

THE NONLINEAR SCHRÖDINGER EQUATION ON HYPERBOLIC SPACE

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ABSTRACT. In this article we study some aspects of dispersive and concentration phenomena for the Schrödinger equation posed on hyperbolic space \mathbb{H}^n , in order to see if the negative curvature of the manifold gets the dynamics more stable than in the Euclidean case. It is indeed the case for the dispersive properties : we prove that the dispersion inequality is valid, in a stronger form than the one on \mathbb{R}^n . However, the geometry does not have enough of an effect to avoid the concentration phenomena and the picture is actually worse than expected. The critical nonlinearity power for blow-up turns out to be the same as in the euclidean case, and we prove that there are more explosive solutions for critical and supercritical nonlinearities.

Keywords : Nonlinear Schrödinger equations on manifolds, hyperbolic space, representations of solutions, dispersion estimates, blow-up.

Mathematics Subject Classification : Primary 35Q55, Secondary 58JXX, 35CXX, 35BXX.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold and let Δ_M be the Laplace-Beltrami operator. The study of the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_M u = V(|u|^2)u, \\ u(0) = u_0, \end{cases}$$

where u is a space-time function with complex values, and V is a real function with controlled growth at infinity, was motivated by number of problems coming from Physics.

It is known that the geometry influences the dynamics of the equation. Instability phenomena appear, even in the defocusing case.

For instance, on the one hand, for the cubic defocusing Schrödinger equation on the sphere \mathbb{S}^2

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{S}^2} u = |u|^2 u, \\ u(0, x) \in H^s(\mathbb{S}^2) \end{cases},$$

the critical regularity index for having the uniform continuity of the flow on the bounded sets of H^s is $s = \frac{1}{4}$, as it was proved recently by Burq-Gérard-Tzvetkov ([7], [9], see also [3]).

On the other hand Bourgain ([6]) and Cazenave-Weissler ([11]) have proved that the Cauchy problem of the same equation, considered on \mathbb{T}^2 and on \mathbb{R}^2 respectively, is H^ϵ well-posed for all positive ϵ , meanwhile for negative s instability phenomena

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Research supported by HARP (HPRN-CT-2001-00273) and HYKE (HPRN-CT-2002-00282) research projects.

appear ([7],[12]). It follows that the critical regularity index for the flat torus and for \mathbb{R}^2 is zero.

Hence these results point out the importance played by the geometry of the manifold in the dynamics of the equation.

It is expected that the positive curvature generates the differences, since for having the instability result on the sphere the dynamics of spherical harmonic concentrated on closed geodesics are studied.

In this article we study dispersive and concentration phenomena for the Schrödinger equation posed on hyperbolic space \mathbb{H}^n , manifold of negative curvature, expecting that the dynamics are stabler than in the Euclidean case.

We shall define in §2 the hyperbolic space and the tools used on it. For the moment, let us introduce some notations. We denote by 0 the origin of the hyperbolic space, $0 = (1, 0, \dots, 0)$. In the sequel we shall use the L^p spaces on \mathbb{H}^n

$$L^p = L^p(\mathbb{H}^n) = L^p(d\Omega),$$

and weighted spaces, defined by

$$L^p(w) = L^p(w d\Omega),$$

where $d\Omega$ denotes the measure on hyperbolic space \mathbb{H}^n and w is a function on \mathbb{H}^n .

First, we shall treat the linear equation. We obtain an explicit representation of the solutions.

Theorem 1.1. *The solution of the linear Schrödinger equation posed on hyperbolic space \mathbb{H}^n*

$$(SL) \begin{cases} i\partial_t u + \Delta_{\mathbb{H}^n} u = 0 \\ u(0, x) = u_0 \end{cases},$$

is, for $n \geq 3$ odd,

$$(1) \quad u(t, \Omega) = \frac{c}{|t|^{\frac{1}{2}}} e^{-it\frac{(n-1)^2}{4}} \int_{\mathbb{H}^n} u_0(\Omega') \left(\frac{\partial_\rho}{\sinh \rho} \right)^{\frac{n-1}{2}} e^{i\frac{\rho^2}{4t}} d\Omega',$$

and for $n \geq 2$ even

$$(2) \quad u(t, \Omega) = \frac{c}{|t|^{\frac{3}{2}}} e^{-it\frac{(n-1)^2}{4}} \int_{\mathbb{H}^n} u_0(\Omega') \left(\frac{\partial_\rho}{\sinh \rho} \right)^{\frac{n-2}{2}} \int_\rho^\infty \frac{e^{i\frac{s^2}{4t}}}{\sqrt{\cosh s - \cosh \rho}} ds d\Omega',$$

where by ρ we denote the hyperbolic distance between the points Ω and Ω' .

As harmonic analysis can be done on \mathbb{H}^n , the proof is based on the representation of the solution via the Fourier transform and on calculus of oscillatory integrals. There are many points in common with the proof of the inverse Fourier formula on hyperbolic space.

This explicit representation of the solution allows us to study the dispersive properties and to obtain the following results.

Theorem 1.2. *i) For all dimension $n \geq 2$, the solution satisfies the following local dispersion inequality*

$$(3) \quad |u(t, \Omega)| \leq c \frac{1}{|t|^{\frac{n}{2}}} \int_{\mathbb{H}^n} |u_0(\Omega')| \left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-1}{2}} d\Omega'.$$

For large times, the following dispersion estimate holds for all $n \geq 3$ odd,

$$(4) \quad |u(t, \Omega)| \leq c \frac{1}{|t|^{\frac{3}{2}}} \int_{\mathbb{H}^n} |u_0(\Omega')| \left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-1}{2}} d\Omega',$$

and for all $n \geq 2$ even

$$(5) \quad |u(t, \Omega)| \leq c \frac{1}{|t|^{\frac{3}{2}}} \int_{\mathbb{H}^n} |u_0(\Omega')| \left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-1}{2}} \frac{1+\rho}{\sqrt{\rho}} d\Omega',$$

ii) Moreover, for $n \geq 3$ and radial initial data, we have a weighted-space local dispersion inequality

$$(6) \quad \|u(t)\|_{L^\infty(w)} \leq \frac{c}{|t|^{\frac{n}{2}}} \|u_0\|_{L^1(w^{-1})},$$

where the weight is

$$w(\Omega) = \frac{\sinh d(0, \Omega)}{d(0, \Omega)}.$$

Finally, for $n \geq 3$, for a finite time T and for radial initial data, we have the local Strichartz weighted estimates

$$(7) \quad \|u\|_{L^p([0, T], L^q(w^{q-2}))} \leq c \|u_0\|_{L^2},$$

for all pairs (p, q) satisfying $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$, $(p, q, n) \neq (2, \infty, 2)$ and $2 \leq p, q$.

The new term that appears in (3),

$$\frac{\rho}{\sinh \rho},$$

is specific to hyperbolic space ; it comes from the Harish-Chandra coefficient and from a kernel that involves the eigenfunctions of the Laplace-Beltrami operator. This new term informs us that, apart from the classical decay in time, we have a new one, in space.

Remark 1.3. In dimension 3, all the dispersion estimates from Theorem 1.2 are global in time, which is not the case for other dimensions.

Remark 1.4. For dimensions allowed in Theorem 1.2, away from the support of the initial data, the decay is stronger than in the \mathbb{R}^n case. More precisely, for initial data with support a domain A of \mathbb{H}^n , the solution satisfies, for any point Ω not included in A , at small times,

$$|u(t, \Omega)| \left(\frac{\sinh d(\Omega, A)}{d(\Omega, A)} \right)^{\frac{n-1}{2}} \leq \frac{c}{|t|^{\frac{n}{2}}} \int_{\mathbb{H}^n} |u_0(\Omega')| \left(\frac{d(\Omega', A)}{\sinh d(\Omega', A)} \right)^{\frac{n-1}{2}} d\Omega'.$$

Remark 1.5. In dimension 3, for radial initial data in a more restrictive space, that is $L^1(\tilde{w}^{-1})$ with the weight

$$\tilde{w}(\Omega) = \sinh d(\Omega, 0),$$

we obtain a local dispersion-type estimate stronger in time,

$$\|u(t)\|_{L^\infty(\tilde{w})} \leq \frac{c}{|t|^{\frac{1}{2}}} \|u_0\|_{L^1(\tilde{w}^{-1})}.$$

However, it is only in (6), in the radial case, that we obtained the dispersion stated in weighted spaces, independently of the initial data.

Let us also notice that the improvements in the dispersive estimates stated in Theorem 1.2 are better decays away from the origin. Near the origin -where the Laplace-Beltrami operator on hyperbolic space is almost the one on \mathbb{R}^n - we have the same decays as in the euclidean case. The infinite speed of propagation of the Schrödinger operator is not sufficient to impose on the dispersion estimate at the origin an influence from the metric at infinity.

As a consequence of (3), we obtain the classical local dispersion inequality

$$\|u(t)\|_{L^\infty} \leq \frac{c}{|t|^{\frac{n}{2}}} \|u_0\|_{L^1}.$$

From the dispersion inequality, by using the TT* functional analysis argument ([26]), the local Strichartz estimates are obtained too. We recall here that for the wave equation on hyperbolic space, the Strichartz estimates have been proved recently by Tataru ([24]).

Let us now remark that on the sphere, the local Strichartz-type estimates are known to hold with a loss of $\frac{1}{p}$ derivatives, that is the $L^p L^q$ norms of the solutions are controlled by the $H^{\frac{1}{p}}$ norm of the initial data, instead of the L^2 norm ([8]). This shows that on hyperbolic space, the dispersion estimates are stronger.

Remark 1.6. *An important problem related to the linear Schrödinger equation is the problem of the potential. The radial Schrödinger equation perturbed with a rough time dependent potential on hyperbolic space was recently treated by Pierfalice in ([23]). By using Theorem 1.2 ii) and fixed point arguments she obtains for the radial perturbed Cauchy problem the Strichartz estimates with the same weight, pointing out again the influence of the negative curvature on the dispersive properties. These estimates hold even in the case of more general nonlinearities.*

Remark 1.7. *It is expected that the results obtained should hold on all symmetric spaces of rank 1.*

Moreover, let us notice that the dispersion kernel obtained is close to the kernel of the heat operator $e^{t\Delta}$, which has been studied intensively on hyperbolic space, usually for t real, but also for complex t with $\Re t > 0$ ([13], [2], [14]). In view of this work, we expect that the dispersion estimate for large time can be improved by an additional decay, for all $n \geq 2$:

$$|u(t, \Omega)| \leq c \frac{1}{|t|^{\frac{3}{2}}} \int_{\mathbb{H}^n} |u_0(\Omega')| \left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-1}{2}} \left(\frac{1}{1+\rho} \right)^{\frac{n-3}{2}} d\Omega'.$$

Let us turn now our attention to the concentration phenomena for the nonlinear equation, and more precisely to blow-up of solutions in the sense of the explosion in finite time of the L^2 norm of their gradient.

We consider the focusing Schrödinger equation with power nonlinearity

$$(S) \begin{cases} i\partial_t u + \Delta_{\mathbb{H}^n} u + |u|^{p-1} u = 0 \\ u(0, x) = u_0 \in H^1(\mathbb{H}^n) \end{cases}.$$

First, let us notice that, as for the Euclidean case, since integration by parts works on hyperbolic space, the mass of a solution of (S)

$$\int_{\mathbb{H}^n} |u|^2 d\Omega,$$

and its energy

$$E(u) = \frac{1}{2} \int_{\mathbb{H}^n} |\nabla u|^2 d\Omega - \frac{1}{p+1} \int_{\mathbb{H}^n} |u|^{p+1} d\Omega,$$

are conserved in time.

By using the classical Strichartz estimates, derived from the dispersion estimate (3) on the linear equation, one can obtain, as in the Euclidean case ([16],[27]), the H^1 local existence of solutions of the equation (S) with $p \leq 1 + \frac{4}{n-2}$.

The argument used on \mathbb{R}^n in the most of the blow-up results is the scale invariance. The construction of explicit blow-up solutions is done using the pseudo-conformal invariance. Informations on the blow-up solutions are obtained by studying the dilatations of the solution and by using the virial identity ([22]). The main difficulty when working on hyperbolic space is that these techniques and notions do not have an obvious analogue.

One can expect that maybe there will be no blow-up solutions, due to the negative curvature of the manifold. In fact it turns out that the geometry does not have enough of an effect to avoid the concentration phenomena. Moreover, we have more types of initial data generating blow-up in finite time than in the Euclidean case.

Theorem 1.8. *For $p < 1 + \frac{4}{n}$, the solutions of the equation (S) are global in $H^1(\mathbb{H}^n)$. The global existence still holds for the power $p = 1 + \frac{4}{n}$ and for initial data of mass smaller than a certain constant.*

However, for $p \geq 1 + \frac{4}{n}$, blow-up solutions exist. More precisely, if the initial data is radial, of finite variance

$$\int_{\mathbb{H}^n} |u_0(\Omega)|^2 d^2(0, \Omega) d\Omega < +\infty,$$

and its energy satisfies

$$E(u_0) < c_n \|u_0\|_2^2,$$

then the solution blows up in finite time. Here c_n is a geometric positive constant given by

$$c_n = \frac{\inf \Delta_{\mathbb{H}^n}^2 d(0, \cdot)}{16}.$$

The proof of the global existence up to the critical power $p = 1 + \frac{4}{n}$ -which is also the one of the Euclidean space- involves the Sobolev embeddings on \mathbb{H}^n ([18]), together with the conservation laws of the equation. The existence of blow-up solutions is shown by adapting the arguments of Glassey and Zakharov on \mathbb{R}^n ([17], [28]) and of Kavian on a star-shaped domain of \mathbb{R}^n ([20]), to the particular metric on \mathbb{H}^n .

Remark 1.9. *We obtain a new virial identity adapted to hyperbolic space that allows us to conclude that solutions of null energy and finite virial blow up in finite time. Probably this new concentration phenomena is the consequence of the improved dispersion of the solutions. We recall that this is not the case on \mathbb{R}^n for the critical power, since the ground state, that is the unique positive solution of the elliptic equation*

$$\Delta_{\mathbb{R}^n} Q + Q^p = Q,$$

gives us the global solution $e^{it}Q$ of null energy.

Moreover, on \mathbb{R}^n , the mass of the ground state is the critical mass for having explosive solutions in critical power. In our case, the Theorem 1.8 implies that the ground state on hyperbolic space has positive energy and mass smaller than the expected critical mass for blow-up on \mathbb{H}^n .

It should be interesting to see what is the minimal blow-up mass, and to get informations about the blow-up speed and about the profiles of the explosive solutions.

This article is organized as follows. In the second section we give the definition of hyperbolic spaces and of the tools used on it, namely the Laplace-Beltrami operator and the Fourier transform. In §3.1 we recall the proof of the representation of the solutions and the dispersion inequality in the Euclidean case, then we give in §3.2 the proof of Theorem 1.1. In §4 we prove the first point of Theorem 1.2, concerning the $L^1 - L^\infty$ estimates. The section §5 concerns the weighted dispersion inequalities of the second point of Theorem 1.2, and of the Remarks 1.4 and 1.5. The last section contains the proof of Theorem 1.8. In Appendix A is proved a technical proposition, crucial for obtaining the dispersion estimates in §4.

Acknowledgments : I am very grateful to V. Georgiev, to P. Gérard and to F. Ricci for having encouraged my research in this direction. I thank T.Duyckaerts and V. Pierfelice for having carefully read some parts of the article.

2. HYPERBOLIC SPACE

2.1. Definition. We shall use here the model of hyperbolic space given by the upper branch of the hyperboloid.

We define the hyperbolic space as being the following surface of \mathbb{R}^{n+1} , given by the parametrization :

$$\mathbb{H}^n = \{\Omega = (t, x) \in \mathbb{R}^{n+1}, (t, x) = (\cosh r, \sinh r \omega), r \geq 0, \omega \in \mathbb{S}^{n-1}\}.$$

Let us introduce the inner product on \mathbb{R}^{n+1}

$$(8) \quad [x, y] = x_0 y_0 - \dots - x_n y_n.$$

An alternative definition for hyperbolic space is

$$\mathbb{H}^n = \{x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}, [x, x] = 1, x_0 > 0\}.$$

This space is invariant under $SO(1, n)$, the group of Lorentz transformations of \mathbb{R}^{n+1} that preserve this inner product.

One has

$$dt = \sinh r dr, dx = \cosh r \omega dr + \sinh r d\omega,$$

and the metric induced on \mathbb{H}^n by the Lorenzian metric on \mathbb{R}^{n+1}

$$dl^2 = -dt^2 + dx^2,$$

is

$$ds^2 = dr^2 + \sinh^2 r d\omega^2,$$

where $d\omega^2$ is the metric on the sphere \mathbb{S}^{n-1} .

The volume element is

$$\int_{\mathbb{H}^n} f(\Omega) d\Omega = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(r, \omega) \sinh^{n-1} r dr d\omega.$$

The length of a curve

$$\gamma(t) = (\cosh r(t), \sinh r(t) \omega(t)),$$

with t varying from a to b , is defined to be as usual,

$$L(\gamma) = \int_a^b \sqrt{|r'(t)|^2 + |\sinh r(t)|^2 |\omega'(t)|^2} dt.$$

The distance between two points of X will be the infimum of the lengths of the paths connecting the points. By direct calculus, one has that the distance of a point to the origin of the hyperboloid $O = (1, 0, \dots, 0)$ is

$$d((\cosh r, \sinh r \omega), O) = r.$$

More generally, the distance between two arbitrary points is

$$(9) \quad d(\Omega, \Omega') = \cosh^{-1}([\Omega, \Omega']).$$

Starting from the general definition of the Laplace-Beltrami operator, one can find its expression on \mathbb{H}^n

$$\Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1) \frac{\cosh r}{\sinh r} \partial_r + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{n-1}}.$$

In Section 6 we will denote $\Delta = \Delta_{\mathbb{H}^n}$.

2.2. The Fourier transform.

For $\theta \in \mathbb{S}^{n-1}$ and λ a real number, the functions of the type

$$(10) \quad h_{\lambda, \theta}(\Omega) = [\Omega, \Lambda(\theta)]^{i\lambda - \frac{n-1}{2}},$$

are generalized eigenfunctions of the Laplacian-Beltrami operator. Here we denoted by $\Lambda(\theta)$ the point of \mathbb{R}^{n+1} given by $(1, \theta)$. Indeed, we have

$$-\Delta_{\mathbb{H}^n} h_{\lambda, \theta} = \left(\lambda^2 + \frac{(n-1)^2}{4} \right) h_{\lambda, \theta}.$$

By analogy with the \mathbb{R}^n case, the definition of the Fourier transform is, for a function on \mathbb{H}^n ,

$$\widehat{f}(\lambda, \theta) = \int_{\mathbb{H}^n} h_{\lambda, \theta}(\Omega) f(\Omega) d\Omega.$$

It turns out that this is the good definition, and one has the Fourier inversion formula for function on \mathbb{H}^n

$$f(\Omega) = \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \bar{h}_{\lambda, \theta}(\Omega) \widehat{f}(\lambda, \theta) \frac{d\theta d\lambda}{|c(\lambda)|^2},$$

where $c(\lambda)$ is the Harish-Chandra coefficient

$$\frac{1}{|c(\lambda)|^2} = \frac{1}{2(2\pi)^n} \frac{|\Gamma(i\lambda + \frac{n-1}{2})|^2}{|\Gamma(i\lambda)|^2}.$$

Fore more details on hyperbolic space, see Helgason ([19]) and Terras ([25]).

3. THE REPRESENTATION OF THE SOLUTIONS

3.1. The Euclidean case.

For the linear equation on \mathbb{R}^n

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{R}^n} u = 0, \\ u(0) = u_0 \in L^2(\mathbb{R}^n), \end{cases},$$

the solution can be written explicitly. By applying the Fourier transform to the equation, one gets

$$\widehat{u}(\xi) = e^{-it|\xi|^2} \widehat{u_0}(\xi),$$

so the solution is

$$u(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-it|\xi|^2 + ix\xi} \widehat{u_0}(\xi) d\xi.$$

By the Plancherel formula, the solution can also be written

$$u(t, x) = \frac{1}{(4\pi it)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y) dy,$$

and therefore one obtains the following estimate

$$\|u(t)\|_{\infty} \leq \frac{c}{|t|^{\frac{n}{2}}} \|u_0\|_1,$$

called dispersion estimate.

3.2. Proof of Theorem 1.1.

For the hyperbolic space we will try the same approach, by using the Fourier transform introduced previously. By Fourier transforming on \mathbb{H}^n , one has

$$\left(i\partial_t - \left(\lambda^2 + \frac{(n-1)^2}{4} \right) \right) \widehat{u}(t, \lambda, \theta) = 0,$$

so that

$$\widehat{u}(t, \lambda, \theta) = e^{-it\left(\lambda^2 + \frac{(n-1)^2}{4}\right)} \widehat{u}_0(\lambda, \theta).$$

By applying the inverse Fourier transform,

$$u(t, \Omega) = \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} e^{-it\left(\lambda^2 + \frac{(n-1)^2}{4}\right)} \bar{h}_{\lambda, \theta}(\Omega) \widehat{u}_0(\lambda, \theta) \frac{d\theta d\lambda}{|c(\lambda)|^2}.$$

Finally, by making explicit the Fourier transform of the initial data,

$$(11) \quad u(t, \Omega) = e^{-it\frac{(n-1)^2}{4}} \int_{\mathbb{H}^n} u_o(\Omega') \int_{-\infty}^{\infty} e^{-it\lambda^2} L_\lambda(\Omega, \Omega') \frac{d\lambda}{|c(\lambda)|^2} d\Omega',$$

where

$$L_\lambda(\Omega, \Omega') = \int_{\mathbb{S}^{n-1}} \bar{h}_{\lambda, \theta}(\Omega) h_{\lambda, \theta}(\Omega') d\theta.$$

Lemma 3.1. *For all $g \in SO(1, n)$,*

$$L_\lambda(\Omega, \Omega') = L_\lambda(g\Omega, g\Omega').$$

Proof. Using the definition (10) of the functions $h_{\lambda, \theta}$, and the fact that g preserves the inner product, we have

$$(12) \quad L_\lambda(\Omega, \Omega') = \int_{\mathbb{S}^{n-1}} [g\Omega, g\Lambda(\theta)]^{-i\lambda - \frac{n-1}{2}} [g\Omega', g\Lambda(\theta)]^{i\lambda - \frac{n-1}{2}} d\theta.$$

Consider now the map F on \mathbb{S}^{n-1} ,

$$F(\theta) = \tilde{\theta},$$

given by the relation

$$(13) \quad g\Lambda(\theta) = \mu(\theta)\Lambda(\tilde{\theta}),$$

for some real $\mu(\theta)$. Then, one can get ([15], Prop. 8.4.1) that this application is unique, is a diffeomorphism,

$$\mu(\theta) = \cosh r_g + \sinh r_g \theta_1,$$

for some constant r_g depending only on g , and the relation between the volume elements $d\tilde{\theta}$ and $d\theta$ on \mathbb{S}^{n-1} is

$$(14) \quad d\tilde{\theta} = \frac{d\theta}{(\cosh r_g + \sinh r_g \theta_1)^{n-1}} = \frac{d\theta}{(\mu(\theta))^{n-1}}.$$

Here we denoted by θ_1 the first \mathbb{R}^n -component of θ .

Then, by making in the right hand side of (12) the change of variable F , and by using (13) and (14), we obtain

$$L_\lambda(\Omega, \Omega') = \int_{\mathbb{S}^{n-1}} [g\Omega, \Lambda(\tilde{\theta})]^{-i\lambda - \frac{n-1}{2}} [g\Omega', \Lambda(\tilde{\theta})]^{i\lambda - \frac{n-1}{2}} d\tilde{\theta} = L_\lambda(g\Omega, g\Omega'),$$

so the Lemma is proved. \square

Let us remark that Lemma 3.1 can also be proved by using the classical argument of the fundamental point pair invariant ([13], p.177).

Lemma 3.2. *We have the following identity*

$$L_\lambda(\Omega, \Omega') = c(\sinh \rho)^{-\frac{n-2}{2}} P_{-\frac{1}{2}+i\lambda}^{-\frac{n-2}{2}}(\cosh \rho),$$

where $\rho = d(\Omega, \Omega')$, and by $P_\nu^\mu(z)$ we denoted the Legendre function, solution of the equation

$$(1-z^2)\partial_z^2 w - 2z\partial_z w + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right)w = 0.$$

Proof. Let us choose a transformation $g \in SO(1, n)$, which maps Ω on the origin of \mathbb{H}^n . Then g will send Ω' in a point of \mathbb{H}^n of coordinates

$$\Omega^* = (\cosh \rho, \sinh \rho \gamma).$$

Since g preserves the product, the radius of Ω^* can be calculated,

$$\rho = \cosh^{-1}[\Omega, \Omega'] = d(\Omega, \Omega').$$

So, by using Lemma 3.1,

$$L_\lambda(\Omega, \Omega') = L_\lambda(0, \Omega^*) = \int_{\mathbb{S}^{n-1}} [\Omega^*, \Lambda(\theta)]^{i\lambda - \frac{n-1}{2}} d\theta,$$

therefore

$$(15) \quad L_\lambda(\Omega, \Omega') = \int_{\mathbb{S}^{n-1}} (\cosh \rho - \sinh \rho \gamma \cdot \theta)^{i\lambda - \frac{n-1}{2}} d\theta.$$

Consider now a transformation $T \in SO(n)$ such that

$$T(1, 0, 0, \dots) = \gamma.$$

Then for $\theta \in \mathbb{S}^{n-1}$, we obtain an $\alpha \in [0, \pi[$ and a $\theta' \in \mathbb{S}^{n-2}$, well defined by the relation

$$\theta = T(\cos \alpha, \sin \alpha \theta').$$

Moreover, the angle between γ and θ is α ,

$$\gamma \cdot \theta = \cos \alpha.$$

The change of variable from θ to α and θ' is

$$d\theta = \sin^{n-2} \alpha d\alpha d\theta'.$$

Let us now apply this change of variables in (15), and obtain

$$L_\lambda(\Omega, \Omega') = c \int_0^\pi (\cosh \rho - \sinh \rho \cos \alpha)^{i\lambda - \frac{n-1}{2}} \sin^{n-2} \alpha d\alpha.$$

Therefore, in view of the integral form of $P_\nu^\mu(\cosh r)$ ([4]), we obtain the identity of the Lemma. \square

We shall now use the following Lemma, corresponding to Lemma 8.5.2 and 8.5.3 of [15].

Lemma 3.3. *The function defined by*

$$L_\lambda^n(\rho) = c(\sinh \rho)^{-\frac{n-2}{2}} P_{-\frac{1}{2}+i\lambda}^{-\frac{n-2}{2}}(\cosh \rho),$$

is, for $\rho > 0$ and for $n \geq 1$ odd,

$$(16) \quad L_\lambda^n(\rho) = c \frac{|\Gamma(i\lambda)|^2}{|\Gamma(i\lambda + \frac{n-1}{2})|^2} \left(\frac{\partial_\rho}{\sinh \rho} \right)^{\frac{n-1}{2}} \cos \lambda \rho,$$

and for $n \geq 2$ even,

$$(17) \quad L_\lambda^n(\rho) = c \frac{|\Gamma(i\lambda)|^2}{|\Gamma(i\lambda + \frac{n-1}{2})|^2} \int_\rho^\infty \frac{\sinh s}{\sqrt{\cosh s - \cosh \rho}} \left(\frac{\partial_s}{\sinh s} \right)^{\frac{n}{2}} \cos \lambda s ds.$$

By using in the formula (11) the expression of $L_\lambda(\Omega, \Omega')$ given by the Lemma 3.2 and Lemma 3.3, we obtain that

$$u(t, \Omega) = ce^{-it\frac{(n-1)^2}{4}} \int_{\mathbb{H}^n} u_o(\Omega') K^n(t, d(\Omega, \Omega')) d\Omega',$$

where the kernel K^n is, for $\rho > 0$ and for $n \geq 3$ odd,

$$K^n(t, \rho) = \int_{-\infty}^{\infty} e^{-it\lambda^2} \left(\frac{\partial_\rho}{\sinh \rho} \right)^{\frac{n-1}{2}} \cos \lambda \rho d\lambda,$$

and for $n \geq 2$ even,

$$(18) \quad K^n(t, \rho) = \int_{-\infty}^{\infty} e^{-it\lambda^2} \int_{\rho}^{\infty} \frac{\sinh s}{\sqrt{\cosh s - \cosh \rho}} \left(\frac{\partial_s}{\sinh s} \right)^{\frac{n}{2}} \cos \lambda s ds d\lambda.$$

In the case of odd dimensions, the kernel can be written

$$(19) \quad K^n(t, \rho) = \left(\frac{\partial_\rho}{\sinh \rho} \right)^{\frac{n-1}{2}} \int_{-\infty}^{\infty} e^{-it\lambda^2} \cos \lambda \rho d\lambda = \frac{c}{|t|^{\frac{1}{2}}} \left(\frac{\partial_\rho}{\sinh \rho} \right)^{\frac{n-1}{2}} e^{i\frac{\rho^2}{4t}},$$

so the representation (1) of the Theorem 1.1 is obtained.

In the case of even dimensions, the kernel can be expressed by induction ([15], formula (8.5.23)) in terms of the kernel of the dimension 2

$$K^n(t, \rho) = \left(\frac{\partial_\rho}{\sinh \rho} \right)^{\frac{n-2}{2}} K^2(t, \rho).$$

As K^2 can be written, after applying the Fubini theorem in (18), in the form

$$K^2(t, \rho) = \int_{\rho}^{\infty} \frac{1}{\sqrt{\cosh s - \cosh \rho}} \int_{-\infty}^{\infty} e^{-it\lambda^2} \sin \lambda s \lambda d\lambda ds = \frac{c}{|t|^{\frac{3}{2}}} \int_{\rho}^{\infty} \frac{e^{i\frac{s^2}{4t}}}{\sqrt{\cosh s - \cosh \rho}} ds,$$

we get that

$$(20) \quad K^n(t, \rho) = \frac{c}{|t|^{\frac{3}{2}}} \left(\frac{\partial_\rho}{\sinh \rho} \right)^{\frac{n-2}{2}} \int_{\rho}^{\infty} \frac{e^{i\frac{s^2}{4t}}}{\sqrt{\cosh s - \cosh \rho}} ds,$$

so we obtain also the representation (2) and the first point of Theorem 1.1 is proved.

4. THE $L^1 - L^\infty$ DISPERSION ESTIMATES

We shall use the following Proposition, which shall be proved in the Appendix A.

Proposition 4.1. *For all $m \geq 1$ integer, the following identity holds*

$$(21) \quad \left(\frac{\partial_s}{\sinh s} \right)^m e^{i\frac{s^2}{4t}} = \sum_{k=1}^m \frac{e^{i\frac{s^2}{4t}}}{t^k} F_k^m(s),$$

where, modulo constants,

$$(22) \quad F_m^m(s) = \left(\frac{s}{\sinh s} \right)^m,$$

and, for $1 \leq k < m$, $F_k^m(s)$ can be written

$$(23) \quad F_k^m(s) = \sum_{r=0}^k \left\{ \begin{array}{l} \sum_{i_1 + \dots + i_r = m-k} \left(\frac{s}{\sinh s} \right)^{k-r} \left(\frac{\partial_s}{\sinh s} \right)^{i_1} \frac{s}{\sinh s} \dots \left(\frac{\partial_s}{\sinh s} \right)^{i_r} \frac{s}{\sinh s} \\ i_j \geq 1 \end{array} \right.$$

Moreover, for $\alpha \in \{0, 1\}$ and $1 \leq k \leq m$

$$(24) \quad \left| \partial_s^\alpha \left(\frac{\sinh s}{s} F_k^m(s) \right) \right| \leq c s^\alpha \left(\frac{s}{\sinh s} \right)^{m-1},$$

and the second derivative of $\frac{\sinh s}{s} F_k^m(s)$ is bounded for $m \geq 2$.

4.1. Dispersive estimates for odd dimensions. From the expression (19), for odd dimensions, the kernel is for $\rho > 0$

$$K^n(t, \rho) = \frac{c}{|t|^{\frac{1}{2}}} \left(\frac{\partial_\rho}{\sinh \rho} \right)^{\frac{n-1}{2}} e^{i \frac{\rho^2}{4t}},$$

so by (21) it can be developed in powers of t as follows

$$(25) \quad K^n(t, \rho) = \sum_{k=1}^{\frac{n-1}{2}} \frac{e^{i \frac{\rho^2}{4t}}}{t^{k+\frac{1}{2}}} F_k^{\frac{n-1}{2}}(\rho).$$

Combining this with (24) for $\alpha = 0$ we obtain the dispersion estimates (3) and (4) for all $n \geq 3$ odd.

4.2. Local dispersion for even dimensions. Between two kernels of consecutive order, there is a induction relation, namely

$$K^n(t, \rho) = c \int_\rho^\infty \frac{\sinh s}{\sqrt{\cosh s - \cosh \rho}} K^{n+1}(t, s) ds,$$

(see for example formula (5.7.4) in the book of Davies ([13])).

Therefore, by using the development (25) of the kernels of odd order, we can get the one for kernels of even order for $\rho > 0$

$$(26) \quad K^n(t, \rho) = \sum_{k=1}^{\frac{n}{2}} \frac{1}{t^{k+\frac{1}{2}}} \int_\rho^\infty \frac{e^{i \frac{\rho^2}{4t}} s}{\sqrt{\cosh s - \cosh \rho}} \frac{\sinh s}{s} F_k^{\frac{n}{2}}(s) ds.$$

Therefore the local dispersion (3) is obtained for $n = 2$ by (22) and the following Proposition. For all even dimensions $n \geq 4$, the estimates (34) and (35) of Corollary 4.4 give us

$$|K^n(t, \rho)| \leq \left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-1}{2}} \left(\frac{1}{t^{\frac{n}{2}}} + \sum_{k=1}^{\frac{n-2}{2}} \frac{1}{t^{k+\frac{1}{2}}} \right),$$

and the local dispersion (3) follows.

Proposition 4.2. *The integral*

$$I(t, \rho) = \int_\rho^\infty \frac{e^{i \frac{\rho^2}{4t}} s}{\sqrt{\cosh s - \cosh \rho}} ds$$

can be estimated, for small times, by

$$(27) \quad |I(t, \rho)| \leq c \sqrt{t} \sqrt{\frac{\rho}{\sinh \rho}}.$$

Proof. In the following we shall use frequently the estimate

$$(28) \quad \frac{1}{\sqrt{\cosh s - \cosh \rho}} \leq \frac{c}{(s - \rho) \sqrt{\cosh \rho}} \leq \frac{c}{s - \rho}$$

for $s > \rho \geq 0$, and

$$(29) \quad \frac{1}{\sqrt{\cosh s - \cosh \rho}} \leq \frac{c}{\sqrt{(s - \rho) \sinh \rho}}$$

for $s > \rho > 0$.

For small time, in order to get the decay (27), we have to get advantage of the imaginary phase. Let $t < \frac{1}{2}$.

The case $\rho \geq \frac{\sqrt{t}}{2}$.

By doing in the integral $I(t, \rho)$ the change of variable $s = \tau \frac{t}{\rho} + \rho$, we obtain

$$\begin{aligned} I(t, \rho) &= \frac{te^{i\frac{\rho^2}{4t}}}{\rho} \int_0^\infty \frac{e^{i\tau^2 \frac{t}{4\rho^2} + i\frac{\tau}{2}} (\tau \frac{t}{\rho} + \rho)}{\sqrt{\cosh(\tau \frac{t}{\rho} + \rho) - \cosh \rho}} d\tau = \\ &= 2e^{i\frac{\rho^2}{4t}} \sqrt{t} \sqrt{\frac{\rho}{\sinh \rho}} \int_0^\infty \frac{e^{i\tau^2 \frac{t}{4\rho^2} + i\frac{\tau}{2}} (\tau \frac{t}{2\rho^2} + \frac{1}{2})}{\sqrt{\frac{\rho}{t \sinh \rho} (\cosh(\tau \frac{t}{\rho} + \rho) - \cosh \rho)}} d\tau. \end{aligned}$$

We split the integral into two pieces

$$(30) \quad I(t, \rho) = 2e^{i\frac{\rho^2}{4t}} \sqrt{t} \sqrt{\frac{\rho}{\sinh \rho}} \left(\int_0^1 + \int_1^\infty \right) = 2e^{i\frac{\rho^2}{4t}} \sqrt{t} \sqrt{\frac{\rho}{\sinh \rho}} (J_1 + J_2).$$

We shall show that J_1 and J_2 are bounded independently of t and ρ .

By using (29),

$$|J_1| \leq \frac{1}{2} \int_0^1 \frac{\tau \frac{t}{\rho^2} + 1}{\sqrt{\frac{\rho}{t \sinh \rho} \tau \frac{t}{\rho} \sinh \rho}} = \frac{1}{2} \int_0^1 \frac{\tau \frac{t}{\rho^2} + 1}{\sqrt{\tau}}.$$

Since we are in the case $\rho \geq \frac{\sqrt{t}}{2}$, it follows that

$$\frac{t}{\rho^2} \leq 4,$$

and therefore

$$|J_1| \leq \frac{1}{2} \int_0^1 \frac{5}{\sqrt{\tau}} = 5,$$

The second term can be written

$$J_2 = \int_1^\infty e^{i\phi(\tau)} \phi'(\tau) \alpha(\tau) d\tau,$$

where

$$\phi(\tau) = \tau^2 \frac{t}{4\rho^2} + \frac{\tau}{2},$$

and

$$\alpha(\tau) = \frac{1}{\sqrt{\frac{\rho}{t \sinh \rho} (\cosh(\tau \frac{t}{\rho} + \rho) - \cosh \rho)}}.$$

By integrating by parts in J_2 we get

$$J_2 = \frac{e^{i\phi(\tau)}}{i} \alpha(\tau) \Big|_1^\infty - \int_1^\infty \frac{e^{i\phi(\tau)}}{i} \alpha'(\tau) d\tau,$$

so

$$(31) \quad |J_2| \leq 2 \sup_{1 \leq \tau} |\alpha| + \int_1^\infty |\alpha'(\tau)| d\tau.$$

Let us notice that by (29)

$$\alpha(1) \leq \frac{1}{\sqrt{\frac{\rho}{t \sinh \rho} \frac{t}{\rho} \sinh \rho}} = 1,$$

and that α is a decreasing function. Therefore,

$$|J_2| \leq 2 + \int_1^\infty |\alpha'(\tau)| d\tau.$$

Since the derivative of α is negative,

$$|J_2| \leq 2 + \left| \int_1^\infty \alpha'(\tau) d\tau \right| = 2 + |\alpha(\tau)|_1^\infty \leq 4.$$

Therefore we have obtained that J_1 and J_2 are bounded independently of t and ρ . In view of (30) the decay (27) is obtained in the region $\rho \geq \frac{\sqrt{t}}{2}$.

The case $\rho < \frac{\sqrt{t}}{2}$.

Since $t < \frac{1}{2}$, then $\rho < \frac{1}{2\sqrt{2}}$, and by noticing that the quotient $\frac{\rho}{\sinh \rho}$ is bounded near zero, it will be sufficient to prove that

$$(32) \quad \int_\rho^\infty \frac{e^{i\frac{s^2}{4t}} s}{\sqrt{\cosh s - \cosh \rho}} ds \leq c\sqrt{t}.$$

Let us split this integral in three parts :

$$\int_\rho^\infty \frac{e^{i\frac{s^2}{4t}} s}{\sqrt{\cosh s - \cosh \rho}} ds = \int_\rho^{2\rho} + \int_{2\rho}^{\sqrt{t}} + \int_{\sqrt{t}}^\infty = I_1 + I_2 + I_3.$$

If $\rho = 0$ then I_1 is zero, otherwise by (29)

$$\begin{aligned} |I_1| &\leq \int_\rho^{2\rho} \frac{s}{\sqrt{\cosh s - \cosh \rho}} ds \leq 2\rho \int_\rho^{2\rho} \frac{1}{\sqrt{(s-\rho)\sinh \rho}} ds = \\ &= 4\rho \frac{\sqrt{\rho}}{\sqrt{\sinh \rho}}. \end{aligned}$$

Since we are in the case $\rho < \frac{\sqrt{t}}{2}$ we get that

$$|I_1| \leq 2\sqrt{t} \sqrt{\frac{\rho}{\sinh \rho}} \leq c\sqrt{t}.$$

If $s \geq 2\rho$, which is the case in I_2 , then by (28)

$$\frac{s}{\sqrt{\cosh s - \cosh \rho}} \leq \frac{s}{s-\rho} \leq 3,$$

and we obtain

$$|I_2| \leq 3(\sqrt{t} - 2\rho) \leq 3\sqrt{t}.$$

By performing in I_3 the change of variable $s = \sqrt{t}\tau$,

$$I_3 = \sqrt{t} \int_1^\infty e^{i\frac{\tau^2}{4}} a(\sqrt{t}\tau) d\tau,$$

where

$$a(s) = \frac{s}{\sqrt{\cosh s - \cosh \rho}}$$

for $s \geq \sqrt{t}$. By integrating by parts two times the integral I_3 , we get

$$|I_3| \leq c\sqrt{t},$$

provided that the following lemma holds. (see Theorem I.8.1 of [1]).

Lemma 4.3. *The function $a(\sqrt{t}\tau)$, its first and second derivative in τ , are all bounded independently of t , for all $\tau \geq 1$.*

Proof. By using the same argument as in the estimate of I_2 , we have that $a(\sqrt{t}\tau)$ is upper-bounded by 3 for $\tau \geq 1$.

The derivative of a is

$$a'(s) = \frac{1}{\sqrt{\cosh s - \cosh \rho}} - \frac{s \sinh s}{2(\cosh s - \cosh \rho)^{\frac{3}{2}}}.$$

Since we are in the region $s \geq \sqrt{t}$, $\rho < \frac{\sqrt{t}}{2}$, the first term can be estimated by (28)

$$\frac{1}{\sqrt{\cosh s - \cosh \rho}} \leq \frac{c}{s - \rho} \leq \frac{c}{\sqrt{t} - \frac{\sqrt{t}}{2}} \leq \frac{c}{\sqrt{t}}.$$

The second term in the derivative of a can be written as

$$-\frac{a(s)}{2} \frac{\sinh s}{\cosh s - \cosh \rho}.$$

Since $s \geq \sqrt{t} > \rho$, the second fraction is positive, and as a was already proved to be bounded, we get

$$|a'(s)| \leq \frac{c}{\sqrt{t}} + c \sup_{s \geq \sqrt{t}} \frac{\sinh s}{\cosh s - \cosh \rho}.$$

We have

$$\left(\frac{\sinh s}{\cosh s - \cosh \rho} \right)' = \frac{1 - \cosh s \cosh \rho}{(\cosh s - \cosh \rho)^2} \leq 0,$$

so it follows that

$$\sup_{s \geq \sqrt{t}} \frac{\sinh s}{\cosh s - \cosh \rho} = \frac{\sinh \sqrt{t}}{\cosh \sqrt{t} - \cosh \rho}.$$

As $t < \frac{1}{2}$ and $\rho < \frac{\sqrt{t}}{2}$,

$$(33) \quad \sup_{s \geq \sqrt{t}} \frac{\sinh s}{\cosh s - \cosh \rho} \leq c \frac{\sqrt{t}}{(\sqrt{t} - \rho)^2} \leq c \frac{\sqrt{t}}{(\sqrt{t} - \frac{\sqrt{t}}{2})^2} \leq \frac{c}{\sqrt{t}},$$

and we have obtained that for $s \geq \sqrt{t}$

$$|a'(s)| \leq \frac{c}{\sqrt{t}}.$$

It follows that

$$\partial_\tau a(\sqrt{t}\tau) = \sqrt{t} a'(\sqrt{t}\tau)$$

is bounded independently of t for $\tau \geq 1$.

Finally, the second derivative of a is

$$a''(s) = -\frac{\sinh s}{2(\cosh s - \cosh \rho)^{\frac{3}{2}}} - \frac{s \cosh s}{(\cosh s - \cosh \rho)^{\frac{3}{2}}} + \frac{3s \sinh^2 s}{4(\cosh s - \cosh \rho)^{\frac{5}{2}}}.$$

By using (33) and then (28) we can treat the first term

$$\frac{\sinh s}{2(\cosh s - \cosh \rho)^{\frac{3}{2}}} \leq \frac{c}{\sqrt{t}} \frac{1}{\sqrt{\cosh s - \cosh \rho}} \leq \frac{c}{\sqrt{t}} \frac{c}{s - \rho} \leq \frac{c}{t}.$$

The second term in the derivative can be written

$$\frac{s \cosh s}{(\cosh s - \cosh \rho)^{\frac{3}{2}}} = a(s) \frac{\cosh s}{\cosh s - \cosh \rho} = a(s) \left(1 + \frac{\cosh \rho}{\cosh s - \cosh \rho} \right).$$

As a was proved to be bounded and as the fraction in the brackets is decreasing, for $s \geq \sqrt{t}$

$$\frac{s \cosh s}{(\cosh s - \cosh \rho)^{\frac{3}{2}}} \leq c \left(1 + \frac{\cosh \rho}{(\sqrt{t} - \rho)^2} \right).$$

We are in the region $\rho < \frac{\sqrt{t}}{2} < \frac{1}{2\sqrt{2}}$, so the second term in the derivative can be estimated

$$\frac{s \cosh s}{(\cosh s - \cosh \rho)^{\frac{3}{2}}} \leq c \left(1 + \frac{c}{t}\right) \leq \frac{c}{t}.$$

Finally, the third term in the derivative can be written as

$$\frac{3a(s)}{4} \left(\frac{\sinh s}{\cosh s - \cosh \rho} \right)^2,$$

and can be upper-bounded using (33) by $\frac{c}{t}$. In conclusion, for $s \geq \sqrt{t}$

$$|a''(s)| \leq \frac{c}{t}.$$

So we get that

$$\partial_\tau^2 a(\sqrt{t}\tau) = t a''(\sqrt{t}\tau)$$

is bounded independently of t for $\tau \geq 1$. \square

In conclusion, we have obtained that the three integrals I_1, I_2 and I_3 are upper-bounded by $c\sqrt{t}$, and (32) follows. Therefore the estimate (27) is proved and Proposition 4.2 follows. \square

Corollary 4.4. *For small times and $n \geq 4$ the following estimates hold*

$$(34) \quad \left| \int_\rho^\infty \frac{e^{i\frac{s^2}{4t}}}{\sqrt{\cosh s - \cosh \rho}} \frac{\sinh s}{s} F_k^{\frac{n}{2}}(s) ds \right| \leq c\sqrt{t} \left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-1}{2}},$$

and for $1 \leq k < \frac{n}{2}$

$$(35) \quad \left| \int_\rho^\infty \frac{e^{i\frac{s^2}{4t}}}{\sqrt{\cosh s - \cosh \rho}} \frac{\sinh s}{s} F_k^{\frac{n}{2}}(s) ds \right| \leq c \left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-1}{2}}.$$

Proof. We redo the proof of Proposition 4.2, which is the particular case $n = 2$ of its Corollary. The only delicate point will be the integral J_2 which will generate the two different estimates (34) and (35).

By using the upper-bound (24) for $\alpha = 0$ and $1 \leq k \leq \frac{n}{2}$, and the fact that the integration is done for $s \geq \rho$,

$$\left| \frac{\sinh s}{s} F_k^{\frac{n}{2}}(s) \right| \leq \left(\frac{s}{\sinh s} \right)^{\frac{n-2}{2}} \leq \left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-2}{2}}.$$

Therefore everytime the imaginary phase is ignored, we find ourselves in the same situation as in Proposition 4.2, and we obtain the desired estimates.

The imaginary phase has been taken in account only in the terms J_2 and I_3 .

For estimating I_3 we have to prove Lemma 4.3 with

$$\tilde{a}(s) = a(s) \frac{\sinh s}{s} F_k^{\frac{n}{2}}(s).$$

We had that a is bounded by a constant, its first derivative bounded by $ct^{-\frac{1}{2}}$, and its second derivative bounded by ct^{-1} . We want the same for \tilde{a} . As t is small, it will be sufficient to prove that $\frac{\sinh s}{s} F_k^{\frac{n}{2}}(s)$ is bounded, and also are its first and second derivative. Since we are in the case $n \geq 4$, Proposition 4.1 tells us already that its second derivative is bounded. Moreover, for $1 \leq k \leq \frac{n}{2}$, by (24) for $\alpha = 0$

$$\left| \frac{\sinh s}{s} F_k^{\frac{n}{2}}(s) \right| \leq \left(\frac{s}{\sinh s} \right)^{\frac{n-2}{2}},$$

and by (24) for $\alpha = 1$

$$\left| \partial_s \frac{\sinh s}{s} F_k^{\frac{n}{2}}(s) \right| \leq s \left(\frac{s}{\sinh s} \right)^{\frac{n-2}{2}} = \frac{s^2}{\sinh s} \left(\frac{s}{\sinh s} \right)^{\frac{n-4}{2}}.$$

Again, since we are in the case $n \geq 4$, these two quantities are bounded, so Lemma 4.3 still holds in the context of the Corollary.

In conclusion, up to the term J_2 , our integrals have the decay of Proposition 4.2

$$\sqrt{t} \sqrt{\frac{\rho}{\sinh \rho}}$$

improved by the extra term

$$\left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-2}{2}}.$$

Let us now first treat J_2 for obtaining (34). By (22),

$$\frac{\sinh s}{s} F_k^{\frac{n}{2}}(s) = \left(\frac{s}{\sinh s} \right)^{\frac{n-2}{2}}.$$

Therefore the same arguments as in Proposition 4.2 can be performed in J_2 , by replacing the function $\alpha(\tau)$ by

$$\alpha(\tau) \left(\frac{\tau \frac{t}{\rho} + \rho}{\sinh \left(\tau \frac{t}{\rho} + \rho \right)} \right)^{\frac{n-2}{2}},$$

which is still a decreasing function, bounded at 1 by $\left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-2}{2}}$. So J_2 will be also upper-bounded by $\left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-2}{2}}$ and (34) is proved.

In view to obtain (35), we restart the argument performed on J_2 in Proposition 4.2, with $\alpha(\tau)$ replaced by

$$\alpha(\tau) \frac{\sinh s}{s} F_k^{\frac{n}{2}}(s),$$

where

$$s = \tau \frac{t}{\rho} + \rho.$$

When getting to (31) we need to estimate

$$I_4 = \int_1^\infty \left| \partial_\tau \left(\alpha(\tau) \frac{\sinh s}{s} F_k^{\frac{n}{2}}(s) \right) \right| d\tau.$$

Now the function $\frac{\sinh s}{s} F_k^{\frac{n}{2}}(s)$ is not necessarily decreasing, so we are not able to get rid of the modulus in the integral, as easily as before. We will be able to estimate

$$(36) \quad I_4 \leq \frac{1}{\sqrt{t}} \left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-2}{2}},$$

that is with a loss of \sqrt{t} .

In the derivation of the product,

$$I_4 = \int_1^\infty \left| \alpha'(\tau) \frac{\sinh s}{s} F_k^{\frac{n}{2}}(s) + \alpha(\tau) \frac{t}{\rho} \partial_s \left(\frac{\sinh s}{s} F_k^{\frac{n}{2}}(s) \right) \right| d\tau,$$

we can use the estimate (24) for $\alpha \in \{0, 1\}$ and then

$$I_4 \leq \left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-2}{2}} \int_1^\infty |\alpha'(\tau)| d\tau + \left(\frac{\rho}{\sinh \rho} \right)^{\frac{n-2}{2}} \int_1^\infty \left| \alpha(\tau) \frac{t}{\rho} \left(\tau \frac{t}{\rho} + \rho \right) \right| d\tau.$$

The first integral is exactly the one in Proposition 4.2, so it has been proved that it is bounded independently of t and ρ . For treating the second integral, we shall use that for all odd N ,

$$\alpha(\tau) = \frac{1}{\sqrt{\cosh\left(\tau\frac{t}{\rho} + \rho\right) - \cosh\rho}} \leq \frac{1}{\sqrt{\sinh\rho}} \left(\tau\frac{t}{\rho}\right)^{-\frac{N}{2}}.$$

Using this estimate for $N = 5$ and for $N = 3$,

$$\int_1^\infty \left| \alpha(\tau) \frac{t}{\rho} \left(\tau\frac{t}{\rho} + \rho\right) \right| d\tau \leq \frac{1}{\sqrt{\sinh\rho}} \int_1^\infty \left(\left(\tau\frac{t}{\rho}\right)^{-\frac{5}{2}} \tau \left(\frac{t}{\rho}\right)^2 + \left(\tau\frac{t}{\rho}\right)^{-\frac{3}{2}} \frac{t}{\rho} \right) d\tau.$$

The integrals in τ are bounded, so

$$\int_1^\infty \left| \alpha(\tau) \frac{t}{\rho} \left(\tau\frac{t}{\rho} + \rho\right) \right| d\tau \leq \frac{1}{\sqrt{\sinh\rho}} \left(\frac{t}{\rho}\right)^{-\frac{1}{2}} (1 + \rho) \leq \frac{c}{\sqrt{t}},$$

so estimate (36) follows and the (35) is proved. \square

4.3. Large time dispersion for even dimensions. By using the development (26) of the dispersion kernel and the estimates (24) for $\alpha = 0$ we obtain

$$(37) \quad |K^n(t, \rho)| \leq \sum_{k=1}^{\frac{n}{2}} \frac{1}{t^{k+\frac{1}{2}}} \left(\frac{\rho}{\sinh\rho}\right)^{\frac{n-2}{2}} \left| \int_\rho^\infty \frac{s}{\sqrt{\cosh s - \cosh\rho}} ds \right|.$$

Let us notice that we have not taken into account the oscillatory phase in the integral. We shall split the remaining integral into two pieces

$$\left| \int_\rho^\infty \frac{s}{\sqrt{\cosh s - \cosh\rho}} ds \right| = \left| \int_0^\infty \frac{s + \rho}{\sqrt{\cosh(s + \rho) - \cosh\rho}} ds \right| = \int_0^1 + \int_1^\infty.$$

For $\rho > 0$, the first part is upper-bounded by

$$\int_0^1 \frac{s + \rho}{\sqrt{s \sinh\rho}} ds \leq c \frac{1 + \rho}{\sqrt{\sinh\rho}},$$

and the second, for a N odd integer sufficiently large, by

$$\int_1^\infty \frac{s + \rho}{\sqrt{s^N \sinh\rho}} ds \leq c \frac{1 + \rho}{\sqrt{\sinh\rho}}.$$

Now, by (37) we get for $\rho > 0$

$$|K^n(t, \rho)| \leq \sum_{k=1}^{\frac{n}{2}} \frac{1}{t^{k+\frac{1}{2}}} \left(\frac{\rho}{\sinh\rho}\right)^{\frac{n-2}{2}} \frac{1 + \rho}{\sqrt{\sinh\rho}}.$$

Therefore we have obtained the dispersion estimate (5) for large times, and the last part of Theorem 1.2 i) follows.

5. THE WEIGHTED DISPERSION ESTIMATES

Let us prove now the weighted dispersion (6). From (3) we have that for small times and $n \geq 3$

$$(38) \quad |u(t, \Omega)| \leq \frac{c}{|t|^{\frac{n}{2}}} \int_{\mathbb{H}^n} |u_0(\Omega')| \left(\frac{\rho}{\sinh\rho}\right)^{\frac{n-1}{2}} d\Omega' \leq \frac{c}{|t|^{\frac{n}{2}}} \int_{\mathbb{H}^n} |u_0(\Omega')| \frac{\rho}{\sinh\rho} d\Omega'.$$

The initial data is considered here radial, that is

$$u_0(\Omega') = u_0(\cosh r', \sinh r' \omega') = u_0(r').$$

Also in hyperbolic coordinates we can write, using (8) and (9)

$$\rho = d(\Omega, \Omega') = \cosh^{-1}[\Omega, \Omega'] = \cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r' \omega \cdot \omega').$$

Therefore, by passing in hyperbolic coordinates in (38),

$$(39) \quad |u(t, \Omega)| = |u(t, r)| \leq \frac{c}{|t|^{\frac{n}{2}}} \int_0^\infty |u_0(r')| K(t, r, r') \sinh^{n-1} r' dr',$$

where

$$K(t, r, r') = \int_{\mathbb{S}^{n-1}} \frac{\cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r' \omega \cdot \omega')}{\sinh \cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r' \omega \cdot \omega')} d\omega'.$$

By doing a rotation as in the proof of the Lemma 3.2,

$$K(t, r, r') = c \int_0^\pi \frac{\cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r' \cos \alpha)}{\sinh \cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r' \cos \alpha)} \sin^{n-2} \alpha d\alpha.$$

Now let us do the change of variable $\cos \alpha = x$ and get

$$K(t, r, r') = c \int_{-1}^1 \frac{\cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r' x)}{\sinh \cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r' x)} (1-x^2)^{\frac{n-3}{2}} dx.$$

Since here n is larger or equal to 3,

$$K(t, r, r') \leq c \int_{-1}^1 \frac{\cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r' x)}{\sinh \cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r' x)} dx.$$

Finally, let us do the change of variable

$$\cosh r \cosh r' - \sinh r \sinh r' x = \cosh y,$$

and then

$$K(t, r, r') \leq \frac{c}{\sinh r \sinh r'} \int_{|r-r'|}^{r+r'} y dy = c \frac{r r'}{\sinh r \sinh r'}.$$

Using now (39), we obtain that

$$|u(t, \Omega)| = |u(t, r)| \leq \frac{c}{|t|^{\frac{n}{2}}} \int_0^\infty |u_0(r')| \frac{r r'}{\sinh r \sinh r'} \sinh^{n-1} r' dr'.$$

Remembering that $r = d(\Omega, 0)$ we can go back to the Ω -coordinates and

$$|u(t, \Omega)| \frac{\sinh d(\Omega, 0)}{d(\Omega, 0)} \leq \frac{c}{|t|^{\frac{n}{2}}} \int_{\mathbb{H}^n} |u_0(\Omega')| \frac{d(\Omega', 0)}{\sinh d(\Omega', 0)} d\Omega',$$

so the weighted local dispersion estimate (6) is proved.

For proving the weighted Strichartz estimates we will apply the general lemma of Keel and Tao (Theorem 10.1 of [21]), in the case

$$H = B_0 = L_{rad}^2, \quad B_1 = L_{rad}^1(w^{-1}), \quad \sigma = \frac{n}{2}, \quad 2 \leq \frac{4}{n\theta}, \quad 0 \leq \theta \leq 1, \quad \left(\frac{4}{n\theta}, \theta, \sigma\right) \neq (2, 1, 1).$$

We obtain that for radial u

$$\|u\|_{L^{\frac{4}{n\theta}}([0, T], (L^2, L^1(w^{-1}))_{\theta, 2}^*)} \leq c \|u_0\|_{L^2},$$

where $(L^2, L^1(w^{-1}))_{\theta, 2}^*$ is the dual of the real interpolation space between L^2 and $L^1(w^{-1})$. The conditions on θ can be rewritten, for $n \geq 2$,

$$0 \leq \theta \leq \frac{2}{n}, \quad (\theta, n) \neq (1, 2).$$

For any $r \leq 2$ we have

$$(L^2, L^1(w^{-1}))_{\theta, r} \subseteq (L^2, L^1(w^{-1}))_{\theta, 2},$$

so that

$$(40) \quad \|u\|_{L^{\frac{4}{n\theta}}([0,T], (L^2, L^1(w^{-1}))_{\theta,r}^*)} \leq c \|u_0\|_{L^2}.$$

By the Theorem 5.5.1. of [5], for $0 < \theta < 1$,

$$(L^2, L^1(w^{-1}))_{\theta,r} = L^r(\tilde{w}),$$

with

$$\frac{1}{r} = \frac{1-\theta}{2} + \frac{\theta}{1}, \quad \tilde{w} = w^{-r\theta}.$$

So the value of r has to be $\frac{2}{1+\theta}$, which implies indeed that $r \leq 2$, and we have

$$(L^2, L^1(w))_{\theta,r} = L^{\frac{2}{1+\theta}}(w^{-\frac{2\theta}{1+\theta}}).$$

By using the classical formula

$$L^p(w)^* = L^{p'}(w^{-\frac{p'}{p}})$$

(see for example (7.4.15) of [15] with $s = 0$ and the definition of $L^p(w)$ corresponding to $L^p(w^{\frac{1}{p}})$ in our notations) we get that

$$(L^2, L^1(w))_{\theta,r}^* = L^{\frac{2}{1-\theta}}(w^{\frac{2\theta}{1-\theta}}).$$

Now the relation (40) gives us the weighted Strichartz estimates

$$\|u\|_{L^{\frac{4}{n\theta}}([0,T], L^{\frac{2}{1-\theta}}(w^{\frac{2\theta}{1-\theta}}))} \leq c \|u_0\|_{L^2},$$

for all θ in $]0, \frac{2}{n}]$ satisfying $(\theta, n) \neq (1, 2)$. Let us notice that we are able to include the value $\theta = 0$, since in this case the estimate corresponds to the mass conservation.

By denoting

$$p = \frac{4}{n\theta}, \quad q = \frac{2}{1-\theta},$$

we have that

$$\frac{2\theta}{1-\theta} = -2 + \frac{2}{1-\theta} = q - 2.$$

Also, the couple (p, q) satisfies $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$, $(p, q, n) \neq (2, \infty, 2)$ and $2 \leq p, q$, therefore the weighted Strichartz estimates (7) are found and the Theorem 1.2 is completely proved.

5.1. Proof of the Remark 1.4.

Let us choose a point Ω outside the support A of the initial data, and a point Ω' in A . Let us denote M a point of the intersection of the boundary of A with the part of geodesic relying Ω and Ω' . Then

$$d(\Omega, \Omega') = d(\Omega, M) + d(M, \Omega'),$$

and

$$d(\Omega, \Omega') \geq d(\Omega, {}^c A) + d(\Omega', A),$$

As the function $\frac{r}{\sinh r}$ is a decreasing function,

$$\frac{d(\Omega, \Omega')}{\sinh d(\Omega, \Omega')} \leq \frac{d(\Omega, {}^c A) + d(\Omega', A)}{\sinh(d(\Omega, {}^c A) + d(\Omega', A))}.$$

By using that

$$\frac{a+b}{\sinh(a+b)} \leq c \frac{a}{\sinh a} \frac{b}{\sinh b},$$

we finally obtain

$$\frac{d(\Omega, \Omega')}{\sinh d(\Omega, \Omega')} \leq c \frac{d(\Omega, {}^c A)}{\sinh d(\Omega, {}^c A)} \frac{d(\Omega', A)}{\sinh d(\Omega', A)}.$$

By using now the inequality (3) we obtain the estimate of the Remark 1.4.

5.2. Proof of the Remark 1.5.

We shall follow the same approach as in §3.2 for proving the weighted dispersion estimate (6).

In dimension 3, by the explicit form (1), the relation (38) becomes

$$u(t, \Omega) = \frac{c}{|t|^{\frac{3}{2}}} \int_{\mathbb{H}^n} u_0(\Omega') \frac{\rho}{\sinh \rho} e^{i\frac{\rho^2}{4t}} d\Omega',$$

and the kernel K given by

$$u(t, \Omega) = \frac{c}{|t|^{\frac{3}{2}}} \int_0^\infty |u_0(r')| K(t, r, r') \sinh^{n-1} r' dr',$$

is exactly

$$K(t, r, r') = \frac{c}{\sinh r \sinh r'} \int_{|r-r'|}^{r+r'} e^{i\frac{y^2}{4t}} y dy = t \frac{c}{\sinh r \sinh r'} e^{i\frac{r^2+r'^2}{4t}} \cos \frac{rr'}{2t}.$$

Therefore

$$|u(t, \Omega)| \sinh d(\Omega, 0) \leq \frac{c}{|t|^{\frac{1}{2}}} \int_{\mathbb{H}^n} |u_0(\Omega')| \frac{d\Omega'}{\sinh d(\Omega', 0)},$$

so the dispersion estimate of the Remark 1.5 is proved.

6. GLOBAL EXISTENCE AND BLOW-UP SOLUTIONS

6.1. Global existence.

The Sobolev embeddings have their analogue on hyperbolic space ([18])

$$\|v\|_{2^*} \leq K(n, 2) \|\nabla v\|_2 - \omega_n^{-\frac{2}{n}} \|v\|_2,$$

where $K(n, 2)$ is the best constant for the Sobolev embeddings on \mathbb{R}^n , and ω_n is the volume of the sphere \mathbb{S}^n . By interpolation between the L^2 and the L^{2^*} norms, we get the Gagliardo-Nirenberg inequality for functions on \mathbb{H}^n

$$\|v\|_{p+1}^{p+1} \leq C_{p+1} \|v\|_2^{2+(p-1)\frac{2-n}{2}} \|\nabla v\|_2^{(p-1)\frac{n}{2}}.$$

This inequality implies that the energy of the solution u of the equation (S) is bounded from below by

$$\|\nabla u\|_2^2 \left(\frac{1}{2} - \frac{C_{p+1}}{p+1} \|u\|_2^{2+(p-1)\frac{2-n}{2}} \|\nabla u\|_2^{(p-1)\frac{n}{2}-2} \right).$$

As a consequence, if $p < 1 + \frac{4}{n}$, since the mass is conserved, the gradient of u is controlled by the energy. Therefore the solution does not blow up and global existence occurs.

In the case $p = 1 + \frac{4}{n}$, if the mass of the initial condition is small enough so that

$$\|u\|_2^{\frac{4}{n}} < \frac{2 + \frac{4}{n}}{2C_{2+\frac{4}{n}}},$$

then the energy controls the gradient and again, the global existence is proved for the equation (S).

Remark 6.1. *The best constant in the Gagliardo-Nirenberg inequality can be proved to be larger or equal to the one on \mathbb{R}^n , but it is not obvious that it is exactly equal to it.*

6.2. Blow-up solutions.

The power $p = 1 + \frac{4}{n}$ shall be proved to be the critical power, in the sense that the nonlinearity is strong enough to generate solutions blowing up in a finite time. In the following we shall show the existence of blow-up solutions, by analyzing an appropriate virial function on hyperbolic space.

Let u be a radial solution of (S) and let h be a radial $C_0^\infty(\mathbb{H}^n)$ function. Then, by using the fact that u satisfies (S), we obtain the first derivative in time of a virial-type function

$$\partial_t \int_{\mathbb{H}^n} |u(t)|^2 h \, d\Omega = 2 \int_{\mathbb{H}^n} \Re(u(t)\bar{u}_t(t)) h \, d\Omega = 2 \int_{\mathbb{H}^n} \Im(u(t)\Delta\bar{u}(t)) h \, d\Omega.$$

By integrating by parts

$$(41) \quad \partial_t \int_{\mathbb{H}^n} |u(t)|^2 h \, d\Omega = -2 \int_{\mathbb{H}^n} \Im(u(t)\nabla\bar{u}(t)) \nabla h \, d\Omega.$$

By using again the equation (S) we obtain

$$\begin{aligned} \partial_t^2 \int_{\mathbb{H}^n} |u|^2 h &= -2 \int_{\mathbb{H}^n} \Re((\Delta u + |u|^{p-1}u)\nabla\bar{u}) \nabla h + 2 \int_{\mathbb{H}^n} \Re(u\nabla(\Delta\bar{u} + |u|^2\bar{u})) \nabla h \\ &= \int_{\mathbb{H}^n} 2|\nabla u|^2 \Delta h - \left(2 - \frac{4}{p+1}\right) |u|^{p+1} \Delta h - 4\Re(\Delta u \nabla\bar{u}) \nabla h - |u|^2 \Delta^2 h. \end{aligned}$$

Since u and h are radial functions, it follows that

$$\partial_t^2 \int_{\mathbb{H}^n} |u|^2 h = \int_{\mathbb{H}^n} 4|\nabla u|^2 \partial_r^2 h - 2\frac{p-1}{p+1} |u|^{p+1} \Delta h - |u|^2 \Delta^2 h.$$

By performing a density argument in the spirit of ([10], Lemmas 6.4.3-6), the previous identities are valid for

$$h(\Omega) = h(\cosh r, \sinh r\omega) = r^2.$$

By direct computation we obtain

$$\partial_r^2 r^2 = 2,$$

and

$$\Delta r^2 = 2 + 2(n-1) \frac{\cosh r}{\sinh r} r.$$

Therefore we can display the energy in the second derivative of the virial

$$(42) \quad \begin{aligned} \partial_t^2 \int_{\mathbb{H}^n} |u|^2 r^2 &= 16E(u) - \int_{\mathbb{H}^n} |u|^2 \Delta^2 r^2 - 2\frac{p-1}{p+1} \int_{\mathbb{H}^n} |u|^{p+1} 2(n-1) \left(\frac{\cosh r}{\sinh r} r - 1\right) \\ &\quad - \left(-\frac{16}{p+1} + 2\frac{p-1}{p+1} 2n\right) \int_{\mathbb{H}^n} |u|^{p+1}. \end{aligned}$$

We have

$$(43) \quad \Delta^2 r^2 = 4(n-1) \frac{1}{\sinh^3 r} (r \cosh r - \sinh r) + 2(n-1)^2 \frac{\cosh r}{\sinh^3 r} (\cosh r \sinh r - r),$$

nonnegative for all $r \geq 0$, since

$$\begin{cases} r \cosh r - \sinh r \geq 0 \\ \cosh r \sinh r - r \geq 0 \end{cases}.$$

Therefore, since p is considered larger than 1, the second and the third term in the right hand side of (42) are negative. The last one is also negative, provided that

$$-\frac{16}{p+1} + 4n \frac{p-1}{p+1} \geq 0,$$

that is exactly

$$p \geq 1 + \frac{4}{n}.$$

Let us notice, by (43), that $\Delta^2 r^2$ is bounded between two positive constants depending only on the dimension n ,

$$0 < k_n < \Delta^2 r^2 < K_n.$$

In conclusion, if u is a solution of (S) of initial data satisfying

$$16E(u_0) < k_n \|u_0\|_2^2,$$

there exists a constant C such that for any t

$$(44) \quad \partial_t^2 \int_{\mathbb{H}^n} |u(t)|^2 r^2 < C < 0.$$

It follows that there is a finite time T for which

$$\lim_{t \rightarrow T} \int_{\mathbb{H}^n} |u(t)|^2 r^2 = 0.$$

Then, using the uncertainty principle

$$\left(\int_{\mathbb{H}^n} |u|^2 \right)^2 \leq c \left(\int_{\mathbb{H}^n} |u|^2 r^2 \right) \left(\int_{\mathbb{H}^n} |\nabla u|^2 \right),$$

tells us that

$$\lim_{t \rightarrow T} \int_{\mathbb{H}^n} |\nabla u(t)|^2 = +\infty,$$

so the Theorem 1.8 is proved.

Finally, let us mention that in the Euclidean case, one has

$$\partial_t^2 \int_{\mathbb{R}^n} |u|^2 |x|^2 = 16E(u),$$

so the argument used before does not work for any solutions of null energy.

7. APPENDIX A

In this Appendix the following Proposition shall be proved.

Proposition 4.1 *For all $m \geq 1$ integer, the following identity holds*

$$(21) \quad \left(\frac{\partial_s}{\sinh s} \right)^m e^{i \frac{s^2}{4t}} = \sum_{k=1}^m \frac{e^{i \frac{s^2}{4t}}}{t^k} F_k^m(s),$$

where, modulo constants,

$$(22) \quad F_m^m(s) = \left(\frac{s}{\sinh s} \right)^m,$$

and, for $1 \leq k < m$, $F_k^m(s)$ can be written

$$(23) \quad F_k^m(s) = \sum_{r=0}^k \sum_{\substack{i_1 + \dots + i_r = m - k \\ i_j \geq 1}} \left(\frac{s}{\sinh s} \right)^{k-r} \left(\frac{\partial_s}{\sinh s} \right)^{i_1} \frac{s}{\sinh s} \dots \left(\frac{\partial_s}{\sinh s} \right)^{i_r} \frac{s}{\sinh s}.$$

Moreover, for $\alpha \in \{0, 1\}$ and $1 \leq k \leq m$

$$(24) \quad \left| \partial_s^\alpha \left(\frac{\sinh s}{s} F_k^m(s) \right) \right| \leq c s^\alpha \left(\frac{s}{\sinh s} \right)^{m-1},$$

and the second derivative of $\frac{\sinh s}{s} F_k^m(s)$ is bounded for $m \geq 2$.

Proof. Each time we differentiate the exponential $e^{i\frac{s^2}{4t}}$ we obtain a term with one negative power of t times $\frac{s}{\sinh s}$. Therefore the terms $F_k^m(s)$ are the ones corresponding to the derivation of $e^{i\frac{s^2}{4t}}$ k times, and they contain k times the function $\frac{s}{\sinh s}$ or its derivatives.

We have denoted in (23) by $k - r$ the number of times $\frac{s}{\sinh s}$ appears, where $0 \leq r \leq k$; the other r terms are derivatives of $\frac{s}{\sinh s}$. Since only k derivatives have fallen on the exponential term, there are $m - k$ left derivatives which act on the remainder term. That is the reason for which in (23) we have the sum

$$i_1 + \dots + i_r = m - k.$$

For showing the estimates of $F_k^m(s)$ we shall need the following Lemma.

Lemma 7.1. *The following property*

$$(45) \quad \exists c_l > 0 \quad \text{such that} \quad \left| \left(\frac{\partial_\rho}{\sinh \rho} \right)^l e^{i\rho^2} \right| \leq c_l \left(\frac{\rho}{\sinh \rho} \right)^l, \forall \rho \geq 0,$$

is valid for all positive integer l . From it follows also that

$$(46) \quad P(l) : \exists c_l > 0 \quad \text{such that} \quad \left| \left(\frac{\partial_\rho}{\sinh \rho} \right)^l \frac{\rho}{\sinh \rho} \right| \leq c_l \left(\frac{\rho}{\sinh \rho} \right)^{l+1}, \forall \rho \geq 0.$$

Proof. We shall split the proof into two cases, one when ρ is small, and one when it is large.

The case $\rho < 1$.

The function

$$G_l(\rho) = \left(\frac{\partial_\rho}{\sinh \rho} \right)^l e^{i\rho^2}$$

is a \mathcal{C}^∞ even function. It is obvious for $l = 0$, and for other values it can be proved by induction as follows. Supposing that the fact is true for l , we have the existence of a \mathcal{C}^∞ odd function H_l such that

$$G_l'(\rho) = H_l(\rho).$$

Therefore

$$G_{l+1}(\rho) = \frac{\partial_\rho}{\sinh \rho} G_l(\rho) = \frac{G_l'(\rho)}{\sinh \rho} = \frac{H_l(\rho)}{\sinh \rho},$$

and the induction argument is done.

As ρ is small, and G_l is a \mathcal{C}^∞ even function, we obtain the existence of a constant c such that

$$|G_l(\rho)| \leq c.$$

Again because ρ is small, we have the quotient $\frac{\rho}{\sinh \rho}$ bounded, and we can find for all integer l a constant c_l such that

$$|G_l(\rho)| \leq c_l \left(\frac{\rho}{\sinh \rho} \right)^l.$$

So in the case $\rho < 1$ the property (45) is satisfied.

The case $\rho \geq 1$.

Let us perform the change of variable $r = \cosh \rho$. Then

$$\frac{\partial_\rho}{\sinh \rho} = \partial_r,$$

and ρ can be written

$$\rho = \log(r + \sqrt{r^2 - 1}).$$

With these notations, the property (45) becomes

$$\left| \partial_r^l e^{i(\log(r + \sqrt{r^2 - 1}))^2} \right| \leq c_l \left(\frac{\log(r + \sqrt{r^2 - 1})}{\sqrt{r^2 - 1}} \right)^l, \forall \rho \geq 1.$$

Let us denote

$$a(r) = r + \sqrt{r^2 - 1},$$

and

$$b(r) = \log a(r).$$

Since r is larger than $\cosh 1$, then for proving the property (45) it suffices to prove that

$$(47) \quad \left| \partial_r^l e^{ib^2(r)} \right| \leq c_l \frac{b^l(r)}{r^l}, \forall r \geq 3.$$

In the following we shall use the following general formula

$$(48) \quad \partial_r^l f(g(r)) = \sum_{q=1}^l c_q f^{(q)}(g(r)) \sum_{p_1 + \dots + p_q = l} c_{p_1, \dots, p_q} \partial_r^{p_1} g(r) \dots \partial_r^{p_q} g(r).$$

This is a weaker form of the Faa di Bruno formula, in which the constants are described.

Let us notice first that

$$(49) \quad |\partial_r^p a(r)| \leq cr^{1-p},$$

and that, since r is large enough,

$$(50) \quad |\log^{(q)}(a(r))| \leq cr^{-q},$$

for all $q \geq 1$. Then, by using (48), for $l \geq 1$,

$$\partial_r^l \log(a(r)) = \sum_{q=1}^l c_q \log^{(q)}(a(r)) \sum_{p_1 + \dots + p_q = l} c_{p_1, \dots, p_q} \partial_r^{p_1} a(r) \dots \partial_r^{p_q} a(r).$$

Now the estimates (49) and (50) tell us that

$$|\partial_r^l \log(a(r))| \leq c \sum_{q=1}^l r^{-q} \sum_{p_1 + \dots + p_q = l} r^{1-p_1} \dots r^{1-p_q} = cr^{-l}.$$

Therefore we have obtained that for $l \geq 1$,

$$(51) \quad |\partial_r^l b(r)| \leq cr^{-l}.$$

Let us compute now the derivatives of b^2

$$\partial_r^l b^2(r) = \sum_{q=0}^l \partial_r^q b(r) \partial_r^{l-q} b(r) = 2b(r) \partial_r^l b(r) + \sum_{q=1}^{l-1} \partial_r^q b(r) \partial_r^{l-q} b(r).$$

By using (51) we get

$$|\partial_r^l b^2(r)| \leq c|b(r)|r^{-l} + \tilde{c} \sum_{q=1}^{l-1} r^{-q} r^{-(l-q)} \leq (c|b(r)| + c)r^{-l}.$$

As the variable r is large, it follows that

$$b(r) = \log(r + \sqrt{r^2 - 1}) \geq c,$$

and therefore

$$(52) \quad |\partial_r^l b^2(r)| \leq c|b(r)|r^{-l}.$$

Finally, by using again the general formula (48),

$$\left| \partial_r^l e^{ib^2(r)} \right| \leq c \sum_{q=1}^l \sum_{p_1+\dots+p_q=l} \partial_r^{p_1} b^2(r) \dots \partial_r^{p_q} b^2(r),$$

so the estimates (52) allows us to write

$$\left| \partial_r^l e^{ib^2(r)} \right| \leq c \sum_{q=1}^l b^q(r) r^{-l}.$$

We have already noticed that $b(r)$ is large, therefore

$$\left| \partial_r^l e^{ib^2(r)} \right| \leq c b^l(r) r^{-l},$$

and we finally proved (47), which implies the wanted property (45).

Property (46) can be proved by induction. $P(0)$ is obvious. Suppose that the properties $P(1), \dots, P(l-1)$ are true. Then the development (21) for $m = l+1$ will contain, by (22) and (23) only one term of the type

$$\left(\frac{\partial_\rho}{\sinh \rho} \right)^l \frac{\rho}{\sinh \rho},$$

all others being of order less than l . More precisely, we have, modulo constants,

$$\begin{aligned} & \left(\frac{\partial_s}{\sinh s} \right)^{l+1} e^{i\frac{s^2}{4t}} = \sum_{k=1}^{l+1} \frac{e^{i\frac{s^2}{4t}}}{t^k} F_k^{l+1}(s) = \\ & = \frac{e^{i\frac{s^2}{4t}}}{t} \left(\frac{\partial_\rho}{\sinh \rho} \right)^l \frac{\rho}{\sinh \rho} + \frac{e^{i\frac{s^2}{4t}}}{t^{l+1}} \left(\frac{s}{\sinh s} \right)^{l+1} + \\ & + \sum_{k=2}^l \frac{e^{i\frac{s^2}{4t}}}{t^k} \sum_{r=0}^k \sum_{\substack{i_1+\dots+i_r=l+1-k \\ i_j \geq 1}} \left(\frac{s}{\sinh s} \right)^{k-r} \left(\frac{\partial_s}{\sinh s} \right)^{i_1} \frac{s}{\sinh s} \dots \left(\frac{\partial_s}{\sinh s} \right)^{i_r} \frac{s}{\sinh s}. \end{aligned}$$

In the last term all index i_j are less than l , and the properties $P(1), \dots, P(l-1)$ tell us that the term is upper-bounded by

$$\left(\frac{s}{\sinh s} \right)^{l+1}.$$

Therefore, by using (45) we obtain that also the first term in the right-hand-side is bounded by the same quantity

$$\left| \left(\frac{\partial_\rho}{\sinh \rho} \right)^{l+1} \frac{\rho}{\sinh \rho} \right| \leq c_k \left(\frac{\rho}{\sinh \rho} \right)^{l+1},$$

so the property $P(l)$ is proven and the induction argument is complete. \square

Let us return now to the estimates of Proposition 4.1.

For $k = m$, estimate (24) when $\alpha = 0$ is obvious using (22). For $0 \leq k < m$, by (23) and we get that

$$\left| \frac{\sinh s}{s} F_k^m(s) \right| \leq \sum_{r=0}^k \sum_{\substack{i_1+\dots+i_r=m-k \\ i_j \geq 1}} \left(\frac{s}{\sinh s} \right)^{k-r-1} \left(\frac{s}{\sinh s} \right)^{i_1+1} \dots \left(\frac{s}{\sinh s} \right)^{i_r+1},$$

so estimate (24) when $\alpha = 0$ follows by using (46).

We start now to look at the derivatives. One has

$$(53) \quad \partial_s \left(\frac{\sinh s}{s} F_k^m(s) \right) = \frac{s \cosh s - \sinh s}{s^2} F_k^m(s) + \frac{\sinh^2 s}{s} \frac{\partial_s}{\sinh s} F_k^m(s)$$

The forms (22) and (23) of $F_k^m(s)$, together with estimate (46), imply

$$\left| \partial_s \left(\frac{\sinh s}{s} F_k^m(s) \right) \right| \leq \frac{s \cosh s - \sinh s}{s^2} \left(\frac{s}{\sinh s} \right)^m + \frac{\sinh^2 s}{s} \left(\frac{s}{\sinh s} \right)^{m+1}.$$

We deduce that

$$\left| \partial_s \left(\frac{\sinh s}{s} F_k^m(s) \right) \right| \leq s \left(\frac{s}{\sinh s} \right)^{m-1} \left(\frac{s \cosh s - \sinh s}{s^2 \sinh s} + 1 \right),$$

and estimate (24) for $\alpha = 1$ follows.

Knowing (53) one can get the expression of the second derivative

$$\begin{aligned} \partial_s^2 \left(\frac{\sinh s}{s} F_k^m(s) \right) &= \frac{s^2 \sinh s - 2(s \cosh s - \sinh s)}{s^3} F_k^m(s) + \\ &+ \frac{3s \cosh s \sinh s - 2 \sinh^2 s}{s^2} \frac{\partial_s}{\sinh s} F_k^m(s) + \frac{\sinh^3 s}{s} \left(\frac{\partial_s}{\sinh s} \right)^2 F_k^m(s). \end{aligned}$$

By using again the forms (22) and (23) of $F_k^m(s)$ and the estimate (46)

$$\begin{aligned} \left| \partial_s^2 \left(\frac{\sinh s}{s} F_k^m(s) \right) \right| &\leq \left(\frac{s}{\sinh s} \right)^{m-2} \left(\frac{s^2 \sinh s - 2(s \cosh s - \sinh s)}{s \sinh^2 s} + \right. \\ &\left. + s \frac{3s \cosh s - 2 \sinh s}{\sinh^2 s} + \frac{s^3}{\sinh s} \right). \end{aligned}$$

As the term in the brackets of the right-hand-side is bounded, for $m \geq 2$ we obtain that the second derivative of $\frac{\sinh s}{s} F_k^m(s)$ is bounded, and Proposition 4.1 is completely proved. \square

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