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ON SCATTERING FOR NLS: FROM EUCLIDEAN TO HYPERBOLIC SPACE

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Abstract. We prove asymptotic completeness in the energy space for the nonlinear Schrödinger equation posed on hyperbolic space $\mathbb{H}^n$ in the radial case, for $n \geqslant 4$, and any energy-subcritical, defocusing, power nonlinearity. The proof is based on simple Morawetz estimates and weighted Strichartz estimates. We investigate the same question on spaces which sort of interpolate between Euclidean space and hyperbolic space, showing that the family of short range nonlinearities becomes larger and larger as the space approaches the hyperbolic space. Finally, we describe the large time behavior of radial solutions to the free dynamics.

1. **Introduction.** Consider the defocusing nonlinear Schrödinger equation on Euclidean space

$$i\partial_t u + \Delta u = |u|^{2\sigma} u, \quad x \in \mathbb{R}^n, n \geqslant 3 \quad ; \quad u_{|t=0} = u_0 \in H^1(\mathbb{R}^n), \tag{1.1}$$

where Δ stands for the usual Laplacian. For $0 < \sigma \leqslant 2/(n-2)$, the solution to (1.1) is global in time, in the class of finite energy solutions [17, 13, 23, 26]. If in addition $\sigma > 2/n$, then there is scattering in H^1 [18, 13, 23, 26]:

$$\exists u_\pm \in H^1(\mathbb{R}^n), \quad \left\| u(t) - e^{it\Delta} u_\pm \right\|_{H^1} \underset{t\to\pm\infty}{\longrightarrow} 0.$$

On the other hand, if σ is too small, then long range effects are present, and the above result holds only in the trivial case [7, 24]: if $\sigma \leqslant 1/n$ and $u_+ \in L^2(\mathbb{R}^n)$, $u \in C(\mathbb{R}; L^2(\mathbb{R}^n))$ are such that

$$\left\| u(t) - e^{it\Delta} u_+ \right\|_{L^2} \underset{t\to+\infty}{\longrightarrow} 0,$$

then necessarily $u_+ = u = 0$ (even if the functions are supposed to be radial). In other words, linear and nonlinear dynamics are not comparable for large time if $\sigma \leqslant 1/n$. In this paper, we show that this phenomenon disappears for radial solutions, when the space variable belongs to the hyperbolic space instead of the

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Euclidean space. Such a phenomenon was established in [5] in the three-dimensional case, with partial results in other dimensions. We prove asymptotic completeness in the case of higher dimensions. Moreover, we consider rotationally symmetric manifolds, which may be viewed as interpolations between Euclidean and hyperbolic spaces, as introduced in [6]. We show that asymptotic completeness holds for radial solutions and $\sigma_0(n) < \sigma$, for some explicit value $\sigma_0(n)$, going to zero as the space approaches the hyperbolic space. The proof relies on simple Morawetz estimates (as opposed to interaction Morawetz estimates, as introduced in [12]), and weighted Strichartz estimates [22, 6]. The energy-critical case is not considered: we always assume $\sigma < 2/(n-2)$. So far, scattering in the energy-critical case is known in the Euclidean case ([13, 23, 26]), but is open even for radial solutions on the hyperbolic space.

We begin with the nonlinear Schrödinger equation on hyperbolic space

$$i\partial_t u + \Delta_{\mathbb{H}^n} u = |u|^{2\sigma} u, \quad x \in \mathbb{H}^n \quad ; \quad u_{|t=0} = u_0 \in H^1(\mathbb{H}^n), \tag{1.2}$$

where $x = (\cosh r, \omega \sinh r) \in \mathbb{H}^n \subset \mathbb{R}^{n+1}$, $r \geqslant 0$, $\omega \in \mathbb{S}^{n-1}$ and

$$\Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1)\frac{\cosh r}{\sinh r}\partial_r + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}.$$

In [5] it has been proved that for small radial initial data, there is asymptotic completeness in L^2 for all $0 < \sigma < 2/n$ and $n \geqslant 2$. At the H^1 level, wave operators were proved to exist for all $0 < \sigma < 2/(n-2)$ and $n \geqslant 2$, without restriction on the size of the radial data. The main ingredient were radial Strichartz estimates similar to those used on $\mathbb{R}^d$, with arbitrary $d \geqslant n$; when working on $\mathbb{H}^n$, we can pretend that we are on $\mathbb{R}^d$, where $d \geqslant n$ is arbitrary. Typically, the assumption $\sigma > 2/d$, which is made on $\mathbb{R}^d$ to have scattering in the energy space, boils down to $\sigma > 0$, since $d \geqslant n$ is arbitrary. Such estimates stem from weighted Strichartz estimates on $\mathbb{H}^n$ in the radial case (see [4] for $n = 3$, [22] for $n \geqslant 4$, and [5] for $n = 2$). Finally, asymptotic completeness was proved for all $0 < \sigma < 2/(n-2)$, without restriction on the size of the radial data, but only in dimension $n = 3$. The latter result used in addition interaction Morawetz estimates, valid also in the non-radial case, in all dimensions $n \geqslant 3$. The issue for $n \geqslant 4$ was that the passage from the interaction Morawetz estimates to global in time estimates in mixed spaces is done *via* a delicate Fourier argument on $\mathbb{R}^n$ [25], difficult to adapt to hyperbolic space. Moreover, the historical approach based on simple Morawetz estimates relies on a precise dispersive rate for the free Schrödinger group (see e.g. [11]),

$$\left\| e^{it\Delta_{\mathbb{R}^n}} \right\|_{L^1(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \leqslant \frac{C}{|t|^{n/2}}, \quad \forall t \neq 0.$$

Such an estimate is known on $\mathbb{H}^3$, and on $\mathbb{H}^n$ with $n \neq 4$ locally in time (say, for $|t| \leqslant 1$); see [4]. For large time, dispersion estimates are proven with the decay rate $|t|^{-3/2}$ (see also [5] for the global exact dispersion in radial setting for $n = 2$, and [1], [20] for exact large time dispersion in all even dimensions). So for $n \geq 4$ the global dispersion is not available. Let us notice that in [22], global weighted Strichartz estimates are proven in the radial setting, thanks to a change of unknown function after which these estimates stem from [9], where dispersion is not used either.

In this paper we cover the cases $n \geqslant 4$ by using simple Morawetz estimate and weighted Strichartz estimates. We focus on the radial case.

Remark 1.1. Quite simultaneously to this work, the existence of scattering operators in $H^1(\mathbb{H}^n)$ for $n \geqslant 2$ and $0 < \sigma < 2/(n-2)$ was established in [20] (see also [1]), without the radial symmetry assumption that we make in this paper. The authors have derived new Morawetz estimates, which overcome the difficulties pointed out above, thanks also to new Strichartz estimates. Our point of view in the present paper is rather to insist on the transition between Euclidean to hyperbolic geometry, as explained below. Also, the proof of the asymptotic completeness in the radial case is naturally shorter, and serves as a basis to study the case of intermediary metrics, where in addition no Fourier analysis seems to available.

Theorem 1.2. *Let $n \geqslant 4$ and*

$$0 < \sigma < \frac{2}{n-2}.$$

Then asymptotic completeness holds in $H^1_{\mathrm{rad}}(\mathbb{H}^n)$ for (1.2): for all $u_0 \in H^1_{\mathrm{rad}}(\mathbb{H}^n)$, there exists $u_+ \in H^1_{\mathrm{rad}}(\mathbb{H}^n)$ such that

$$\left\| u(t) - e^{it\Delta_{\mathbb{H}^n}} u_+ \right\|_{H^1(\mathbb{H}^n)} \underset{t\to+\infty}{\longrightarrow} 0,$$

where u is the solution to (1.2).

In view of [5], the wave operators $W_\pm$ are well-defined on $H^1_{\mathrm{rad}}(\mathbb{H}^n)$ for this range of σ. The above result shows that the wave operators are invertible on $H^1_{\mathrm{rad}}(\mathbb{H}^n)$ ($u_+ = W_+^{-1} u_0$), so we infer the existence of a scattering operator for arbitrarily large data, with no long range effect. This extends the result of [5], established for $n = 3$ only.

Corollary 1.3. *For $n \geqslant 3$, and*

$$0 < \sigma < \frac{2}{n-2},$$

the scattering operator $S = W_+^{-1} W_-$ associated to (1.2) is well-defined from $H^1_{\mathrm{rad}}(\mathbb{H}^n)$ to $H^1_{\mathrm{rad}}(\mathbb{H}^n)$: for all $u_- \in H^1_{\mathrm{rad}}(\mathbb{H}^n)$, there exists $u \in C(\mathbb{R}; H^1_{\mathrm{rad}}(\mathbb{H}^n))$ solution to

$$i\partial_t u + \Delta_{\mathbb{H}^n} u = |u|^{2\sigma} u,$$

such that

$$\left\| u(t) - e^{it\Delta_{\mathbb{H}^n}} u_- \right\|_{H^1(\mathbb{H}^n)} \underset{t\to-\infty}{\longrightarrow} 0,$$

and a unique $u_+ = Su_- \in H^1_{\mathrm{rad}}(\mathbb{H}^n)$ such that

$$\left\| u(t) - e^{it\Delta_{\mathbb{H}^n}} u_+ \right\|_{H^1(\mathbb{H}^n)} \underset{t\to+\infty}{\longrightarrow} 0.$$

The absence of long range effects is of course an effect of the geometry of the hyperbolic space. Typically, the usual algebraic decay on $\mathbb{R}^n$ is replaced by an exponential decay. This vague statement can be compared to the phenomenon studied in [10], where instead of changing the geometry of the space, an external potential was added:

$$i\partial_t u + \Delta u = -|x|^2 u + |u|^{2\sigma} u, \quad x \in \mathbb{R}^n \quad ; \quad u_{|t=0} = u_0 \in \Sigma = H^1 \cap \mathcal{F}(\mathbb{H}^1).$$

The effect of this repulsive harmonic potential (as opposed to the usual harmonic potential $+|x|^2$) is to accelerate the particle which goes to infinity exponentially fast, so that asymptotic completeness holds in Σ for any $0 < \sigma < 2/(n-2)$ (no long range effect). In [8] *linear* scattering theory was considered for perturbations of the Hamiltonian $-\Delta - |x|^\alpha$, for $0 < \alpha \leqslant 2$. It is shown that the borderline between short range and long range moves as α varies from 0 to 2. Essentially, a potential V is short range as soon as $|V(x)| \lesssim \langle x\rangle^{-1+\alpha/2-\varepsilon}$ when $\alpha < 2$, and

$|V(x)| \lesssim (1 + \log \langle x \rangle)^{-1-\varepsilon}$ when $\alpha = 2$, for some $\varepsilon > 0$; the dynamics generated by $-\Delta - |x|^\alpha$ accelerates the particles, from an algebraic decay with a larger and larger power, to the limiting exponential case (if $\alpha > 2$, the underlying operator is not even essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$, due to infinite speed of propagation, see e.g. [15]). Note that nonlinear perturbations of $-\Delta - |x|^\alpha$ for $0 < \alpha < 2$ have not been studied, due to a lack of suitable technical tools. In the present paper, we analyze what can be considered as the geometrical counterpart of this problem.

Notation 1.4. Let $k \in \mathbb{N}$ and

$$\phi(r) = \sum_{j=0}^{k} \frac{1}{(2j+1)!} r^{2j+1}.$$

We denote by M_k^n (or simply M when there is no possible confusion) the n-dimensional rotationally symmetric manifold with metric

$$ds^2 = dr^2 + \phi(r)^2 d\omega^2,$$

where $d\omega^2$ stands for the metric on $\mathbb{S}^{n-1}$.

The Laplace–Beltrami operator on M_k^n is

$$\Delta_M = \partial_r^2 + (n-1)\frac{\phi'(r)}{\phi(r)}\partial_r + \frac{1}{\phi(r)^2}\Delta_{\mathbb{S}^{n-1}}.$$

Remark 1.5. If $k = 0$, we recover the Euclidean case. The hyperbolic case corresponds to $k = \infty$. The manifold M_k^n can thus be viewed as an interpolation between these two cases. Note also that for finite k, the volume element of M behaves at infinity like the one on $\mathbb{R}^N$, for $N = (2k+1)(n-1)+1$, a parameter which turns out to play an crucial role in scattering theory, as shown below.

Theorem 1.6. *Let $n \geqslant 4$. For $k \in \mathbb{N}$, consider the nonlinear Schrödinger equation*

$$i\partial_t u + \Delta_M u = |u|^{2\sigma} u, \quad x \in M_k^n \quad ; \quad u_{|t=0} = u_0 \in H^1_{\rm rad}(M_k^n). \tag{1.3}$$

Set $N = (2k+1)(n-1)+1$. For $2/N < \sigma < 2/(n-2)$, asymptotic completeness holds in $H^1_{\rm rad}(M)$: for all $u_0 \in H^1_{\rm rad}(M)$, there exists $u_+ \in H^1_{\rm rad}(M)$ such that

$$\left\| u(t) - e^{it\Delta_M} u_+ \right\|_{H^1(M)} \underset{t\to+\infty}{\longrightarrow} 0.$$

From the above example, we see that this result is a transition between Euclidean ($k = 0$) and hyperbolic ($k \to \infty$) cases. In view of the results of [6], we infer

Corollary 1.7. *For $n \geqslant 4$, $k \in \mathbb{N}$, $N = (2k+1)(n-1)+1$ and*

$$\frac{2}{N} < \sigma < \frac{2}{n-2},$$

the scattering operator $S = W_+^{-1} W_-$ associated to (1.3) is well-defined from $H^1_{\rm rad}(M)$ to $H^1_{\rm rad}(M)$: for all $u_- \in H^1_{\rm rad}(M)$, there exists $u \in C(\mathbb{R}; H^1_{\rm rad}(M))$ solution to

$$i\partial_t u + \Delta_M u = |u|^{2\sigma} u,$$

such that

$$\left\| u(t) - e^{it\Delta_M} u_- \right\|_{H^1(M)} \underset{t\to-\infty}{\longrightarrow} 0,$$

and a unique $u_+ = S u_- \in H^1_{\rm rad}(M)$ such that

$$\left\| u(t) - e^{it\Delta_M} u_+ \right\|_{H^1(M)} \underset{t\to+\infty}{\longrightarrow} 0.$$

Remark 1.8. We see that as soon as $k \geqslant 1$, $2/N < 1/n$. In view of the results of [7, 24], this shows that the curved geometry already changes the short range/long range borderline. In §5, we present a formal argument (rigorous justification is left out), relying on the description of the free dynamics (see Proposition 1.12 below) indicating that for $\sigma \leqslant 1/N$, long range effects are present (see Remark 5.2).

Remark 1.9. The proof we present still works for other functions ϕ. We choose to restrict our attention to such spaces M_k^n in order to emphasize the transition between Euclidean and hyperbolic spaces.

Remark 1.10. The existence of a "scattering" dimension $N = (2k+1)(n-1)+1$ can be compared to Sobolev embeddings on the Heisenberg group. It is shown in [3] that the indices for Sobolev embeddings on the $(2n+1)$-dimensional Heisenberg group correspond to their counterparts on $\mathbb{R}^{2n+2}$.

Remark 1.11. Such a scattering dimension appears in [6] under more general (and more geometrically relevant) assumptions on the manifold M: the function ϕ does not necessarily have the precise form we study in the present paper. These assumptions are related to the growth of the volume element and to the sectional curvature of the manifold M, and imply the existence of wave operators for a larger range of nonlinearities than in the Euclidean case [6]. However, generalizing the Morawetz estimates proved in Lemma 2.3 below (which is a crucial step to prove asymptotic completeness) under these more general assumptions remains an interesting open question.

To conclude this introduction, and give a rather general picture of large time dynamics of solutions to Schrödinger equations, we describe the asymptotic behavior of the free dynamics in the radial setting. It seems that the analogous result in the non-radial case, even on hyperbolic space, is not available so far.

Proposition 1.12. *Let $n \geqslant 2$.*
(1) *Consider the linear equation*

$$i\partial_t u + \Delta_{\mathbb{H}^n} u = 0, \quad x \in \mathbb{H}^n \quad ; \quad u_{|t=0} = u_0 \in L^2_{\rm rad}(\mathbb{H}^n).$$

There exists a linear operator $\mathcal{L}$, unitary from $L^2_{\rm rad}(\mathbb{H}^n)$ to $L^2_{\rm rad}(\mathbb{R}^n)$, such that

$$\|u(t) - v(t)\|_{L^2(\mathbb{H}^n)} \underset{t\to+\infty}{\longrightarrow} 0,$$
$$\text{where } v(t,r) = \frac{e^{-i(n-1)t/2 + ir^2/(4t)}}{t^{n/2}} \left(\frac{r}{\sinh r}\right)^{\frac{n-1}{2}} (\mathcal{L}u_0)\left(\frac{r}{t}\right).$$

(2) *Let $k \geqslant 1$. Consider the linear equation*

$$i\partial_t u + \Delta_M u = 0, \quad x \in M_k^n \quad ; \quad u_{|t=0} = u_0 \in L^2_{\rm rad}(M_k^n).$$

There exists a linear operator $\mathcal{L}$, unitary from $L^2_{\rm rad}(M_k^n)$ to $L^2_{\rm rad}(\mathbb{R}^n)$, such that

$$\|u(t) - v(t)\|_{L^2(M)} \underset{t\to+\infty}{\longrightarrow} 0,$$
$$\text{where } v(t,r) = \frac{e^{ir^2/(4t)}}{t^{n/2}} \left(\frac{r}{\phi(r)}\right)^{\frac{n-1}{2}} (\mathcal{L}u_0)\left(\frac{r}{t}\right).$$

Remark 1.13. In the Euclidean case $k = 0$, $\mathcal{L}$ is, up to a multiplicative constant and a dilation, the usual Fourier transform. In the case of $\mathbb{H}^3$, the first point of Proposition 1.12 was established in [5]. There again, $\mathcal{L}$ is essentially the Fourier transform. It is not clear whether the same holds in the case of $\mathbb{H}^n$, for $n \neq 3$, where Fourier analysis is well developed. See Remark 5.1.

The rest of the paper is organized as follows. In the next paragraph, we recall the general approach for Morawetz inequalities, and give applications for the case of defocusing nonlinear Schrödinger equations on $\mathbb{H}^n$ or M. We prove Theorems 1.2 and 1.6 in §3 and §4, respectively. Proposition 1.12 is established in §5.

2. **Morawetz inequality.** We first recall the general virial computation on a manifold M, where technical ingredients such as integration by parts work as in the Euclidean case. Typically, M can be chosen to be $\mathbb{R}^n$, $\mathbb{R}^n\times\mathbb{R}^n$, $\mathbb{H}^n$ or M_k^n, with no restriction on the dimension. The homogeneous contribution is treated in [19], and the inhomogeneous case is easily inferred.

Lemma 2.1 (Virial inequality). *Let a be a real function on M with positive Hessian. If v is a global $L^\infty(\mathbb{R},H^1(M))$ solution of*

$$i\partial_t v+\Delta_M v=Fv \quad ; \quad v_{|t=0}=v_0\in H^1(M), \tag{2.1}$$

then there exists a positive constant C such that

$$\begin{aligned}\int_0^T \Big(\int_M(-\Delta^2 a)\frac{|v|^2}{2}+\mathrm{Re}\int_M 2Fv\nabla\overline{v}\cdot\nabla a+\overline{F}|v|^2\Delta a\Big)\leqslant&\\ \leqslant C\sup_{t\in[0,T]}\int_M|\overline{v}\nabla v\cdot\nabla a|.&\end{aligned} \tag{2.2}$$

Lemma 2.2 (Morawetz inequality on $\mathbb{H}^n$). *Let $n\geqslant 4$. All solutions u of equation* (1.2) *(not necessarily radial) satisfy to*

$$\int_0^T\int_{\mathbb{H}^n}\frac{\cosh r}{\sinh^3 r}|u(t,x)|^2dx\,dt\leqslant C\sup_{t\in[0,T]}\|u(t)\|_{H^1}^2, \tag{2.3}$$

where $r=d_{\mathbb{H}^n}(O,x)$.

Proof. We apply Lemma 2.1, with $M=\mathbb{H}^n$, $u=v$, $F=|u|^{2\sigma}$ and $a(x)=r=d_{\mathbb{H}^n}(O,x)$. In the left hand side of (2.2), the contribution of the nonlinearity is

$$\mathcal{N}=\int_0^T\left(\int_{\mathbb{H}^n}\frac{2}{2\sigma+2}\nabla(|u|^{2\sigma+2})\cdot\nabla a+|u|^{2\sigma+2}\Delta a\right),$$

so by integrating by parts the first term,

$$\mathcal{N}=\int_0^T\left(\int_{\mathbb{H}^n}-\frac{1}{\sigma+1}|u|^{2\sigma+2}\Delta a+|u|^{2\sigma+2}\Delta a\right)=\int_0^T\int_{\mathbb{H}^n}\frac{\sigma}{\sigma+1}|u|^{2\sigma+2}\Delta a.$$

Since $\Delta a=(n-1)\frac{\cosh r}{\sinh r}$, the nonlinear contribution is non-negative (defocusing nonlinearity), and we get

$$\int_0^T\int_M(-\Delta^2 a)\frac{|u|^2}{2}\leqslant C\sup_{t\in[0,T]}\|u(t)\|_{H^1}^2.$$

By computing $-\Delta^2 a(x)=(n-1)(n-3)\dfrac{\cosh r}{\sinh^3 r}$, the lemma follows. □

Lemma 2.3 (Morawetz inequality on M). *Let $n\geqslant 4$ and $k\in\mathbb{N}$. All solutions to* (1.3) *(not necessarily radial) satisfy to*

$$\int_0^T\int_M\frac{1}{r^3}|u(t,x)|^2dxdt\leqslant C\sup_{t\in[0,T]}\|u(t)\|_{H^1(M)}^2, \tag{2.4}$$

where $r=d_M(O,x)$.

Proof. The proof follows the same lines as above. For $k=0$, this is the standard Morawetz estimate; see e.g. [11]. We therefore assume $k \geqslant 1$. The manifold M_k^n has a negative sectional curvature, so the Hessian of the function distance to the origin is positive (Theorem 3.6 of §6 in [21]). For $a(x) = r = d_M(O,x)$, we compute

$$\Delta_M a = (n-1)\frac{\phi'}{\phi},$$
$$\Delta_M^2 a = \frac{n-1}{\phi^3}\left(\phi^2\phi^{(3)} + (n-4)\phi\phi'\phi'' - (n-3)\left(\phi'\right)^3\right).$$

Since $\Delta_M a$ is non-negative, the nonlinear term is neglected, just like in the proof of Lemma 2.2. We check

$$-\Delta_M^2 a \underset{r\to 0}{\sim} (n-1)(n-3)\frac{1}{r^3},$$
$$-\Delta_M^2 a \underset{r\to \infty}{\sim} (n-1)(2k+1)\left(2k(n-1)+n-3\right)\frac{1}{r^3}.$$

To establish the lemma, it suffices to prove that $-\Delta_M^2 a > 0$ for $r > 0$. Write the numerator of $-\Delta_M^2 a$ as

$$(n-1)\left((n-3)\phi'\left(\left(\phi'\right)^2 - \phi\phi''\right) + \phi\left(\phi'\phi'' - \phi\phi^{(3)}\right)\right).$$

We claim that for all $r > 0$ (and $k \geqslant 1$),

$$\phi'(r)\left(\left(\phi'(r)\right)^2 - \phi(r)\phi''(r)\right) > 1 \quad ; \quad \phi(r)\left(\phi'(r)\phi''(r) - \phi(r)\phi^{(3)}(r)\right) > 0.$$

Since

$$\phi'\phi'' - \phi\phi^{(3)} = \left(\left(\phi'\right)^2 - \phi\phi''\right)',$$

and $\phi(0) = \phi''(0) = 0$ and $\phi'(0) = 1$, it suffices to show that the above quantity is non-negative. From

$$\phi''(r) = \phi(r) - \frac{1}{(2k+1)!}r^{2k+1}, \quad \phi^{(3)}(r) = \phi'(r) - \frac{1}{(2k)!}r^{2k},$$

we infer

$$\left(\phi'\phi'' - \phi\phi^{(3)}\right)(r) = -\frac{\phi'(r)}{(2k+1)!}r^{2k+1} + \frac{\phi(r)}{(2k)!}r^{2k}$$
$$= \sum_{j=0}^{k} \frac{1}{(2j)!(2k)!}\left(\frac{1}{2j+1} - \frac{1}{2k+1}\right) r^{2k+2j+1} > 0.$$

The estimate announced above follows, hence the lemma. □

Remark 2.4. If the function ϕ is replaced by

$$\phi(r) = \sum_{j=0}^{k} \frac{a_j}{(2j+1)!} r^{2j+1}$$

for some $a_j > 0$, then the results of [6] show that weighted Strichartz estimates are available in the radial setting, showing the existence of wave operators with the same algebraic conditions as in Theorem 1.6. However, for $k \geqslant 2$ and a general family $(a_j)_{0\leqslant j\leqslant k}$ of positive numbers, it is not clear whether the analogue of the above lemma is valid or not: it may very well happen that with our choice for a, $-\Delta_M^2 a$ has some zero for $0 < r < \infty$, thus ruining the above argument.

3. **Asymptotic completeness in hyperbolic space.** In this paragraph, we prove Theorem 1.2.

Since we are in a defocusing case, we have a global in time *a priori* estimate for the H^1-norm of u, hence the following control, without radial assumption:

$$\left\| \sqrt{\frac{\cosh r}{\sinh^3 r}} u(t,x) \right\|_{L^2(\mathbb{R},L^2(\mathbb{H}^n))} \leqslant C(u_0). \tag{3.1}$$

This global control will allow us to prove that u belongs globally in time to certain weighted mixed spaces, yielding asymptotic completeness. We set:

$$\mathtt{w}_n = \mathtt{w}_n(r) = \left(\frac{\sinh r}{r}\right)^{\frac{n-1}{2}},$$

and we denote by $d\Omega$ the measure on $\mathbb{H}^n$. We recall that (p,q) is n-admissible if

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad p \geqslant 2, \quad (p,q,n) \neq (2,\infty,2). \tag{3.2}$$

We shall use the following global Strichartz estimates for the radial free evolution, established in [22] for $n \geqslant 4$:

$$\left\| e^{it\Delta_{\mathbb{H}^n}} f(\cdot) \right\|_{L^p(\mathbb{R},L^q(\mathtt{w}_n^{q-2} d\Omega))} \leqslant C \|f\|_{L^2}, \tag{3.3}$$

$$\left\| \int_{I\cap\{s\leqslant t\}} e^{i(t-s)\Delta_{\mathbb{H}^n}} F(s) ds \right\|_{L^p(I,L^q(\mathtt{w}_n^{q-2} d\Omega))} \leqslant C \left\| F \right\|_{L^{r'}\left(I,L^{s'}(\mathtt{w}_n^{s'-2} d\Omega)\right)}, \tag{3.4}$$

for all radial functions $f \in L^2_{\text{rad}}(\mathbb{H}^n)$, $F \in L^{r'}\left(I; L^{s'}_{\text{rad}}\left(\mathbb{H}^n, \mathtt{w}_n^{s'-2} d\Omega\right)\right)$ and every n-admissible pairs (p,q) and (r,s). If A is a derivative in space of order one, similar estimates hold with the operator A in front of f and of the integral in (3.4). The constants are independent of the time interval I.

Let $A \in \{\text{Id}, \nabla\}$. In view of the above Strichartz estimates, we wish to control

$$\mathtt{w}_n^{1-2/q'} A\left(|u|^{2\sigma} u\right) \text{ in } L^{p'}\left(I; L^{q'}\right)$$

by some power of $\|u\|_{X(I)}$, where

$$\begin{aligned} X(I) = \Big\{ v \in L^\infty\left(I, H^1(d\Omega)\right) \cap L^2\left(I, W^{1,2^*}(\mathtt{w}_n^{2^*-2} d\Omega)\right), \\ \|v\|_{X(I)} = \|v\|_{L^\infty(I,H^1(d\Omega))} + \|v\|_{L^2\left(I,W^{1,2^*}(\mathtt{w}_n^{2^*-2} d\Omega)\right)} < \infty \Big\}, \end{aligned} \tag{3.5}$$

and $2^* = \frac{2n}{n-2}$ (the pair $(2,2^*)$ is admissible, since $n \geqslant 3$). This is achieved in the following lemma.

Lemma 3.1. *Fix $n \geqslant 4$ and $0 < \sigma < \frac{2}{n-2}$. Let u be a radial solution to (1.2), and $A \in \{\text{Id}, \partial_r\}$. There exist an n-admissible pair (p,q), $0 < \alpha < 2\sigma$, and $C > 0$ such that for all time interval I,*

$$\left\| \mathtt{w}_n^{1-2/q'} A\left(|u|^{2\sigma} u\right) \right\|_{L^{p'}(I;L^{q'})} \leqslant C \|f\|_{L^1(I\times\mathbb{H}^n)}^{\alpha/2} \|u\|_{X(I)}^{2\sigma+1-\alpha}, \tag{3.6}$$

where $X(I)$ is defined in (3.5), and

$$f(t,r) = \frac{\cosh r}{\sinh^3 r} |u(t,r)|^2.$$

Proof. First, note that we have the uniform point-wise estimate

$$\left|A\left(|u|^{2\sigma}u\right)\right| \lesssim |u|^{2\sigma}\,|Au|\,.$$

We want to apply Hölder's inequality, after the following splitting:

$$\mathtt{w}_n^{1-2/q'}|u|^{2\sigma}\,|Au| = \left(\sqrt{\frac{\cosh r}{\sinh^3 r}}|u|\right)^{\alpha} \times |u|^{2\sigma-\alpha}$$

$$\times\, \mathtt{w}_n^{2/n}\,|Au|\, \times \mathtt{w}_n^{1-2/q'}\mathtt{w}_n^{-2/n}\left(\frac{\sinh^3 r}{\cosh r}\right)^{\alpha/2}. \quad (3.7)$$

The first term will be estimated in $L^{2/\alpha}_{t,x}$, the second in $L^\infty_t L_x^{2^*/(2\sigma-\alpha)}$, the third in $L^2_t L^{2^*}_x$, and the last in $L^\infty_t L^\theta_x$, where $2^* = \frac{2n}{n-2}$. Write

$$\frac{1}{q'} = \frac{\alpha}{2} + \frac{2\sigma-\alpha}{2^*} + \frac{1}{2^*} + \frac{1}{\theta} \quad ; \quad \frac{1}{p'} = \frac{\alpha}{2} + \frac{1}{2}. \quad (3.8)$$

Thanks to Sobolev embedding, the third term will be controlled by

$$\|u\|^{2\sigma-\alpha}_{L^\infty_t L^{2^*}_x} \lesssim \|u\|^{2\sigma-\alpha}_{L^\infty_t H^1_x} \lesssim \|u\|^{2\sigma-\alpha}_{X(I)}\,.$$

Since the pair $(2, 2^*)$ is n-admissible (endpoint), the lemma will follow if we can choose (p, q) and α such that:

- (p, q) is n-admissible.
- $\alpha > 0$, with $2\sigma - \alpha > 0$. (Morally, $0 < \alpha \ll 1$.)
- The last factor in (3.7) is in $L^\theta(\mathbb{H}^n)$, for some $\theta \in [1, \infty[$.

If p is imposed in view of (3.8), that is

$$\frac{1}{p} = \frac{1}{2} - \frac{\alpha}{2},$$

then (p, q) is n-admissible if

$$\frac{1}{q} = \frac{1}{2^*} + \frac{\alpha}{n}. \quad (3.9)$$

This is consistent with the first equality of (3.8) provided that

$$\frac{1}{\theta} = \frac{2}{n} - \frac{n-2}{n}\sigma - \frac{2\alpha}{n}. \quad (3.10)$$

Examine the last condition of the three listed above. Working in radial coordinates, recall that the measure element is $\sinh^{n-1} r dr$. Integrability near $r = 0$ is not a problem. Integrability as $r \to \infty$ follows from an exponential decay, provided that:

$$\theta\left(\alpha + \frac{n-1}{2}\left(1 - \frac{2}{q'} - \frac{2}{n}\right)\right) + n - 1 < 0.$$

Using (3.9) and (3.10), this becomes:

$$\frac{\alpha}{n-1} - \frac{\alpha}{2} - \frac{2\sigma-\alpha}{2^*} < 0. \quad (3.11)$$

Consider the extreme case $\alpha = 0$. The above condition is obviously fulfilled, and θ is finite since $\sigma < \frac{2}{n-2}$, with $\theta > 1$ since $\sigma > 0$.

Since the conditions $\theta \in]1, \infty[$ and (3.11) are open, by continuity, we can find $\alpha > 0$ such that they remain valid. So we have fulfilled all the conditions listed above, and the lemma follows from Hölder's inequality. □

Proof of Theorem 1.2. Let I be some time interval, and $t_0 \in I$. For (p,q) the n-admissible pair of Lemma 3.1, weighted Strichartz estimates (3.3)–(3.4) yield

$$\|u\|_{X(I)} \leqslant C \left(\|u(t_0)\|_{H^1(\mathbb{H}^n)} + \|f\|^{\alpha/2}_{L^1(I\times\mathbb{H}^n)} \|u\|^{2\sigma+1-\alpha}_{X(I)} \right).$$

We have seen in the proof of Lemma 3.1 that $\alpha > 0$ is such that $2\sigma > \alpha$. Therefore, the exponent $2\sigma+1-\alpha$ is larger than one. Recall the standard bootstrap argument (see e.g. [2]).

Lemma 3.2 (Bootstrap argument)**.** *Let $\gamma = \gamma(t)$ be a nonnegative continuous function on $[0,T]$ such that, for every $t \in [0,T]$,*

$$\gamma(t) \leqslant \varepsilon_1 + \varepsilon_2 \gamma(t)^\theta,$$

where $\varepsilon_1, \varepsilon_2 > 0$ and $\theta > 1$ are constants such that

$$\varepsilon_1 < \left(1 - \frac{1}{\theta}\right) \frac{1}{(\theta\varepsilon_2)^{1/(\theta-1)}} \ , \quad \gamma(0) \leqslant \frac{1}{(\theta\varepsilon_2)^{1/(\theta-1)}}.$$

Then, for every $t \in [0,T]$, we have

$$\gamma(t) \leqslant \frac{\theta}{\theta-1}\ \varepsilon_1.$$

Let $\varepsilon > 0$. Since $f \in L^1(\mathbb{R}\times\mathbb{H}^n)$, we can split $\mathbb{R}_+$ into a *finite* family

$$\mathbb{R}_+ = \bigcup_{j=1}^{J} I_j, \quad I_j = [T_j, T_{j+1}[, \text{with } T_1 = 0 \text{ and } T_{J+1} = +\infty,$$

so that $\|f\|_{L^1(I_j\times\mathbb{H}^n)} \leqslant \varepsilon$. Choosing $\varepsilon > 0$ sufficiently small and summing up over the I_j's, we conclude:

$$u \in X(\mathbb{R}).$$

Using weighted Strichartz inequality again, we see that $\left(e^{-it\Delta_{\mathbb{H}^n}} u(t,\cdot)\right)_{t>0}$ is a Cauchy sequence in $H^1(\mathbb{H}^n)$ as $t \to +\infty$. So, there is scattering at the H^1 level:

$$\exists u_+ \in H^1(\mathbb{H}^n), \quad \left\|u(t) - e^{it\Delta_{\mathbb{H}^n}} u_+\right\|_{H^1(\mathbb{H}^n)} \underset{t\to+\infty}{\longrightarrow} 0.$$

This completes the proof of Theorem 1.2. In view of [5], Corollary 1.3 follows. $\square$

4. **Asymptotic completeness in intermediary manifolds.** The proof of Theorem 1.6 follows the same strategy as above. Introduce

$$\mathtt{w}_n = \mathtt{w}_n(r) = \left(\frac{\phi(r)}{r}\right)^{\frac{n-1}{2}},$$

and denote by $d\Omega$ the measure on M^n_k. The following weighted Strichartz estimates are established in [6]:

$$\left\|e^{it\Delta_M} f(\cdot)\right\|_{L^p(\mathbb{R},L^q(\mathtt{w}_n^{q-2}d\Omega))} \leqslant C\|f\|_{L^2},$$

$$\left\|\int_{I\cap\{s\leqslant t\}} e^{i(t-s)\Delta_M} F(s)ds\right\|_{L^p(I,L^q(\mathtt{w}_n^{q-2}d\Omega))} \leqslant C\,\|F\|_{L^{r'}\left(I,L^{s'}(\mathtt{w}_n^{s'-2}d\Omega)\right)},$$

for all radial functions $f \in L^2_{\rm rad}(M)$, $F \in L^{r'}\left(I; L^{s'}_{\rm rad}\left(\mathbb{H}^n, \mathtt{w}_n^{s'-2}d\Omega\right)\right)$ and every n-admissible pairs (p,q) and (r,s). If A is a derivative in space of order one, similar estimates hold with the operator A in front of f and of the above retarded integral. The constants are independent of the time interval I. Mimicking the proof of Theorem 1.2, it suffices to prove the following

Lemma 4.1. *Fix $n \geqslant 4$, $k \geqslant 1$ and $2/N < \sigma < \frac{2}{n-2}$, where $N = (2k+1)(n-1)+1$. Let u be a radial solution to (1.3), and $A \in \{\mathrm{Id}, \partial_r\}$. There exist an n-admissible pair (p,q), $0 < \alpha < 2\sigma$, and $C > 0$ such that for all time interval I,*

$$\left\| \mathtt{w}_n^{1-2/q'} A\left(|u|^{2\sigma} u\right)\right\|_{L^{p'}(I;L^{q'})} \leqslant C\|f\|^{\alpha/2}_{L^1(I\times\mathbb{H}^n)} \|u\|^{2\sigma+1-\alpha}_{X(I)}, \tag{4.1}$$

where $X(I)$ is defined as in (3.5), and

$$f(t,r) = \frac{1}{r^3}|u(t,r)|^2.$$

Proof. The proof is very similar to the proof of Lemma 3.1. We want to apply Hölder's inequality, after the following splitting:

$$\mathtt{w}_n^{1-2/q'}|u|^{2\sigma}|Au| = \left(\frac{1}{r^{3/2}}|u|\right)^{\alpha} \times |u|^{2\sigma-\alpha} \times \mathtt{w}_n^{2/n}|Au| \times \mathtt{w}_n^{1-2/q'}\mathtt{w}_n^{-2/n} r^{3\alpha/2}.$$

Write

$$\frac{1}{q'} = \frac{\alpha}{2} + \frac{2\sigma-\alpha}{a} + \frac{1}{2^*} + \frac{1}{\theta} \quad ; \quad \frac{1}{p'} = \frac{\alpha}{2} + \frac{1}{2}, \tag{4.2}$$

where $2^* = \frac{2n}{n-2}$. The lemma will follow if we can choose (p,q), a, and α, such that:

- (p,q) is n-admissible.
- $\alpha > 0$, with $2\sigma - \alpha > 0$.
- $a \in [2, 2^*]$ (to control the second term thanks to Sobolev embedding).
- The last factor in the above splitting is in $L^\theta(M_k^n)$, for some $\theta \in [1,\infty[$.

With p given by the second equation in (4.2), the first condition is equivalent to

$$\frac{1}{q} = \frac{1}{2^*} + \frac{\alpha}{n}. \tag{4.3}$$

This is consistent with the first equality of (4.2) provided that

$$\frac{1}{\theta} = \frac{2}{n} - \frac{2\sigma}{a} - \alpha\left(\frac{1}{a} + \frac{1}{n} + \frac{1}{2}\right). \tag{4.4}$$

Let us examine the last condition of the four listed above. Working in radial coordinates, recall that the measure element is $\phi(r)^{n-1}dr$. Integrability near $r=0$ is not a problem. Integrability as $r \to \infty$ holds if:

$$\theta\left(\frac{3\alpha}{2} + k(n-1)\left(1 - \frac{2}{q'} - \frac{2}{n}\right)\right) + (2k+1)(n-1) < -1,$$

that is, thanks to (4.3),

$$\theta\left(\frac{3\alpha}{2} + \frac{N-n}{n}\alpha - \frac{2(N-n)}{n}\right) + N < 0 \Longleftrightarrow \frac{1}{\theta} < \frac{2}{n} - \frac{2}{N} - \alpha\left(\frac{1}{2N} + \frac{1}{n}\right).$$

Consider the extreme case $\alpha = 0$. The above condition on θ yields

$$\frac{2}{N} < \frac{2\sigma}{a} \Longleftrightarrow \frac{a}{N} < \sigma.$$

In view of (4.4), θ is finite if

$$\sigma < \frac{a}{n}.$$

Moreover θ is always larger than 1 ($n \geqslant 4$ and $\sigma > 0$). Since $k \geqslant 1$, $N > n$, and since $\frac{2}{N} < \sigma < \frac{2}{n-2}$, we can find $a \in [2, 2^*]$ such that

$$\frac{a}{N} < \sigma < \frac{a}{n}.$$

Fix the parameter a. Since the requirements we have made are open conditions, by continuity, we can find $0 < \alpha \ll 1$ such that they are still satisfied. So we have fulfilled all the conditions listed above, and the lemma follows from Hölder's inequality. □

5. **The free dynamics in the radial case.** To prove Proposition 1.12, we first reduce the analysis to the Euclidean case, as in [22, 6]. Consider the equation

$$i\partial_t u + \Delta u = 0, \tag{5.1}$$

where Δ stands for the Laplace–Beltrami associated to an n-dimensional rotationally symmetric manifold with metric

$$ds^2 = dr^2 + \phi(r)^2 d\omega^2, \text{ where } \phi(r) = \sum_{j=0}^{k} \frac{1}{(2j+1)!} r^{2j+1},$$

and k is possibly infinite. Introduce $\widetilde{u}$, given by

$$\widetilde{u}(t,r) = u(t,r)\left(\frac{\phi(r)}{r}\right)^{\frac{n-1}{2}}.$$

In the case of radial solutions, (5.1) is equivalent to

$$\begin{aligned} &i\partial_t \widetilde{u} + \Delta_{\mathbb{R}^n}\widetilde{u} = V\widetilde{u},\\ &\text{where } V(r) = \frac{n-1}{2}\frac{\phi''(r)}{\phi(r)} + \frac{(n-1)(n-3)}{4}\left(\left(\frac{\phi'(r)}{\phi(r)}\right)^2 - \frac{1}{r^2}\right). \end{aligned}$$

We check easily the following dichotomy:

- If k is finite, then V is smooth, with $V(r) = \mathcal{O}(r^{-2})$ as $r \to \infty$.
- If $k = \infty$ (case of hyperbolic space), then $\phi'' = \phi$, and $V = (n-1)/2 + \widetilde{V}$, where $\widetilde{V}$ is as above.

Up to replacing $\widetilde{u}$ with $e^{i(n-1)t/2}\widetilde{u}$ when k is infinite, we see that it suffices to study the first case. The potential V is a short range potential, as far as linear scattering theory is concerned (see e.g. [14, 27]). Therefore, there exists $\widetilde{u}_+ \in L^2_{\rm rad}(\mathbb{R}^n)$ such that

$$\left\|\widetilde{u}(t) - e^{it\Delta_{\mathbb{R}^n}}\widetilde{u}_+\right\|_{L^2(\mathbb{R}^n)} \underset{t\to+\infty}{\longrightarrow} 0.$$

Moreover, the map $\widetilde{u}_{|t=0} \mapsto \widetilde{u}_+$ is linear and continuous from $L^2_{\rm rad}(\mathbb{R}^n)$ to $L^2_{\rm rad}(\mathbb{R}^n)$. Recalling that the volume element is $r^{n-1}dr$ on $\mathbb{R}^n$, and $\phi(r)^{n-1}dr$ on the manifold that we consider, we infer

$$\|u(t) - v_1(t)\|_{L^2(M)} \underset{t\to+\infty}{\longrightarrow} 0,$$

where

$$v_1(t,r) = \left(\frac{r}{\phi(r)}\right)^{\frac{n-1}{2}} e^{it\Delta_{\mathbb{R}^n}}\left(\widetilde{u}_+(r)\right).$$

Proposition 1.12 then follows from the standard large time asymptotics for $e^{it\Delta_{\mathbb{R}^n}}$,

$$\left\|e^{it\Delta_{\mathbb{R}^n}}\varphi - \Lambda(t)\right\|_{L^2(\mathbb{R}^n)} \underset{t\to+\infty}{\longrightarrow} 0, \text{ where } \Lambda(t,x) = \frac{e^{i|x|^2/(4t)}}{t^{n/2}}\left(\mathcal{F}\varphi\right)\left(\frac{x}{2t}\right), \tag{5.2}$$

and the Fourier transform $\mathcal{F}$ is normalized so the above relation holds true. This asymptotics is, for instance, a straightforward consequence of the factorization $e^{it\Delta_{\mathbb{R}^n}} = \mathcal{M}D\mathcal{F}\mathcal{M}$, where $\mathcal{M}$ is the multiplication by an exponential, and D is the L^2-unitary dilation at scale t.

Remark 5.1. We notice that $V = 0$ if ϕ' is constant: in the Euclidean case, $\widetilde{u} = cu$. If $n = 3$ and $\phi'' = c\phi$, V is constant: for radial solutions on $\mathbb{H}^3$, and up to a purely time dependent phase shift, there is no external potential. In the two cases distinguished here, we have $\widetilde{u}_+ = \widetilde{u}_{|t=0}$. Then (5.2) shows why the Fourier transform is present in the description of the asymptotic behavior of radial solutions to (5.1). Recall that the Fourier transform for radially symmetric functions on $\mathbb{H}^n$ is much simpler when $n = 3$; see [4] and references therein.

Remark 5.2. Following the formal argument given in [16], Proposition 1.12 suggests that for $\sigma \leqslant 1/N$, long range effects are present in (1.3). Suppose that $n \geqslant 2$ and $0 < \sigma \leqslant 1/N$, where $N = (2k+1)(n-1)+1$. Let $u \in C([T,\infty[; L^2_{\rm rad}(M^n_k))$ be a solution of (1.3) such that there exists $u_+ \in L^2_{\rm rad}(M^n_k)$ with

$$\left\| u(t) - e^{it\Delta_M} u_+ \right\|_{L^2} \underset{t\to+\infty}{\longrightarrow} 0.$$

Formal computations indicate that necessarily, $u_+ \equiv 0$ and $u \equiv 0$: the linear and nonlinear dynamics are no longer comparable, due to long range effects. To see this, let $\psi \in C_0^\infty(M)$ be radial, and $t_2 \geqslant t_1 \geqslant T$. By assumption,

$$\left\langle \psi, e^{-it_2\Delta_M} u(t_2) - e^{-it_1\Delta_M} u(t_1) \right\rangle = -i \int_{t_1}^{t_2} \left\langle e^{it\Delta_M}\psi, \left(|u|^{2\sigma} u\right)(t) \right\rangle dt$$

goes to zero as $t_1, t_2 \to +\infty$. Proposition 1.12 implies that for $t \to +\infty$, we have

$$\left\langle e^{it\Delta_M}\psi, \left(|u|^{2\sigma} u\right)(t) \right\rangle \approx \frac{1}{t^{n\sigma+n}} \int_0^\infty \left(\frac{r}{\phi(r)}\right)^{(n-1)(\sigma+1)} \varphi\left(\frac{r}{t}\right) \phi(r)^{n-1} dr,$$

for $\varphi = \mathcal{L}\psi |\mathcal{L}u_+|^{2\sigma} \overline{\mathcal{L}u_+}$. With the change of variable $r \mapsto tr$, the above integral is equal to

$$\frac{1}{t^{n\sigma+n-1}} \int_0^\infty \left(\frac{tr}{\phi(tr)}\right)^{(n-1)(\sigma+1)} \varphi\left(r\right) \phi(tr)^{n-1} dr.$$

For $r \geqslant 1$ and large t, the function at stake behaves like

$$\frac{1}{t^{n\sigma+n-1}} \left(\frac{tr}{(tr)^{2k+1}}\right)^{(n-1)(\sigma+1)} \varphi(r) \left(tr\right)^{(n-1)(2k+1)} = \frac{r^{-(N-n)\sigma+n-1}}{t^{N\sigma}} \varphi(r).$$

This function of t is not integrable, unless $\varphi \equiv 0$. This means that $\mathcal{L}u_+ = 0 = u_+$ ($\operatorname{Ker}\mathcal{L} = \{0\}$). The assumption and the conservation of mass then imply $u \equiv 0$.

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