REMARKS ON THE BLOW-UP FOR THE SCHRÖDINGER EQUATION WITH CRITICAL MASS ON A PLANE DOMAIN

VALERIA BANICA

ABSTRACT. In this paper we concentrate on the analysis of the critical mass blowing-up solutions for the cubic focusing Schrödinger equation with Dirichlet boundary conditions, posed on a plane domain. We bound the blow-up rate from below, for bounded and unbounded domains. If the blow-up occurs on the boundary, the blow-up rate is proved to grow faster than $(T-t)^{-1}$, the expected one. Moreover, we show that blow-up cannot occur on the boundary, under certain geometric conditions on the domain. *2000 Mathematics Subject Classification.* 35Q55, 35B33, 35B40, 35Q40.

1. INTRODUCTION

Let us first recall the known results for the $\mathbb{R}^n$ case.
Consider the nonlinear Schrödinger equation on $\mathbb{R}^n$, for $p \geq 1$,

$$(S) \begin{cases} i\partial_t u + \Delta u + |u|^{p-1}u = 0, \\ \qquad\qquad u(0) = u_0. \end{cases}$$

The associated Cauchy problem is locally well posed in $\mathbb{H}^1$ for $p < 1 + \frac{4}{n-2}$ ([8], [10], [5]). In the endpoint case $p = 1 + \frac{4}{n-2}$, the local well-posedness was treated in [6].

The Gagliardo-Nirenberg inequality

$$\|v\|_{p+1}^{p+1} \leq C_{p+1}\|v\|_2^{2+(p-1)\frac{2-n}{2}}\|\nabla v\|_2^{(p-1)\frac{n}{2}}$$

implies that the energy of the solution u of the equation (S),

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx,$$

is bounded from below by

$$\|\nabla u\|_2^2\left(\frac{1}{2} - \frac{C_{p+1}}{p+1}\|u\|_2^{2+(p-1)\frac{2-n}{2}}\|\nabla u\|_2^{(p-1)\frac{n}{2}-2}\right).$$

As a consequence, if $p < 1+\frac{4}{n}$, since the mass is conserved, the gradient of u is controlled by the energy. Therefore the solution does not blow up and global existence occurs.

The power $p = 1 + \frac{4}{n}$ is a critical power, in the sense that the nonlinearity is strong enough to generate solutions blowing up in a finite time. However, even in this case, we

1

have a global result for small initial conditions.

Indeed, in the case $p = 1 + \frac{4}{n}$, if the mass of the initial condition is small enough so that

$$\frac{C_{2+\frac{4}{n}}}{2 + \frac{4}{n}} \|u\|_2^{\frac{4}{n}} < \frac{1}{2},$$

then the energy controls the gradient and again, the global existence is proved for the equation (S).

For this particular value of p, Weinstein has given a sharpening of the Gagliardo-Nirenberg inequality ([27]). By variational methods using Lions concentration-compacity lemma ([13], [14]), he obtained the existence of a minimizer Q for the optimal constant of Gagliardo-Nirenberg's inequality

$$\frac{1}{C_{2+\frac{4}{n}}} = \inf_{v \in \mathbb{H}^1(\mathbb{R}^n)} \frac{\|v\|_2^{\frac{4}{n}} \|\nabla v\|_2^2}{\|v\|_{2+\frac{4}{n}}^{2+\frac{4}{n}}}.$$

This minimizer satisfies the Euler-Lagrange equation

$$\Delta Q + Q^{1+\frac{4}{n}} = \frac{2}{n} Q.$$

Such a positive function, called ground state of the nonlinear Schrödinger equation, is radial, exponentially decreasing at infinity and regular. Recently, Kwong has shown that it is unique up to a translation ([12]). Moreover, it verifies Pohozaev's identities

$$\begin{cases} \|\nabla Q\|_2^2 - \|Q\|_{2+\frac{4}{n}}^{2+\frac{4}{n}} + \frac{2}{n}\|Q\|_2^2 = 0, \\ (n-2)\|\nabla Q\|_2^2 - \frac{n^2}{n+2}\|Q\|_{2+\frac{4}{n}}^{2+\frac{4}{n}} + 2\|Q\|_2^2 = 0, \end{cases}$$

which lead to the following relations between the norms of Q

(1) $$\begin{cases} \|Q\|_{2+\frac{4}{n}}^{2+\frac{4}{n}} = \frac{n+2}{n}\|Q\|_2^2, \\ \|\nabla Q\|_2^2 = \|Q\|_2^2. \end{cases}$$

Then the optimal value for the constant of the Gagliardo-Nirenberg inequality is

$$C_{2+\frac{4}{n}} = \frac{n+2}{n} \frac{1}{\|Q\|_2^{\frac{4}{n}}}.$$

In conclusion, if $p = 1 + \frac{4}{n}$, the solutions of the equation (S) with initial condition of mass smaller than the one of the ground state

$$\|u\|_2 < \|Q\|_2,$$

are global in time.

The mass $\|Q\|_2$ is critical, in the sense that we can construct as follows solutions of mass equal to $\|Q\|_2$, which blows up in finite time. Since $p = 1 + \frac{4}{n}$, the pseudo-conformal transform of a solution u of (S)

$$\frac{1}{t^{\frac{n}{2}}} e^{i\frac{|x|^2}{4t}} u\left(-\frac{1}{t}, \frac{x}{t}\right),$$

is also a solution of (S) ([4]). So, from a stationary solution on $\mathbb{R}^n$

$$e^{it}Q(x),$$

for all positive T,

$$u(t,x) = \frac{e^{\frac{i}{T-t}}}{(T-t)^{\frac{n}{2}}} e^{-i\frac{|x|^2}{4(T-t)}} Q\left(\frac{x}{T-t}\right),$$

is a solution blowing up at the time T. Moreover, Merle proved in [17] that all blowing-up solutions on $\mathbb{R}^n$ with critical mass $\|Q\|_2$ are of this type, up to the invariants of the equation. The proof is based on a result of concentration of Weinstein ([28], see Lemma 1.2) and on the study of the first order momentum

$$f(t) = \int_{\mathbb{R}^n} |u(t,x)|^2 x\,dx,$$

and of the virial

$$g(t) = \int_{\mathbb{R}^n} |u(t,x)|^2 |x|^2 dx$$

associated to a solution u of the equation (S). The conservative properties of these two quantities on $\mathbb{R}^n$, in the case of the critical power $1+\frac{4}{n}$, play an important role in Merle's proof. The derivative of the first order momentum is constant in time

$$\partial_t^2 f = 0,$$

and g satisfies the virial identity ([4])

$$\partial_t^2 g = 16E(u) - 4\frac{n(p-1)-4}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx = 16E(u).$$

In certain cases of initial conditions with mass larger than $\|Q\|_2$ recent achievements were done by Merle and Raphaël, concerning the blow-up rate and the blow-up profile ([19], [20]).

For the equation (S_p) with $p \geq 1 + \frac{4}{n}$, Zakharov [30] and Glassey [9] had obtained that the solutions of negative energy are blowing up in finite time. The same result for solutions of nonnegative energy is valid under certain conditions on the derivatives of the virial ([25]). The proof is based on an upper bound of the virial in terms of its first and second derivative, which implies the cancellation of the virial at a finite time T. Since the mass is conserved, it follows that the solution must blow up at the time T.

In this paper we are concerned with the nonlinear Schrödinger equation posed on a regular domain Ω of $\mathbb{R}^n$, with Dirichlet boundary conditions

$$\begin{cases} i\partial_t u + \Delta u + |u|^{p-1}u = 0, \\ \qquad u_{|\mathbb{R}\times\partial\Omega} = 0, \\ \qquad u(0) = u_0. \end{cases}$$

The Cauchy problem is locally well posed on $\mathbb{H}^2 \cap \mathbb{H}^1_0(\Omega)$ in dimension 2 and 3. In dimension 2, for nonlinearities less than cubic, Vladimirov [26] and Ogawa and Ozawa [22] have shown the well-posedness of the Cauchy problem on $\mathbb{H}^1_0(\Omega)$, but without the uniform continuity of the flow on bounded sets of $\mathbb{H}^1_0(\Omega)$. For nonlinearities stronger

than cubic in dimension 2, or for any power nonlinearity p, in dimension higher than 2, the Cauchy problem on $\mathbb{H}_0^1(\Omega)$ is open.

For the equation with power $p < 1 + \frac{4}{n}$, one can show as for the case $\mathbb{R}^n$ that the $\mathbb{H}_0^1(\Omega)$ solutions are global in time. For the equation with power $p \geq 1 + \frac{4}{n}$, posed on a star-shaped domain of $\mathbb{R}^n$, Kavian has proved the blow-up in finite time of the $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$ solutions of negative energy or of positive energy but under some conditions on the first and second derivatives of the virial ([11]). His proof follows the one on $\mathbb{R}^n$ ([9]), by estimating via the geometric condition on Ω the boundary terms which appear in the second derivative of the virial.

From now on we will analyze the cubic equation on Ω

$$(S_\Omega) \quad \begin{cases} i\partial_t u + \Delta u + |u|^2 u = 0, \\ \quad u_{|\mathbb{R} \times \partial\Omega} = 0, \\ \quad u(0) = u_0. \end{cases}$$

Let us first notice that the conservations of the mass and of the energy of the solutions are still valid. The Cauchy problem is locally well posed on $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$, and also on $\mathbb{H}_0^1(\Omega)$ apart from the property of uniform continuity of the flow, not known to hold. The usual Strichartz inequalities are no longer valid and the loss of derivatives is stronger than in the case of a compact manifold ([3]).

As in the case of the plane, for initial conditions with mass smaller than the one of the ground state, the Cauchy problem is globally well-posed on $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$. The proof, given by Brézis and Gallouët, is based on logarithmic type estimates ([2]). This result has been extended to the natural space $\mathbb{H}_0^1(\Omega)$, apart from the uniform continuity of the flow ([26],[22],[4]).

The critical mass for blow-up is $\|Q\|_2$, as in the case of the equation posed on $\mathbb{R}^2$. More precisely, the following result holds.

Theorem 1.1. *(Burq-Gérard-Tzvetkov [3]) Let Ω be a regular bounded domain of $\mathbb{R}^2$. Let $x_0 \in \Omega$ and $\psi \in \mathcal{C}_0^\infty(\Omega)$ be a function equal to 1 near x_0. Then there exist positive numbers κ and α_0 such that for all $\alpha > \alpha_0$, there exists a time T_α and a function r_α defined on $[0, T_\alpha[\times\Omega$, satisfying*

$$\|r_\alpha(t)\|_{\mathbb{H}^2(\Omega)} \leq c e^{-\frac{\kappa}{T_\alpha - t}},$$

such that

$$u(t, x) = \psi(x)\frac{e^{\overline{\alpha^2(T_\alpha - t)}}}{\alpha(T_\alpha - t)}e^{-i\frac{|x - x_0|^2}{4\alpha(T_\alpha - t)}}Q\left(\frac{x - x_0}{\alpha(T_\alpha - t)}\right) + r_\alpha(t, x),$$

is a critical mass solution of (S_Ω), blowing up at x_0 at the time T_α with the blow-up rate $\frac{1}{T_\alpha - t}$.

The proof, following an idea of Ogawa and Tsutsumi ([23]), is based on a fixed point method which allows to complete the cut-off of the explicit blowing up solution on $\mathbb{R}^2$ at x_0 to a blowing up solution on Ω at x_0. Theorem 4.1.1 implies in particular that at every point of Ω there are explosive solutions. Moreover, the proof is still valid for

the torus $\mathbb{T}^2$ and for a larger class of subsets of the plane, which satisfy the property of 2-continuation, from $\mathbb{H}^2 \cap \mathbb{H}_0^1(\Omega)$ to $\mathbb{H}^2(\mathbb{R}^2)$, and for which the Laplacian domain

$$D(-\Delta_\Omega) = \{u \in \mathbb{H}_0^1(\Omega), \Delta u \in \mathbb{L}^2(\Omega)\},$$

is $\mathbb{H}^2 \cap \mathbb{H}_0^1$. Such subsets are for example the domains with compact regular boundary and convex polygons bounded or unbounded.

As in the $\mathbb{R}^n$ case, the following lemma, due to Weinstein, will give us the general behavior of a blowing-up solution of critical mass on a domain.

Lemma 1.2. *(Weinstein [28]) Let $u_k \in \mathbb{H}^1(\mathbb{R}^n)$ be a sequence of functions of critical mass satisfying*

$$\begin{cases} \beta_k = \|\nabla u_k\|_2 \underset{k\to\infty}{\longrightarrow} \infty, \\ E(u_k) \underset{k\to\infty}{\longrightarrow} c < \infty. \end{cases}$$

Then there exist points $x_k \in \mathbb{R}^d$ and $\theta_k \in \mathbb{R}$ such that in $\mathbb{H}^1(\mathbb{R}^n)$

$$\frac{e^{i\theta_k}}{\beta_k^{\frac{n}{2}}} u_k \left(\frac{x}{\beta_k} + x_n \right) \underset{k\to\infty}{\longrightarrow} \frac{1}{\omega^{\frac{n}{2}}} Q \left(\frac{x}{\omega} \right),$$

where $\omega = \|\nabla Q\|_2$.

Let u be a solution of (S_Ω) that blows up at the finite time T, that is

$$\lambda(t) = \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \underset{t\to T}{\longrightarrow} \infty.$$

Consider u to be extended by zero outside Ω. By combining Lemma 1.2 for families $u_k = u(t_k)$ with t_k sequences convergent to T with the result of Kwong on the uniqueness of the ground state ([12]), there exist $\theta(t)$ real numbers and $x(t) \in \mathbb{R}^2$ such that in $\mathbb{H}^1(\mathbb{R}^2)$

$$\text{(2)} \qquad \frac{e^{i\theta(t)}}{\lambda(t)} u \left(t, \frac{x}{\lambda(t)} + x(t) \right) \underset{t\to T}{\longrightarrow} Q(x).$$

Then, in the space of distributions,

$$\text{(3)} \qquad |u(t, \cdot + x(t))|^2 \underset{t\to T}{\longrightarrow} \|Q\|_2^2 \, \delta_0.$$

In this paper we concentrate on the further analysis of the blowing-up solutions with critical mass on a plane domain. The results are the following.

Theorem 1.3. *Let u be a $\mathcal{C}([0, T[, \mathbb{H}_0^1)$ solution of the Schrödinger equation (S_Ω), which has critical mass and blows up at the finite time T.*
  i) For bounded domains, the blowing-up rate is lower bounded by

$$\frac{1}{T-t} \lesssim \|\nabla u(t)\|_2.$$

ii) If there exist solutions u of critical mass blowing up at a finite time T on the boundary of Ω, that is if the concentration parameter $x(t)$ converges as $t \to T$ to a point on the boundary, then the blowing-up rate satisfies

$$\lim_{t \to T}(T - t)\|\nabla u(t)\|_2 = \infty.$$

The main difficulty for the Schrödinger equation posed on a domain is that the conservation of the derivative of the first momentum and the virial identity fail.

In order to avoid this difficulty, we shall use systematically in the proof of Theorem 1.3 a Cauchy-Schwarz type inequality derived from Weinstein's inequality. Precisely, we show that if v is a $\mathbb{H}^1(\mathbb{R}^2)$ function of critical or subcritical mass, then

$$\left| \int \Im(v \nabla \overline{v}) \nabla \theta dx \right| \leq \left(2E(v) \int |v|^2 |\nabla \theta|^2 dx \right)^{\frac{1}{2}}$$

for all real function θ. This inequality allows us to estimate the virial, that we shall assume to be localized if Ω is unbounded (see Remark 1.6). The lower bound for the blowing-up rate is the same as the one found by Antonini on the torus ([1]).

By following the approach of Weinstein in [29], and the recent results of Maris in [16], we analyze the convergence to the ground state of the modulations of the solutions (2), and we obtain, for bounded domains, the following additional informations.

Proposition 1.4. *i) The blow-up rate verifies*

$$\int_\Omega |u(t,x)|^2 |x - x(t)|^2 dx \approx \frac{1}{\|\nabla u(t)\|_2^2}.$$

ii) The concentration parameter $x(t)$ can be chosen to be as the first order momentum

$$x(t) = \frac{\int_\Omega |u(t,x)|^2 x dx}{\|Q\|_2^2}.$$

Corollary 1.5. *If Ω is a disc centered at 0 and if equation (S_Ω) is considered to be invariant under rotations, then $x(t)$ can be chosen to be 0, and we have*

$$g(t) \approx \frac{1}{\|\nabla u(t)\|_2^2}.$$

Remark 1.6. *For unbounded domains, it is expected that a blowing-up solution of critical mass concentrates at one point, that is the concentration parameter $x(t)$ converges as $t \to T$. Under this assumption, the first assertion of Theorem 1.3 is also true for unbounded domains, and so are the assertions of Proposition 1.4, for the virial and the first order momentum localized at the blow-up point (see §4).*

There is no known example of a solution of nonlinear Schrödinger equation with a blow-up rate larger than $\frac{1}{T-t}$, neither in the case of supercritical mass, nor in the case of supercritical nonlinearities.

Therefore we expect that the blowing-up rate grows exactly like $\frac{1}{T-t}$ and that the profiles are the ones on $\mathbb{R}^2$ modulo an exponentially decreasing in $\mathbb{H}^1$ function.

Since it is not likely that the blowing-up rate at the boundary grows strictly faster than $\frac{1}{T-t}$, we also expect that there are no solutions blowing-up on the boundary of a domain. This is confirmed for certain simple cases by the following result.

Theorem 1.7. *If Ω is a half-plane or a plane sector, then there are no solutions blowing-up in a finite time on the boundary of the half-plane or in the corner of the sector respectively.*

Indeed, under these geometric hypotheses on Ω, the boundary terms which appear in the second derivative of the virial associated to a blowing-up solution of (S_Ω) cancel, so we have, as on $\mathbb{R}^n$, the virial identity

$$\partial_t^2 g = 16 E(u).$$

The proof then follows the one by Merle in [17] for the equation posed on $\mathbb{R}^n$, and we obtain that all explosive solutions on Ω must be of the type

$$\frac{e^{\frac{i}{T-t}}}{(T-t)^{\frac{n}{2}}} \, e^{-i\frac{|x|^2}{4(T-t)}} \, Q\left(\frac{x}{T-t}\right),$$

up to the invariants of the equation. Therefore we arrive at a contradiction by looking at the support of the solution.

Remark 1.8. *The results of this paper are valid also in higher dimension, but are presented in the 2-dimensional case. The reason is that the only existence result of blowing-up solutions with critical mass on a domain is Theorem 1.1.*

Remark 1.9. *In the case of Neumann boundary conditions, the situation changes radically for blow up at the boundary. For blow up inside the domain, the situation is expected to be similar to the Dirichlet case.*

Blow-up can appear at the boundary of a half-plane, as it can easily be seen by taking the restriction to the half-plane of the explicit solution on $\mathbb{R}^2$, blowing-up at zero. Notice also that this is a blowing-up solution of mass $\frac{\|Q\|_2^2}{2}$. By the same method one can construct solution of mass $\frac{\|Q\|_2^2}{n}$, blowing-up at the corner of a plane sector of angle $\frac{2\pi}{n}$. So for unbounded domains the critical mass for blow up on the boundary probably depends on the domain.

By using the techniques of Theorem 1.1, blowing-up solution of this mass $\frac{\|Q\|_2^2}{2}$ can be contructed to blow up on the boundary of a bounded set, in a point around which the boundary is locally a line. For bounded domains, we expect that the critical mass for blow up on the boundary should be $\frac{\|Q\|_2^2}{2}$.

One of the difficulties is that the best constant in the corresponding Gagliardo-Nirenberg inequality

$$\|v\|_4^4 \leq A\|v\|_2^2(\|\nabla v\|_2^2 + B\|v\|_2^2),$$

is not known for the case of Neumann boundary conditions.

The paper is organized as follows. Section §2 contains some results on general domains. We prove a Cauchy-Schwarz type inequality for critical and subcritical mass functions, which we will use to show Theorem 1.3. The nature of the convergence to the ground

state of the modulations of the solutions is analyzed, by spectral theory techniques given in the Appendix. These concentration results will be used later to prove Theorem 1.7 and Proposition 1.4. Moreover, we calculate the derivatives in time for a virial type function. In §3, by studying the virial, the lower-bound of the blowing-up rate is proved for bounded domains Ω. In this section, we also give the proof of Proposition 1.4. In §4, by introducing a localized virial, we find the same lower-bound for the blowing-up rate for unbounded domains (see Remark 1.6). Section §5 contains the results regarding the explosion on the boundary of Ω.

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2. Results on general domains

2.1. A Cauchy-Schwarz inequality for subcritical mass functions.

Lemma 2.1. *Let θ be a real valued function. All $v \in \mathbb{H}^1(\mathbb{R}^2)$ with critical or subcritical mass satisfy*

$$(*) \quad \left| \int \Im(v\nabla\overline{v})\nabla\theta dx \right| \leq \left(2E(v) \int |v|^2 |\nabla\theta|^2 dx \right)^{\frac{1}{2}}.$$

Proof. The precised version of the Gagliardo-Niremberg inequality, presented in the introduction, is, for function w in $\mathbb{H}^1(\mathbb{R}^2)$,

$$\|w\|_4^4 \leq \frac{2}{\|Q\|_2^2} \|w\|_2^2 \|\nabla w\|_2^2.$$

As a consequence, if

$$\|w\|_2 \leq \|Q\|_2,$$

then the energy of w is nonnegative.

Therefore on the one hand,

$$0 \leq E(e^{i\alpha\theta}v)$$

for every real number α and for all real function θ, since $e^{i\alpha\theta}v$ is still a function of critical or subcritical mass. On the other hand

$$E(e^{i\alpha\theta}v) = \frac{1}{2} \int |i\alpha\nabla\theta\, v + \nabla v|^2 dx - \frac{1}{4} \int |v|^4 dx$$

$$= \frac{\alpha^2}{2} \int |v|^2 |\nabla\theta|^2 dx - \alpha \int \Im(v\nabla\overline{v})\nabla\theta dx + E(v)$$

Thus the discriminant of the equation in α must be negative or null and we obtain the claimed Cauchy-Schwarz type inequality $(*)$.

$\square$

2.2. **The concentration of the solution.** In this subsection we shall give a refined description of a critical mass blowing-up solution u of (S_Ω), by following the approach of Weinstein in [29].

In order to deal with real functions, we shall analyze the modulus of u. However, the same arguments below can be used to get the corresponding results on u (see Remark 6.1).

One can write the convergence (2)

$$u(t,x) = e^{-i\theta(t)}\lambda(t)(Q + R(t))(\lambda(t)(x - x(t))),$$

with R a complex function such that

$$\|R(t)\|_{\mathbb{H}^1(\mathbb{R}^2)} \xrightarrow[t\to T]{} 0.$$

Since the modulus is a continuous function on $\mathbb{H}^1(\mathbb{R}^2)$ ([15]), this implies

$$|u(t,x)| = \lambda(t)|Q + R(t)|(\lambda(t)(x - x(t))) = \lambda(t)(Q + \tilde{\tilde{R}}(t))(\lambda(t)(x - x(t))),$$

with $\tilde{\tilde{R}}$ a real function strongly converging to 0 in $\mathbb{H}^1(\mathbb{R}^2)$.

Let us set

$$\tilde{\lambda}(t) = \frac{\||\nabla|u(t)|\|_2}{\|\nabla Q\|_2}.$$

By noticing that $|u(t)|$ is also of critical mass, its energy is nonnegative, and

(4) $$0 \le \|\nabla u(t)\|_2^2 - \big\||\nabla|u(t)|\big\|_2^2 = 2E(u) - 2E(|u(t)|) \le 2E(u),$$

which implies

$$(\lambda(t) + \tilde{\lambda}(t))(\lambda(t) - \tilde{\lambda}(t)) = O(1).$$

Since $0 \le \tilde{\lambda}(t) \le \lambda(t)$,

$$\lambda(t) - \tilde{\lambda}(t) = O\left(\frac{1}{\lambda(t)}\right),$$

and we have

(5) $$|u(t,x)| = \tilde{\lambda}(t)(Q + \tilde{R}(t))(\tilde{\lambda}(t)(x - x(t))),$$

with $\tilde{R}(t)$ a real function such that

$$\|\tilde{R}(t)\|_{\mathbb{H}^1(\mathbb{R}^2)} \xrightarrow[t\to T]{} 0.$$

Proposition 2.2. *The remainder term $\tilde{R}$ has the decay*

(6) $$\|\tilde{R}(t)\|_{\mathbb{H}^1} \le \frac{\tilde{C}}{\tilde{\lambda}(t)} \le \frac{C}{\lambda(t)}.$$

The proof follows Merle's one in [18]. However, for the sake of completness, we give in the Appendix a proof by a slightly different method.

Finally, let us give the following property of decay of the solution.

Lemma 2.3. *Let u be a critical mass solution of (S_Ω), blowing up at the finite time T, at one point $x_0 \in \Omega$, which means that the concentration parameter $x(t)$ converges to x_0. Then, the gradient of $u(t)$ restricted outside any neighborhood V of x_0 satisfies*

$$\sup_{t \in [0,T[} \int_{c V} |\nabla u(t)|^2 dx < \infty.$$

Proof. The inequality (4) implies

$$\sup_{t \in [0,T[} \int_{c V} |\nabla u(t)|^2 dx \leq 2E(u) + \sup_{t \in [0,T[} \int_{c V} \left|\nabla |u(t)|\right|^2 dx.$$

By using (5),

$$\int_{c V} \left|\nabla |u(t)|\right|^2 dx = \tilde{\lambda}^2(t) \int_{c \tilde{\lambda}(t)(V - x(t))} |\nabla Q + \nabla \tilde{R}(t)|^2.$$

Since $x(t)$ converges to x_0 and Q is exponentially decreasing,

$$\tilde{\lambda}^2(t) \int_{c \tilde{\lambda}(t)(V - x(t))} |\nabla Q|^2 = o(1).$$

Then it follows that

$$\int_{c V} \left|\nabla |u(t)|\right|^2 dx \lesssim \tilde{\lambda}^2(t) \int_{c \tilde{\lambda}(t)(V - x(t))} |\nabla \tilde{R}|^2,$$

and the decay (6) of R implies

$$\int_{c V} \left|\nabla |u(t)|\right|^2 dx = O(1),$$

so the lemma is proved. $\square$

Remark 2.4. *Another proof of this lemma can be done by using the approach of Merle in [17]. However, we shall need the full strength of Proposition 2.2 later in §3.3 and §3.4.*

2.3. **Derivatives of virial type functions.** Let u be a solution of (S_Ω) and let h be a $\mathcal{C}^\infty(\mathbb{R}^2)$ function with bounded first and second derivatives. Then, by using the fact that u satisfies (S_Ω), we obtain

$$\partial_t \int_\Omega |u(t)|^2 h dx = 2 \int_\Omega \Re\left(u(t)\overline{u}_t(t)\right) h dx = 2 \int_\Omega \Im\left(u(t)\Delta\overline{u}(t)\right) h dx.$$

Since u cancels on the boundary of Ω, by integration by parts

$$(7) \qquad \partial_t \int_\Omega |u(t)|^2 h dx = -2 \int_\Omega \Im\left(u(t)\nabla\overline{u}(t)\right) \nabla h dx.$$

By using again the equation (S_Ω)

$$\partial_t^2 \int_\Omega |u|^2 h = -2 \int_\Omega \Re\left((\Delta u + |u|^2 u)\nabla\overline{u}\right) \nabla h + 2 \int_\Omega \Re\left(u\nabla(\Delta\overline{u} + |u|^2\overline{u})\right) \nabla h$$

$$= \int_\Omega -|u|^2 \Delta^2 h - |u|^4 \Delta h + 2|\nabla u|^2 \Delta h - 4\Re\left(\Delta u \nabla \overline{u}\right) \nabla h.$$

It follows that

$$\partial_t^2 \int_\Omega |u|^2 h = \int_\Omega -|u|^2 \Delta^2 h - |u|^4 \Delta h + 4\Re \sum_{i,j} \partial_i u \partial_j \overline{u} \partial_{ij} h - 2\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 \frac{\partial h}{\partial \nu} d\sigma.$$

Therefore, by making the energy of the solution appear, we have the following identity.

Lemma 2.5. *For a solution u of (S_Ω) and a $\mathcal{C}^\infty(\mathbb{R}^2)$ function h with bounded derivatives $\partial_{ij} h$ and $\Delta^2 h$, we have*

$$\partial_t^2 \int_\Omega |u|^2 h = 16E(u) - \int_\Omega (2|\nabla u|^2 - |u|^4)(4 - \Delta h) - \int_\Omega |u|^2 \Delta^2 h$$

$$+ \int_\Omega \left(4\Re \sum_{i,j} \partial_i u \partial_j \overline{u} \partial_{ij} h - 2|\nabla u|^2 \Delta h\right) - 2\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 \frac{\partial h}{\partial \nu} d\sigma.$$

Corollary 2.6. *For a solution u of (S_Ω) and a $\mathcal{C}^\infty(\mathbb{R}^2)$ function h equal to $|x|^2$ on $B(0, R)$, with bounded derivatives $\partial_{ij} h$ and $\Delta^2 h$, we have the estimate*

$$\left|\partial_t^2 \int_\Omega |u(t)|^2 h dx - 16E(u)\right| \leq C \int_{\{|x|\geq R\}\cap\Omega} (|u(t)|^2 + |\nabla u(t)|^2) dx$$

$$+ \int_{\partial\Omega} \left|\frac{\partial u(t)}{\partial \nu}\right|^2 \left|\frac{\partial h}{\partial \nu}\right| d\sigma.$$

3. The blow-up rate on bounded plane domains

3.1. The convergence of the concentration points $x(t)$.

Lemma 3.1. *Let Ω be a bounded domain and let u be a critical mass solution of (S_Ω), blowing up at the finite time T. Then the concentration parameter $x(t)$ has a limit at the time T.*

Proof. From (3) it follows that for a test function ψ,

$$\int_{\Omega - x(t)} |u(t, x + x(t))|^2 \psi(x) dx \xrightarrow[t\to T]{} \|Q\|_2^2 \psi(0).$$

If ψ is chosen such that $\psi(0) \neq 0$ then, since the set Ω is bounded, it follows that

$$(8) \qquad\qquad \limsup_{t\to T} |x(t)| < \infty.$$

The first order momentum

$$f(t) = \int_\Omega |u(t, x)|^2 x dx,$$

stays finite in time since Ω is bounded and u conserves its mass. By using the formula (7) for vector-valued functions h, one can calculate the derivative

$$f'(t) = -2 \int_\Omega \Im(u(t)\nabla\overline{u}(t))dx.$$

The inequality $(*)$ in the special case $\theta_i(x) = x_i$ implies that this derivative is bounded in time

$$|f'(t)|^2 \le 4 \sum_{i\in\{1,2\}} \left|\int_\Omega \Im(u(t)\nabla\overline{u}(t))\nabla\theta_i dx\right|^2 \le 16E(u)\|u\|_2^2.$$

Therefore f admits a limit at the time T. Let us define x_0 by

$$f(T) = x_0\|Q\|_2^2.$$

Using the convergence (3) and (8) which implies that $\Omega - x(t)$ is a uniformly bounded set, one has

$$f(t) - x(t)\|Q\|_2^2 = \int_{\Omega-x(t)} |u(t, x + x(t))|^2 x dx \xrightarrow[t\to T]{} 0.$$

Therefore the point x_0 is the limit of $x(t)$, and the square of the solution behaves like a Dirac function

(9) $$|u(t,\cdot)|^2 \xrightarrow[t\to T]{} \|Q\|_2^2 \delta_{x_0}.$$

$\square$

In the following, we shall suppose, up to a translation, that the solution blows up at the point $0 \in \overline{\Omega}$.

3.2. **Lower bound for the blow-up rate.** The derivative in time of the the virial of the solution u,

$$g(t) = \int_\Omega |u(t,x)|^2 |x|^2 dx,$$

can be calculated with the formula (7) with $h(x) = |x|^2$, and

$$g'(t) = -4 \int_\Omega \Im(u(t)\nabla\overline{u}(t))x dx.$$

Therefore the inequality $(*)$ in the case $\theta(x) = |x|^2$ implies that

$$|g'(t)| \le 4\sqrt{2E(u)g(t)}.$$

The concentration result (9) of the former subsection gives

$$g(T) = 0,$$

and one can now write

$$\sqrt{g(t)} = -\int_t^T \frac{g'(\tau)}{2\sqrt{g(\tau)}}d\tau \le \int_t^T 2\sqrt{2E(u)} = 2\sqrt{2E(u)}(T - t),$$

and obtain

$$g(t) \le 8E(u)(T - t)^2.$$

Then the uncertainty principle

$$\left(\int_{\mathbb{R}^2} |u|^2\right)^2 \leq \left(\int_{\mathbb{R}^2} |u|^2 |x|^2\right)\left(\int_{\mathbb{R}^2} |\nabla u|^2\right)$$

gives us a lower bound of the blow-up rate

$$\frac{\|Q\|_2^2}{2\sqrt{2E(u)}(T-t)} \leq \|\nabla u(t)\|_2,$$

so the first assertion of Theorem 1.3 is proved.

3.3. **Equivalence between the virial and the blow-up rate.** By using (5),

$$\int_\Omega |u(t)|^2 |x - x(t)|^2 dx = \frac{1}{\tilde{\lambda}^2(t)} \int_{\tilde{\lambda}(t)(\Omega - x(t))} (Q + \tilde{R}(t))^2 |x|^2 dx.$$

Since $x(t)$ tends to 0 and Q is exponentially decreasing,

$$\frac{1}{\tilde{\lambda}^2} \int_{\tilde{\lambda}(t)(\Omega - x(t))} Q^2 |x|^2 dx = O\left(\frac{1}{\tilde{\lambda}^2(t)}\right),$$

so

$$\int_\Omega |u(t)|^2 |x - x(t)|^2 dx \lesssim \frac{1}{\tilde{\lambda}^2(t)} \int_{\tilde{\lambda}(t)(\Omega - x(t))} \tilde{R}^2(t) |x|^2 dx + \frac{1}{\tilde{\lambda}^2(t)}.$$

The domain Ω is considered bounded, so one can write

$$\int_\Omega |u(t)|^2 |x - x(t)|^2 dx \lesssim \int \tilde{R}^2(t) dx + \frac{1}{\tilde{\lambda}^2(t)},$$

and by using the decay (6) of $\tilde{R}$, we obtain

$$\int_\Omega |u(t)|^2 |x - x(t)|^2 dx \lesssim \frac{1}{\lambda^2(t)}.$$

As we did in the previous subsection, by the uncertainy principle for $u(t, x + x(t))$,

$$\|u\|_2^4 \lesssim \lambda^2(t) \int |u(t)|^2 |x - x(t)|^2 dx,$$

and so the first assertion of Proposition 1.4 follows,

$$\int_\Omega |u(t)|^2 |x - x(t)|^2 dx \approx \frac{1}{\lambda^2(t)}.$$

3.4. **A differentiable choice for** $x(t)$**.** Let us set

$$y(t) = \frac{\int |u(t)|^2 x dx}{\|Q\|^2}.$$

By using the conservation of the mass, which is critical,

$$x(t) - y(t) = \frac{1}{\|Q\|_2^2} \int |u(t)|^2 (x - x(t)) dx.$$

Then by (5) one has

$$x(t) - y(t) = \frac{1}{\tilde{\lambda}(t)\|Q\|_2^2} \int_{\tilde{\lambda}(t)(\Omega - x(t))} (Q + R(t))^2 x dx.$$

Therefore, by the same arguments as in the previous subsection, and the by using the fact that since Q is radially symmetric,

$$\int Q^2(x) x dx = 0,$$

then

$$|x(t) - y(t)| \le \frac{C}{\tilde{\lambda}^2(t)}.$$

If we define S by

$$|u(t,x)| = \tilde{\lambda}(Q + \tilde{R}(t))(\tilde{\lambda}(x - x(t))) = \tilde{\lambda}(Q + S(t))(\tilde{\lambda}(x - y(t))),$$

one has

$$\|S(t)\|_{\mathbb{H}^1} \le 2\|\tilde{R}(t)\|_{\mathbb{H}^1} + \|Q(\cdot + \lambda(x(t) - y(t)) - Q(\cdot)\|_{\mathbb{H}^1}.$$

The decay of the difference between $x(t)$ and $y(t)$, together with (6), implies

$$\|S(t)\|_{\mathbb{H}^1} \le \frac{C}{\tilde{\lambda}(t)}.$$

So, by changing $x(t)$ into

$$\frac{\int |u(t)|^2 x dx}{\|Q\|^2},$$

we have the convergence corresponding to (5)

$$|u(t,x)| = \tilde{\lambda}(t)(Q + S(t))(\tilde{\lambda}(t)(x - y(t))),$$

with S decreasing in $\mathbb{H}^1$ as does R, and so the second assertion of Proposition 1.4 follows.

The interest of this choice of the concentration parameter is that $y(t)$ is a differentiable function, and, moreover, in the radial case we obtain Corollary 1.5.

4. The blow-up rate on unbounded plane domains

Consider now the equation (S_Ω) on an unbounded domain of the plane or on a surface. Let u be a critical mass solution that blows up at an interior point x_0 of Ω, that is

$$x(t) \xrightarrow[t \to T]{} x_0.$$

Modulo a translation, we can suppose that x_0 is zero and so,

$$|u(t,x)|^2 \xrightarrow[t \to T]{} \|Q\|_2^2 \, \delta_0.$$

Let ϕ be a $\mathcal{C}_0^\infty$ function, equal to 1 on $B(0,R)$. Let us introduce the localized virial of the solution

$$g_\phi(t) = \int |u(t,x)|^2 \phi^2(x)|x|^2 dx.$$

Then, using (7) with $h(x) = \phi^2(x)|x|^2$, one has

$$g_\phi'(t) = -2 \int \Im(u(t)\nabla\overline{u}(t))\nabla(\phi^2|x|^2)dx.$$

The inequality $(*)$ with $\theta(x) = \phi^2(x)|x|^2$ gives us

$$|g_\phi'(t)|^2 \leq 8E(u) \int |u|^2 |\nabla(\phi^2|x|^2)|^2 dx$$

Since $\nabla(\phi^2|x|^2)$ is a $\mathcal{C}_o^\infty(\mathbb{R}^2)$ function cancelling at 0, and since the square of $|u|$ behaves like a Dirac distribution, it follows that

$$g_\phi'(T) = 0.$$

Then, as in the former section, and using the existence of a positive constant C such that

$$|\nabla(\phi^2|x|^2)|^2 \leq C\psi^2|x|^2,$$

one has

$$g_\phi(t) \lesssim (T-t)^2.$$

The uncertainty principle reads

$$\left(\int |u|^2\phi^2 dx\right)^2 \leq \left(\int |u|^2\phi^2|x|^2 dx\right)\left(\int |\nabla(u\phi)|^2 dx\right).$$

By integrating by parts the last term and by using the fact that ϕ is equal to 1 on $B(0,R)$, it follows that

$$\left(\int_{B(0,R)} |u(t)|^2\right)^2 \leq g_\phi(t)\left(\int |\nabla u|^2\phi^2 dx - \int |u|^2\phi\Delta\phi dx\right).$$

Since ϕ is a $\mathcal{C}_0^\infty$ function,

$$\left(\int_{B(0,R)} |u(t)|^2\right)^2 \leq g_\phi(t)\left(C\int |\nabla u|^2 dx - \int |u|^2\phi\Delta\phi dx\right).$$

On the one hand the $\mathbb{L}^2$ norm of u is conserved. On the other hand, the behavior of $|u|^2$ as a Dirac distribution implies that the norm of its restriction outside a neighborhood of zero tends to 0 in time. So we have

$$\begin{cases} \int_{B(0,R)} |u(t)|^2 = O(1), \\[2mm] \int |u(t)|^2 \phi \Delta\phi \, dx = o(1), \end{cases}$$

and since g_ϕ is bounded in time,

$$1 \lesssim \sqrt{g_\phi(t)} \|\nabla u(t)\|_2$$

Then the decay of g_ϕ gives us the lower bound of the blow-up speed

$$\frac{1}{T-t} \lesssim \|\nabla u(t)\|_2.$$

5. Blow-up on the boundary

5.1. **Necessary condition for blow-up on the boundary.** Let us first introduce a notion of limit of sets, as in [7].

Definition 5.1. *A sequence of open sets M_m is said to tend to an open set M of $\mathbb{R}^2$ if the following conditions are satisfied.*
i) For all compact $K \subset M$, there exists $n_K \in \mathbb{N}$, such that for all $n \geq n_K$, $K \subset M_n$.
ii) For all compact $K \subset^c \overline{M}$, there exists $n_K \in \mathbb{N}$, such that for all $n \geq n_K$, $K \subset^c \overline{M_n}$.

Let us suppose that there exists an explosive solution u of the equation (S_Ω) at $0 \in \partial\Omega$. The convergence (2) implies that

$$\lambda(t)(\Omega - x(t)) \xrightarrow[t \to T]{} \mathbb{R}^2.$$

As in [7], the limit set depends on the position of $x(t)$ with respect to the boundary of Ω. If there is a positive number C such that for all t

$$\lambda(t) d(x(t), \partial\Omega) \leq C,$$

then $\lambda(t)(\Omega - x(t))$ tends to a half-plane and blow-up cannot occur. Also, if

$$\lambda(t) d(x(t), \partial\Omega) \xrightarrow[t \to T]{} \infty,$$

and $x(t)$ is not in Ω, then, by Definition 5.1, $\lambda(t)(\Omega - x(t))$ tends to the empty set. Therefore the only possibility to have explosion on the boundary is that $x(t) \in \Omega$ and

$$\lambda(t) d(x(t), \partial\Omega) \xrightarrow[t \to T]{} \infty.$$

In particular, since 0 is on the boundary,

(10) $$|\lambda(t) x(t)| \xrightarrow[t \to T]{} \infty.$$

We have

$$|x(t)|^2 \int_{B(x(t), \frac{C}{\lambda(t)})} |u|^2 \leq 2 \int_{B(x(t), \frac{C}{\lambda(t)})} |u|^2 |x - x(t)|^2 + 2 \int_{B(x(t), \frac{C}{\lambda(t)})} |u|^2 |x|^2.$$

On the one hand, by using the Weinstein relation (2), one has

$$|x(t)|^2 \int_{B(x(t), \frac{C}{\lambda(t)})} |u|^2 \approx |x(t)|^2.$$

On the other hand, using again (2),

$$\int_{B(x(t), \frac{C}{\lambda(t)})} |u|^2 |x - x(t)|^2 \lesssim \frac{1}{\lambda(t)^2}.$$

In view of (10), these two facts imply

$$|x(t)|^2 \lesssim \int_{B(x(t), \frac{C}{\lambda(t)})} |u|^2 |x|^2 \lesssim g_\psi,$$

where g_ψ is the localized virial function defined in §4. In the same section it was proved that

$$g_\psi \lesssim (T - t)^2,$$

so it follows that

$$|x(t)| \lesssim T - t.$$

By using again (10),

$$\frac{1}{T - t} \ll \lambda(t),$$

and the second assertion of Theorem 1.3 is proved.

5.2. **Results of non-explosion.** From now on we assume that Ω be a half plane whose boundary contains 0 or a plane sector with corner 0. Suppose there exists an explosive solution u of critical mass such that $|u|^2$ behaves like a Dirac mass at 0 in the sense

$$|u(t, \cdot)|^2 \xrightarrow[t \to T]{} \|Q\|_2^2 \delta_0.$$

For a radial function $f \in \mathcal{C}^\infty(\mathbb{R}^2)$, the result of Lemma 2.5 becomes

$$\partial_t^2 \int_\Omega |u|^2 f = 16 E(u) - \int_\Omega (2|\nabla u|^2 - |u|^4)(4 - \Delta f) - \int_\Omega |u|^2 \Delta^2 f$$

$$(11) \qquad\qquad + \int_\Omega 4\Re \sum_{i,j} \partial_i u \partial_j \overline{u} \partial_{ij} f - 2|\nabla u|^2 \Delta f,$$

since from the choice of Ω

$$x.\nu = 0 \text{ on } \partial\Omega.$$

It follows that for a radial function $f \in \mathcal{C}^\infty(\mathbb{R}^2)$, equal to $|x|^2$ on $B(0, R)$, with bounded derivatives $\partial_{i,j} f$ and $\Delta^2 f$, the estimate of Corollary 2.6 becomes

$$(12) \qquad \left| \partial_t^2 \int_\Omega |u(t)|^2 f - 16 E(u) \right| \leq C \int_{\{|x| \geq R\} \cap \Omega} |u(t)|^2 + |\nabla u(t)|^2.$$

Arguing as in [17], we obtain the following lemmas.

Lemma 5.2. *The initial condition is of finite variance*

$$\int_\Omega |u_0|^2 |x|^2 dx < \infty.$$

Proof. Let us consider ψ a $\mathcal{C}_0^\infty(\mathbb{R})$ positive radial function which is equal to $|x|^2$ on $B(0,1)$. Notice that

$$|\nabla\psi|^2 \le C\psi.$$

For all entire n, we introduce the localized virial functions

$$g_n(t) = \int_\Omega |u(t)|^2 \psi_n dx,$$

where

$$\psi_n(x) = n^2 \psi\left(\frac{x}{n}\right).$$

The Taylor formula in zero for the function $g_n(t)$ gives us

$$|g_n(t) - g_n(0)| \le t|g_n'(0)| + C \sup_t |g_n''(t)|.$$

Since ψ_n are equal to $|x|^2$ on $B(0,1)$, and the derivatives $\partial_{ij}\psi_n$ and $\Delta^2\psi_n$ are uniformly bounded, we can estimate by (12)

$$|g_n''(t) - 16E(u)| \le C \int_{|x|>1} (|u(t)|^2 + |\nabla u(t)|^2) dx.$$

Then, in view of Lemma 2.3, the quantity $g_n''(t)$ is bounded uniformly on n. So we have

$$|g_n(t) - g_n(0)| \le T|g_n'(0)| + C.$$

By using the inequality $(*)$,

$$|g_n'(t)| = \left|\int_\Omega \Im(u(t)\nabla\overline{u}(t))\nabla\psi_n\right| \le \left(2E(u)\int_\Omega |u(t)|^2 |\nabla\psi_n|^2\right)^{\frac{1}{2}}.$$

The choice of ψ_n gives us

$$|\nabla\psi_n|^2 \le \psi_n,$$

and it follows that

$$|g_n'(t)| \le C \left(\int_\Omega |u(t)|^2 \psi_n\right)^{\frac{1}{2}} = C\sqrt{g_n(t)}.$$

Therefore

$$g_n(0) - 2C\sqrt{g_n(0)} - C \le g_n(t)$$

for any time t, and

$$g_n(0) - 2C\sqrt{g_n(0)} - C \le \lim_{t\to T} g_n(t).$$

The concentration of the solution as a Dirac distribution implies that for fixed n

$$\lim_{t\to T} g_n(t) = 0,$$

and therefore

$$\lim_{n\to\infty} (g_n(0) - 2C\sqrt{g_n(0)} - C) \le 0.$$

As a consequence, $g_n(0)$ is bounded as n tends to infinity. Since the supports of ψ_n cover Ω when n tends to infinity, it follows that the initial condition is of finite variance

$$\int_\Omega |u_0|^2 |x|^2 dx < \infty.$$

$\square$

Remark 5.3. *When Ω is a bounded domain, and u is a critical mass function blowing up at a point of Ω or of its boundary, it is easy to see that the initial condition is of finite variance*

$$\int_\Omega |u|^2 |x|^2 dx \le C\|u\|_2^2 < \infty.$$

Lemma 5.4. *The limit in time of the virial function is*

$$g(T) = 0.$$

Proof. Let us consider a $\mathcal{C}^\infty$ positive function ϕ which is null on $B(0,1)$ and verifies

$$\frac{|x|}{2} \le \phi(x) \le |x|,$$

on $^cB(0,2)$. Suppose also that the derivatives $\partial_{ij}\psi$ and $\Delta^2\psi$ are bounded. We denote

$$\phi_n(x) = n\,\phi\left(\frac{x}{n}\right),$$

so ϕ_n are supported on $^cB(0,n)$ and verify

$$\frac{|x|}{2} \le \phi_n(x) \le |x|$$

on $^cB(0,2n)$.

Taylor's formula together with (7) and the estimate (12) gives us

$$\int |u(t)|^2 \phi_n \le \int_{|x|>n} |u_0|^2 |x|^2 + T \left| \int_{|x|>n} \Im(u_0 \nabla \overline{u_0}) \nabla \phi_n^2 \right|$$

$$(13) \qquad\qquad + C(T-t)^2 + C \int_0^T (T-\tau) \int_{|x|>n} (|\nabla u(\tau)|^2 + |u(\tau)|^2) d\tau.$$

The Lemma 5.2 ensures us that the initial data is of finite variance, therefore

$$\int_{|x|>n} |u_0|^2 |x|^2 \xrightarrow[n\to\infty]{} 0,$$

Also, by Lemma 2.3,

$$\left| \int_{|x|>n} \Im(u_0 \nabla \overline{u_0}) \nabla \phi_n^2 \right| \xrightarrow[n\to\infty]{} 0.$$

Then, using again Lemma 2.3 and the conservation of the mass, for all τ and for all n there exist a positive constant C such that

$$\int_{|x|>n} (|\nabla u(\tau)|^2 + |u(\tau)|^2) \le C.$$

One also has, for every τ,

$$\int_{|x|>n} (|\nabla u(\tau)|^2 + |u(\tau)|^2) \xrightarrow[n\to\infty]{} 0.$$

Then by the dominated convergence theorem

$$\int_0^T (T-\tau) \int_{|x|>n} (|\nabla u(\tau)|^2 + |u(\tau)|^2) d\tau \xrightarrow[n\to\infty]{} 0.$$

Therefore it follows from (13) that for all t,

$$\int |u(t)|^2 \phi_n^2 \le \epsilon(n) + C(T-t)^2,$$

with

$$\epsilon(n) \xrightarrow[n\to\infty]{} 0.$$

On the one hand, in view of the choice of ϕ_n, this gives us

$$\int_{|x|>2n} |u(t)|^2 |x|^2 \le 2\epsilon(n) + C(T-t)^2.$$

On the other hand, for fixed n, the concentration of the solution as a Dirac distribution implies

$$\lim_{t\to T} \int_{|x|<2n} |u(t)|^2 |x|^2 = 0.$$

Therefore, for all n

$$\lim_{t\to T} \int |u(t)|^2 |x|^2 \le \epsilon(n).$$

By letting n to tend to infinity one has

$$\lim_{t\to T} \int |u(t)|^2 |x|^2 = 0,$$

that is

$$g(T) = 0,$$

and the Lemma 5.4 is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

This lemma and the same arguments as in §3.2 give us also

$$g'(T) = 0.$$

By using the formula (11) with $f(x) = |x|^2$, the second derivative of the virial is exactly

$$g''(t) = 16E(u).$$

Then it follows that

$$g(t) = 8E(u)(T-t)^2,$$

and by the same calculation as in §2.1

$$E(e^{i\frac{|x|^2}{4(T-t)}} u(t,x)) = E(u) + \frac{1}{4(T-t)} g'(t) + \frac{1}{16(T-t)^2} g(t) = 0.$$

For fixed t, by the variational characterization of the ground state Q, there exists real numbers θ and ω such that

$$u(t,x) = e^{-i\frac{|x|^2}{4(T-t)}} e^{i\theta} \omega \, Q(\omega(x-x_0))$$

for some $x_0 \in \mathbb{R}^2$ ([4]). This means that the support of u is the entire $\mathbb{R}^2$ that is a contradiction, and the proof of Theorem 1.7 is complete.

6. Appendix

In this Appendix we give a proof for Proposition 2.2. Let us recall the notations of §2.2. We have defined

$$\tilde{\lambda}(t) = \frac{\||\nabla|u(t)|\|_2}{\|\nabla Q\|_2},$$

and the solution u was written (5)

$$|u(t,x)| = \tilde{\lambda}(t)(Q + \tilde{R}(t))(\tilde{\lambda}(t)(x - x(t))),$$

with $\tilde{R}(t)$ a real function such that

$$\|\tilde{R}(t)\|_{\mathbb{H}^1(\mathbb{R}^2)} \xrightarrow[t \to T]{} 0.$$

We shall prove in the following the decay (6) asserted in Proposition 2.2

$$\|\tilde{R}(t)\|_{\mathbb{H}^1} \leq \frac{\tilde{C}}{\tilde{\lambda}(t)} \leq \frac{C}{\lambda(t)}.$$

The fact that u is of critical mass gives us

(14)
$$\int \tilde{R}^2 = -2 \int Q\tilde{R},$$

and the choice of $\tilde{\lambda}$ implies

$$\int |\nabla \tilde{R}|^2 = -2 \int \nabla Q \nabla \tilde{R}.$$

Let us calculate the energy of $|u|$,

$$\frac{2E(|u|)}{\tilde{\lambda}^2} = \int |\nabla Q + \nabla \tilde{R}|^2 - \frac{1}{2} \int (Q + \tilde{R})^4.$$

The energy of Q is zero, so

$$\frac{2E(|u|)}{\tilde{\lambda}^2} = \int |\nabla \tilde{R}|^2 + 2\nabla Q \nabla \tilde{R} - \frac{\tilde{R}^4}{2} - 2Q\tilde{R}^3 - 3Q^2\tilde{R}^2 - 2Q^3\tilde{R}.$$

The ground state Q verifies the equation

$$\Delta Q + Q^3 = Q,$$

and therefore, by using the relation (14) on $\tilde{R}$,

$$\int 2\nabla Q \nabla \tilde{R} - 2Q^3\tilde{R} = -2 \int Q\tilde{R} = \int \tilde{R}^2.$$

So finally

$$< L\tilde{R}, \tilde{R} > = \frac{2E(|u|)}{\tilde{\lambda}^2} + \frac{1}{2} \int \tilde{R}^4 + \int 2Q\tilde{R}^3,$$

where L is the operator

$$L = -\Delta + (1 - 3Q^2).$$

Since $\tilde{R}$ tends to 0 in $\mathbb{H}^1$, by using the Sobolev embeddings, the cubic and quadratic terms in $\tilde{R}$ are negligible with respect to the $\mathbb{H}^1$ norm of $\tilde{R}$. Also, the energy of $|u|$ is bounded by the constant energy of u, so for having (6) it is sufficient to prove the existence of a positive constant δ such that for t close enough to T

$$\delta\|\tilde{R}(t)\|_{\mathbb{H}^1}^2 \leq < L\tilde{R}(t), \tilde{R}(t) > .$$

Remark 6.1. *The initial complex function R can be analyzed in the same manner, and one has*

$$< L_-\Im R, \Im R > + < L\Re R, \Re R > \leq \frac{2E(u)}{\lambda^2} + \frac{1}{2} \int |R|^4 + \int 2Q|R|^3,$$

where L_- is the operator

$$L_- = -\Delta + (1 - Q).$$

This operator is non-negative and its kernel is spanned by Q. So once the decay (6) is obtained, by decomposing $\Im R$ with respect to Q, we also have

$$\|R(t)\|_{\mathbb{H}^1} \leq \frac{C}{\lambda(t)}.$$

Following the ideas of Weinstein in [29], we shall look for the nature of the negative eigenvalues of L.

Lemma 6.2. *The second eigenvalue of L is 0.*

Proof. Let us consider the functional

$$J(f) = \frac{\|u\|_2^2 \|\nabla u\|_2^2}{\|u\|_4^4},$$

which is minimized by Q (see the introduction). Then, for a test function f,

$$\partial_\epsilon^2 J(Q + \epsilon f)|_{\epsilon=0} \geq 0.$$

By explicitly calculating this second derivative and using (1) in the calculus, one has

$$2\|Q\|_2^2 < Lf, f > \geq -8 < Q, f > < \nabla Q, \nabla f > .$$

If we take f to be orthogonal to Q, then

$$< Lf, f > \geq 0,$$

and by the Min-Max Principle ([24]), the second eigenvalue of L is non-negative. By noticing that the two partial derivatives of Q verify

$$L\partial_i Q = 0,$$

we obtain that 0 is an eigenvalue of L of order grater than one, so the first eigenvalue is negative. Therefore the second eigenvalue of L is 0. $\qquad\square$

We shall use the following theorem.

Theorem 6.3. *(Maris [16]). Let $g \in \mathcal{C}^1([0, \infty))$, with $g(0) = 0$, $g'(0) > 0$ and $|g'(s) - g'(0)| \le C|s|^\alpha$, for small s and some $C, \alpha > 0$. Let $a_0 = \sup\{a > 0 | g(s) > 0, \forall s \in (0, a)\}$, and let u_0 be a ground state of the operator*

$$-\Delta u + g(u).$$

We define

$$I(u, \lambda) = \lambda u g'(u) - (\lambda + 2)g(u),$$

and we will make the following assumptions : $a_0 < u_0(0)$ and there exists a continuous function $\lambda : (a_0, u_0(0)] \rightarrow (0, \infty)$ such that for any $U \in (a_0, u_0(0)]$ we have

$$\begin{cases} I(u, \lambda(U)) \le 0, \forall u \in [0, U], \\ I(u, \lambda(U)) \ge 0, \forall u \in [U, u_0(0)]. \end{cases}$$

Then

$$Ker(-\Delta + g'(u_0)) = \{\partial_1 u_0, \partial_2 u_0\}.$$

Next we show that the operator L satisfies the hypothesis of the theorem.

Lemma 6.4. *The kernel of L has dimension 2.*

Proof. In we take function g to be

$$g(s) = s - s^3,$$

then $a_0 = 1$, the ground state u_0 is Q,

$$-\Delta + g'(u_0) = L,$$

and

$$I(u, \lambda) = 2u((1 - \lambda)u^2 - 1).$$

Let us consider the integral of g,

$$G(s) = \frac{s^2}{2} - \frac{s^4}{4}.$$

By using the relation (1) between the $\mathbb{L}^2$ and the $\mathbb{L}^4$ norms of Q

$$\int G(Q(x))dx = 0.$$

The positivity of $G(s)$ on $[0, \sqrt{2}[$ implies the existence of points x such that $Q(x) > \sqrt{2}$, and in particular $Q(x) > 1$. Let us recall that Q is a radial positive decreasing function. It follows that $Q(0) > 1$, and the first assumption of the theorem 6.3 is satisfied. The second assumption is satisfied for the function

$$\lambda(U) = 1 - \frac{1}{U^2},$$

and we can conclude that

$$KerL = \{\partial_1 Q, \partial_2 Q\}.$$

$\square$

We return now to the study of $\tilde{R}$. We impose a choice of $x(t)$ which will yield an orthogonality property of $\tilde{R}$. Since

$$\frac{1}{\tilde{\lambda}(t)}|u|\left(t, \frac{x}{\tilde{\lambda}(t)} + x(t)\right) \xrightarrow[t \to T]{} Q(x),$$

we can choose $x(t)$ such that the functional

$$I(z) = \left\|\frac{1}{\tilde{\lambda}(t)}|u|\left(t, \frac{\cdot + z}{\tilde{\lambda}(t)} + x(t)\right) - Q(\cdot)\right\|_{\mathbb{H}^1}^2$$

reaches its minimum for $z = 0$. By using (5), this implies that the derivative in z of

$$\partial_z\|(Q + \tilde{R}(t))(\cdot) - Q(\cdot - z)\|_{\mathbb{H}^1}^2,$$

must be zero at $z = 0$. It follows that

$$\int \tilde{R}\,\partial_i Q + \int \nabla\tilde{R}\,\partial_i\nabla Q = 0.$$

One can then integrate by parts and obtain

$$\int \tilde{R}\,\partial_i Q - \int \tilde{R}\,\partial_i\Delta Q = 0.$$

By recalling that the ground state Q verifies

$$\Delta Q + Q^3 = Q,$$

it follows that $\tilde{R}$ has the orthogonality property

(15) $< \partial_i Q^3, \tilde{R} >= 0.$

Let us recall that for having the decay property (6) of $\tilde{R}$, it is sufficient to prove that the operator L controls its $\mathbb{H}^1$ norm.

Lemma 6.5. *There exist a positive constant δ such that for t close enough to T,*

$$\delta\|\tilde{R}(t)\|_{\mathbb{H}^1}^2 \leq< L\tilde{R}(t), \tilde{R}(t) > .$$

Proof. We denote by $R_{\shortparallel}$ the projection of $\tilde{R}$ on the space spanned by Q, and by $R_{\perp}$ the remainder term, orthogonal to Q. Since the operator L is self-adjoint,

$$< L\tilde{R}, \tilde{R} >=< LR_{\shortparallel}, R_{\shortparallel} > +2 < LR_{\shortparallel}, R_{\perp} > + < LR_{\perp}, R_{\perp} > .$$

The first term reads

$$< LR_{\shortparallel}, R_{\shortparallel} >=< LQ, Q > \frac{< Q, \tilde{R} >^2}{\|Q\|_2^4},$$

and by using (14)

$$< LR_{\shortparallel}, R_{\shortparallel} >= C\|\tilde{R}\|_2^4.$$

The second term is

$$< LR_{\shortparallel}, R_{\perp} >= \frac{< Q, \tilde{R} >}{\|Q\|_2^2} < LQ, R_{\perp} >,$$

and since $LQ = -2Q^3$, by using the Cauchy-Schwarz inequality,

$$< LR_\perp, R_{\shortparallel} >= -2\frac{< Q, \tilde{R} >}{\|Q\|_2^2} < Q^3, R_\perp >\le C\|\tilde{R}\|_2^3.$$

Now we have to estimate the third term. Let us notice that the orthogonality relation (15) yields

$$< \partial_i Q^3, R_\perp >= 0.$$

We will show that

$$\inf_{f \in \perp\{Q, \partial_i Q^3\}} \frac{< Lf, f >}{\|f\|_2^2} = I > 0.$$

From the proof of Lemma 6.2 we have $I \ge 0$. Consider now a sequence of functions f_j, normalized in $\mathbb{L}^2$, which minimize I

$$< Lf_j, f_j >\underset{j\to\infty}{\longrightarrow} I.$$

The gradients of f_j are also bounded in $\mathbb{L}^2$, so we can extract a subsequence converging weakly in $\mathbb{H}^1$ to a function f

$$f_{j_n} \rightharpoonup f.$$

In particular,

$$< f_{j_n}^2, Q^2 >\underset{n\to\infty}{\longrightarrow}< f^2, Q^2 >,$$

and it follows that f is a minimizer for I,

$$Lf = If.$$

If $I = 0$, then f must be in the kernel of L. Lemma 6.4 ensures us that the kernel contains only the derivatives of Q, and since f is orthogonal to the derivatives of Q^3, it follows that $f = 0$. This is in contradiction with the positive $\mathbb{L}^2$ norm of f, so $I > 0$.

Therefore, since $R_\perp$ is orthogonal to Q and to the two derivatives of Q^3,

$$< LR_\perp, R_\perp >\ge I\|R_\perp\|_2^2 = I(\|\tilde{R}\|_2^2 - \|R_{\shortparallel}\|_2^2).$$

Arguing as for the first term,

$$\|R_{\shortparallel}\|_2^2 \le C\|\tilde{R}\|_2^4,$$

and we finally have

$$< L\tilde{R}, \tilde{R} >\ge I\|\tilde{R}\|_2^2 - C\|\tilde{R}\|_2^4 - C\|\tilde{R}\|_2^3.$$

Since $\tilde{R}$ tends to 0 in $\mathbb{L}^2$ norm, there exist a positive constant C such that for t close enough to T,

$$< L\tilde{R}, \tilde{R} >\ge C\|\tilde{R}\|_2^2.$$

For a positive number ϵ,

$$< L\tilde{R}, \tilde{R} >= \epsilon \left(\int |\nabla\tilde{R}|^2 + \int (1 - 3Q^2)\tilde{R}^2 \right) + (1 - \epsilon) < L\tilde{R}, \tilde{R} >,$$

so, using the control of the $\mathbb{L}^2$ norm by L and the boundeness of Q,

$$< L\tilde{R}, \tilde{R} >\ge \epsilon \left(\int |\nabla\tilde{R}|^2 - C_Q \int \tilde{R}^2 \right) + (1 - \epsilon)C\|\tilde{R}\|_2^2.$$

By choosing ϵ small enough to have

$$(1 - \epsilon)C - \epsilon\, C_Q > 0,$$

we get the existence of a positive constant δ such that

$$< L\tilde{R}, \tilde{R} > \geq \delta \|\tilde{R}\|^2_{\mathbb{H}^1}.$$

$\square$

Therefore the proof of Proposition 2.2 is complete.

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V. Banica, Universita di Pisa, Dipartimento di Matematica, Via Buonarroti 2, 56127 Pisa, Italy

E-mail address: Valeria.Banica@mail.dm.unipi.it