SIAM J. MATH. ANAL.
Vol. 35, No. 4, pp. 868–883

DISPERSION AND STRICHARTZ INEQUALITIES FOR SCHRÖDINGER EQUATIONS WITH SINGULAR COEFFICIENTS*

VALERIA BANICA†

Abstract. In this paper we prove the global dispersion and the Strichartz inequalities for a class of one-dimensional Schrödinger equations with step-function coefficients having a finite number of discontinuities. The local and global dispersion and Strichartz inequalities are discussed for certain Schrödinger equations with low regularity coefficients oscillating at infinity.

Key words. Schrödinger equation, nonsmooth coefficients, dispersion and Strichartz inequalities, Bloch waves

AMS subject classifications. 35J10, 35R05, 35B45, 35C

DOI. 10.1137/S0036141002415025

1. Introduction.

Strichartz estimates [7], [11] are an important tool for the understanding of nonlinear evolution equations. In the study of the dispersive properties of the Schrödinger equation with variable coefficients, the absence of the property of finite speed of propagation raises more difficulties than in the case of the wave equation. A way to "replace" this property is to impose a nontrapping condition on the trajectories. There are many results of wellposedness and smoothing effect for Schrödinger operators with smooth coefficients which are asymptotically flat and satisfy a nontrapping condition [4], [5], [8]. Staffilani and Tataru [10] proved the Strichartz estimates under the same conditions, but for lower regularity coefficients, only of $\mathcal{C}^2$-class. However, in order to have wellposedness for nonlinear Schrödinger equations (NLS), the nontrapping condition can be dropped. In their recent paper [2], Burq, Gérard, and Tzvetkov have obtained Strichartz estimates with fractional loss of derivative for metrics on $\mathbb{R}^d$ with uniformity assumptions at infinity, without geometric conditions. These new dispersive estimates imply local and global existence results for the Cauchy problem.

In this paper we study the dispersion property and the Strichartz inequalities for the one-dimensional Schrödinger equation

$$\text{(S)}\qquad \begin{cases} (i\,\partial_t + \partial_x a(x)\partial_x)\, u(t,x) = 0 \text{ for } (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x) \in \mathbb{L}^2(\mathbb{R}) \end{cases}$$

for certain rough coefficients $a(x)$ without any geometric nontrapping condition.

In section 2 we prove global dispersion in the case of positive lamina coefficients, i.e., step functions with a finite number of singularities. Let us note in this situation the existence of trapped trajectories.

THEOREM 1.1. *Consider a partition of the real axis*

$$-\infty = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = \infty$$

and a step function

$$a(x) = b_i^{-2} \text{ for } x \in (x_{i-1}, x_i),$$

*Received by the editors September 24, 2002; accepted for publication (in revised form) February 7, 2003; published electronically October 14, 2003.
http://www.siam.org/journals/sima/35-4/41502.html

†Université de Paris Sud, Mathématiques, Bât. 425, 91405 Orsay Cedex, France (Valeria.Banica@math.u-psud.fr).

where b_i are positive numbers.

The solution of the Schrödinger equation (S) *satisfies the dispersion inequality*

$$\|u(t,\cdot)\|_{\mathbb{L}^\infty(\mathbb{R})} \leq \frac{C_n}{\sqrt{t}}\|u_0\|_{\mathbb{L}^1(\mathbb{R})}$$

and the Strichartz inequalities

$$\|u\|_{\mathbb{L}^p(\mathbb{R},\mathbb{L}^q(\mathbb{R}))} \leq C_n\|u_0\|_{\mathbb{L}^2(\mathbb{R})}$$

for every pair (p,q) *verifying*

$$\frac{2}{p}+\frac{1}{q}=\frac{1}{2}\,.$$

The proof consists of writing the solution by using the resolvent of the operator $-\partial_x a(x)\partial_x$. The resolvent is calculated and expressed in terms of series of exponentials. In order to get global dispersion, we discuss these series within the framework of the theory of Wiener's almost periodic functions.

We can also prove a similar result for the operator

$$i\,\partial_t + \frac{1}{\rho(x)}\partial_x a(x)\partial_x\,,$$

where $\rho(x)$ is a step function of the same type as $a(x)$.

Moreover, if $v(t,x)$ is the solution of the associated wave system

$$\text{(O)}\qquad \left\{\begin{array}{c} \left(\partial_t^2 - \partial_x a(x)\partial_x\right)v(t,x) = 0 \text{ for } x\in\mathbb{R},\\ v(0,x)=u_0(x)\in\mathbb{L}^2(\mathbb{R}),\\ \partial_t v(0,x)=0, \end{array}\right.$$

the same method gives us the following estimate:

$$\sup_{x\in\mathbb{R}}\int_{-\infty}^{\infty}|v(t,x)|dt \leq C_n\|u_0\|_{\mathbb{L}^1(\mathbb{R})}.$$

Dispersion is not satisfied if the step function coefficients are periodic. In section 3, by using the Krönig–Penney model, we show that the local dispersion fails in the case of 2-valued periodic step function coefficients.

THEOREM 1.2. *Let* $x_0 \in (0,1)$ *and let* b_0, b_1 *be positive numbers satisfying* $b_0x_0 = b_1(1-x_0)$. *Consider the* 1*-periodic function*

$$a(x) = \left\{\begin{array}{l} b_0^{-2} \text{ for } x\in[0,x_0),\\ b_1^{-2} \text{ for } x\in[x_0,1). \end{array}\right.$$

The local dispersion estimate fails for the Schrödinger equation (S)*.*

The proof is based on the representation of the solution by its Floquet decomposition.

The fact that the coefficient a is not very oscillating at infinity seems to be essential for having dispersion. Applying the method used by Avellaneda, Bardos, and Rauch in [1], we can construct counterexamples for global dispersion and Strichartz's inequalities in the case of certain continuous coefficients oscillating at infinity.

Also, as Castro and Zuazua have recently shown in [3], even if the coefficients are flat at infinity, but rough ($C^{0,\alpha}$) and locally very oscillating, the local Strichartz inequalities fail.

All these results suggest the conjecture that the one-dimensional Schrödinger equations with strictly positive BV (bounded variation) coefficients satisfy the dispersion property.

2. Laminar media.

2.1. Representation of the resolvent of $-\partial_x a(x)\partial_x$. The operator $-\partial_x a(x)\partial_x$, defined from

$$\{h \in \mathbb{H}^1(\mathbb{R}), a\,\partial_x\, h \in \mathbb{H}^1(\mathbb{R})\}$$

to $\mathbb{L}^2(\mathbb{R})$, is self-adjoint. For $\omega \geq 0$ let R_ω be its resolvent

$$R_\omega g = (-\partial_x a(x)\partial_x + \omega^2 I)^{-1} g.$$

In order to obtain the expression of the resolvent on the intervals where a is constant, the second-order equations

$$\frac{1}{b_i^2}(R_\omega g)'' = \omega^2 R_\omega g - g$$

must be solved. Then, for $x \in (x_{i-1}, x_i)$, we have

$$R_\omega g(x) = c_{2i-1} e^{\omega b_i x} + c_{2i} e^{-\omega b_i x} + \int_{-\infty}^{\infty} \frac{g(y)}{2\omega} b_i e^{-\omega b_i |x-y|} dy.$$

Since $R_\omega g$ belongs to $\mathbb{L}^2(\mathbb{R})$ the coefficients c_2 and c_{2n-1} are zero. The conditions of continuity of $R_\omega g$ and of $a\,\partial_x R_\omega g$ at the points x_i give a system of $2n-2$ equations on the c_i's. The matrix D_n of this system is

$$\begin{pmatrix} e^{\omega b_1 x_1} & -e^{\omega b_2 x_1} & -e^{-\omega b_2 x_1} & 0 & 0 & 0 & 0 & 0 \\ b_2 e^{\omega b_1 x_1} & -b_1 e^{\omega b_2 x_1} & b_1 e^{-\omega b_2 x_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\omega b_2 x_2} & e^{-\omega b_2 x_2} & -e^{\omega b_3 x_2} & -e^{-\omega b_3 x_2} & 0 & 0 & 0 \\ 0 & b_3 e^{\omega b_2 x_2} & -b_3 e^{-\omega b_2 x_2} & -b_2 e^{\omega b_3 x_2} & b_2 e^{-\omega b_3 x_2} & 0 & 0 & 0 \\ : & : & : & : & : & : & : & : \\ 0 & 0 & 0 & 0 & 0 & e^{\omega b_{n-1} x_{n-1}} & e^{-\omega b_{n-1} x_{n-1}} & -e^{-\omega b_n x_{n-1}} \\ 0 & 0 & 0 & 0 & 0 & b_n e^{\omega b_{n-1} x_{n-1}} & -b_n e^{-\omega b_{n-1} x_{n-1}} & b_{n-1} e^{-\omega b_n x_{n-1}} \end{pmatrix}.$$

The right-hand side of the system is

$$T_n = \begin{pmatrix} t_1 \\ : \\ t_{n-1} \end{pmatrix},$$

with

$$t_i = \begin{pmatrix} \int_{-\infty}^{\infty} \frac{g(y)}{2\omega}(-b_i e^{-\omega b_i |xi-y|} + b_{i+1} e^{-\omega b_{i+1}|x_i - y|}) dy \\ \int_{-\infty}^{\infty} \frac{g(y)}{2\omega} b_{i+1} b_i (-e^{-\omega b_i |x_i-y|} + e^{-\omega b_{i+1}|x_i-y|}) \mathrm{sign}(x_i - y) dy \end{pmatrix}.$$

Therefore the resolvent on each interval (x_i, x_{i+1}) is a finite sum of terms

$$(1) \qquad R_\omega g(x) = \sum_{finite} C e^{\omega \beta(x)} \int_{I(x_i)} \frac{g(y)}{2\omega} \frac{e^{\pm \omega b_i y}}{\det D_n(\omega)} dy + \int_{-\infty}^{\infty} \frac{g(y)}{2\omega} b_i e^{-\omega b_i |x-y|} dy,$$

where $\beta(x)$ are real functions depending on $\{x, x_i, b_i\}$, C is a constant depending of $\{b_i\}$ and bounded by $(\max b_i^{-2})^n$, and $I(x_i)$ is either $(-\infty, x_i)$ or (x_i, ∞). Let $\widetilde{D}_n$ be the same matrix as D_n, with the last two terms of the last column replaced by

$$\begin{pmatrix} -e^{\omega b_n x_{n-1}} \\ -b_{n-1} e^{\omega b_n x_{n-1}} \end{pmatrix}.$$

The development of the determinants of D_n and $\widetilde{D}_n$ with respect to the last column gives the following induction relations:

$$\begin{cases} \det D_n = e^{-\omega b_n x_{n-1}}\Big[(b_{n-1}-b_n)e^{-\omega b_{n-1}x_{n-1}}\det\widetilde{D_{n-1}}- \\ \qquad -(b_{n-1}+b_n)e^{\omega b_{n-1}x_{n-1}}\det D_{n-1}\Big], \\ \det\widetilde{D_n} = e^{\omega b_n x_{n-1}}\Big[(b_{n-1}-b_n)e^{\omega b_{n-1}x_{n-1}}\det D_{n-1}- \\ \qquad -(b_{n-1}+b_n)e^{-\omega b_{n-1}x_{n-1}}\det\widetilde{D_{n-1}}\Big]. \end{cases}$$

Let us define for $n \geq m \geq 2$

$$Q_m(\omega) = e^{-2\omega b_m x_m}\frac{\det\widetilde{D_m}}{\det D_m}.$$

By denoting

$$d_{m-1} = \frac{b_{m-1}-b_m}{b_{m-1}+b_m},$$

we have for $n \geq 3$

$$\det D_n(\omega) = (b_1+b_2)e^{-\omega(b_2-b_1)x_1}\prod_{i=2\ldots n-1}(b_i+b_{i+1})e^{\omega(b_i-b_{i+1})x_i}(1-d_iQ_i(\omega)), \tag{2}$$

and for $n = 2$

$$\det D_2(\omega) = (b_1+b_2)e^{-\omega(b_2-b_1)x_1}. \tag{3}$$

Also, we obtain an induction formula on the Q_m's:

$$Q_m(\omega) = e^{-2\omega b_m(x_m-x_{m-1})}\frac{-d_{m-1}+Q_{m-1}(\omega)}{1-d_{m-1}Q_{m-1}(\omega)}. \tag{4}$$

Note that a Möbius transform on the unit disc occurs in this expression.

Let $\epsilon_n > 0$ be such that for every complex ω with

$$\Re\omega > -\epsilon_n,$$

the estimate

$$|Q_2(\omega)| = |d_1 e^{-2\omega b_2(x_2-x_1)}| < 1$$

holds and gives by induction

$$|Q_m(\omega)| < 1.$$

Hence $(\det D_n(\omega))^{-1}$ is uniformly bounded and well defined in this region, which contains the imaginary axis. Therefore $\omega R_\omega u_0(x)$ can be analytically continued, and we can use the following spectral theory lemma.

Lemma 2.1. *The solution of the Schrödinger equation* (S) *verifies*

$$u(t,x) = \int_{-\infty}^{\infty} e^{it\tau^2}\tau R_{i\tau}u_0(x)\frac{d\tau}{\pi}. \tag{5}$$

2.2. The algebra of Wiener's almost-periodic functions. Let us recall the structure of the Banach algebra of Wiener's almost-periodic functions:

$$B = \left\{ h : \mathbb{R} \mapsto \mathbb{C}\,,\; h(t) = \sum_{\lambda\in\mathbb{R}} c(\lambda) e^{i\lambda t} \text{ with } \|h\|_B = \sum_{\lambda\in\mathbb{R}} |c(\lambda)| < \infty \right\}.$$

We define for $h \in B$

$$\|h\|_\infty = \sup_{t\in\mathbb{R}} |h(t)|$$

and

$$\rho(h) = \inf\{r > 0 \;\mid\; \exists\, C_r > 0 \text{ for all } k \in \mathbb{N}\,,\; \|h^k\|_B \leq C_r r^k\}.$$

The following classical result, which is a consequence of Theorems 6§4 and 2§29 of [6], will be used.

THEOREM 2.2. *For all $h \in B$ we have*

$$\rho(h) = \|h\|_\infty.$$

COROLLARY 2.3. *Let $h \in B$ with $\|h\|_\infty < 1$ and let α be a complex number on the open unit disc. Then*

$$g = \frac{h-\alpha}{1-\overline{\alpha}h}$$

also belongs to B and

$$\rho(g) < 1.$$

Proof. The function $\overline{\alpha}h$ belongs to B and

$$\|\overline{\alpha}h\|_\infty < |\alpha| < 1.$$

By using Theorem 2.2 we have

$$\|(\overline{\alpha}h)^k\|_B \leq C|\alpha|^k.$$

Since

$$\frac{h-\alpha}{1-\overline{\alpha}h} = (h-\alpha)\sum_{k=0}^{\infty} (\overline{\alpha}h)^k,$$

it follows that g belongs to B. Moreover, by the maximum principle,

$$\|g\|_\infty < 1.$$

By again applying Theorem 2.2, the corollary is proved. ∎

2.3. The dispersion inequality. The $Q_m(i\tau)$'s are series of complex exponentials. In this subsection we will show that they belong to B with respect to the real variable τ. The estimates of their norm in this algebra will imply the dispersion for the Schrödinger equation (S).

Let us define

$$r_2 = |d_1|, \ \ r_m = \frac{|d_{m-1}| + r_{m-1}}{1 - |d_{m-1}| r_{m-1}}.$$

Obviously $Q_2 \in B$ and

$$\|Q_2\|_\infty = r_2.$$

Therefore Theorem 2.2 gives us

$$\rho(Q_2) = r_2 < 1.$$

By using Corollary 2.3 and the Möbius transform which occurs in formula (4), one can show by induction that $Q_m \in B$ and

$$\rho(Q_m) \leq r_m < 1.$$

Then formulae (2) and (3) lead us to the estimate

$$\|(\det D_n(i\tau))^{-1}\|_B < K_n, \tag{6}$$

where K_n is a constant depending on b_i.

In order to prove dispersion, it is sufficient, using (1) and (5), to estimate terms of the following type:

$$J_i(t,x) = \int_{-\infty}^{\infty} e^{it\tau^2} C e^{i\tau\beta(x)} \int_{I(x_i)} \frac{u_0(y)}{2i\tau} \frac{e^{\pm i\tau b_i y}}{\det D_n(i\tau)} dy \, \tau \frac{d\tau}{2\pi}.$$

By performing a change of variable in τ,

$$|J_i(t,x)| \leq C \int_{I(x_i)} \frac{|u_0(y)|}{4\pi\sqrt{t}} \left| \int_{-\infty}^{\infty} e^{is^2} \frac{e^{i\frac{s}{\sqrt{t}}(\beta(x) \pm b_i y)}}{\det D_n(i\frac{s}{\sqrt{t}})} ds \right| dy$$

$$\leq C \frac{\|u_o\|_{\mathbb{L}^1(\mathbb{R})}}{\sqrt{t}} \|(\det D_n(i\xi))^{-1}\|_B.$$

Then (6) implies that

$$\sup_x |J_i(t,x)| \leq K_n \frac{\|u_0\|_{\mathbb{L}^1(\mathbb{R})}}{\sqrt{t}},$$

so the dispersion inequality for the Schrödinger equation (S) is satisfied.

Remark 2.4. The finite sum in (1) contains $n2^n$ terms. Therefore, by estimating the solution as above, term by term, we cannot obtain the dispersion for equation (S) if $a(x)$ has an infinite number of steps. Therefore the method is too rough to prove dispersion for an arbitrary strictly positive BV coefficient $a(x)$.

Strichartz inequalities follow from the dispersion inequality by the classical duality argument TT^* [12], so the proof of Theorem 1.1 is complete.

Since we can express the solution of the wave equation (O) as

$$v(t,x) = \int_{-\infty}^{\infty} e^{it\tau} R_{i\tau} u_0(x) i\tau \frac{d\tau}{2\pi},$$

the property

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |v(t,x)| dt \leq C\|u_0\|_{\mathbb{L}^1(\mathbb{R})}$$

follows similarly to the dispersion inequality for the solution of (S).

3. Periodic laminar media.

3.1. General theory of periodic-coefficient equations.

Let θ be a number in $[0,2\pi]$ and consider the operator on $\mathbb{L}^2(\mathbb{S}^1)$

$$A_\theta = -(i\theta + \partial_x)a(x)(i\theta + \partial_x).$$

This operator is self-adjoint with a compact resolvent, hence the eigenvalues form a sequence of strictly positive numbers $\{\omega_{\theta,n}^2\}_{n\in\mathbb{N}}$. Moreover, the set of the corresponding eigenfunctions $p_n(\theta,x)$ is an orthonormal basis of $\mathbb{L}^2(\mathbb{S}^1)$.

Let us provide a way to construct the elements of this basis. Finding the eigenfunction $p_n(\theta,x)$ is equivalent to finding the function

$$\Psi_n(\theta,x) = e^{i\theta x} p_n(\theta,x)$$

that satisfies

$$(\mathrm{H}_{\theta,n}) \qquad -\partial_x a(x)\partial_x \Psi_n(\theta,x) = \omega_{\theta,n}^2 \Psi_n(\theta,x).$$

Note that this new function has the quasi-periodic property

$$\Psi_n(\theta,x+1) = e^{i\theta}\Psi_n(\theta,x).$$

Equation $(\mathrm{H}_{\theta,n})$ is of the type

$$(\mathrm{H}) \qquad -\partial_x a(x)\partial_x \Psi(x) = \lambda^2 \Psi(x)$$

on

$$\{\Psi \in \mathbb{H}^1_{loc}(\mathbb{R}),\, a\,\partial_x\Psi \in \mathbb{H}^1_{loc}(\mathbb{R})\}.$$

This equation can be treated similarly to Hill's equation [9]. Let T be an operator acting on the solution space as follows:

$$T(\Psi)(x) = \Psi(x+1).$$

On the one hand, the eigenvalues of T verify

$$x^2 - x\mathrm{Tr}(T) + \det T = 0.$$

On the other hand, the generalized Wronskian

$$W = \Psi_1 a\partial_x \Psi_2 - \Psi_2 a\partial_x \Psi_1$$

associated with (Ψ_1, Ψ_2), a normalized basis of solutions of (H), i.e.,

$$\Psi_1(0) = (a\,\partial_x \Psi_2)(0) = 1, \qquad (a\,\partial_x \Psi_1)(0) = \Psi_2(0) = 0,$$

is constant. Therefore

$$\det T = W(1) = W(0) = 1,$$

and the eigenvalues are $e^{i\xi}$ and $e^{-i\xi}$ for some complex ξ. If $|\mathrm{Tr}(T)|$ is larger than 2, then ξ is purely imaginary and there exists a basis of solutions of exponential growth. In this case λ^2 belongs to an instability interval of the equation. Otherwise, if $|\mathrm{Tr}(T)|$ is less than or equal to 2, ξ is real and λ^2 belongs to a stability interval. Moreover, if $\xi \in \pi\mathbb{Z}$, periodic solutions exist. If $\xi \in \mathbb{R}\backslash\pi\mathbb{Z}$, the existence of a basis of quasi-periodic solutions is assured.

So, the eigenvalues of A_θ are exactly the values λ^2 for which the operator T associated with (H) admits $e^{i\theta}$ and $e^{-i\theta}$ as eigenvalues. If $\theta \in (0,\pi) \cup (\pi, 2\pi)$, then these eigenvalues are simple. Therefore, in order to construct the $\mathbb{L}^2(\mathbb{S}^1)$ basis made of the eigenfunctions of A_θ, one has to find all λ for which the operator T associated with (H) verifies

$$\mathrm{Tr} T = 2\cos\theta.$$

For such a λ, we consider (Ψ_1, Ψ_2) a normalized basis of solutions of (H). If $\Psi_2(1) \neq 0$, then

$$\Psi(x) = \Psi_1(x) - \frac{\Psi_1(1) - e^{i\theta}}{\Psi_2(1)}\Psi_2(x) \tag{7}$$

is a solution of (H) and an eigenfunction of T for the eigenvalue $e^{i\theta}$. Finally,

$$p(x) = \Psi(x)e^{-i\theta x}$$

is an eigenfunction of the operator A_θ, associated with the eigenvalue λ^2.

3.2. Representation of solutions. In order to find the representation of the solution of (S), we decompose the initial data as follows:

$$u_0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\,\widehat{u_0}(\xi)d\xi = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{2k\pi}^{2(k+1)\pi} e^{ix\xi}\,\widehat{u_0}(\xi)d\xi$$

$$= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_0^{2\pi} e^{i(2k\pi+\theta)x}\,\widehat{u_0}(2k\pi+\theta)d\theta.$$

Thus u_0 can be written

$$u_0(x) = \frac{1}{2\pi}\int_0^{2\pi} v(\theta, x)\,d\theta,$$

with

$$v(\theta,x)=\sum_{k\in\mathbb{Z}}e^{i(2k\pi+\theta)x}\,\widehat{u_0}(2k\pi+\theta). \tag{8}$$

Moreover,

$$\|u_0\|^2_{\mathbb{L}^2(\mathbb{R})}=\frac{1}{2\pi}\|\widehat{u_0}\|^2_{\mathbb{L}^2(\mathbb{R})}=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{2k\pi}^{2(k+1)\pi}|\widehat{u_0}(x)|^2dx=\sum_{k\in\mathbb{Z}}\int_0^{2\pi}|\widehat{u_0}(2k\pi+\theta)|^2d\theta$$

$$=\int_0^{2\pi}\int_0^1|e^{-i\theta x}v(\theta,x)|^2dxd\theta=\int_0^{2\pi}\int_0^1|v(\theta,x)|^2dxd\theta.$$

Since v satisfies the quasi-periodicity property

$$v(\theta,x+1)=e^{i\theta}v(\theta,x),$$

then $v(\theta,x)e^{-i\theta x}$ is 1-periodic. Therefore we can decompose it with respect to the $\mathbb{L}^2(\mathbb{S}^1)$ basis of eigenfunctions of the operator A_θ introduced in section 3.1. If $\theta\in(0,\pi)\cup(\pi,2\pi)$, the eigenvalues of A_θ are simple and we can write

$$v(\theta,x)e^{-i\theta x}=\sum_{n\in\mathbb{N}}c_n(\theta)p_n(\theta,x);$$

that is,

$$v(\theta,x)=\sum_{n\in\mathbb{N}}c_n(\theta)\Psi_n(\theta,x). \tag{9}$$

Finally,

$$u(t,x)=\frac{1}{2\pi}\int_0^{2\pi}\sum_{n\in\mathbb{N}}e^{it\omega^2_{\theta,n}}c_n(\theta)\Psi_n(\theta,x)d\theta \tag{10}$$

is the solution of the Schrödinger equation (S). Moreover, using the above link between the $\mathbb{L}^2$ norms of the initial datum u_0 and of v,

$$\|u_0\|^2_{\mathbb{L}^2(\mathbb{R})}=\sum_{n\in\mathbb{N}}\|c_n\|^2_{\mathbb{L}^2(0,2\pi)}.$$

Let us now express the solution u in terms of the initial datum u_0. By using the definitions (8) and (9),

$$c_n(\theta)=\langle v(\theta,\cdot),\Psi_n(\theta,\cdot)\rangle=\sum_{k\in\mathbb{Z}}\widehat{u_0}(2k\pi+\theta)\langle e^{i(2k\pi+\theta)\,\cdot},\Psi_n(\theta,\cdot)\rangle.$$

Since $e^{-i\theta x}\Psi_n(\theta,x)$ is 1-periodic, its Fourier decomposition contains only even exponentials:

$$e^{-i\theta x}\Psi_n(\theta,x)=\sum_{k\in\mathbb{Z}}d_{n,k}(\theta)e^{i2\pi kx}.$$

Therefore

$$c_n(\theta) = \sum_{k\in\mathbb{Z}} \widehat{u_0}(2k\pi+\theta)\overline{d}_{n,k}(\theta) = \int_{-\infty}^{\infty} u_0(y)e^{-iy\theta}\sum_{k\in\mathbb{Z}} e^{-i2k\pi y}\overline{d}_{n,k}(\theta)dy$$

$$= \int_{-\infty}^{\infty} u_0(y)\overline{\Psi}_n(\theta,y)dy.$$

In conclusion, for any initial datum u_0, the solution of the Schrödinger equation (S) is

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0(y)\int_0^{2\pi}\sum_{n\in\mathbb{N}} e^{it\omega_{\theta,n}^2}\Psi_n(\theta,x)\overline{\Psi}_n(\theta,y)d\theta dy.$$

3.3. Explicit solutions for the Krönig–Penney model. Let

$$a(x) = \begin{cases} b_0^{-2} \text{ for } x\in[0,x_0), \\ b_1^{-2} \text{ for } x\in[x_0,1) \end{cases}$$

as defined in the statement of Theorem 1.2. Fix $\theta\in(0,\pi)\cup(\pi,2\pi)$. Following the approach presented in section 3.1, in this subsection we will explicitly find the functions $\Psi_n(\theta,x)$.

The basis of normalized solutions associated with the (H) is

$$\begin{cases} \Psi_1(x) = \begin{cases} \frac{1}{2}e^{i\lambda b_0 x} + \frac{1}{2}e^{-i\lambda b_0 x} \text{ for } x\in(0,x_0), \\ a_j^1 e^{i\lambda b_1 x} + b_j^1 e^{-i\lambda b_1 x} \text{ for } x\in(x_0,1), \end{cases} \\ \Psi_2(x) = \begin{cases} -\frac{ib_0}{2\lambda}e^{i\lambda b_0 x} + \frac{ib_0}{2\lambda}e^{-i\lambda b_0 x} \text{ for } x\in(0,x_0), \\ a_j^2 e^{i\lambda b_1 x} + b_j^2 e^{-i\lambda b_1 x} \text{ for } x\in(x_0,1) \end{cases} \end{cases}$$

with

$$\begin{cases} a_j^1 = \frac{1}{4b_0}[(b_0+b_1)e^{i\lambda x_0(b_0-b_1)} + (b_0-b_1)e^{-i\lambda x_0(b_0+b_1)}], \\ b_j^1 = \frac{1}{4b_1}[(b_0+b_1)e^{-i\lambda x_0(b_0-b_1)} + (b_0-b_1)e^{i\lambda x_0(b_0+b_1)}], \\ a_j^2 = \frac{i}{4\lambda}[-(b_0+b_1)e^{i\lambda x_0(b_0-b_1)} + (b_0-b_1)e^{-i\lambda x_0(b_0+b_1)}], \\ b_j^2 = \frac{i}{4\lambda}[(b_0+b_1)e^{-i\lambda x_0(b_0-b_1)} - (b_0-b_1)e^{i\lambda x_0(b_0+b_1)}]. \end{cases}$$

The trace of the shift operator T is

$$\mathrm{Tr}T = \Psi_1(1) + \frac{1}{b_1^2}\,\partial_x\Psi_2(1).$$

One can calculate

$$\mathrm{Tr}(T) = (r+1)\cos[\lambda(x_0b_0 + (1-x_0)b_1)] - (r-1)\cos[\lambda(x_0b_0-(1-x_0)b_1)],$$

where

$$r = \frac{b_0^2+b_1^2}{2b_0b_1}.$$

By setting the conditions

$$\mathrm{Tr}(T) = 2\cos\theta, \ \ x_0 b_0 = (1 - x_0) b_1,$$

it follows that

$$2\cos\theta = (r+1)\cos(\lambda 2 x_0 b_0) - (r-1).$$

Hence we have

$$\lambda \in \left\{ \frac{2\pi j + f(\theta)}{2x_0 b_0}\ , \ j \in \mathbb{Z} \right\},$$

where $f(\theta)$ is the analytic function

$$f(\theta) = \arccos \frac{r - 1 + 2\cos\theta}{r+1}.$$

As the solutions Ψ_1 and Ψ_2 are the same for λ and for $-\lambda$, we have to check if there exist different integers j and k such that

$$2\pi j + f(\theta) = \pm(2\pi k + f(\theta)).$$

If this is true, it follows that

$$j + k = \frac{f(\theta)}{\pi}.$$

Since $r > 1$ gives $f(\theta) < \pi$ and $\theta \neq 0$ gives $f(\theta) \neq 0$, then j and k must satisfy

$$0 < |j + k| < 1.$$

In conclusion, the values

$$\left| \frac{2\pi j + f(\theta)}{2x_0 b_0} \right|$$

are different, so we can consider the eigenvalues of the operator A_θ indexed by $j \in \mathbb{Z}$ as follows:

$$\omega_{\theta,j} = \frac{2\pi j + f(\theta)}{2x_0 b_0}. \tag{11}$$

Note that since θ has been fixed in $(0, \pi) \cup (\pi, 2\pi)$,

$$\omega_{\theta,j} \neq 0 \text{ for all } j \in \mathbb{Z}.$$

By using (7), we obtain a quasi-periodic solution for equation $(\mathrm{H}_{\theta,j})$:

$$\widetilde{\Psi}_j(\theta, x) = \left(\frac{1}{2} + h_j(\theta)\right) e^{i\omega_{\theta,j} b_0 x} + \left(\frac{1}{2} - h_j(\theta)\right) e^{-i\omega_{\theta,j} b_0 x} \text{ for } x \in (0, x_0) \tag{12}$$

with

$$h_j(\theta) = i\frac{(b_0 + b_1)\cos(2\omega_{\theta,j} b_0 x_0) + (b_0 - b_1) - e^{i\theta}}{(b_0 + b_1)\sin(2\omega_{\theta,j} b_0 x_0)}.$$

The definition (11) of $\omega_{\theta,j}$ gives

$$h_j(\theta) = h(\theta) = i\frac{(b_0+b_1)\cos f(\theta) + (b_0-b_1) - e^{i\theta}}{(b_0+b_1)\sin f(\theta)}.$$

Then we can calculate for $x \in (0, x_0)$

$$\widetilde{\Psi}_j(\theta,x) = \cos(\omega_{\theta,j}\, b_0 x) + 2h(\theta)\sin(\omega_{\theta,j}\, b_0 x),$$

and for $x \in (x_0, 1)$

$$\widetilde{\Psi}_j(\theta,x) = \left(a_j^1 - a_j^2\, h(\theta)\frac{2\omega_{\theta,j}}{ib_0}\right)e^{i\omega_{\theta,j}b_1 x} + \left(b_j^1 - b_j^2\, h(\theta)\frac{2\omega_{\theta,j}}{ib_0}\right)e^{-i\omega_{\theta,j}b_1 x}$$

$$= \frac{b_0+b_1}{4b_0}(1+2h(\theta))e^{i\omega_{\theta,j}(x_0(b_0-b_1)+b_1 x)} + \frac{b_0-b_1}{4b_0}(1-2h(\theta))e^{-i\omega_{\theta,j}(x_0(b_0+b_1)-b_1 x)}$$

$$+\frac{b_0+b_1}{4b_0}(1-2h(\theta))e^{-i\omega_{\theta,j}(x_0(b_0-b_1)+b_1 x)} + \frac{b_0-b_1}{4b_0}(1+2h(\theta))e^{i\omega_{\theta,j}(x_0(b_0+b_1)-b_1 x)}.$$

It follows that

$$\int_0^1 |\widetilde{\Psi}_j(\theta,x)|^2 dx = \alpha_j(\theta) = \beta(\theta) + \frac{\gamma(\theta)}{2\pi j + f(\theta)},$$

with $\beta(\theta)$ strictly positive. Let $\Psi_j(\theta,x)$ be the $\mathbb{L}^2$ normalization of $\widetilde{\Psi}_j(\theta,x)$:

$$\Psi_j(\theta,x) = \frac{\widetilde{\Psi}_j(\theta,x)}{\sqrt{\alpha_j(\theta)}}.$$

We are now in the context described in section 3.2.

3.4. The failure of local dispersion. Let $\mathcal{X}$ be a 2π-periodic function whose restriction to $(0,2\pi)$ is $\mathcal{C}_0^\infty$. One can write

$$\mathcal{X}(\xi) = \sum_{k\in\mathbb{Z}} s_k e^{ik\xi}.$$

Let v_0 be the Fourier localization outside $2\pi\mathbb{Z}$ points of the initial data u_0

$$\widehat{v_0}(\xi) = \widehat{u_0}(\xi)\mathcal{X}(\xi).$$

By applying Plancherel's theorem one has

$$v_0(x) = \int_{-\infty}^{\infty} e^{ix\xi}\widehat{u_0}(\xi)\mathcal{X}(\xi)\frac{d\xi}{2\pi} = \sum_{k\in\mathbb{Z}} u_0(x+k)s_k.$$

Since $\mathcal{X}|_{(0,2\pi)}$ is in $\mathcal{C}_0^\infty$,

$$\sum_{k\in\mathbb{Z}} |s_k| = S < \infty,$$

so the localization preserves the regularity $\mathbb{L}^1(\mathbb{R}) \cap \mathbb{L}^2(\mathbb{R})$ with

$$\begin{cases} \|v_0\|_{\mathbb{L}^1(\mathbb{R})} \leq C\|u_0\|_{\mathbb{L}^1(\mathbb{R})}, \\ \|v_0\|_{\mathbb{L}^2(\mathbb{R})} \leq C\|u_0\|_{\mathbb{L}^2(\mathbb{R})}. \end{cases}$$

For such an initial datum v_0, the coefficients $c_j(\theta)$ defined in section 3.2 are

$$c_j(\theta) = \sum_{k\in\mathbb{Z}} \widehat{u_0}(2k\pi+\theta)\mathcal{X}(2k\pi+\theta)\overline{d}_{j,k}(\theta)$$

$$= \mathcal{X}(\theta)\int_{-\infty}^{\infty} u_0(y)e^{-iy\theta}\sum_{k\in\mathbb{Z}} e^{-i2k\pi y}\overline{d}_{j,k}(\theta)dy = \mathcal{X}(\theta)\int_{-\infty}^{\infty} u_0(y)\overline{\Psi}_{\theta,j}(y)dy.$$

Then, by the representation formula (10), the solution $v(t,x)$ of the equation (S) with initial datum v_0 can be written as

$$v(t,x) = \int_{-\infty}^{\infty} u_0(y)K_t(x,y)dy,$$

where

$$K_t(x,y) = \frac{1}{2\pi}\int_0^{2\pi}\sum_{j\in\mathbb{Z}} e^{it\omega_{\theta,j}^2}\Psi_{\theta,j}(x)\overline{\Psi}_{\theta,j}(y)\mathcal{X}(\theta)d\theta.$$

Since

$$\|v_0\|_{\mathbb{L}^1(\mathbb{R})} \leq C\|u_0\|_{\mathbb{L}^1(\mathbb{R})},$$

in order to have the dispersion inequality

$$\|v(t,\cdot)\|_{\mathbb{L}^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}}\|v_0\|_{\mathbb{L}^1(\mathbb{R})},$$

the dispersion kernel must satisfy

$$\|K_t\|_{\mathbb{L}^\infty(x,y)} \leq \frac{C}{\sqrt{t}}.$$

We will show that there exist times t, arbitrarily small, for which K_t is not an $\mathbb{L}^\infty(x,y)$ function.

Let us change t in $\frac{t}{4b_o^2x_0^2}$ and x in $\frac{x}{2x_0}$. By using definition (11) of $\omega_{\theta,j}$ and formula (12) for $\widetilde{\Psi}_j(\theta,x)$, we have that $K_t(x,y)$ is, for $x < x_0$, equal to

$$\frac{1}{4\pi}\sum_{j\in\mathbb{Z}}\int_0^{2\pi} e^{it(2\pi j+f(\theta))^2}\left(e^{ix(2\pi j+f(\theta))}(1+2h(\theta)) + e^{-ix(2\pi j+f(\theta))}(1-2h(\theta))\right)$$

$$\times\left(e^{-iy(2\pi j+f(\theta))}(1+2\overline{h}(\theta)) + e^{iy(2\pi j+f(\theta))}(1-2\overline{h}(\theta))\right)\frac{\mathcal{X}(\theta)}{\alpha_j(\theta)}d\theta.$$

It follows that the kernel is the sum of four terms of the following type:

$$J_t(x,y) = \frac{1}{4\pi}\sum_{j\in\mathbb{Z}}\int_0^{2\pi} e^{it(2\pi j+f(\theta))^2}e^{i(x-y)(2\pi j+f(\theta))}(1+2h(\theta))(1+2\overline{h}(\theta))\frac{\mathcal{X}(\theta)}{\alpha_j(\theta)}d\theta.$$

In view of the forthcoming applications of the stationary phase formula, we can consider that $J_t(x,y)$ is, modulo an $\mathbb{L}^\infty$ function, the same sum as above, with α_0 replaced by α_1. Since $|f(\theta)| < \pi$, one can choose a function $\alpha_\xi(\theta)$ which is strictly positive, bounded, and $\mathcal{C}^\infty$ with respect to the variable ξ such that

$$\alpha_\xi(\theta) = \beta(\theta) + \frac{\gamma(\theta)}{\xi + f(\theta)} \text{ for } |\xi| > \pi.$$

This allows us to apply the Poisson formula, so $J_t(x,y)$ can be written as

$$\frac{1}{2}\sum_{l\in\mathbb{Z}} e^{i\xi l}\int_{-\infty}^{\infty}\int_0^{2\pi} e^{it(\xi+f(\theta))^2} e^{i(x-y)(\xi+f(\theta))}(1+2h(\theta))(1+2\overline{h}(\theta))\frac{\mathcal{X}(\theta)}{\alpha_\xi(\theta)}d\theta d\xi.$$

By changing $\xi + f(\theta)$ into ζ,

$$J_t(x,y) = \frac{1}{2}\sum_{l\in\mathbb{Z}}\int_0^{2\pi} e^{-if(\theta)l}\, I_l(t, x-y, \theta)\, d\theta,$$

where

$$I_l(t, x-y, \theta) = \mathcal{X}(\theta)(1+2h(\theta))(1+2\overline{h}(\theta))\int_{-\infty}^{\infty} e^{it\zeta^2} e^{i(x-y+l)\zeta}\frac{d\zeta}{\alpha_{\zeta-f(\theta)}(\theta)}$$

verifies

$$|\partial\theta^k I_l(t, x-y, \theta)| \leq C \text{ for all } k \in \mathbb{N}.$$

The only critical point of $f|_{(0,2\pi)}$ is π, which is nondegenerate, so we can apply the stationary phase formula for large l. In view of the definition of $\alpha_\zeta(\pi)$, $J_t(x,y)$ is, modulo an $\mathbb{L}^\infty$ function,

$$J_t(x,y) = \sum_{l\in\mathbb{Z}^*}\left(\frac{e^{-if(\pi)l}}{\sqrt{|l|}}\, I_l(t, x-y)\,\frac{1}{2}\mathcal{X}(\pi)(1+2h(\pi))(1+2\overline{h}(\pi)) + O(|l|^{-\frac{3}{2}})\right)$$

with

$$I_l(t, x-y) = \int_{-\infty}^{\infty} e^{it\zeta^2} e^{i(x-y+l)\zeta}\frac{d\zeta}{\beta(\pi) + \frac{\gamma(\pi)}{\zeta}}.$$

We have used the known result that the sum of exponentials

$$F(\alpha) = \sum_{l\in\mathbb{Z}^*}\frac{e^{-i\alpha l}}{\sqrt{|l|}} \tag{13}$$

blows up as

$$\frac{1}{\sqrt{|\alpha|}}$$

if α tends to zero, and otherwise the sum is finite. Here $f(\pi) \in (0, \pi)$.

By changing ζ in $\frac{x-y+l}{\sqrt{t}}$ and by considering that (x,y) lies in a compact set, we have

$$I_l(t, x-y) = \frac{x-y+l}{\sqrt{t}}\int_{-\infty}^{\infty} e^{i(x-y+l)^2(\zeta^2+\frac{\zeta}{\sqrt{t}})}\,\frac{d\zeta}{\beta(\pi) + \frac{\gamma(\pi)\sqrt{t}}{(x-y+l)\zeta}}.$$

The stationary phase formula applied again for $\zeta = -\frac{1}{2\sqrt{t}}$ gives

$$I_l(t, x-y) = \frac{1}{\sqrt{t}} e^{-i\frac{(x-y+l)^2}{4t}} \frac{1}{\beta(\pi) - \frac{2\gamma(\pi)t}{x-y+l}} + \frac{O((x-y+l)^{-2})}{\sqrt{t}}.$$

Thus, modulo an $\mathbb{L}^\infty$ function, we obtain that

$$J_t(x,y) = \frac{C}{\sqrt{t}} \sum_{l \in \mathbb{Z}^*} \frac{e^{-if(\pi)l}}{\sqrt{|l|}} \, e^{-i\frac{(x-y+l)^2}{4t}},$$

with $C \neq 0$. Let t verify

$$\frac{1}{4t} \in 2\pi\mathbb{Z}.$$

Note that t can be chosen arbitrary small. Also,

$$J_t(x,y) = \frac{Ce^{-i\frac{(x-y)^2}{4t}}}{\sqrt{t}} \sum_{l \in \mathbb{Z}^*} \frac{e^{-i(\frac{x-y}{2t}+f(\pi))l}}{\sqrt{|l|}}.$$

It follows then that $K_t(x,y)$ is, modulo an $\mathbb{L}^\infty$ function,

$$\frac{e^{-i\frac{(x-y)^2}{4t}}}{\sqrt{t}} \left(C_1 \, F\left(\frac{x-y}{2t} + f(\pi)\right) + C_2 \, F\left(-\frac{x-y}{2t} + f(\pi)\right)\right)$$

$$+\frac{e^{-i\frac{(x+y)^2}{4t}}}{\sqrt{t}} \left(C_3 \, F\left(\frac{x+y}{2t} + f(\pi)\right) + C_4 \, F\left(-\frac{x+y}{2t} + f(\pi)\right)\right).$$

Since $f(\pi) \neq 0$, in view of the behavior of F presented above (see (13)), the kernel $K_t(x,y)$ is not in $\mathbb{L}^\infty(x,y)$. Therefore the local dispersion for the Schrödinger equation (S) fails and Theorem 1.2 is proved.

Acknowledgment. I thank my advisor Patrick Gérard for having guided this work.

REFERENCES

[1] M. Avellaneda, C. Bardos, and J. Rauch, *Contrôlabilité exacte, homogénéisation et localisation d'ondes dans un milieu non-homogène*, Asymptot. Anal., 5 (1992), pp. 481–494.
[2] N. Burq, P. Gérard, and N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. Math., to appear.
[3] C. Castro and E. Zuazua, *Concentration and lack of observability of waves in highly heterogeneous media*, Arch. Ration. Mech. Anal., 164 (2002), pp. 39–72.
[4] W. Craig, T. Kappeler, and W. Strauss, *Microlocal dispersive smoothing for the Schrödinger equation*, Comm. Pure Appl. Math., 48 (1995), pp. 769–860.
[5] S. Doi, *Remarks on the Cauchy problem for Schrödinger-type equations*, Comm. Partial Differential Equations, 21 (1996), pp. 163–178.
[6] I.M. Gelfand, D.A. Raikov, and G.E. Chilov, *Les anneaux normés commutatifs*, Monographies internationales de mathématiques modernes, Gauthier-Villars, Paris, 1964.
[7] J. Ginibre and G. Velo, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal., 133 (1995), pp. 50–68.
[8] L. Kapitanski and Y. Safarov, *Dispersive smoothing for Schrödinger equations*, Math. Res. Lett., 3 (1996), pp. 77–91.

[9] W. Magnus and S. Winkler, *Hill's Equation*, Interscience Tracts in Pure and Applied Mathematics 20, Interscience–John Wiley, New York, 1966.

[10] G. Staffilani and D. Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients*, Comm. Partial Differential Equations, 27 (2002), pp. 1337–1372.

[11] R.S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J., 44 (1977), pp. 705–714.

[12] P. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc., 81 (1975), pp. 177–178.