

SELSIMILAR SOLUTIONS OF THE BINORMAL FLOW AND THEIR STABILITY

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Abstract

We review some recent results concerning the evolution of a vortex filament and its relation to the cubic non-linear Schrödinger equation. Selfsimilar solutions and questions related to their stability are studied.

Keywords: vortex filaments, selfsimilarity, non-linear Schrödinger.

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1 Introduction

1.1 The binormal flow

This paper is mainly concerned with the so called Localized Induction approximation for the evolution of a vortex filament. In geometrical terms it is given by the binormal flow (BF): consider the 3d curve $\chi(s, t) \in \mathbb{R}^3$ with $t \in \mathbb{R}$ denoting time and $s \in \mathbb{R}$ the arclength parameter, then the flow is given by

$$\begin{cases} \chi_t = \chi_s \wedge \chi_{ss} \\ \chi(s, 0) = \chi_0(s). \end{cases} \quad (1)$$

Denoting c the curvature and b the binormal vector we get from the Frenet equations

$$\begin{cases} \chi_t = cb \\ \chi(s, 0) = \chi_0(s). \end{cases} \quad (2)$$

This flow appeared for the first time in a work in 1906 by L.S Da Rios [DaR] as a model which describes the flow of a vortex filament. That is to say a vortex tube of infinitesimal cross section which propagates according to Euler equations. Let us call $\chi(s, t)$ a curve in \mathbb{R}^3 which can be either closed, and then we will consider $s \in [0, 2\pi]$ and 2π -periodic functions, or which extends to infinity in both directions. In that case $s \in \mathbb{R}$. Let us consider for simplicity of the exposition this second possibility.

Assume that the vorticity w of a given fluid of velocity u is supported along the filament described by the curve $\chi(s, t)$. By that we mean w is the singular vectorial measure

$$w = \Gamma T ds,$$

where Γ stands for the circulation,

$$T = \chi_s,$$

s is arclength parametrization and Γ is constant along the filament. The velocity can be recovered from the vorticity through the Biot-Savart integral

$$u(P) = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{\chi(s) - P}{|\chi(s) - P|^3} \wedge T(s) ds. \quad (3)$$

Typical examples of vortex structures are:

1. The straight line vortex, $\chi(s) = (0, 0, s)$. The velocity at a point $P = (x, y, z)$ with $z \neq 0$ can be easily computed from (3)

$$u(x, y, z) = \frac{\Gamma}{4\pi} (-y, x, 0) \int_{-\infty}^{\infty} \frac{ds}{(x^2 + y^2 + (z - s)^2)^{\frac{3}{2}}} = (-y, x, 0) \frac{C}{x^2 + y^2}.$$

Hence the velocity around the filament is just a rotation of speed a constant times the inverse of the distance to the filament. As a consequence the filament remains stationary.

2. The vortex ring, $\chi(s, 0) = (\cos s, \sin s, 0)$.

3. The helical vortex, $\chi(s, 0) = (\cos s/\sqrt{2}, \sin s/\sqrt{2}, / \sqrt{2})$,.-see [LF] and [Har]

The computation of the velocity field outside of the filament is much more delicate in the last two examples and in fact, it is better to compute the so called vector potential using cylindric variables. We refer the reader to [AKO], sections 2.5 and 2.6 for the details.

Notice that the Biot–Savart integral is perfectly defined for points P outside of the filament. The idea is to compute $u(P)$ as P approaches the filament to eventually identify that limit with the velocity of the filament. Also observe that the term inside of the integral in (3) has enough decay at infinity. So in order to understand the leading behaviour we can just look at a neighbourhood of $\chi(s_0)$, the closest point in the filament to P . A quick look at the integral is enough to show that the limit diverges as the inverse of the distance. In fact we see that the first approximation is

$$u(P) \sim \frac{\chi(s_0) - P}{|\chi(s_0) - P|} \wedge T(s_0).$$

This term behaves as in the case of example 1 of the straight filament. As we said the overall effect on the movement of the filament is null. Doing the Taylor expansion around that point, calling ϵ the distance of P to the filament and removing the first term we get (see for example [Sa])

$$u(P) \sim -c\Gamma \log \epsilon \chi_s(s_0) \wedge \chi_{ss}(s_0) + \dots$$

Hence we see we have a logarithmic dependence on the distance to the filament on the direction of

$$\chi_s \wedge \chi_{ss} = b(s_0).$$

After a renormalization of time we get

$$\chi_t = \chi_s \wedge \chi_{ss}. \tag{4}$$

This computation is extremely crude and at best indicates that close to the filament the leading term is in the direction of the binormal. The general consense (see [Sa]) about this approximation also known as local induction approximation, is that it is specially suitable for numerical integration but it has serious flaws and is of limited physical interest. In any case it would be extremely useful to find a way of desingularizing the integral in (3).

Our next step is to establish the connection between the binormal flow and non-linear Schrödinger equation. A first hint of such a connection comes by identifying the normal plane to the curve $\chi(s)$ at a given point with the complex plane. Then if n and b are the normal and the binormal vector respectively we can write the binormal as

$$b = in.$$

Hence the binormal flow can be understood as

$$\frac{1}{i} \chi_t = cn.$$

Finally if we remember that the “curvature flow” has as a linear approximation the heat flow, we can “guess” that the underlying linear equation for (4) will be Schrödinger equation. In fact something even stronger is true.

H. Hasimoto in [Has] observed that if we define the complex valued function u as

$$u(s, t) = c(s, t) e^{i \int_0^s \tau(s', t) ds'},$$

where τ stands for the torsion of the curve and c for the the curvature, and if $\chi(s, t)$ solves (4) then u solves

$$i u_t + u_{ss} + \frac{1}{2} (|u|^2 + A(t)) u = 0, \quad (5)$$

with

$$A(t) = \left(2 \frac{c_{ss} - c \tau^2}{c} + c^2 \right) (0, t). \quad (6)$$

Conversely, let $A(t)$ be a time depending function, and u a regular non vanishing solution of (5). If we write

$$u(s, t) = c(s, t) e^{i \phi(s, t)},$$

by defining $\tau(s, t) = \phi_s(s, t)$, we have

$$u(s, t) = c(s, t) e^{i \int_0^s \tau(s', t) ds' + \phi(0, t)}.$$

We obtain then a filament function

$$\tilde{u}(s, t) = c(s, t) e^{i \int_0^s \tau(s', t) ds'},$$

satisfying (5) with $A(t)$ replaced by $A(t) + \phi_t(0, t)$. The real functions c and τ verify then the intrinsic equations and the identity (6) with $A(t)$ replaced by $A(t) + \phi_t(0, t)$.

Notice that taking $\tilde{u} = u \exp\left(-\frac{i}{2} \int_0^t A(t') dt'\right)$ we get

$$i \tilde{u}_t + \tilde{u}_{ss} + \frac{1}{2} |\tilde{u}|^2 \tilde{u} = 0, \quad (7)$$

which is the well known cubic non linear Schrödinger equation.

In doing this H. Hasimoto uses the Frenet frame (T, n, b) and the corresponding Frenet equations

$$\begin{aligned} T_s &= c n, \\ n_s &= -c T + \tau b, \\ b_s &= -\tau n. \end{aligned} \quad (8)$$

There is no reason why to do this and more general frames (T, e_1, e_2) can be used so that the assumption $c > 0$ is not necessary, see for example [Ko]. The final conclusion is still the same and equation (5) is also obtained.

Also it is important to observe that if in (4) we differentiate with respect to s in both sides we get, since $\chi_s = T$,

$$T_t = \chi_s \wedge \chi_{sss} = T \wedge T_{ss}. \quad (9)$$

Hence

$$\frac{d}{dt} |\chi_s|^2 = 2 \frac{d}{dt} \chi_s \cdot \chi_s = 2 (\chi_s \wedge \chi_{ss})_s \cdot \chi_s = 0,$$

and the arclength parametrization is preserved. This is generally admitted as one of the flaws of BF as an approximation of (3).

Notice that $|T|^2 = 1$ and $T \wedge T_{ss} = JD_s T_s$ where J is the complex structure on the sphere and D_s the covariant derivative. Therefore we can write (5) as

$$T_t = JD_s T_s$$

which is nothing but the Schrödinger map onto the sphere in the one dimensional case. For higher dimensions we get

$$T_t = T \wedge \Delta T \tag{10}$$

which also appeared in ferromagnetism where is referred as the Lipschitz–Landau equation.

Let us finish this short introduction to the binormal flow observing that Hasimoto transformation allows us to exhibit some particular examples of (4):

1. The straight line is just $c(s, t) = 0, \quad A(t) = 0.$

2. The circle is just $c(s, t) = 1 \quad A(t) = -1.$

3. The helix is just

$$u(s, t) = e^{-itN^2} e^{iNs} \quad A(t) = -1.$$

4. The travelling wave solutions

$$u(s, t) = e^{-itN^2} e^{iNs} Q(s - 2Nt), \quad A(t) = -1,$$

with $Q(s) = \frac{1}{2\sqrt{2}} \operatorname{sech}(s)$. This last example is particularly interesting because suggests the possibility of the existence of travelling wave solutions along vortex filaments. In fact a few years after H. Hasimoto's paper, E.J. Hopfinger and F.K. Browand [HB] (see also [MHR]) gave experimental evidence of such a possibility.

It is also interesting to recall some results by R. Klein, A. Majda and K. Damodaran [KMD] about the interaction of more than one filament. They propose a mathematical model based on a system of coupling NLS. This model is obtained as an equilibrium between the interaction of a filament with itself, that is described just by the linear part of BF, and with the others, that is measured by a sum of coulombic type potentials, see [MB] for the details. This system allows for helices as solutions. In the case of two or three helices with the same vorticity and at the same distance the configuration is stable, see [KPV1]. There is some experimental evidence that the case of four helices could be unstable, see [AKOS].

Finally recall that the L^2 norm is conserved in NLS. In our setting this becomes

$$\int_{-\infty}^{\infty} c^2(s, t) ds = \int_{-\infty}^{\infty} |u(s, t)|^2 ds, \tag{11}$$

which can be understood as the kinetic energy. Notice that in the case of the helix we obtain a solution of infinite energy eventhough it has finite L^2 -norm when consider as a periodic solution. As a filament one has to look at the complete helix because any vortex filament either ends in the boundary which confines the fluid or it has to go to infinity. This suggests as a natural question to consider solutions of (11) with infinite energy.

1.2 The mKdV flow

The connection between NLS, which is a completely integrable system, and a geometric flow raised the natural question about the existence of more examples with similar connections. G. Lamb in the 70's ([La]) was the first one in extending Hasimoto's idea, obtaining a remarkable generalization which include NLS and mKdV (even as complex valued functions) as possible limits. It is what is known as Hirota equation. Y. Fukumoto and T. Miyazaki [FM] obtained the same flow when in da Rios approximation is included the existence of an axial flow along the filament. So it is still a 3d-flow.

Almost at the same time R.E. Goldstein and D.M. Petrich [GP1] proposed the mKdV hierarchy as the PDE of the curvature $k(s, t)$ of plane curve flows which preserve the enclosed area. In a later paper and in a complete formal fashion they relate it to the dynamics of a vortex patch in the plane. Recall that for plane fluids the vorticity is a scalar, see [GP2]. A vortex patch describes the situation of a compactly supported vorticity with a jump discontinuity through the boundary. Assume for simplicity that the vorticity is a constant Γ in the patch Ω which is interior to the boundary $\partial\Omega$. Then N.J. Zabusky, M.H. Hughes and K.V. Roberts [ZHR] obtain an equation for the boundary which is given by a singular integral coming from Biot-Savart law. After doing a formal Taylor expansion in this integral (see [GP2]) the first term is just a translation along the boundary. The second term is

$$z_t = -z_{sss} + \frac{3}{2}\bar{z}_s z_{ss}^2 \quad z_t = k_s n - \frac{1}{2}k^2 T.$$

If we call k the curvature one gets

$$k_t = -k_{sss} - \frac{3}{2}k^2 k_s$$

which is mKdV equation.

The biggest flaw of this model is that it is also arclength preserving which is something that one cannot assume in the vortex-patch motion due to the filamentation phenomena. In fact, since the early simulations of Deem and Zabusky [DZ] there is a strong numerical evidence (see [Ar] and [Bu2]) that small perturbations of a circular patch can generate long and very thin fingers, and as a consequence a big increasing of the total length of the boundary of the patch is produced. As far as we know, nothing has been rigorously proved so far that can support the connection of this geometric flow with the one of the vortex patches, (see [WD] for a discussion about this point and also about the connection of the travelling wave solutions of the mKdV with the V-states of Deem and Zabusky introduced in [DZ]). However it would be interesting to know if under some assumptions, and possibly with some extra equations that keep track of the evolution of the length of the curve, the mKdV can be useful to describe some of the phenomena that appears in the dynamics of the vortex patches.

The rest of the paper will be devoted to the BF and its connection with NLS. In the next section we shall present some results on NLS equations with rough data, that will lead to some intuition on the existence of selfsimilar solutions for the binormal flow. Section §3 is devoted to the construction of some selfsimilar solutions which are examples of regular curves which develop a singularity in a later time. This singularity is either a corner or a 3d logarithmic spiral. Analogous examples but of plane curves

can be given for the mKdV flow, at least with some smallness condition in the curvature. We refer the reader to the paper by G. Perelman and L. Vega [PeV] for the corresponding results, and to the one by de la Hoz, [Ho2] for some numerical simulations concerning these examples. In the last section we shall study the stability of the selfsimilar solutions constructed in §3.1-§3.3.

2 Cubic NLS with infinite energy

2.1 Some symmetries

Recall the cubic NLS equation

$$\begin{cases} iu_t + u_{ss} + \frac{1}{2}|u|^2u & = 0 \\ u(s, 0) & = u_0(s). \end{cases} \quad (12)$$

This equation is scaling invariant:

- If $\lambda > 0$ then $u_\lambda = \lambda u(\lambda s, \lambda^2 t)$ is a solution if so is u .

It is also galilean invariant:

- If $N \in \mathbb{R}$ and u is a solution then

$$u_N(s, t) = e^{-itN^2 + iNs} u(s - 2Nt, t) \quad (13)$$

is also a solution.

Notice that if \widehat{u} denotes the Fourier transform then

$$\widehat{u}_N(\xi) = \widehat{u}(\xi - N).$$

Hence if u belongs to the Sobolev space H^k ,

$$\begin{aligned} \lim_{N \rightarrow \infty} \int |\widehat{u}_N|^2 (1 + |\xi|)^{2k} d\xi &= \lim_{N \rightarrow \infty} \int |\widehat{u}(\xi)|^2 (1 + |\xi - N|)^{2k} d\xi \\ &= \begin{cases} 0 & \text{if } k < 0 \\ \infty & \text{if } k > 0. \end{cases} \end{aligned}$$

Assume that we have a result that assures that if $u_0 \in H^k$, then there exists a unique solution of (12) up to time $t_0 = t_0(\|u_0\|_{H^k})$ with $t_0(r) \rightarrow \infty$ as $r \rightarrow 0$. Then if $k < 0$ we will immediately obtain a global existence result from a local existence one just by using galilean transformations. This suggests to consider (12) a supercritical equation from the point of view of galilean transformations and within the class of Sobolev spaces H^k if $k < 0$.

2.2 NLS with rough data

In this section we want to review three results about NLS.

First of all we want to address the issue of selfsimilar solutions with respect to scaling within a general framework. This was first studied by T. Cazenave and F.B. Weissler [CW] and then extended by F. Planchon [P11], [P12] to NLS equations with

nonlinearities $|u|^\alpha u$ which are L^2 critical or supercritical, in the Besov spaces $\dot{B}_2^{\frac{n}{2}-\frac{2}{\alpha},\infty}$. But recall we are interested in 1D cubic NLS, which is subcritical in L^2 . At this respect T. Cazenave, L. Vega and M.C. Vilela [CVV] were able to extend F. Planchon's result to nonlinearities which are subcritical in L^2 although the 1d cubic NLS is not included in their result.

Let us consider the power like NLS

$$\begin{cases} iu_t + \Delta u + \lambda|u|^\alpha u = 0 & s \in \mathbb{R}^d, \quad t \in \mathbb{R}, \\ u(s, 0) = u_0(s), \end{cases} \quad (14)$$

and focus on $d = 1$ and $d = 2$ and $\lambda = \pm 1$. In [CVV] the following result was proved.

Theorem 1 *Suppose*

$$\frac{4(d+1)}{d(d+2)} < \alpha < \frac{4(d+1)}{d^2}.$$

There exists $\delta > 0$ such that if $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ satisfies $\|u_0\|_{\mathcal{Y}} < \delta$ then there exists a unique solution $u \in L^{\frac{\alpha(d+2)}{2},\infty}(\mathbb{R}^{d+1})$ of (14) such that

$$\|u\|_{L^{\frac{\alpha(d+2)}{2},\infty}(\mathbb{R}^{d+1})} \leq 2\delta.$$

Here \mathcal{Y} is defined as

$$\mathcal{Y} = \left\{ \varphi \in \mathcal{S}'(\mathbb{R}^d); e^{it\Delta}\varphi \in L^{\frac{\alpha(d+2)}{2},\infty}(\mathbb{R}^{d+1}) \right\},$$

and (14) is understood as the associated integral equation:

$$u(t) = e^{it\Delta}u_0 + i\lambda \int_0^t e^{i(t-\tau)\Delta}|u|^\alpha u d\tau.$$

Notice that

- For $d = 1$, $\frac{8}{3} < \alpha < 8$ and we would like to obtain $\alpha = 2$.
- For $d = 2$, $\frac{3}{2} < \alpha < 3$.

The key issue is as we see to choose a suitable space of initial data which is defined through the free evolution, namely

$$u_0 \in \mathcal{Y} \quad \text{iff} \quad e^{it\Delta}u_0 \in L^{p,\infty}(\mathbb{R}^{d+1}),$$

where $L^{p,\infty}(\mathbb{R}^{d+1})$ denotes the Lorentz space. These spaces are naturally defined by real interpolation which behaves poorly within the setting of mixed norm spaces $L^p(\mathbb{R}, L^q(\mathbb{R}^d))$ and that is the reason for assuming $p = q$ in Theorem 1.

The other key ingredient is to use some Strichartz estimates for the operator

$$\mathcal{T}(F) = \int_0^t e^{i(t-\tau)\Delta}F(\cdot, \tau)d\tau \quad (15)$$

within (L^{p_1}, L^{p_2}) spaces which were unknown and involve pairs (p_1, p_2) which are non admissible in the sense of J. Ginibre and G. Velo [GV]. The possibility of the existence

of this type of inequalities was first observed by T. Cazenave and F.B. Weissler in [CW]. Based on the work of M. Keel and T. Tao [KT], D. Foschi [Fo] and M.C. Vilela [Vi] independently proved a general result at this respect although there are still open questions in this direction.

However the inequalities for the Duhamel operator are easier to prove if one relates \mathcal{T} with Bochner–Riesz operators. These operators have been widely studied in Fourier analysis and are given in terms of the multiplier

$$m_\gamma(\xi) = (1 - |\xi|^2)_+^\gamma, \quad \xi \in \mathbb{R}^d$$

with $(\cdot)_+$ denoting the positive part. Then the Bochner–Riesz operator of order γ is defined as

$$BR_\gamma(f)\widehat{f}(\xi) = m_\gamma(\xi)\widehat{f}(\xi).$$

The index of interest is $\gamma = -1$, which is closely related to the more natural multiplier

$$H(\xi) = (1 - |\xi|^2)^{-1}, \quad \xi \in \mathbb{R}^d \tag{16}$$

understood in the sense of the principal value, and which is linked to Helmholtz equation

$$\Delta u(x) + u(x) = f(x), \quad x \in \mathbb{R}^d. \tag{17}$$

The connection with (15) is clear because $\mathcal{T}F$ is a solution of

$$i\partial_t u + \Delta u = F,$$

and therefore, at least formally

$$\widehat{u} = \frac{1}{-\tau - |\xi|^2} \widehat{F}.$$

By scaling considerations one can always consider that $\text{supp } \widehat{F}(\xi, \tau) \subset \{|\xi| \leq 1\}$ and then one mimics the available proof of the corresponding inequalities for solutions of (17) using the multiplier (16).

The final remark we want to make about Theorem 1 is that the condition on the initial datum is not so easy to verify. In fact to check whether or not the condition is satisfied is quite a delicate question in Fourier Analysis. In the case of the free Schrödinger operator this question is nothing but to study L^p bounds for the dual operator to the free propagator $e^{it\Delta}u_0$. In terms of this latter operator the question is if the inequalities

$$\|e^{it\Delta}u_0\|_{L^p(\mathbb{R}^{d+1})} \leq c \|\widehat{u}_0\|_{L^q(\mathbb{R}^d)} \tag{18}$$

with

- $\frac{d+2}{p} + \frac{d}{q} = d$
- $p > d$,

hold true. The answer is positive if $d = 1$ and it was established by C. Fefferman and E. Stein (see [Fe]). In higher dimensions $d \geq 2$ this was proved by E. Stein and P.A. Tomas for $q \leq 2$ (see [St]). Recently, the case $d = 2$ was extended by T. Tao in [Ta] to $q \leq \frac{5}{2}$. As a consequence if μ is small enough and

$$\|\widehat{u}_0\|_{L^{q,\infty}} \leq \mu \tag{19}$$

for $q = \frac{d\alpha}{d\alpha - 2}$ with $\frac{8}{3} < \alpha < 8$ if $d = 1$ and $\frac{5}{3} < \alpha < 3$ if $d = 2$ we can apply Theorem 1. Moreover in [CVV] it is proved that if u_0 verifies (19) then

$$\sup_t \|\widehat{u}(\cdot, t)\|_{L^{q,\infty}} \leq C\mu$$

for a fixed constant C and with q as above.

This was the starting point of the work by A. Grünrock [Gr1][Gr2]. In [Gr2] he proves the following theorem

Theorem 2 *Let $1 < r \leq 2$ and $\widehat{u}_0 \in L^{r'}(\mathbb{R})$. Then the I.V.P.*

$$iu_t + u_{ss} = \lambda|u|^2u \quad u(0) = u_0 \quad \lambda = \pm 1, \quad (20)$$

is locally well posed and

$$\widehat{u}(\cdot, t) \in L^{r'}(\mathbb{R}) \quad (21)$$

for all t .

So in particular if

$$|\widehat{u}_0(\xi)| \leq \frac{1}{(1 + |\xi|)^\gamma}, \quad \gamma > 0,$$

the conclusion holds. As it shall be explained later, the selfsimilar solutions of BF will lead to the case $\gamma = 0$.

Finally we would like to mention another work, by A. Vargas and L. Vega [VV], related to (20) and which precedes to [Gr2]. First, the following local result is proved in [VV].

Proposition 1 *Assume $u_0 \in \mathcal{S}'(\mathbb{R})$ such that there exists $\tilde{T} > 0$ with*

$$\left\| e^{it\partial_x^2} u_0 \right\|_{L^3([- \tilde{T}, \tilde{T}], L^6(\mathbb{R}))} < +\infty. \quad (22)$$

Then there exists $T \leq \tilde{T}$ and a unique solution u of (20) such that

$$\|u\|_{L^3([-T, T], L^6(\mathbb{R}))} < +\infty.$$

We say that $u_0 \in \mathcal{Y}_2$ if (22) is satisfied uniformly in \tilde{T} . As a global result, the following theorem is proved in [VV].

Theorem 3 *Let $u_0 \in \mathcal{S}'(\mathbb{R})$ that can be decomposed as $u_0 = v_0 + w_0$ with*

$$\|v_0\|_{L^2(\mathbb{R})} \leq M^r \quad , \quad \|w_0\|_{\mathcal{Y}_2} \leq M^{-1} \quad \text{for some } r < 1 \text{ and } M > 1. \quad (23)$$

Then there exists a unique solution u of (20) for

$$0 \leq t \leq T = T(M),$$

that satisfies

$$\|u\|_{L^3([0, T], L^6(\mathbb{R}))} \leq c.$$

Moreover

$$\lim_{M \rightarrow \infty} T(M) = \infty.$$

As a consequence if for all $M > 0$, u_0 can be decomposed as $u_0 = v_0^M + w_0^M$ satisfying (23), then there is a unique global solution of (20).

Again the difficulty relies in giving good conditions on u_0 such that (22) is satisfied. In terms of the pointwise decay of the Fourier transform it is worse than the result by Grünrock, because a decay $r > 1/6$ is necessary. However in terms of the L^p local integrability of \hat{u}_0 the results are stronger. In fact it is sufficient to assume the following. Assume $\hat{u}_0 \in X_{p,\gamma}$ which is given as the functions f such that

$$\|f\|_{p,\gamma} = \sum_{j \geq 0} \left(\sum_{\mathcal{I} \in \mathcal{D}_j} 2^{-p\gamma j} \|f|_{\mathcal{I}}\|_{L^2}^p \right)^{1/p}$$

where $\mathcal{D}_j = \{[k2^j, (k+1)2^j) : k \in \mathbb{Z}\}$, the family of dyadic intervals of length 2^j in \mathbb{R} . Then (22) holds $p = 3$, $\gamma = \frac{1}{6}$. The above norms were firstly introduced by J. Bourgain and naturally appear in the study of the free solutions of Schrödinger equation - see also [PIV] equations, (3.1) and (3.2).

Notice that the advantage of Proposition 1 with respect to Theorem 2 is that the solution obtained in the former case is $L^1_{loc}(\mathbb{R}^{d+1})$. This property is fundamental in order to integrate the Frenet equations to obtain a solution of the BF from cubic NLS. In any case it would be interesting to know if the methods by A. Vargas and L. Vega can be extended to the same range of decay as the ones obtained by A. Grünrock.

2.3 The δ -function as initial datum for NLS

Assume next that $u_0 = a\delta$, for $a \in \mathbb{C}$. Then for all N

$$u_{0,N}(s) := e^{iNs}u_0 = u_0.$$

If uniqueness holds then we conclude from the galilean transform (13) that

$$u(s, t) = u_N(s, t) = e^{-itN^2 + iNs}u(s - 2Nt, t) \quad \text{for all } N.$$

After differentiation with respect to N in the above identity we get

$$0 = i(s - 2Nt)u(s - 2Nt, t) - 2t\partial_s u(s - 2Nt, t).$$

Hence

$$u(s, t) = g(t)e^{i\frac{s^2}{4t}}.$$

But u has to solve NLS. This gives

$$g(t) = \frac{a}{\sqrt{t}} \exp(\pm i|a|^2 \ln t + ia_1), \quad a, a_1 \in \mathbb{C}, \quad (24)$$

which has no limit as $t \uparrow 0^+$. Therefore (12) is ill posed with $u_0 = a\delta$ (see the article of Kenig, Ponce and Vega [KPV2]).

However notice that if we change NLS from (12) into

$$iu_t + u_{ss} + \frac{1}{2}(|u|^2 + A(t))u = 0,$$

we see that taking $A(t) = -\frac{|a|^2}{t}$, then $u_a(s, t) = a \frac{e^{i\frac{s^2}{4t}}}{\sqrt{t}}$ is a solution of

$$\begin{cases} iu_t + u_{ss} + \frac{1}{2} \left(|u|^2 - \frac{|a|^2}{t} \right) u = 0, \\ u(s, 0) = a\delta. \end{cases}$$

This suggests the possibility of finding one solution of the binormal flow. The corresponding curve should be determined by the property that curvature and torsion are

$$c(s, t) = \frac{a}{\sqrt{t}} \quad , \quad \tau(s, t) = \frac{s}{2t}, \quad (25)$$

for some $a \in \mathbb{R}^+$.

3 Selfsimilar solutions for the binormal flow

In this section we review some of the results proved in [GRV] and [GuV] concerning the self-similar solutions of the binormal flow. We also include in Propositions 4 and 5 some results for curves in \mathbb{R}^3 that as far as we know are new. These results are an extension to general curves of those proved in [GRV] for the particular case in which the curve is precisely a self-similar solution of the binormal flow.

3.1 A V-shape as initial datum for the binormal flow

The BF solution corresponding to (25) should have as initial condition

$$\chi(s, 0) = \begin{cases} sA^+ & \text{if } s \geq 0 \\ -sA^- & \text{if } s < 0, \end{cases}$$

with $|A^+| = |A^-| = 1$.

Frenet equations (8) allow us to compute $\chi(s, t)$ from (25) up to a rigid motion, obtaining a real analytic curve for $t > 0$ which should have a singularity at $t = 0$. Notice that the binormal flow is reversible in time: if $\chi(s, t)$ is a solution so is $\tilde{\chi}(s, t) = -\chi(s, -t)$. Recall that the energy in this case is infinity, although it is locally finite as for the helix.

We gather the results proven on selfsimilar solution of the binormal flow in the following theorem.

Theorem 4 *The self-similar solutions are parametrized by a positive number a , and satisfy*

$$|\chi_a(s, t) - sA_a^+ \mathbb{I}_{[0, \infty)}(s) - sA_a^- \mathbb{I}_{(-\infty, 0]}(s)| \leq 2a\sqrt{t},$$

for $t \geq 0$, where A_a^\pm are two distinct non opposite unit vectors. In particular,

$$\chi_a(s, 0) = \begin{cases} A_a^+ s & s > 0 \\ A_a^- s & s < 0. \end{cases} \quad (26)$$

Moreover, there exists $\tilde{a}, \tilde{\tilde{a}}$ such that if $a < \tilde{a}$, $\chi_a(s, 1)$ has no self-intersection and if $a > \tilde{\tilde{a}}$ it has finitely many. Finally, the relation between a and A_a^\pm is

$$\sin \frac{\theta}{2} = e^{-\frac{\pi a^2}{2}},$$

where θ is the angle between the vectors A_a^+ and $-A_a^-$.

The behaviour of $\chi_a(s, t)$ for $t > 0$ was exhibited by the physicists M. Lakshmanan and M. Daniel in [LD] and a numeric study on it was done by T.F. Buttke in [Bu1]. The rigorous description of these solutions up to $t = 0$ was done by S. Guttierrez, J. Rivas and the second author in [GRV], and will be presented in the two next sections. Let us mention that the link between the parameter a and the angle between the vectors A_a^+ and $-A_a^-$ is an original material, and we dedicate to its proof the third subsection §3.3.

Before trying to compute χ is worthy to stress the point that galilean transformations act on the set of solutions of the BF. In fact writting the flow in intrinsic coordinates we get -see [DaR], [Be]

$$\begin{aligned} c_t &= -(c\tau)_s - c_s\tau \\ \tau_t &= \left(\frac{c_{ss} - c\tau^2}{c} \right)_s + c_s c. \end{aligned} \quad (27)$$

It is straightforward to observe that for $N \in \mathbb{R}$

$$(c_N, \tau_N) = (c(s - 2Nt, t), \tau(s - 2Nt, t) + N)$$

is a solution of (27) if so is (c, τ) . Also

$$(c_N, \tau_N) = (c, \tau) \quad \forall N \quad \text{iff} \quad (c, \tau) = \left(\frac{a}{\sqrt{t}}, \frac{s}{2t} \right) \quad t > 0.$$

3.2 The computation of the selfsimilar solutions and some remarks

The simplest way to construct $\chi(s, t)$ is to observe that from (25) we can expect it to have a selfsimilar behaviour also with respect to scaling. Hence we settle the ansatz:

$$\chi(s, t) = \sqrt{t} G \left(\frac{s}{\sqrt{t}} \right) \quad t > 0,$$

which gives from the BF equation ($\chi_t = \chi_s \wedge \chi_{ss}$)

$$\frac{1}{2}G - \frac{s}{2}G' = G' \wedge G''. \quad (28)$$

Since $G' = T$, a new differentiation gives using Frenet equations

$$\begin{aligned} -\frac{s}{2}T' = T \wedge T'' &= (T \wedge (c_s n + c n_s)) \\ &= c_s b + c (T \wedge (-cT + \tau b)). \end{aligned}$$

Hence (see [LD] and [Bu1]),

$$\frac{s}{2}c n = -c_s b + c \tau n \quad (29)$$

and therefore

$$c_s = 0 \quad \tau = \frac{s}{2}. \quad (30)$$

In particular, there exists $a \in \mathbb{R}^+$ such that $c = a$. Now the Frenet equations write

$$\begin{aligned} T' &= a n, \\ n' &= -a T + \frac{s}{2} b, \\ b' &= -\frac{s}{2} n, \end{aligned} \quad (31)$$

and we get that $G' = T$ is a real analytic function. Moreover

$$T(s, t) = T\left(\frac{s}{\sqrt{t}}\right). \quad (32)$$

Recall that BF is invariant under rotations so we can assume without loss of generality that

$$\begin{pmatrix} T \\ n \\ b \end{pmatrix}(0, 0) = \mathbb{I}_{3\wedge 3} \quad \text{and from (32)} \quad \begin{pmatrix} T \\ n \\ b \end{pmatrix}(0, t) = \mathbb{I}_{3\wedge 3}.$$

We have to determine $\chi(0, t)$. But from

$$\chi_t = cb$$

we conclude

$$\chi_t(0, t) = \frac{a}{\sqrt{t}}(0, 0, 1).$$

As a conclusion $\chi(0, t) = 2a\sqrt{t}(0, 0, 1)$. We could have got it also from (28).

Therefore

$$\chi(s, t) = \sqrt{t} G\left(\frac{s}{\sqrt{t}}\right)$$

and G is determined up to a rotation by the solution of (31) with

$$\begin{aligned} G(0) &= 2a(0, 0, 1) \\ T(0) &= (1, 0, 0) \\ n(0) &= (0, 1, 0). \end{aligned}$$

It is easy to do the numerical analysis of G . We start with (28):

$$\frac{1}{2}G - \frac{s}{2}G' = G' \wedge G''$$

Recall the identity:

$$F \wedge (G \wedge H) = (F \cdot H) \cdot G - (F \cdot G) \cdot H. \quad (33)$$

Then

$$G' \wedge (G' \wedge G'') = (G' \cdot G'') G' - (G' \cdot G') G''$$

But on the other hand from

$$-\frac{s}{2}T' = T \wedge T''$$

we get

$$-\frac{s}{2}T' \cdot T = 0 = -\frac{s}{2}G' \cdot G''$$

and $|T|^2 = 1$.

Hence from (28), (33)

$$\begin{cases} \frac{1}{2}G \wedge G' = G'' \\ G'(0) = (1, 0, 0) \end{cases} \quad G(0) \cdot G'(0) = 0, \quad (34)$$

which can be easily computed numerically. We refer the reader to the web page <http://ehu.es/luisvega/vortex/files/esquina.mov> to see the corresponding flow of curves. In these pictures is clearly seen that the profile G tends to two straight lines at infinity, and as a consequence that

$$\chi(s, 0) = \begin{cases} A^+s & s > 0 \\ A^-s & s < 0. \end{cases} \quad (35)$$

In order to prove it we will have to work a little bit. We follow the arguments of S. Gutierrez, J. Rivas and L. Vega [GRV]. Recall that from (28):

$$\frac{1}{2}G - \frac{s}{2}G' = T \wedge T' = T \wedge an = ab.$$

Hence, for $s > 0$

$$\left(\frac{G(s)}{s}\right)' = \frac{sG' - G}{s^2} = -\frac{2a}{s^2}b.$$

It follows that

$$\frac{G(s)}{s} = G(1) + \int_1^s -\frac{2a}{s'^2}b(s')ds', \quad (36)$$

and therefore, since b is unitary, we conclude that there exists

$$\lim_{s \rightarrow \infty} \frac{G(s)}{s} = A^+. \quad (37)$$

Analogously we obtain that there exists A^- such that $\lim_{s \rightarrow -\infty} \frac{G(s)}{s} = A^-$.

Moreover,

$$A^+ - \frac{G(s)}{s} = \frac{G(s)}{s} \Big|_s^\infty = -2a \int_s^\infty \frac{b(s')}{s'^2} ds',$$

and

$$G(s) = sA^+ + 2as \int_s^\infty \frac{b(s')}{s'^2} ds' \quad s > 0 \quad (38)$$

$$G(s) = sA^- - 2as \int_{-\infty}^s \frac{b(s')}{s'^2} ds' \quad s < 0. \quad (39)$$

We have the following proposition.

Proposition 2 *For all s*

$$|\chi(s, t) - sA^+\mathbb{I}_{[0, \infty)}(s) - sA^-\mathbb{I}_{(-\infty, 0]}(s)| \leq 2a\sqrt{t} \quad t > 0,$$

where \mathbb{I} denotes the characteristic function.

Proof If $s = 0$ it is trivial because $\chi(0, t) = 2a\sqrt{t}$. We will assume that $s > 0$, the other case been proved in a similar way. Then

$$\chi(s, t) = \sqrt{t}G\left(\frac{s}{\sqrt{t}}\right) = sA^+ + 2as \int_{s/\sqrt{t}}^{\infty} \frac{b(s')}{s'^2} ds',$$

and then

$$|\chi(s, t) - sA^+| \leq 2as \left| \int_{s/\sqrt{t}}^{\infty} \frac{b(s')}{s'^2} ds' \right|.$$

The proof follows using that b is unitary.

Remark 1 *It can be also proved that (see [GRV])*

$$\lim_{t \rightarrow 0} T(s, t) = \begin{cases} A^+ & s > 0, \\ A^- & s < 0. \end{cases}$$

Our next purpose is to prove that $A^+ \neq A^-$. Recall that we are dealing with a one parameter family of solutions with a as the free parameter. We will make explicit the dependence on a in the following proposition in order to clarify the exposition.

Proposition 3 (i) $A_a^+ \neq -A_a^- \quad \forall a$.

(ii) $A_a^+ = A_a^- \iff a = 0$.

(iii) $\exists \tilde{a}, \tilde{\tilde{a}}$ such that if $a < \tilde{a}$, $G_a(s)$ has no selfintersection and if $a > \tilde{\tilde{a}}$ it has finitely many.

Proof Recall that $G_a - sG'_a = 2ab$. Also $G''_a = T' = an$. Hence

$$\begin{aligned} G'''_a = an' &= a \left(-aT + \frac{s}{2}b \right) \\ &= -a^2T + \frac{s}{4}(G_a - sG'_a). \end{aligned}$$

Then

$$G'''_a + \left(a^2 + \frac{s^2}{4} \right) G'_a - \frac{s}{4} G_a = 0 \quad (40)$$

$$\begin{cases} G_a(0) &= 2a(0, 0, 1) \\ G'_a(0) &= (1, 0, 0) \\ G''_a(0) &= a(0, 1, 0) \end{cases} \quad (41)$$

Also,

- $G_a(s) = (x_a(s), y_a(s), z_a(s))$ with $\begin{pmatrix} x_a(0) \\ x'_a(0) \\ x''_a(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
- In (40) if $G_a(s)$ is a solution then $G_a(-s)$ is a solution.

We conclude that if $a \neq 0$ then

$$\begin{aligned} x_a(s) &= -x_a(-s), \\ y_a(s) &= y_a(-s), \\ z_a(s) &= z_a(-s), \end{aligned} \tag{42}$$

and therefore x_a, y_a, z_a are three linearly independent solutions of the scalar ODE

$$w''' + \left(a^2 + \frac{s^2}{4}\right) w' - \frac{s}{4} w = 0 \tag{43}$$

which are tempered distributions (G_a, G'_a, G''_a, G'''_a grow polynomially). Applying the Fourier transform we get:

$$\xi \widehat{w}'' + 3\widehat{w}' + 4\xi^2 \widehat{w} - 4\xi a^2 \widehat{w} = 0. \tag{44}$$

This is a singular regular O.D.E. with $r = 0$ and $r = -2$ the solutions of the indicial equation

$$r(r-1) + 3r = 0.$$

Therefore the regular solutions have to be even, and the odd solutions have to behave as $\omega_0 \delta'$ at the origin. But we know that x_a is odd and non trivial. Therefore $w_a \neq 0$ which amounts to say that $A_{a,1}^+ \neq 0$, where we use the vectorial notation $A_a^\pm = (A_{a,1}^\pm, A_{a,2}^\pm, A_{a,3}^\pm)$. From (42) we get

$$\begin{aligned} A_{a,1}^+ &= A_{a,1}^-, \\ A_{a,2}^+ &= -A_{a,2}^-, \\ A_{a,3}^+ &= -A_{a,3}^-. \end{aligned}$$

Hence $A_a^+ + A_a^- \neq 0$.

Assume now that $A_a^+ = A_a^-$. Then $A_{a,2}^+ = A_{a,2}^- = A_{a,3}^+ = A_{a,3}^- = 0$. But then the even functions $\widehat{y}_a(\xi), \widehat{z}_a(\xi)$ should be regular at the origin and linearly dependent. Hence $a = 0$ and $A_a^+ = (1, 0, 0) = A_a^-$.

Only (iii) remains to be proved. Recall that $G_a = sT + 2ab$. Hence $|G_a|^2 = s^2 + a^2$. Therefore if $G_a(s_1) = G_a(s_2)$ for different s_1 and s_2 , then $s_1^2 = s_2^2$ and $s_1 = -s_2$. But from (42) this implies that $x(s_1) = 0 = x(s_2)$. Recall that

$$\begin{aligned} x_a''' + \left(a^2 + \frac{s^2}{4}\right) x_a' - \frac{s}{4} x_a &= 0, \\ x_a(0) &= 0, \\ x_a'(0) &= 1, \\ x_a''(0) &= 0. \end{aligned}$$

Hence $x_a(s) = s + O(a^2 s^3)$ and on the other hand the map $a \mapsto \lim_{s \rightarrow \infty} \frac{x_a(s)}{s} = A_{1,a}^+$ is continuous from the proof of Proposition 2. Hence if a is small enough there is no selfintersection. On the other hand it is easy to see that $x_a'(s) = \cos(as) + O\left(\frac{s^3}{a}\right)$ and therefore

$$x_a(s) = \frac{1}{a} \sin(as) + O\left(\frac{s^4}{a}\right) \quad 0 < s < 1.$$

Hence for a large there are selfintersections.

Notice that if $\chi(s, t)$ is a solution $-\chi(s, -t)$ is also a solution. Hence

$$\begin{cases} \chi_t = \chi_s \wedge \chi_{ss} \\ \chi(s, -1) = -G_a(s). \end{cases}$$

is a solution that develops a singularity at time $t = 0$ although G_a is real analytic.

A few remarks are in order.

Remark 2 *The kinetic energy $(\int c^2(s, 1) ds)$ is infinity but locally finite as in the case of the helix.*

Remark 3 *The solution $\chi_a(s, t) = \sqrt{t} G_a\left(\frac{s}{\sqrt{t}}\right)$ has as torsion $\frac{s}{2t}$ which is selfsimilar of the type*

$$\tau_\lambda(s, t) = \lambda \tau(\lambda s, \lambda^2 t).$$

This scaling leaves invariant the following conserved quantity

$$\int \tau(s, 0) ds = \int \tau(s, t) ds.$$

Notice that in our situation the above invariant has to be understood in the sense

$$\lim_{R \rightarrow \infty} \int_{-R}^R \tau(s, t) ds = 0.$$

Remark 4 *It is interesting to analyze the type of singularity we get from the point of view of Euler equations and vortex filaments. In particular let us recall the condition obtained by P. Constantin, C. Fefferman and A. Majda [CFM] which is measured in terms of $\|\nabla_x \xi\|_{L^\infty}$, where ξ stands for the unit direction of the vorticity:*

$$\xi = \frac{\omega}{|\omega|}. \quad (45)$$

Their condition is that

$$\int_0^T \|\nabla_x \xi\|_{L^\infty}^2 dt = +\infty \quad (46)$$

if there is a blow up at time T . In our setting ξ plays the role of T and therefore $|\nabla_x \xi|$ is the curvature. Hence (46) becomes

$$\int_{-1}^0 \frac{a^2}{|t|} dt = \infty. \quad (47)$$

Remark 5 *Recall that in the case of the free Schrödinger equation we have a similar situation. Taking*

$$u(s, t) = \frac{e^{i\frac{s^2}{4t}}}{\sqrt{|t|}} \quad -1 < t < 0,$$

we have a solution of

$$\begin{cases} iu_t + u_{ss} = 0, \\ u(s, -1) = e^{i\frac{s^2}{4}}, \end{cases}$$

such that $u(s, 0) = 2\sqrt{\pi}e^{i\frac{\pi}{4}} \delta$.

Remark 6 Equation (10) of the tangent vector suggests many possible generalizations by considering other targets besides the sphere. Let us then introduce the notation

$$u \wedge_- v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} u \wedge v.$$

Therefore instead of (10) we consider

$$T_t = T \wedge_- T_{ss}, \quad (48)$$

with T a map from \mathbb{R}^2 onto the hyperbolic plane \mathbb{H}^2 . Analogously, since $T = \chi_s$, we can obtain the equation

$$\chi_t = \chi_s \wedge_- \chi_{ss}. \quad (49)$$

Similar calculations and results as in [GRV] were done by de la Hoz [Ho1] for selfsimilar solutions of (49) in the hyperbolic setting. The extra-difficulty is that there is a-priori no control on the size of the euclidean length of the generalized binormal vectors.

Remark 7 Let us finally remark that numerical computations on the selfsimilar curves of BF were done by F. de la Hoz, C. García-Cervera and L. Vega in [HG-CV]. In that paper the authors point out the similarity of these curves with those that appear in the experiments of the flow around a delta wing.

3.3 Proof of the identity $A_{a,1} = e^{-\frac{\pi a^2}{2}}$

Let us try to answer a more delicate question. We have been able to construct non trivial solutions of BF such that the initial condition has a V-shape determined by two unit vectors A_a^+ and A_a^- . The question we propose is if given any pair of vectors (A^+, A^-) there is a such that $(A^+, A^-) = (A_a^+, A_a^-)$. We have seen in Proposition 3 (ii), that the particular case $A^+ = -A^-$ cannot be obtained using the family of curves $\{G_a\}$, $a \in \mathbb{R}$. In this section we shall prove that except for this case the answer to our question is positive.

The proof requires to integrate the Frenet system of equations in a different way. We start defining the complex vector

$$N = n + i b e^{i \int_0^s \tau(s') ds'}$$

and

$$u = c e^{i \int_0^s \tau(s') ds'}.$$

Then we rewrite the Frenet system as:

$$\begin{aligned} N_s &= -u T \\ T_s &= \frac{1}{2} (\bar{u} N + u \bar{N}). \end{aligned}$$

We denote (v_1, v_2, v_3) the components of a \mathbb{R}^3 vector v . Then

$$|n_j + i b_j|^2 + T_j^2 = n_j^2 + b_j^2 + T_j^2 = 1.$$

Next we introduce

$$\varphi_j = \frac{N_j}{1 - T_j}$$

so that φ_j satisfies the Riccati equation

$$\varphi_j' - \frac{1}{2} \bar{u} \varphi_j^2 - \frac{1}{2} u = 0.$$

Now we consider η_j such that

$$\varphi_j(s) = \frac{u}{c} \eta_j(s) = e^{i \int_0^s \tau} \eta_j,$$

$$\varphi_j' = \frac{u}{c} \eta_j' + i\tau \frac{u}{c} \eta_j = \frac{1}{2} u \eta_j^2 + \frac{1}{2} u = \frac{1}{2} u (\eta_j^2 + 1).$$

Hence

$$\eta_j' + i\tau \eta_j = \frac{1}{2} c (1 + \eta_j^2).$$

The next step is to define θ_j as $\theta_j = \exp[-c/2 \int_0^s \eta_j]$, i.e.:

$$\eta_j = -\frac{2}{c} \frac{\theta_j'}{\theta_j}$$

Then

$$\eta_j' = 2 \frac{c'}{c^2} \frac{\theta_j'}{\theta_j} - \frac{2}{c} \frac{\theta_j''}{\theta_j} + \frac{2}{c} \frac{\theta_j'^2}{\theta_j^2}$$

Hence

$$2 \frac{c'}{c^2} \frac{\theta_j'}{\theta_j} - \frac{2}{c} \frac{\theta_j''}{\theta_j} = -i\tau \frac{2}{c} \frac{\theta_j'}{\theta_j} + \frac{1}{2} c.$$

We have arrived at the remarkable fact that θ_j solves

$$\theta_j'' + \left(-\frac{c'}{c} + i\tau\right) \theta_j' + \frac{c^2}{4} \theta_j = 0. \quad (50)$$

Multiplying by $\frac{\theta_j'}{c^2}$ we get that

$$E(s) := \left(\frac{\theta_j'}{c}\right)^2 + \frac{\theta_j^2}{4} = \text{constant} = E(0). \quad (51)$$

Also notice that

$$|N_j|^2 + |T_j|^2 = 1$$

is equivalent to $(n_j - ib_j)(n_j + ib_j) = (1 - T_j)(1 + T_j)$. Hence

$$|\varphi_j|^2 = \varphi_j \bar{\varphi}_j = \frac{n_j + ib_j}{1 - T_j} \cdot \frac{n_j - ib_j}{1 - T_j} = \frac{1 + T_j}{1 - T_j},$$

and

$$T_j = \frac{|\varphi_j|^2 - 1}{|\varphi_j|^2 + 1}.$$

But $\varphi_j = e^{i \int_0^s \tau} \eta_j$ and $\eta_j = -\frac{2\theta'_j}{c\theta_j}$. Then

$$T_j = \frac{|\eta_j|^2 - 1}{|\eta_j|^2 + 1} = \frac{\frac{4}{c}|\theta'_j|^2 - |\theta_j|^2}{\frac{4}{c}|\theta'_j|^2 + |\theta_j|^2} = 1 - \frac{|\theta_j|^2}{2E(0)}.$$

Therefore we have proved the following proposition valid for any curve.

Proposition 4 *Any of the components T_j of the tangent vector T can be written as*

$$T_j = 1 - \frac{|\theta_j|^2}{2E(0)},$$

with θ_j solution of (50) and $E(0)$ as in (51).

Assume for simplicity

$$\begin{pmatrix} T \\ n \\ b \end{pmatrix} (0) = \mathbb{I}_{3 \wedge 3}.$$

Then

$$T_1(0) = 1 \quad \implies \quad \theta_1(0) = 0.$$

Taking $\theta'_1(0) = \frac{c(0)}{\sqrt{2}}$ we get

$$T_1 = 1 - |\theta_1|^2 \quad ; \quad E(0) = \frac{1}{2}. \quad (52)$$

Similarly

$$T_2(0) = 0 \quad \implies \quad \eta_2(0) = 1 \quad , \quad \theta_2(0) = 1 \quad , \quad \theta'_2(0) = -\frac{c(0)}{2}.$$

Then

$$T_2 = 1 - |\theta_2|^2 \quad ; \quad E(0) = \frac{1}{2}. \quad (53)$$

Finally

$$T_3(0) = 0 \quad , \quad \eta_3(0) = i \quad , \quad \theta_3(0) = i \quad , \quad \theta'_3(0) = \frac{c(0)}{2}$$

gives

$$T_3 = 1 - |\theta_3|^2 \quad ; \quad E(0) = \frac{1}{2}. \quad (54)$$

As for n and b we get, dropping the j -index,

- $cn = T_s = (1 - |\theta|^2)_s = -2\text{Re } \theta \bar{\theta}'$
- $T_{ss} = -2|\theta'|^2 - 2\text{Re } \theta \bar{\theta}'' = -2|\theta'|^2 + \frac{c^2}{2}|\theta|^2 - 2\frac{c'}{c}\text{Re } \theta \bar{\theta}' + 2\tau\text{Im } \theta \bar{\theta}'$

From $T_{ss} = (cn)_s$ we get

$$\frac{c}{2}b = \text{Im } \theta \bar{\theta}'.$$

Hence we have proved the following proposition valid for any curve.

Proposition 5 Any of the components n_j and b_j of the normal and binormal vectors n and b can be written as

$$c(n_j - ib_j) = -2\theta_j \bar{\theta}'_j \quad , \quad j = 1, 2, 3, \quad (55)$$

with θ_j solution of (50) verifying (52)-(54).

In our particular case equation (50) reduces to

$$\theta'' + i\frac{s}{2}\theta' + \frac{a^2}{4}\theta = 0. \quad (56)$$

Our final purpose is to find a relation between γ , the angle determined by A_a^+ and $-A_a^-$, and the parameter a .

It is easy to find two linearly independent solutions by just computing the Fourier transform in (56) :

$$\beta_1(s) = \int_{-\infty}^{\infty} e^{i(s\xi + \xi^2)} \frac{d}{d\xi} \left[e^{-i\frac{a^2}{2} \log |\xi|} \chi_{[0, +\infty)}(\xi) \right] d\xi, \quad (57)$$

$$\beta_2(s) = \int_{-\infty}^{\infty} e^{i(s\xi + \xi^2)} \frac{d}{d\xi} \left[e^{-i\frac{a^2}{2} \log |\xi|} \chi_{[(-\infty, 0]}(\xi) \right] d\xi. \quad (58)$$

Recall that $A_a^+ = (A_{a,1}, A_{a,2}, A_{a,3})$, $A_a^- = (A_{a,1}, -A_{a,2}, -A_{a,3})$ so that

$$\cos \gamma = 1 - 2A_{a,1}^2 \quad ; \quad \sin \frac{\gamma}{2} = A_{a,1}.$$

Then it suffices to compute $A_{a,1} = \lim_{s \rightarrow \infty} T_1(s)$.

But

$$T_1(s) = 1 - |\theta_1|^2 = 1 - |a_{11}\beta_1 + a_{21}\beta_2|^2$$

with

$$a_{1,1} = a_{2,1} = \frac{a}{2\sqrt{2}\beta'_1(0)}.$$

Hence

$$1 - T_1 = \frac{a^2}{4(\beta'_1(0))^2} |\beta_1(s) + \beta_2(s)|^2.$$

But notice that

$$\lim_{s \rightarrow \infty} (\beta_1 + \beta_2)(s) = \int_{-\infty}^{\infty} e^{i\xi} \frac{d}{d\xi} \left[e^{-i\frac{a^2}{2} \log |\xi|} \right] d\xi$$

and

$$\beta'_1(0) = \frac{a^2}{2} \int_0^{\infty} e^{i\xi^2 - \frac{a^2}{2} \log |\xi|} d\xi.$$

After some calculations we get

$$A_{a,1} = e^{-\frac{\pi a^2}{2}}. \quad (59)$$

Finally notice that the dispersive relation of β_1 and β_2 is

$$\phi(\xi) = \xi^2 - \frac{a^2}{2} \log |\xi|, \quad (60)$$

which indicates that the non-linearity only makes a logarithmic correction to the free evolution.

3.4 Spirals

In this section we will review some of the results obtained by S. Gutierrez and L. Vega in [GuV]. So far we have dealt with the I.V.P.

$$\begin{cases} iu_t + u_{ss} + \frac{u}{2}(|u|^2 + A(t)) & = 0 \\ u(s, 0) & = a\delta. \end{cases} \quad (61)$$

In this section we will be interested in

$$u(s, 0) = a_1 \text{ p.v. } \frac{1}{s}. \quad (62)$$

On the other hand concerning the BF we have characterized all the possible solutions with respect to the scaling that preserve arc length. They were determined up to rotations by a free parameter, namely the curvature must be $\frac{a}{\sqrt{t}}$ and the torsion $\frac{s}{2t}$.

In order to capture the solutions of the BF which come from the initial condition (62) we have to change the ansatz. Assume \mathcal{A} is a real antisymmetric matrix. Then consider solutions of the BF

$$\chi_t = \chi_s \wedge \chi_{ss} \quad (63)$$

of the type

$$\chi(s, t) = e^{\mathcal{A}/2 \log t} \sqrt{t} G \left(\frac{s}{\sqrt{t}} \right) \quad t > 0. \quad (64)$$

Hence G has to be a solution of ($e^{\mathcal{A}/2 \log t}$ is a rotation)

$$(I + \mathcal{A})G - sG' = 2G' \wedge G'' \quad , \quad |G'|^2 = 1. \quad (G' \cdot G'' = 0). \quad (65)$$

Multiplying on the left by G' we obtain

$$G'' = \frac{1}{2}(I + \mathcal{A})G \wedge G', \quad (66)$$

which together with the conditions

$$(I + \mathcal{A})G(0) \cdot G'(0) = 0, \quad (67)$$

$$|G'(0)|^2 = 1, \quad (68)$$

is equivalent to (65). It is quite straightforward to prove that (66) has global existence. Also it can be easily computed numerically. Again in this case the relevant feature is the behaviour of G at \pm infinity. This follows from the identity

$$\left(e^{-\mathcal{A} \log s} \frac{G(s)}{s} \right)_s = \frac{(-\mathcal{A} + I)G - G'}{s^2} = 2 \frac{c b}{s^2}.$$

Hence if we prove that the curvature is bounded we can proceed as in the case of the kink solutions. This is a consequence of the following elemental lemma. Consider without loss of generality

$$\mathcal{A} = \begin{pmatrix} 0 & -\mu & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mu \in \mathbb{R}.$$

Lemma 1 *If G solves (65) then*

$$|T'|^2(s) = -\mu T_3(s) - \nu \quad \text{with} \quad (69)$$

$$\nu = -\mu T_3(0) - \frac{1}{4} |(I + \mathcal{A})G(0)|^2. \quad (70)$$

Proof Differentiating in (65) we get

$$\mathcal{A}T = sT' + 2T \wedge T'.$$

Then

$$\mathcal{A}T \wedge T = -sT \wedge T' - 2(T \cdot T')T + 2T''$$

$$(\mathcal{A}T \wedge T) \cdot T' = 2T'' \cdot T' = \frac{d}{ds} |T'|^2.$$

But $(\mathcal{A}T \wedge T) \cdot T' = -\mu T_3'$ which concludes the proof.

It turns out that if $a \neq 0$ the frame $\left(T, \frac{\mathcal{A}T}{|\mathcal{A}T|}, \frac{\mathcal{A}T \wedge T}{|\mathcal{A}T \wedge T|}\right)$ is the right one to study this family of solutions in the spirit of what we have seen for the kink solutions. This is a consequence of the identities:

$$c(n - ib) = \left(\frac{dc^2}{ds} - 2ic^2 \left(\tau - \frac{s}{\tau}\right)\right) \left(\frac{\mathcal{A}T \wedge T}{|\mathcal{A}T \wedge T|^2} - i \frac{\mathcal{A}T}{|\mathcal{A}T|^2}\right). \quad (71)$$

Let us consider (y, h) given by

$$y := \frac{dc^2}{ds} = \frac{d|T'|^2}{ds} \quad (72)$$

$$h := c^2 \left(\tau - \frac{s}{2}\right). \quad (73)$$

The key point then is that (y, h) solves the system

$$\begin{cases} y' = sh + g(|T'|^2), & g(|T'|^2) = 2E(0) - \frac{(3|T'|^2 + \nu)(|T'|^2 + \nu)}{2} \\ h' = -\frac{s}{4}y, \end{cases} \quad (74)$$

with ν given in (70).

The relation with NLS of the above comments is also clear. The ansatz (64) implies that c and τ have to be selfsimilar i.e.

$$c(s, t) = \frac{1}{\sqrt{t}} \tilde{c}\left(\frac{s}{\sqrt{t}}\right) \quad ; \quad \tau(s, t) = \frac{1}{\sqrt{t}} \tilde{\tau}\left(\frac{s}{\sqrt{t}}\right)$$

Therefore we have to look for solutions of (5) with $A(t) = \frac{\nu}{t}$ of the type $\frac{1}{\sqrt{t}} \tilde{u}\left(\frac{s}{\sqrt{t}}\right)$.

This gives us ODE

$$\tilde{u}'' - \frac{i}{2} (\tilde{u} + s\tilde{u}') + \frac{\tilde{u}}{2} (|\tilde{u}|^2 + \nu) = 0.$$

Then we consider $\tilde{u}(s) = f(s)e^{is^2/4}$ with

$$f'' + i\frac{s}{2}f' + \frac{f}{2}(|f|^2 + \nu) = 0. \quad (75)$$

The fundamental property of (75) is that it has a natural energy

$$E(s) = |f'|^2 + \frac{1}{4}(|f|^2 + \nu)^2 = E(0). \quad (76)$$

Finally the way to analyze the solutions of (75) is by defining y and h as

$$\frac{y}{2} + ih = \bar{f}f'. \quad (77)$$

It turns out that (y, h) solves the system (74). The results can be found in [GuV].

With respect to the I.V.P. with data (62) we have the following result.

Proposition 6 *Let $\nu \geq 0$ and consider the I.V.P.*

$$\begin{cases} iu_t + u_{ss} + \frac{u}{2} \left(|u|^2 + \frac{\nu}{t} \right) = 0 & t > 0 \quad s \in \mathbb{R}, \\ u(x, 0) = a_1 p.v. \frac{1}{s} & a_1 \in \mathbb{C} \setminus \{0\}. \end{cases} \quad (78)$$

Then either there is no weak solution u for the I.V.P. (78) in the class

$$t^{1/2}u, \quad t^{3/2}|u|^2u \in L^\infty([0, +\infty), \mathcal{S}(\mathbb{R}))$$

with $\lim_{t \downarrow 0} u(s, t) = a_1 p.v. \left(\frac{1}{s} \right)$ in $\mathcal{S}'(\mathbb{R})$ or there is more than one.

4 Stability

In this section we shall present the known results concerning the stability of the BF selfsimilar solutions χ_a , which show that under some appropriate small regular perturbations the corner still remains at time zero. To this purpose we first study the perturbations of the filament function $u_a(s, t) = a \frac{e^{i\frac{s^2}{4t}}}{\sqrt{t}}$, solutions of

$$\begin{cases} iu_t + u_{ss} + \frac{1}{2} \left(|u|^2 - \frac{|a|^2}{t} \right) u = 0, \\ u(s, 0) = a \delta. \end{cases} \quad (79)$$

The formation of a corner for the flow of curves χ_a corresponds to the limit $a \delta$ of u_a as time goes to zero. The informations about the perturbations of u_a will be gathered in the first subsection. Then in the last subsection we shall deal with the construction of new BF solutions. We shall define from a perturbation of u_a a curvature and a torsion function. Then we shall construct a flow of curves, solution of the BF with these curvature and torsion function. Finally, we shall prove that the new flow of curves is close to the selfsimilar one, and in particular a singularity is still generated at time zero.

4.1 Perturbations of NLS solutions with δ -function as initial datum

In an attempt to understand the above equation (79), we considered in [BV1] the equation

$$iu_t + \partial_{ss}u \pm |u|^\alpha u = 0, \quad (80)$$

with an initial data of Dirac type, and its solution

$$u(s, t) = u_{a,\pm\alpha}(s, t) = u_a(s, t) e^{\pm iA_{a,\alpha}(t)},$$

with

$$A_{a,\alpha}(t) = \begin{cases} \frac{|a|^\alpha}{1-\frac{\alpha}{2}} t^{1-\frac{\alpha}{2}} & \text{si } \alpha \neq 2, \\ |a|^2 \log t & \text{si } \alpha = 2. \end{cases}$$

For $\alpha \geq 2$, $u_{a,\pm\alpha}$ does not have a limit as t goes to zero, and this gives an instability result, as explained in the previous subsection 2.3.

A first natural question is to find out if $u_{a,\pm\alpha}$ is a stable solution in the case $\alpha < 2$. Denoting

$$\eta(s, t) = e^{\mp iA_{a,\alpha}(t)}(u - u_{a,\pm\alpha})(s, t),$$

and using the fact that u_a is a solution of the linear equation, we have

$$\begin{cases} i\eta_t + \partial_{ss}\eta \pm (|\eta + u_a|^\alpha - |u_a|^\alpha)(\eta + u_a) = 0, \\ \eta(s, 0) = u_0(s). \end{cases}$$

If $\alpha < 2$ then the term $|u_a|^\alpha$ is integrable at $t = 0$, and we can perform a fixed point argument. We have obtained the following positive answer, valid also in higher dimensions, for $\alpha < \frac{2}{d}$ (a similar result is presented in [BV1] for Sobolev spaces, with a more technical proof).

Theorem 5 *Let $\alpha < 2$ and $u_0 \in L^2(\mathbb{R})$. There exists a time $T = T(a, \|u_0\|_{L^2})$ and a unique solution u of equation (80) with initial data $u_0 + a\delta$, such that*

$$u - u_{a,\pm\alpha} \in L^4([0, T], L^\infty(\mathbb{R})) \cap \mathcal{C}([0, T], L^2(\mathbb{R})).$$

A case of rough perturbation of $a\delta_0$ has been studied by Kita. In [Ki] he has described the solution generated by the sum of two or three Dirac masses.

Finally, we shall do a parallel with the corresponding parabolic problem,

$$\begin{cases} u_t - \Delta u \pm |u|^\alpha u = 0, \\ u(s, 0) = a\delta, \end{cases}$$

which has been intensively studied. F.B. Weissler has shown in [We] that in the focusing case, for $\alpha \leq \frac{2}{d}$, there is no unicity of the solution. In the defocusing case, if only positive solutions are considered, H. Brézis and A. Friedman have proven in [BF] the existence of a unique solution for $\alpha < \frac{2}{d}$, and non-existence for $\alpha \geq \frac{2}{d}$. Therefore for the heat equation there is an important difference between the focusing and the defocusing case, even for $\alpha < \frac{2}{d}$. Theorem 5 shows that this is not the case for the

Schrödinger equation.

The second natural question is to find out if u_a is a stable solution of the cubic 1-D Schrödinger equation modified by the complex phase $e^{\pm ia^2 \log t}$.

Having as a purpose the study of perturbations of the selfsimilar flow χ_a , we shall first consider perturbations at positive times of the filament function

$$u_a(s, t) = a \frac{e^{i \frac{s^2}{4t}}}{\sqrt{t}},$$

and we shall treat more generally the equation

$$iu_t + u_{ss} \pm \left(|u|^2 - \frac{a^2}{t} \right) u = 0$$

around its particular solution u_a . By defining v the pseudo-conformal transformation of u ,

$$u(s, t) = \frac{e^{i \frac{s^2}{4t}}}{\sqrt{t}} \bar{v} \left(\frac{s}{t}, \frac{1}{t} \right), \quad (81)$$

this is equivalent to study the large time behavior of perturbations of the constant solution a of

$$iv_t + v_{ss} \pm \frac{1}{t} (|v|^2 - a^2) v = 0. \quad (82)$$

Let us notice that this equation has a natural energy

$$E(t) = \frac{1}{2} \int |v_s(t)|^2 ds \mp \frac{1}{4t} \int (|v(t)|^2 - a^2)^2 ds.$$

with the law

$$\partial_t E(t) \mp \frac{1}{4t^2} \int (|v|^2 - a^2)^2 ds = 0.$$

In the defocusing case, this has allowed us to obtain in [BV1] a good control in time of the norm $\|v(t) - a\|_{L^2}$ and to conclude global existence.

As the coefficients of the linear terms are of the type $\frac{1}{t}$, and since the perturbation is done around a constant function, there will be connections with the long range effects of 1-D cubic Schrödinger equation (T. Ozawa [Oz], R. Carles [Ca], N. Hayashi and P. Naumkin [HN]) and with the scattering around the constant solution 1 of 2-D Gross-Pitaevskii equation (S. Gustafson, K. Nakanishi and T.-P. Tsai [GNT]).

We shall search for wave operators for the perturbation $w(t) = v(t) - a$, meaning that we shall search for solutions of

$$iw_t + w_{ss} \pm \frac{1}{t} (|w + a|^2 - a^2) (w + a) = 0,$$

behaving at large times as a given profile $w_1(s, t)$. This can be done by a fixed point argument, in a suitable space defined around w_1 , for the application

$$\begin{aligned} Aw &= w_1 \mp i \int_t^\infty e^{i(t-\tau)\partial_s^2} \left(\frac{(|w + a|^2 - a^2)(w + a)}{\tau} - \frac{(|w_1 + a|^2 - a^2)(w_1 + a)}{\tau} \right) d\tau \\ &\mp i \int_t^\infty e^{i(t-\tau)\partial_s^2} \left(\frac{(|w_1 + a|^2 - a^2)(w_1 + a)}{\tau} - (i\partial_\tau + \partial_x^2)w_1 \right) d\tau. \end{aligned}$$

For showing that w behaves like a free evolution $e^{it\partial_s^2} u_+$, one should control the Duhamel terms corresponding to the worse source terms, the linear ones $\frac{a^2}{t} w_1$ and $\frac{a^2}{t} \bar{w}_1$. These Duhamel terms are

$$a^2 \int_t^\infty e^{i(t-\tau)\partial_s^2} e^{i\tau\partial_s^2} u_+ \frac{d\tau}{\tau}, \quad a^2 \int_t^\infty e^{i(t-\tau)\partial_s^2} e^{-i\tau\partial_s^2} \bar{u}_+ \frac{d\tau}{\tau}.$$

We notice that the first integral is not controllable. To avoid this term, we shall look for w close to the long range ansatz

$$w_1(s, t) = e^{\pm ia^2 \log t} e^{it\partial_s^2} u_+(s).$$

By using inhomogeneous Strichartz inequalities we manage to control the remaining linear term

$$\begin{aligned} a^2 \left\| \int_t^\infty e^{i(t-\tau)\partial_s^2} \left(e^{-i\tau\partial_s^2} \frac{\bar{u}_+}{\tau^{1\pm ia^2}} \right) d\tau \right\|_{L_s^2} &= a^2 \left\| \int_t^\infty e^{-i2\tau\partial_s^2} \left(\frac{\bar{u}_+}{\tau^{1\pm ia^2}} \right) d\tau \right\|_{L_s^2} \\ &\leq Ca^2 \left\| \frac{\bar{u}_+}{\tau} \right\|_{L^{p'}((t, \infty), L^{q'})} = Ca^2 \|u_+\|_{L^{q'}} \left\| \frac{1}{\tau} \right\|_{L^{p'}(t, \infty)} = Ca^2 \|u_+\|_{L^{q'}} \frac{1}{t^{\frac{1}{p}}}. \end{aligned}$$

The 1-admissible couples (p, q) are between $(\infty, 2)$ and $(4, \infty)$, therefore if $u_+ \in L^1 \cap L^2$, the best decay we can obtain this way is $t^{-\frac{1}{4}}$. We denote

$$v_1(s, t) = a + w_1(s, t) = a + e^{\pm ia^2 \log t} e^{it\partial_s^2} u_+(s).$$

We get our first result of [BV2], in mixed norm spaces $L^p L^q$.

Theorem 6 *Let $t_0 > 0$. There exists a constant $a_0 > 0$ such that for all $a < a_0$, and for all u_+ small in $L^1 \cap L^2$ with respect to a_0 and t_0 , equation (82) has a unique solution v with*

$$v - v_1 \in \mathcal{C}([t_0, \infty), L^2(\mathbb{R})) \cap L^4([t_0, \infty), L^\infty(\mathbb{R})),$$

satisfying, as t goes to infinity,

$$\|v(t) - v_1(t)\|_{L^2} + \|v - v_1\|_{L^4((t, \infty), L^\infty)} = \mathcal{O}(t^{-\frac{1}{4}}).$$

However, the $t^{-\frac{1}{4}}$ decay, optimal for the methods used in mixed norm spaces, is not enough to construct BF solutions. We shall then exploit more the oscillations of the linear remaining term, by using the Fourier transform and strengthening the conditions on u_+ . Writing

$$\int_t^\infty e^{i(t-2\tau)\partial_s^2} \left(e^{-i\tau\partial_s^2} \frac{\bar{u}_+}{\tau^{1\pm ia^2}} \right) d\tau = \int_t^\infty \int \frac{e^{-i(t-2\tau)\xi^2}}{\tau^{1\pm ia^2}} e^{ix\xi} \widehat{u}_+(-\xi) d\xi d\tau,$$

and doing an integration by parts in τ we get

$$= \int \frac{e^{it\xi^2}}{t^{1\pm ia^2}} e^{is\xi} \frac{\widehat{u}_+(-\xi)}{2i\xi^2} d\xi + (1 \pm ia^2) \int_t^\infty \int \frac{e^{-i(t-2\tau)\xi^2}}{\tau^{2\pm ia^2}} e^{is\xi} \frac{\widehat{u}_+(-\xi)}{2i\xi^2} d\xi d\tau.$$

Using Plancherel formula we obtain the control

$$\left\| a^2 \int_t^\infty e^{i(t-\tau)\partial_s^2} \left(e^{-i\tau\partial_s^2} \frac{\bar{u}_+}{\tau^{1\pm ia^2}} \right) d\tau \right\|_{L^2} \lesssim \frac{a^2}{t} \left\| \frac{\widehat{u}_+(-\xi)}{\xi^2} \right\|_{L^2} = \frac{a^2}{t} \|u_+\|_{\dot{H}^{-2}}.$$

The derivatives ∇^k of this term have a $1/t$ decay, with constants depending on the \dot{H}^{-2+k} norm of u_+ . We obtain in [BV2] a new result, with a better decay.

Theorem 7 *Let $t_0 > 0$ and $m \in \mathbb{N}^*$. There exists a constant $a_0 > 0$ such that for all $a < a_0$ and for all u_+ small in $\dot{H}^{-2} \cap H^m \cap W^{m,1}$ with respect to a_0 and t_0 , equation (82) has a unique solution*

$$v - v_1 \in \mathcal{C}([t_0, \infty), H^m(\mathbb{R})),$$

satisfying, as t goes to infinity, for all entire number $0 < k \leq m$,

$$\|(v - v_1)(t)\|_{L^2} = \mathcal{O}(t^{-\frac{1}{2}}), \quad \|\partial_s^k(v - v_1)(t)\|_{L^2} = \mathcal{O}(t^{-1}). \quad (83)$$

The presence of two different rates of decay is now due to the quadratic terms $\frac{2a}{t}|w_1|^2$ and $\frac{a}{t}w_1^2$. The less oscillating is the first one, which gives the rate of decay $t^{-\frac{1}{2}}$. However, its derivatives can be better upper-bounded. We shall sketch in the following the way we treat the Duhamel term coming from the worse quadratic term, $\frac{|w_1|^2}{t}$. We shall use again the Fourier oscillations,

$$\begin{aligned} \partial_s^k \int_t^\infty e^{i(t-\tau)\partial_s^2} \left(\frac{|e^{i\tau\partial_s^2} u_+(\tau)|^2}{\tau} \right) d\tau &= \int_t^\infty \int |\xi|^k \frac{e^{-i(t-\tau)\xi^2}}{\tau} e^{is\xi} \widehat{e^{i\tau\partial_s^2} u_+} * \widehat{e^{-i\tau\partial_s^2} u_+} d\xi d\tau \\ &= \int_t^\infty \int \int \frac{e^{-it\xi^2 + 2i\tau\xi(\eta-\xi)}}{\tau} |\xi|^k e^{is\xi} \widehat{u_+}(\eta) \widehat{u_+}(\eta-\xi) d\eta d\xi d\tau, \end{aligned}$$

and an integration by parts in τ

$$\begin{aligned} &= \int \int \frac{e^{-it(\xi^2 + 2\xi(\eta-\xi))}}{t} |\xi|^k e^{is\xi} \frac{\widehat{u_+}(\eta) \widehat{u_+}(\eta-\xi)}{2i\xi(\eta-\xi)} d\eta d\xi \\ &+ \int_t^\infty \int \int \frac{e^{-it\xi^2 - 2i\tau\xi(\eta-\xi)}}{\tau^2} |\xi|^k e^{is\xi} \frac{\widehat{u_+}(\eta) \widehat{u_+}(\eta-\xi)}{2i\xi(\eta-\xi)} d\eta d\xi d\tau. \end{aligned}$$

Hence we have obtained that this quadratic Duhamel term can be controlled in \dot{H}^1 by

$$\frac{Ca}{t} \left\| \widehat{u_+} * \frac{\widehat{u_+}}{\cdot} \right\|_{L^2} = \frac{Ca}{t} \left\| u_+ \mathcal{F} \left(\frac{\widehat{u_+}}{\cdot} \right) \right\|_{L^2} \leq \frac{Ca}{t} \|u_+\|_{H^1} \|u_+\|_{\dot{H}^{-1}}.$$

We notice that this argument can be repeated for \dot{H}^k , with $k > 1$, but not for $k = 0$.

At the level of the initial equation before the pseudo-conformal transformation, we obtain that $u(t)$ is close to

$$u_1(s, t) = u_a + e^{\pm ia^2 \log t} \widehat{u_+} \left(-\frac{s}{2} \right).$$

The condition $u_+ \in \dot{H}^{-2}$ is translated on $\widehat{u_+}$ by a canceling condition at the origin, which could be related to the blow-up result of J. Bourgain and W. Wang [BW]. We notice the absence of a limit in L^2 for $u(t) - u_a$ at time zero. However this does not exclude the possibility to construct from u a binormal flow with a trace at zero. This is because the curvature and the torsion defined from u corresponds to the second derivatives of the curves.

4.2 On the stability of selfsimilar solutions of the binormal flow

In this last subsection we sketch how from a solution of Theorem 7 we construct a flow of curves χ , solution of the BF, and get information about its limit as t goes to zero. We prove that the new curve χ is close to the selfsimilar one χ_a , and then obtain the persistence of the formation of a singularity at time zero.

Let v be a solution of Theorem 7 with $m = 3$, and u its pseudo-conformal transform (81). Since v is regular enough and does not vanish, we can define two real functions c and ϕ such that

$$u(s, t) = c(s, t)e^{i\phi(s, t)}. \quad (84)$$

We define $\tau(s, t) := \phi_s(s, t)$, so

$$u(s, t) = c(s, t)e^{i \int_0^s \tau(s, t) ds + \phi(0, t)}.$$

Then, the function

$$\tilde{u}(s, t) = c(s, t)e^{i \int_0^s \tau(s, t) ds}$$

is a filament function, solution of (5) with $A(t)$ replaced by $\frac{a^2}{t} + \phi_t(0, t)$. In particular, (c, τ) satisfy the intrinsic equations (27), and a computation yields the extra-information at $s = 0$

$$\frac{a^2}{t} + \partial_t \phi(0, t) = 2 \left(\frac{c_{ss} - c\tau^2}{c} \right) (0, t) + c^2(0, t). \quad (85)$$

In view of the pseudo-conformal transform, the curvature is

$$c(s, t) = |u(s, t)| = \frac{1}{\sqrt{t}} \left| v \left(\frac{s}{t}, \frac{1}{t} \right) \right|,$$

and the torsion is

$$\tau(s, t) = \Im \frac{u_s(s, t)}{u(s, t)} = \Im \frac{\frac{is}{2t} \bar{v} \left(\frac{s}{t}, \frac{1}{t} \right) + \partial_s \bar{v} \left(\frac{s}{t}, \frac{1}{t} \right)}{\bar{v} \left(\frac{s}{t}, \frac{1}{t} \right)}.$$

As $v \in H^3(\mathbb{R}) \subset C^{\frac{5}{2}}(\mathbb{R})$, the curvature and torsion function will be regular enough.

By definition,

$$v(s, t) = a + e^{\pm ia^2 \log t} e^{it\partial_s^2} u_+(x) + (v - v_1)(s, t).$$

The decay of Theorem 7 combined with the explicit formula of the free Schrödinger evolution allows us to obtain that (c, τ) is close to (c_a, τ_a) , with strong rate of decay as t goes to zero, and constants depending on the norms of u_+ .

Starting from (c, τ) we construct the binormal flow as follows. We fix an initial data $(T, n, b)(0, \tilde{t}_0)$ at time $\tilde{t}_0 = 1/t_0$, to be specified later. For $\tilde{t}_0 > t > 0$ we define $(T, n, b)(0, t)$ by imposing that the derivatives in time satisfy the same system as the tangent, normal and binormal vectors of a binormal flow of curves satisfy in general. Then, we define $(T, n, b)(s, t)$ from $(T, n, b)(0, t)$ by integrating the Frenet system for fixed time t . We therefore impose

$$\begin{pmatrix} T \\ n \\ b \end{pmatrix} (s, t) = \begin{pmatrix} T \\ n \\ b \end{pmatrix} (0, \tilde{t}_0) - \int_t^{\tilde{t}_0} \begin{pmatrix} 0 & -c\tau & c_x \\ c\tau & 0 & \left(\frac{c_{ss} - c\tau^2}{c} \right) \\ -c_s & -\left(\frac{c_{ss} - c\tau^2}{c} \right) & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ b \end{pmatrix} (0, t') dt'$$

$$+ \int_0^s \begin{pmatrix} 0 & c & 0 \\ -c & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ b \end{pmatrix} (s', t) ds'.$$

Proceeding this way, T shall satisfy the equation

$$T_t = T \wedge T_{ss},$$

of the tangent vector of a general binormal flow.

Once T is constructed, for a given $\chi(0, \tilde{t}_0)$ we define

$$\chi(s, t) = \chi(0, \tilde{t}_0) - \int_t^{\tilde{t}_0} (cb)(0, t') dt' + \int_0^s T(s, t) ds.$$

By using Frenet system,

$$T_t = T \wedge T_{ss} = T \wedge (cn)_s = T \wedge (c_s n + c\tau b) = -c\tau n + c_s b,$$

and it follows that χ satisfy the binormal flow.

Finally, $\chi(s, t)$ is obtained this way for all times when (c, τ) are regular, that is $\tilde{t}_0 > t > 0$. Knowing that c is close to $c_a = \frac{a}{\sqrt{t}}$, we get

$$|\chi(s, t_1) - \chi(s, t_2)| = \left| \int_{t_1}^{t_2} c(s, t) b(s, t) ds \right| \leq \int_{t_1}^{t_2} \frac{Ca}{\sqrt{t}} dt \xrightarrow{t_1, t_2 \rightarrow 0} 0.$$

This implies the existence of a limit $\chi_0(s)$ at $t = 0$, and we obtain similarly that for all $s \in (-\infty, \infty)$,

$$|\chi(s, t) - \chi_0(s)| \leq Ca\sqrt{t}.$$

Next, for proving that $\chi(s, 0)$ is close to $\chi_a(s, 0)$, we first show that the tangent $T(s, t)$ is close to $T_a(s, t)$. As a first step, we prove that $T(0, t)$ is close to $T_a(0, t)$, and then we prove that $T(s, t)$ is close to $T_a(s, t)$. For the second step we need to know that $(T, n, b)(0, t)$ is close to $(T_a, n_a, b_a)(0, t)$. In the following we shall sketch the estimates at $(0, t)$, which is the less technical step, but nevertheless a key point of the proof.

From the above definition of (T, n, b) ,

$$\begin{pmatrix} T \\ n \\ b \end{pmatrix}_t (0, t) = \begin{pmatrix} 0 & -c\tau & c_s \\ c\tau & 0 & \left(\frac{c_{ss} - c\tau^2}{c}\right) \\ -c_s & -\left(\frac{c_{ss} - c\tau^2}{c}\right) & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ b \end{pmatrix} (0, t).$$

Now we use the extra-information at $x = 0$ (85),

$$\frac{c_{ss} - c\tau^2}{c} = \frac{c_a^2 - c^2}{2} + \frac{\phi_t}{2}.$$

Therefore we obtain

$$\begin{pmatrix} T \\ n \\ b \end{pmatrix}_t (0, t) = \begin{pmatrix} 0 & -c\tau & c_s \\ c\tau & 0 & \frac{c_a^2 - c^2}{2} + \frac{\phi_t}{2} \\ -c_s & -\frac{c_a^2 - c^2}{2} - \frac{\phi_t}{2} & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ b \end{pmatrix} (0, t).$$

In order to get rid of the term $\phi_t(t, 0)$, we perform the change of functions $\tilde{n} + i\tilde{b} = e^{i\frac{\phi}{2}}(n + ib)$, and we impose as an initial condition $(T, \tilde{n}, \tilde{b})(0, \tilde{t}_0) = (T_a, n_a, b_a)(0, \tilde{t}_0)$. For having $(T, \tilde{n}, \tilde{b})(0, t)$ close to $(T_a, n_a, b_a)(0, t)$, it is then enough to show that the matrix

$$\begin{pmatrix} 0 & |c\tau| & |c_s| \\ |c\tau| & 0 & \left|\frac{c_a^2 - c^2}{2}\right| \\ |c_s| & \left|\frac{c_a^2 - c^2}{2}\right| & 0 \end{pmatrix}(t, 0)$$

is integrable in time near $t = 0$. This is true, in view of the strong rate of decay obtained for $(c, \tau) - (c_a, \tau_a)$. Also, we get a good decay for $|e^{i\frac{\phi}{2}} - 1|$, so $(T, \tilde{n}, \tilde{b})(0, t)$ is close to $(T, n, b)(0, t)$. In conclusion, we have $(T, n, b)(0, t)$ close to $(T_a, n_a, b_a)(0, t)$.

We obtain finally that for $\epsilon > 0$, and u_+ small enough,

$$|(T - T_a)(s, t)| \leq \epsilon.$$

In particular, for $s > 0$,

$$\chi(s, t) - \chi(0, t) = \int_0^s T(s, t) ds = A_a^+ s + \int_0^s (T - T_a)(s, t) ds + \int_0^s T_a(s, t) - A_a^+ ds.$$

The third term tends to zero as t goes to zero, uniformly in s [GRV]. By letting t go to zero, we obtain the desired information

$$|\chi_0(s) - \chi_0(0) - sA_a^+| \leq \epsilon s.$$

More precisely, we have obtained in [BV2] the following result.

Theorem 8 *Let $\epsilon > 0$, $t_0 > 0$, $0 < a < a_0$, where a_0 is the constant in Theorem 7. Let u_+ small in $\dot{H}^{-2} \cap H^3 \cap W^{3,1}$ with $s^2 u_+$ small in H^1 with respect to ϵ , a and t_0 , and let v be the corresponding solution obtained in Theorem 7. By using the Hasimoto transform, we construct from v a family of curves $\chi(s, t)$ which solves the binormal flow for $\tilde{t}_0 > t > 0$, and such that there exists a unique χ_0 satisfying*

$$|\chi(s, t) - \chi_0(s)| < Ca\sqrt{t}$$

uniformly in $s \in (-\infty, \infty)$.

Moreover,

$$|\chi_0(s) - \chi_0(0) - sA_a^+ \mathbb{I}_{[0, \infty)}(s) - sA_a^- \mathbb{I}_{(-\infty, 0]}(s)| < \epsilon |s|,$$

where A_a^\pm are two distinct non opposite unit vectors, such that the angle γ between A_a^+ and $-A_a^-$ is determined by the relation

$$\sin \frac{\gamma}{2} = e^{-\frac{\pi a^2}{2}}.$$

Since

$$\chi_a(s, 0) = sA_a^+ \mathbb{I}_{[0, \infty)}(s) + sA_a^- \mathbb{I}_{(-\infty, 0]}(s),$$

we have that $\chi_0(s)$ lies in the ϵ -cone around $\chi_a(s, 0)$. Therefore we have obtained a large family of curves, solutions of the binormal flow, which are close to the selfsimilar solution χ_a . In particular a corner is still formed for $\chi(s, t)$ at $t = 0$ and at $s = 0$. The angle of this corner can be made as close as desired to the one in between A_a^+ and $-A_a^-$ by taking ϵ small enough. This shows that the formation of a singularity in finite time of χ_a is not an isolated phenomenon. In a work in progress we study, instead of the existence of wave operators in Theorem 7, the asymptotic completeness question, which could lead to a better description of the allowed perturbations.

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