau)| \\ &\leq |\Psi_w(p)(\tau) - \Psi_w(\tilde{p})(\tau)| + |\Psi_w(\tilde{p})(\tau) - \Psi_{\tilde{w}}(\tilde{p})(\tau)|. \end{aligned}$$

By the contraction estimate (5.18) on  $\Psi_w$ , we get

$$\|p - \tilde{p}\|_{3-3\varepsilon', \tau_0} \leq 2 \|\Psi_w(\tilde{p}) - \Psi_{\tilde{w}}(\tilde{p})\|_{3-3\varepsilon', \tau_0}.$$

Therefore, in view of the definition of  $\Psi_w$ ,

$$\begin{aligned} \|p - \tilde{p}\|_{3-3\varepsilon', \tau_0} &\lesssim \sum_{1 \leq j \leq 5} \|(D_j(\tilde{p})(w) - D_j(\tilde{p})(\tilde{w}))\|_{3-3\varepsilon', \tau_0} \\ &\quad + \left\| \int_\tau^{+\infty} (D_6(\tilde{p})(w) - D_6(\tilde{p})(\tilde{w})) \right\|_{3-3\varepsilon', \tau_0}. \end{aligned}$$

By the definition (5.10) of  $D_j(\tilde{p})$ , one has

$$\begin{aligned} |D_j(\tilde{p})(w) - D_j(\tilde{p})(\tilde{w})| &\leq |\langle P_S(R_{\tilde{p}}(w) - R_{\tilde{p}}(\tilde{w})), m_j \rangle| + |\langle P_S Z_{\tilde{p}}(w - \tilde{w}), m_j \rangle| \\ &\leq |\langle R_{NL, \tilde{p}}(w) - R_{NL, \tilde{p}}(\tilde{w}), m_j \rangle| + |\langle R_{L, \tilde{p}}(w - \tilde{w}), m_j \rangle| + |\langle P_S Z_{\tilde{p}}(w - \tilde{w}), m_j \rangle|. \end{aligned}$$

In view of the explicit formulas for  $R_{NL}$  and of the pointwise estimate (5.6) on  $R_L$ , we infer, since  $w, \tilde{w} \in L^\infty H^1$ :

$$\begin{aligned} |(D_j(\tilde{p})(w) - D_j(\tilde{p})(\tilde{w}))(\tau)| &\lesssim \|w(\tau) - \tilde{w}(\tau)\|_{H^1} (\|w(\tau)\|_{H^1} + \|\tilde{w}(\tau)\|_{H^1}) \\ &+ \frac{1}{\tau^3} \|w(\tau) - \tilde{w}(\tau)\|_{H^1} + |\tilde{p}(\tau)| \|w(\tau) - \tilde{w}(\tau)\|_{H^1} \lesssim \frac{1}{\tau^{4-2\varepsilon'}} \|w - \tilde{w}\|_{\delta', \varepsilon', \tau_0}. \end{aligned}$$

In conclusion,

$$(7.4) \quad \|p - \tilde{p}\|_{3-3\varepsilon', \tau_0} \lesssim \frac{1}{\tau_0^{\varepsilon'}} \|w - \tilde{w}\|_{\delta', \varepsilon', \tau_0}.$$

Also, since

$$\Phi_1(w)(\tau) = -n_6 \int_\tau^{+\infty} D_6(p)(w),$$

we get

$$\|\Phi_1(w) - \Phi_1(\tilde{w})\|_{\delta', \varepsilon', \tau_0} \lesssim \frac{1}{\tau_0^{\varepsilon'}} \|w - \tilde{w}\|_{\delta', \varepsilon', \tau_0}.$$

Therefore  $\Phi_1$  is (Lipschitz-)continuous on  $Y(\delta', \varepsilon', \tau_0)$ . It remains to show the continuity of  $\Phi_2$ .

Let  $w \in K$  and  $w_n \in K$  such that  $w_n \rightarrow w$  in  $Y(\delta', \varepsilon', \tau_0)$ . Denote by  $\phi_n = \Phi_2(w_n) = P_M \Phi(w_n)$ , and  $\phi = \Phi_2(w)$ . By Lemma 7.3,  $\Phi(K)$  is relatively compact and there exists a subsequence of  $\phi_n$  which converges in  $Y(\delta', \varepsilon', \tau_0)$  to some  $\tilde{\phi} \in K$ . It remains to show that  $\phi = \tilde{\phi}$ . By (7.4) we have

$$\lim_{n \rightarrow \infty} \|p_n - p\|_{3-3\varepsilon', \tau_0} = 0.$$

By definition of  $\Phi_2$ , we have

$$\partial_\tau \phi_n + iL\phi_n - P_M Z_{p_n}(\phi_n) = P_M R_{p_n}(w_n) + P_M Z_{p_n}(P_S w_n).$$

Letting  $n$  tends to  $\infty$ , we get that  $\tilde{\phi}$  satisfies the following equation in the sense of distributions

$$\partial_\tau \tilde{\phi} + iL\tilde{\phi} - P_M Z_p(\tilde{\phi}) = P_M R_p(w) + P_M Z_p(P_S w).$$

Using that  $\phi$  is, by definition, solution to the same equation, we get

$$\partial_\tau (\tilde{\phi} - \phi) + iL(\tilde{\phi} - \phi) - P_M Z_p(\tilde{\phi} - \phi) = 0,$$

which implies, by Lemma 6.2, that  $\phi = \tilde{\phi}$ . The proof is complete.  $\square$

*Proof of Theorem 3.5.* By Proposition 7.4,  $\Phi$  is a continuous map from  $K$  into itself. By Lemma 7.3,  $\Phi(K)$  is relatively compact in  $Y(\delta', \varepsilon', \tau_0)$ . As  $K$  is a convex closed subset of  $Y(\delta', \varepsilon', \tau_0)$ , we can apply Schauder's Theorem (see e.g. [32, Corollary B.3]) which implies that  $\Phi$  has a fixed point  $w \in K$ . By the definition of  $K$  and Proposition 5.3, Theorem 3.5 follows.  $\square$

#### APPENDIX A. A DIFFERENTIAL INEQUALITY

**Lemma A.1.** *Let  $\mu > 0$  and  $m \in \mathbf{N}$ . Let  $(a_j)_{j=1\dots m}$ ,  $(b_j)_{j=1\dots m}$ , be real constants,  $a, b, \eta > 0$ , and  $(\alpha_j)_{j=1\dots m}$ ,  $(\beta_j)_{j=1\dots m}$  be constants in  $[0, 1]$ . Assume*

$$\forall j \in \{1 \dots m\}, \quad a_j + (b - a)\alpha_j > 0, \quad b_j + (a - b)\beta_j > 0.$$

*There exists  $\tau_0$  such that for any  $M > 0$  and any nonnegative continuous functions  $z_1$  and  $z_2$  on  $[\tau_0, +\infty[$  such that*

$$\sup_{\tau \geq \tau_0} |\tau^a z_1(\tau)| + \sup_{\tau \geq \tau_0} |\tau^b z_2(\tau)| < \infty.$$

and satisfying the following differential inequality on  $[\tau_0, +\infty[$ :

$$(A.1) \quad \begin{cases} z_1(\tau) \leq \int_{\tau}^{\infty} \left( \frac{M}{\sigma^{a+1+\eta}} + C \sum_{j=1}^m \frac{z_1(\sigma)^{1-\alpha_j} z_2(\sigma)^{\alpha_j}}{\sigma^{a_j+1}} \right) d\sigma, \\ z_2(\tau) \leq \int_{\tau}^{\infty} \left( \frac{M}{\sigma^{b+1+\eta}} + C \sum_{j=1}^m \frac{z_1(\sigma)^{\beta_j} z_2(\sigma)^{1-\beta_j}}{\sigma^{b_j+1}} \right) d\sigma, \end{cases}$$

we have

$$\sup_{\tau \geq \tau_0} |\tau^a z_1(\tau)| + \sup_{\tau \geq \tau_0} |\tau^b z_2(\tau)| \leq \mu M.$$

*Proof.* Denote by

$$Z_1(\tau) = \tau^a z_1(\tau), \quad Z_2(\tau) = \tau^b z_2(\tau).$$

Let  $\tilde{a}_j = (1 - \alpha_j)a + \alpha_j b + a_j$ ,  $\tilde{b}_j = \beta_j a + (1 - \beta_j)b + b_j$ . Using Young's inequality,  $Z_1^{1-\theta} Z_2^{\theta} \leq (1 - \theta)Z_1 + \theta Z_2$ , (A.1) and Hölder inequality yield

$$z_1(\tau) \leq M \left\| \frac{1}{\sigma^{a+1+\eta}} \right\|_{L^1} + C \sum_{j=1}^m \left( \|Z_1\|_{L^\infty} + \|Z_2\|_{L^\infty} \right) \left\| \frac{1}{\sigma^{\tilde{a}_j+1}} \right\|_{L^1},$$

where the Lebesgue norms correspond to integration over  $[\tau, \infty[$ . Similarly,

$$z_2(t) \leq M \left\| \frac{1}{\sigma^{b+1+\eta}} \right\|_{L^1} + C \sum_{j=1}^m \left( \|Z_1\|_{L^\infty} + \|Z_2\|_{L^\infty} \right) \left\| \frac{1}{\sigma^{\tilde{b}_j+1}} \right\|_{L^1}.$$

For any  $c > 0$ ,

$$\left\| \frac{1}{\sigma^{c+1}} \right\|_{L^1([\tau, \infty[)} = \frac{1}{c\tau^c},$$

and hence (with a constant  $C$  depending only on the parameters  $\eta, a, b, \tilde{a}_j, \tilde{b}_j$ )

$$\begin{aligned} Z_1(\tau) &\leq \frac{C}{\tau^\eta} M + C \sum_{j=1}^m \frac{\|Z_1\|_{L^\infty([\tau_0, +\infty[)} + \|Z_2\|_{L^\infty([\tau_0, +\infty[)}}{\tau^{\tilde{a}_j-a}}, \\ Z_2(\tau) &\leq \frac{C}{\tau^\eta} M + C \sum_{j=1}^m \frac{\|Z_1\|_{L^\infty([\tau_0, +\infty[)} + \|Z_2\|_{L^\infty([\tau_0, +\infty[)}}{\tau^{\tilde{b}_j-b}}. \end{aligned}$$

By assumption,  $\tilde{a}_j - a$  and  $\tilde{b}_j - b$  are positive. Taking the sup norm of the preceding inequalities and using the triangle inequality, we get

$$\begin{aligned} \|Z_1\|_{L^\infty([\tau_0, +\infty[)} &\leq \frac{C}{\tau_0^\eta} M + C \sum_{j=1}^m \frac{\|Z_1\|_{L^\infty([\tau_0, +\infty[)} + \|Z_2\|_{L^\infty([\tau_0, +\infty[)}}{\tau_0^{\tilde{a}_j-a}}, \\ \|Z_2\|_{L^\infty([\tau_0, +\infty[)} &\leq \frac{C}{\tau_0^\eta} M + C \sum_{j=1}^m \frac{\|Z_1\|_{L^\infty([\tau_0, +\infty[)} + \|Z_2\|_{L^\infty([\tau_0, +\infty[)}}{\tau_0^{\tilde{b}_j-b}}. \end{aligned}$$

Taking  $\tau_0$  large, we obtain

$$\begin{aligned} \|Z_1\|_{L^\infty([\tau_0, +\infty[)} &\leq \frac{\mu}{4} M + \frac{1}{4} \|Z_1\|_{L^\infty([\tau_0, +\infty[)} + \frac{1}{4} \|Z_2\|_{L^\infty([\tau_0, +\infty[)}, \\ \|Z_2\|_{L^\infty([\tau_0, +\infty[)} &\leq \frac{\mu}{4} M + \frac{1}{4} \|Z_1\|_{L^\infty([\tau_0, +\infty[)} + \frac{1}{4} \|Z_2\|_{L^\infty([\tau_0, +\infty[)}. \end{aligned}$$

Summing up, we get the announced result.  $\square$

## APPENDIX B. SOME INTERPOLATION INEQUALITIES

**Lemma B.1.** *Let  $d \geq 1$ . There exists  $C > 0$  such that for all  $f \in \mathcal{S}(\mathbf{R}^d)$ ,*

$$(B.1) \quad \|\langle x \rangle f\|_{L^2} \leq \left\| \langle x \rangle^3 f \right\|_{L^2}^{1/3} \|f\|_{L^2}^{2/3},$$

$$(B.2) \quad \left\| \langle x \rangle^2 f \right\|_{L^2} \leq \left\| \langle x \rangle^3 f \right\|_{L^2}^{2/3} \|f\|_{L^2}^{1/3},$$

$$(B.3) \quad \|f\|_{H^1} \leq \|f\|_{H^3}^{1/3} \|f\|_{L^2}^{2/3},$$

$$(B.4) \quad \|f\|_{H^2} \leq \|f\|_{H^3}^{2/3} \|f\|_{L^2}^{1/3},$$

$$(B.5) \quad \|\langle x \rangle \nabla f\|_{L^2} \leq C \left\| \langle x \rangle^3 f \right\|_{L^2}^{1/3} \|f\|_{L^2}^{1/3} \|f\|_{H^1}^{1/3},$$

$$(B.6) \quad \left\| \langle x \rangle^2 \nabla f \right\|_{L^2} \leq C \left\| \langle x \rangle^3 f \right\|_{L^2}^{2/3} \|f\|_{H^1}^{1/3},$$

$$(B.7) \quad \|\langle x \rangle \nabla^2 f\|_{L^2} \leq C \left\| \langle x \rangle^3 f \right\|_{L^2}^{1/3} \|f\|_{H^1}^{2/3},$$

$$(B.8) \quad \|\langle x \rangle \nabla^4 f\|_{L^2} \leq C \left\| \langle x \rangle^5 f \right\|_{L^2}^{1/5} \|f\|_{H^5}^{4/5},$$

$$(B.9) \quad \left\| \langle x \rangle^4 \nabla f \right\|_{L^2} \leq C \left\| \langle x \rangle^5 f \right\|_{L^2}^{4/5} \|f\|_{H^5}^{1/5}.$$

*Proof.* To prove (B.1), use Hölder's inequality:

$$\begin{aligned} \|\langle x \rangle f\|_{L^2}^2 &= \int_{\mathbf{R}^d} \left( \langle x \rangle^6 |f(x)|^2 \right)^{1/3} (|f(x)|^2)^{2/3} dx \\ &\leq \left\| \left( \langle x \rangle^6 |f(x)|^2 \right)^{1/3} \right\|_{L^3} \left\| (|f(x)|^2)^{2/3} \right\|_{L^{3/2}}. \end{aligned}$$

Inequality (B.2) follows the same way:

$$\begin{aligned} \left\| \langle x \rangle^2 f \right\|_{L^2}^2 &= \int_{\mathbf{R}^d} \left( \langle x \rangle^6 |f(x)|^2 \right)^{2/3} (|f(x)|^2)^{1/3} dx \\ &\leq \left\| \left( \langle x \rangle^6 |f(x)|^2 \right)^{2/3} \right\|_{L^{3/2}} \left\| (|f(x)|^2)^{1/3} \right\|_{L^3}. \end{aligned}$$

Inequalities (B.3) and (B.4) then follow from (B.1) and (B.2), respectively, and Plancherel formula.

Integrating by parts, we have

$$\begin{aligned} \|\langle x \rangle \nabla f\|_{L^2}^2 &= - \int_{\mathbf{R}^d} \bar{f}(x) \nabla \cdot \left( \langle x \rangle^2 \nabla f(x) \right) dx \\ &\lesssim \int_{\mathbf{R}^d} |f(x)| \langle x \rangle |\nabla f(x)| dx + \int_{\mathbf{R}^d} |f(x)| \langle x \rangle^2 |\Delta f(x)| dx \\ &\lesssim \|\langle x \rangle f\|_{L^2} \|f\|_{H^1} + \left\| \langle x \rangle^2 f \right\|_{L^2} \|f\|_{H^2} \\ &\lesssim \left\| \langle x \rangle^3 f \right\|_{L^2}^{1/3} \|f\|_{L^2}^{4/3} \|f\|_{H^1}^{1/3} + \left\| \langle x \rangle^3 f \right\|_{L^2}^{2/3} \|f\|_{L^2}^{2/3} \|f\|_{H^1}^{2/3}, \end{aligned}$$

where we have used (B.1)–(B.4). Inequality (B.5) follows.

Integration by parts also yields

$$\begin{aligned} \left\| \langle x \rangle^2 \nabla f \right\|_{L^2}^2 &= - \int_{\mathbf{R}^d} \bar{f}(x) \nabla \cdot \left( \langle x \rangle^4 \nabla f(x) \right) dx \\ &\lesssim \left\| \langle x \rangle^3 f \right\|_{L^2} \|f\|_{H^1} + \left\| \langle x \rangle^3 f \right\|_{L^2} \|\langle x \rangle \nabla^2 f\|_{L^2} \\ &\lesssim \left\| \langle x \rangle^3 f \right\|_{L^2} \left( \|f\|_{H^3}^{1/3} \|f\|_{L^2}^{2/3} + \|\langle x \rangle \nabla^2 f\|_{L^2} \right). \end{aligned}$$

On the other hand,

$$\begin{aligned}
\|\langle x \rangle \nabla^2 f\|_{L^2}^2 &= - \int_{\mathbf{R}^d} \nabla \bar{f}(x) \nabla \left( \langle x \rangle^2 \nabla^2 f(x) \right) dx \\
&\lesssim \|\langle x \rangle \nabla f\|_{L^2} \|f\|_{H^2} + \|\langle x \rangle^2 \nabla f\|_{L^2} \|f\|_{H^3} \\
\text{(B.10)} \quad &\lesssim \|f\|_{H^3} \left( \|\langle x \rangle^3 f\|_{L^2}^{1/3} \|f\|_{L^2}^{2/3} + \|\langle x \rangle^2 \nabla f\|_{L^2} \right).
\end{aligned}$$

We infer, for instance,

$$\begin{aligned}
\|\langle x \rangle^2 \nabla f\|_{L^2}^2 &\lesssim \|\langle x \rangle^3 f\|_{L^2}^{7/6} \|f\|_{L^2}^{1/3} \|f\|_{H^3}^{1/2} + \|\langle x \rangle^2 \nabla f\|_{L^2}^{1/2} \|\langle x \rangle^3 f\|_{L^2} \|f\|_{H^3}^{1/2} \\
&\leq C \|\langle x \rangle^3 f\|_{L^2}^{7/6} \|f\|_{L^2}^{1/3} \|f\|_{H^3}^{1/2} + \varepsilon \|\langle x \rangle^2 \nabla f\|_{L^2}^2 \\
&\quad + C_\varepsilon \left( \|\langle x \rangle^3 f\|_{L^2} \|f\|_{H^3}^{1/2} \right)^{4/3},
\end{aligned}$$

where we have used Young's inequality, with  $(4, 4') = (4, 4/3)$ . Taking  $\varepsilon < 1$  yields (B.6), and (B.7) then follows from (B.10).

The proof of (B.8) and (B.9) is similar, and we omit it.  $\square$

#### REFERENCES

1. J.-P. Anker and V. Pierfelice, *Nonlinear Schrödinger equation on real hyperbolic spaces*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 5, 1853–1869.
2. V. Banica, *Remarks on the blow-up for the Schrödinger equation with critical mass on a plane domain*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3** (2004), no. 1, 139–170.
3. ———, *The nonlinear Schrödinger equation on the hyperbolic space*, Comm. Partial Differential Equations **32** (2007), no. 10, 1643–1677.
4. V. Banica, R. Carles, and T. Duyckaerts, *On scattering for NLS: from Euclidean to hyperbolic space*, Discrete Contin. Dyn. Syst. **24** (2009), no. 4, 1113–1127.
5. V. Banica, R. Carles, and G. Staffilani, *Scattering theory for radial nonlinear Schrödinger equations on hyperbolic space*, Geom. Funct. Anal. **18** (2008), no. 2, 367–399.
6. V. Banica and T. Duyckaerts, *Weighted Strichartz estimates for radial Schrödinger equation on noncompact manifolds*, Dyn. Partial Differ. Equ. **4** (2007), no. 4, 335–359.
7. M. Beceanu, *A centre-stable manifold for the focussing cubic NLS in  $\mathbb{R}^{1+3}$* , Comm. Math. Phys. **280** (2008), no. 1, 145–205.
8. ———, *A centre-stable manifold in  $H^{1/2}$  for the  $H^{1/2}$  critical NLS*, preprint, archived as arXiv:0902.1643, 2009.
9. J. Bourgain, *Global solutions of nonlinear Schrödinger equations*, American Mathematical Society Colloquium Publications, vol. 46, American Mathematical Society, Providence, RI, 1999.
10. J. Bourgain and W. Wang, *Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 1-2, 197–215 (1998), Dedicated to Ennio De Giorgi.
11. N. Burq, P. Gérard, and P. Raphaël, *Blow-up solutions for the  $L^2$ -critical nonlinear Schrödinger equation on a manifold*, work in progress.
12. N. Burq, P. Gérard, and N. Tzvetkov, *Two singular dynamics of the nonlinear Schrödinger equation on a plane domain*, Geom. Funct. Anal. **13** (2003), no. 1, 1–19.
13. R. Carles, *Nonlinear Schrödinger equations with repulsive harmonic potential and applications*, SIAM J. Math. Anal. **35** (2003), no. 4, 823–843.
14. R. Carles and Y. Nakamura, *Nonlinear Schrödinger equations with Stark potential*, Hokkaido Math. J. **33** (2004), no. 3, 719–729.
15. T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.
16. J. Dereziński and C. Gérard, *Scattering theory of quantum and classical  $N$ -particle systems*, Texts and Monographs in Physics, Springer Verlag, Berlin Heidelberg, 1997.
17. G. Fibich, F. Merle, and P. Raphaël, *Proof of a spectral property related to the singularity formation for the  $L^2$  critical nonlinear Schrödinger equation*, Phys. D **220** (2006), no. 1, 1–13.

18. T. Hmidi and S. Keraani, *Blowup theory for the critical nonlinear Schrödinger equations revisited*, Int. Math. Res. Not. (2005), no. 46, 2815–2828.
19. A. Ionescu and G. Staffilani, *Semilinear Schrödinger flows on hyperbolic spaces: scattering in  $H^1$* , Math. Ann. **345** (2009), no. 1, 133–158.
20. J. Krieger, E. Lenzmann, and P. Raphaël, *On stability of pseudo-conformal blowup for  $L^2$ -critical Hartree NLS*, Ann. Henri Poincaré **10** (2009), no. 6, 1159–1205.
21. J. Krieger and W. Schlag, *Non-generic blow-up solutions for the critical focusing NLS in 1-d*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 1, 1–125.
22. F. Merle, *Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power*, Duke Math. J. **69** (1993), no. 2, 427–454.
23. ———, *Nonexistence of minimal blow-up solutions of equations  $i u_t = -\Delta u - k(x)|u|^{4/N}u$  in  $\mathbf{R}^N$* , Ann. Inst. H. Poincaré Phys. Théor. **64** (1996), no. 1, 33–85.
24. F. Merle and P. Raphaël, *On universality of blow-up profile for  $L^2$  critical nonlinear Schrödinger equation*, Invent. Math. **156** (2004), no. 3, 565–672.
25. ———, *The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation*, Ann. of Math. (2) **161** (2005), no. 1, 157–222.
26. T. Ogawa and Y. Tsutsumi, *Blow-up of  $H^1$  solutions for the one-dimensional nonlinear Schrödinger equation with critical power nonlinearity*, Proc. Amer. Math. Soc. **111** (1991), no. 2, 487–496.
27. G. Perelman, *On the formation of singularities in solutions of the critical nonlinear Schrödinger equation*, Ann. Henri Poincaré **2** (2001), no. 4, 605–673.
28. P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics, vol. 171, Springer-Verlag, New York, 1998.
29. F. Planchon and P. Raphaël, *Existence and stability of the log-log blow-up dynamics for the  $L^2$ -critical nonlinear Schrödinger equation in a domain*, Ann. Henri Poincaré **8** (2007), no. 6, 1177–1219.
30. P. Raphaël, *Stability of the log-log bound for blow up solutions to the critical non linear Schrödinger equation*, Math. Ann. **331** (2005), no. 3, 577–609.
31. J. Simon, *Compact sets in the space  $L^p(0, T; B)$* , Ann. Mat. Pura Appl. (4) **146** (1987), 65–96.
32. M. E. Taylor, *Partial differential equations. III*, Applied Mathematical Sciences, vol. 117, Springer-Verlag, New York, 1997, Nonlinear equations, Corrected reprint of the 1996 original.
33. M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87** (1982/83), no. 4, 567–576.
34. ———, *Modulational stability of ground states of nonlinear Schrödinger equations*, SIAM J. Math. Anal. **16** (1985), no. 3, 472–491.

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