ON THE NONLINEAR SCHRÖDINGER DYNAMICS ON $\mathbb{S}^2$

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ABSTRACT. We analyze the evolution of the highest weight spherical harmonics by the nonlinear Schrödinger equation on $\mathbb{S}^2$. Sharp estimates are proved for the dynamics parallel and orthogonally to the initial data. Also, we give an ansatz of the solution with respect to the spherical harmonics.
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1. Introduction

The nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = F(u)$$

is motivated by many questions raised in Physics (see the recent survey of the subject [17]). On $\mathbb{R}^d$ the Cauchy problem has been largely studied in the past twenty years. In the one-dimensional case the Sobolev embedding suffices to have well-posedness in the energy space. Unfortunately in higher dimensions this argument is no longer effective. The Strichartz estimates ([18]) were then successfully exploited in order to get existence and regularity results ([8],[10], [19]).

For the same problem on a compact Riemannian manifold, with Δ being the associated Laplace-Beltrami operator, it appears that the geometry influences the dynamics of the equation.

The Strichartz estimates with fractional loss of derivative have been proved by Burq, Gérard and Tzvetkov in [2]. If we consider the low regularity equation with defocusing polynomial nonlinearity, these estimates imply local existence results. Moreover, on surfaces in the case of defocusing polynomial nonlinearities and on three-manifolds in the case of defocusing cubic nonlinearities the global existence in the energy space $\mathbb{H}^1$ follows.

However, instability phenomena appear, even in the defocusing case.

On the one hand the same authors proved in [3] that the flow map of the cubic defocusing Schrödinger equation on the sphere $\mathbb{S}^2$

$$\begin{cases} i\partial_t u + \Delta_{S^2} u = |u|^2 u, \\ u(0,x) \in \mathbb{H}^s(S^2) \end{cases}$$

is not uniformly continuous for $s \in [0, \frac{1}{4}[$, that is for Sobolev regularity indices greater than zero, which is the scaling index. Similar results hold also on a plane domain [4].

For $\mathbb{S}^2$, the same authors have proved recently that the index $\frac{1}{4}$ is the critical regularity index, that is for $s > \frac{1}{4}$, the Cauchy problem is well posed ([5]).

On the other hand Cazenave-Weissler [6] and Bourgain [1] proved that the Cauchy problem is $\mathbb{H}^\epsilon$ well-posed on $\mathbb{R}^2$ and on $\mathbb{T}^2$ respectively for all positive ϵ. Moreover, on $\mathbb{T}^2$, the flow is not uniformly continuous for s negative ([3]), therefore zero is the critical regularity index.

Hence these results point out the importance played by the geometry of the manifold in the dynamics of the equation. This is not in contradiction with the positive results on the wave operator, since the Schrödinger equation does not enjoy the property of finite speed of propagation.

Instability phenomena appear for a large class of dispersive equations. In the recent paper [11], Kenig, Ponce and Vega have studied the low regularity properties of the focusing nonlinear Schrödinger and Korteweg-de Vries equations. Then, in [7], Christ, Colliander and Tao have extended this study to the defocusing analogues of these equations. The lack of well-posedness appear also for the defocusing wave equation in $\mathbb{R}^3$, with supercritical nonlinearity, as Lebeau has shown in [13] (see also [14]). Koch and Tzvetkov have shown in [12] that the flow of the Benjamin-Ono equation fails to be uniformly continuous on $\mathbb{H}^s$ for all positive s. All these results are obtained by constructing families of exact solutions of the equations, that contradict the well-posedness.

In order to obtain in [3] the instability result on $\mathbb{S}^2$, the evolution of certain spherical harmonics, concentrated on geodesics, is studied as follows. Let ψ_n be the $\mathbb{H}^s$-normalized spherical harmonic obtained by restricting to the sphere the harmonic polynomial

$$\psi_n(x_1, x_2, x_3) = n^{\frac{1}{4}-s}(x_1 + ix_2)^n.$$

Let us notice that when n tends to infinity, ψ_n concentrates on the circle $x_1^2 + x_2^2 = 1$. By direct calculus one can estimates the $\mathbb{L}^p$ norms of ψ

$$(1) \qquad \begin{cases} \|\psi_n\|_\infty \approx n^{\frac{1}{4}-s}, \\ \|\psi_n\|_2 \approx n^{-s}, \\ \|\psi_n\|_4^4 \approx n^{\frac{1}{2}-4s}, \\ \|\psi_n\|_6^3 \approx n^{\frac{1}{2}-3s}. \end{cases}$$

The equivalents are considered as n goes to infinity, and so shall be in all the paper :

$$f_n \approx g_n \iff \exists\, c, C \in \mathbb{R}^+, c\, g_n \leq f_n \leq C g_n.$$

$$f_n \lesssim g_n \iff \exists\, C \in \mathbb{R}^+, f_n \leq C g_n.$$

Consider now the Schrödinger equation

$$(S) \begin{cases} i\partial_t u + \Delta_{\mathbb{S}^2} u = |u|^2 u, \\ u_n(0, x) = \kappa_n \psi_n(x), \end{cases}$$

where κ_n is a number between $\frac{1}{2}$ and 1. For every real α the rotation R_α defined on $\mathbb{R}^3$ by

$$R_\alpha(x_1, x_2, x_3) = (x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha, x_3)$$

verifies the relation

$$\psi(R_\alpha(x)) = e^{in\alpha}\psi(x).$$

Then the uniqueness of the solution for the same Cauchy problem (S) with initial data $e^{in\alpha}\psi(x)$ gives us the identity

$$u(t, R_\alpha(x)) = e^{in\alpha}u(t,x).$$

Using this fact and an algebraic lemma on the spherical harmonics, in [3] it is shown that the solution u can be decomposed only on ψ_n and on $\{h_{n+j}\}_{j\geq 1}$, the spherical harmonics of order $n+j$ satisfying

$$h_{n+j}(R_\alpha(x)) = e^{in\alpha}h_{n+j}(x).$$

We shall consider these spherical harmonics to be normalized in $\mathbb{L}^2$.
Let $\omega_n\psi_n$ be the orthonormal projection of $|\psi_n|^2\psi_n$ on the space spanned by ψ_n, and r_n the remainder term of this projection

$$|\psi_n|^2\psi_n = \omega_n\psi_n + r_n.$$

By the same arguments above, r_n express only in terms of h_{n+j}'s. One can write the solution of (S)

$$u_n(t,x) = \kappa_n e^{-it(n(n+1)+\kappa_n^2\omega_n)}((1+\widetilde{z_n}(t))\psi_n(x) + q_n(t,x)),$$

with q_n only in terms of h_{n+j}'s.

For $s \in]\frac{3}{20}, \frac{1}{4}[$, it is shown in ([3]) that the $\mathbb{H}^s$ norm of $q_n(t)$ is negligible with respect to the one of ψ_n, and $|\widetilde{z_n}(t)|$ tends to 0 when n tends to infinity. These results imply that the solution behaves like the initial data ψ_n with an oscillating exponential type coefficient. Knowing that ω_n tends to infinity, a good choice of a bounded sequence κ_n gives an important dephasing between the solutions u_n, so the Cauchy problem for the equation (S_Ω) is ill-posed on $\mathbb{H}^s(\mathbb{S}^2)$, in the sense that the flow is not uniformly continuous on bounded sets of $\mathbb{H}^s$.

The purpose of this paper is to provide a further analysis of these solutions. We prove sharp estimates for $|\widetilde{z_n}(t)|$ and for $\|q(t)\|_{H^s}$. In particular these results point out that even in the remainder part $\widetilde{z_n}\psi_n + q_n$, the dynamics orthogonally to ψ_n is weak. We also obtain an ansatz of the solution with respect to the spherical harmonics h_{n+j}.

For simplicity, the indices n of the functions defined above will be ignored from now on. We define

$$\alpha_j = 2nj + j^2 + j - \kappa^2\omega \ , \ k_{j,l} = \kappa^2 < \overline{h_{n+l}}\,\psi^2, h_{n+j} >,$$

$$\mu_j = \sqrt{(3k_{j,j}+\alpha_j)(k_{j,j}+\alpha_j)}.$$

Consider the operator

$$A = -\Delta - n(n+1)$$

and the operator M defined on the space spanned by the h_{n+j}'s by

$$M(h_{n+j}) = \mu_j h_{n+j}.$$

Theorem 1.1. *Let $T>0$. For every $s \in [0, \frac{1}{4}[$, for $t \in [0,T]$, the solution of (S) is*

$$u(t,x) = \kappa e^{-it(n(n+1)+\kappa^2\omega)}(z(t)\psi(x) + q(t,x)),$$

with the sharp estimates

$$\left\{ \begin{array}{c} \sup_{0<t<T} \|q(t)\|_{H^{\frac{1}{2}}} \approx n^{-3s}, \\ |z(t)-1| \approx tn^{-4s}. \end{array} \right.$$

(in the second equivalent, t is present in order to include the case $t=0$, when $z(0)=1$) Moreover,

i) For $s \in [0, \frac{1}{4}[$ the coefficient of ψ is

$$z(t) = e^{-it\frac{3\kappa^4}{\|\psi\|_2^2} < A^{-1}r, r>} + O(n^{-\frac{1}{2}-6s}).$$

ii) For $s \in]\frac{1}{12}, \frac{1}{4}[$ we have the ansatz

$$u(t,x) = \kappa e^{-it(n(n+1)+\kappa^2\omega)} \left(e^{-it\frac{3\kappa^4}{\|\psi\|_2^2} < A^{-1}r, r>} \psi(x) - i \int_0^t e^{isM} r ds \right) + \widetilde{u}(t,x),$$

with

$$\|\widetilde{u}(t)\|_{H^s} \ll n^{-\frac{1}{2}-2s}.$$

The proof is based on a further exploitation of the conservation laws than in [3]. Also, by using Sogge's estimates on the spherical harmonics ([15], [16])

$$\|h_m\|_p \leq C m^{\frac{1}{4}-\frac{1}{2p}} \text{ for } 2 \leq p \leq 6, \tag{2}$$

we give upper bounds for the $\mathbb{L}^p$ norms of q better than the ones obtained by interpolation between $\mathbb{L}^2$ and $\mathbb{H}^1$. By using all these estimates in the study of the equations of z and q, we have the description of z. It follows then that the upper bounds founded before are sharp. Finally, we obtain the ansatz by analyzing the system obtained by projecting the equation (S) on each mode, and by using the important distance between two consecutive eigenvalues of the laplacian.

Remark 1.2. *It will be shown that*

$$\frac{3\kappa^4}{\|\psi\|_2^2} < A^{-1}r, r > \approx n^{-4s},$$

so the oscillation of the solution is stronger as s decreases to zero, that is if the amplitude of the initial data grows faster.

Remark 1.3. *The linear part that comes from the cubic nonlinearity of the equation has an essential contribution in the ansatz of the solution. From it the operator M is defined in terms of μ_j instead of α_j, and the effective dynamics orthogonally to ψ*

$$-i \int_0^t e^{isM} r ds$$

verifies an equation depending on M

$$\left\{ \begin{array}{c} (i\partial_t + M)v(t,x) + ir(x) = 0, \\ v(0,x) = 0. \end{array} \right.$$

Remark 1.4. *In the case $s = \frac{1}{4}$, it is not known if the flow is uniformly continuous or not.*

The paper is organized as follows. In §3.2.1, by using the energy laws, we find upper bounds for the norms of q, for $|z-1|$ and for the coefficients of the spherical harmonics h_{n+l} in the solution u. In §3.2.2 sharp estimates are given for some particular scalar product of spherical harmonics. In §3.2.3, by using these estimates in the study of the equation verified by $z(t)$, we get the description of $z(t)$. In §3.2.4 this description implies that the upper bounds obtained previously on $\|q\|_{H^s}$ and on $|z-1|$ are sharp. By projecting the equation (S) on the space spanned by h_{n+j}, we describe in §3.3 the ansatz of the solution with respect to the spherical harmonics.

I thank my advisor Patrick Gérard for having guided this work.

2. Estimates on the solution

2.1. **Upper bounds on norms of q and on $|z-1|$.** The conservation laws of the equation (S) are

$$\begin{cases} |z(t)|^2\|\psi\|_2^2 + \|q(t)\|_2^2 = \|\psi\|_2^2, \\ |z(t)|^2\|\nabla\psi\|_2^2 + \|\nabla q(t)\|_2^2 + \frac{1}{2\kappa^2}\|u(t)\|_4^4 = \|\nabla\psi\|_2^2 + \frac{\kappa^2}{2}\|\psi\|_4^4. \end{cases}$$

By subtracting from the second conservation law the first one multiplied by $n(n+1)$ we obtain

$$\|\nabla q(t)\|_2^2 - n(n+1)\|q(t)\|_2^2 = \frac{\kappa^2}{2}\|\psi\|_4^4 - \frac{1}{2\kappa^2}\|u(t)\|_4^4. \tag{3}$$

As mentioned in the introduction, one can decompose

$$q(t,x) = \sum_{j\geq 1} z_j(t) h_{n+j}(x).$$

Obviously,

$$\sum_{j\geq 1}(n+j)|z_j(t)|^2 \leq \sum_{j\geq 1}((j+n)(j+n+1) - n(n+1))|z_j(t)|^2,$$

so

$$\|q(t)\|_{H^{\frac{1}{2}}}^2 \lesssim \|\nabla q(t)\|_2^2 - n(n+1)\|q(t)\|_2^2,$$

and the identity (3) gives

$$\|q(t)\|_{H^{\frac{1}{2}}}^2 \lesssim \frac{\kappa^2}{2}\|\psi\|_4^4 - \frac{\kappa^2}{2}\|z(t)\psi + q(t)\|_4^4.$$

The numbers κ will be chosen to be bounded with respect to n, so

$$\|q(t)\|_{H^{\frac{1}{2}}}^2 \lesssim \|\psi\|_4^4 - \|z(t)\psi + q(t)\|_4^4, \tag{4}$$

and by using the estimates (1) on the norms of ψ,

$$\|q(t)\|_{H^{\frac{1}{2}}} \lesssim \|\psi\|_4^2 \lesssim n^{\frac{1}{4}-2s}.$$

Then one has a first upper bound on the $\mathbb{L}^2$ and on the $\mathbb{H}^s$ norm of q

$$\begin{cases} \|q(t)\|_2 \lesssim n^{-\frac{1}{4}-2s}, \\ \|q(t)\|_{H^s} \lesssim n^{-\frac{1}{4}-s}. \end{cases} \tag{5}$$

By exploiting further the inequality (4), better estimations on $q(t)$ are found, namely the ones claimed in Theorem 1.1.

Lemma 2.1. *For $s \in [0, \frac{1}{4}[$ the norms of q are upper-bounded by*

$$\left\{ \begin{array}{l} \|q(t)\|_{H^{\frac{1}{2}}} \lesssim n^{-3s}, \\ \|q(t)\|_2 \lesssim n^{-\frac{1}{2}-3s}, \\ \|q(t)\|_{H^s} \lesssim n^{-\frac{1}{2}-2s}. \end{array} \right. \tag{6}$$

Proof. By developing the right-hand side of (4) and by neglecting the negative terms,

$$\|\psi\|_4^4 - \|z(t)\psi + q(t)\|_4^4 \lesssim \int |\psi|^4 \left|1 - |z(t)|^4\right| + |z(t)|^3 |\psi^3 q(t)| + |z(t)||\psi q^3(t)|.$$

The conservation of the mass gives

$$1 - |z(t)|^2 = \frac{\|q(t)\|_2^2}{\|\psi\|_2^2}, \tag{7}$$

and the preliminary estimates (5) obtained above on $q(t)$ ensures us that

$$|z(t)|^2 = 1 + O(n^{-\frac{1}{2}-2s}) \approx 1. \tag{8}$$

Thus one can write

$$\|\psi\|_4^4 - \|z(t)\psi + q(t)\|_4^4 \lesssim \int |\psi|^4 \frac{\|q(t)\|_2^2}{\|\psi\|_2^2} + |\psi^3 q(t)| + |\psi q^3(t)|.$$

The terms in the right side can be estimated as follows

$$\int |\psi q^3| \lesssim \|\psi\|_\infty \|q\|_3^3 \lesssim n^{\frac{1}{4}-s} \|q\|_{H^{\frac{1}{3}}}^3 \lesssim n^{-\frac{1}{4}-s} \|q\|_{H^{\frac{1}{2}}}^3,$$

$$\int |\psi^3 q| \lesssim \|\psi\|_6^3 \|q\|_2 \lesssim n^{-3s} \|q\|_{H^{\frac{1}{2}}},$$

and

$$\int |\psi|^4 \frac{\|q\|_2^2}{\|\psi\|_2^2} \lesssim \|\psi\|_4^4 n^{2s} \|q\|_2^2 \lesssim n^{-\frac{1}{2}-2s} \|q\|_{H^{\frac{1}{2}}}^2.$$

So for n large enough

$$\|\psi\|_4^4 - \|z(t)\psi + q(t)\|_4^4 \lesssim n^{-3s} \|q(t)\|_{H^{\frac{1}{2}}} + n^{-\frac{1}{2}-2s} \|q(t)\|_{H^{\frac{1}{2}}}^2 + n^{-\frac{1}{4}-s} \|q(t)\|_{H^{\frac{1}{2}}}^3.$$

By using (4),

$$\|q(t)\|_{H^{\frac{1}{2}}}^2 \lesssim n^{-3s} \|q(t)\|_{H^{\frac{1}{2}}} + n^{-\frac{1}{2}-2s} \|q(t)\|_{H^{\frac{1}{2}}}^2 + n^{-\frac{1}{4}-s} \|q(t)\|_{H^{\frac{1}{2}}}^3.$$

Since

$$\|q(0)\|_{H^{\frac{1}{2}}} = 0,$$

the term that gives the behavior of the right-hand side is $n^{-3s}\|q(t)\|_{H^{\frac{1}{2}}}$, and the claimed better estimations (6) are obtained. In particular, we also obtain

$$\int |\psi|^4 \frac{\|q(t)\|_2^2}{\|\psi\|_2^2} + |\psi^3 q(t)| + |\psi q^3(t)| \lesssim n^{-3s} \|q(t)\|_{H^{\frac{1}{2}}}, \tag{9}$$

and by (3)

$$\|\nabla q(t)\|_2^2 - n(n+1)\|q(t)\|_2^2 \approx \|\psi\|_4^4 - \|z(t)\psi + q(t)\|_4^4 \lesssim n^{-6s} \tag{10}$$

□

These new estimates on q will imply the ones on z mentioned in Theorem 1.1.

Lemma 2.2. *For $s \in [0, \frac{1}{4}[$ the coefficient z verifies*

$$|z(t) - 1| \lesssim tn^{-4s}.$$

Proof. The function

$$c(t) := e^{-it(n(n+1)+\kappa^2\omega)}$$

verifies

$$i\partial_t c - n(n+1)c = \kappa^2\omega|c|^2 c.$$

Then, the projection of the equation (S) on the space spanned by ψ is

$$\begin{cases} (i\partial_t z + \kappa^2\omega z)\|\psi\|_2^2 = \kappa^2 \int |z\psi + q|^2(z\psi + q)\overline{\psi}, \\ z(0) = 1. \end{cases}$$

Since

$$\omega = \frac{\|\psi\|_4^4}{\|\psi\|_2^2},$$

the equation of $z - 1$ writes

$$\begin{cases} i\partial_t(z-1) = \frac{\kappa^2}{\|\psi\|_2^2}\left(\int |z\psi + q|^2(z\psi+q)\overline{\psi} - \int z|\psi|^4\right), \\ (z-1)(0) = 0. \end{cases} \tag{11}$$

By integrating in time and by developing the right-hand side of the equation,

$$|z(t)-1| \lesssim \frac{1}{\|\psi\|_2^2}\int_0^t \int |\psi|^4|z(t)||\,|z(t)|^2-1|+|z(t)|^2|\psi^3 q(\tau)|+|z(t)||\psi^2q^2(\tau)|+|\psi q^3(\tau)|dxd\tau.$$

By using again the informations (7), (8) on z, we have

$$|z(t) - 1| \lesssim \frac{1}{\|\psi\|_2^2}\int_0^t \int |\psi|^4\frac{\|q(\tau)\|_2^2}{\|\psi\|_2^2} + |\psi^3 q(\tau)| + |\psi^2q^2(\tau)| + |\psi q^3(\tau)|dxd\tau.$$

The square term on the right-hand side can be upper bounded by estimates (1) and (6)

$$\int \psi^2 q^2(\tau) \lesssim \|\psi\|_\infty^2\|q(\tau)\|_2\, n^{-\frac{1}{2}}\|q(\tau)\|_{\mathbb{H}^{\frac{1}{2}}} \lesssim n^{\frac{1}{2}-2s}n^{-\frac{1}{2}-3s}n^{-\frac{1}{2}}\|q(\tau)\|_{\mathbb{H}^{\frac{1}{2}}} \ll n^{-3s}\|q(\tau)\|_{H^{\frac{1}{2}}},$$

and the others have been upper bounded in (9), therefore

$$|z(t) - 1| \lesssim n^{2s}\, tn^{-3s} \sup_{0<\tau<T} \|q(\tau)\|_{H^{\frac{1}{2}}} \lesssim tn^{-4s}, \tag{12}$$

and Lemma 3.2.2 is proved. □

Finally, let us prove some estimates on the norms of q and on the coefficients z_j of the spherical harmonics in q that will be used in the next sections. We will use Sogge's estimates (2) on the spherical harmonics in order to obtain better estimations on the $\mathbb{L}^p$ norms of q than the ones given by interpolation between $\mathbb{L}^2$ and $\mathbb{H}^1$.

Lemma 2.3. *One has*

$$\left(\sum_{j\geq 1} j|z_j(t)|^2\right)^{\frac{1}{2}} \lesssim n^{-\frac{1}{2}-3s}, \tag{13}$$

and

$$\begin{cases} \|q(t)\|_4^2 \leq n^{-\frac{3}{4}-6s}\log n, \\ \|q(t)\|_6^3 \leq n^{-1-9s}(\log n)^{\frac{3}{2}}. \end{cases} \tag{14}$$

Proof. By repeating the argument on the conservation laws, one has

$$n\sum_{j\geq 1} j|z_j(t)|^2 \leq \sum_{j\geq 1}((j+n)(j+n+1)-n(n+1))|z_j(t)|^2$$
$$\leq \|\nabla q(t)\|_2^2 - n(n+1)\|q(t)\|_2^2,$$

so (13) is obtained by using (10).
Let us decompose

$$q(t,x) = \sum_{j=1..n^\alpha} z_j(t)h_{n+j}(x) + q_\alpha(t,x),$$

where α is a positive number to be fixed later. Here q_α is the part of q containing only the spherical harmonics of order greater than $n+n^\alpha$.
The same argument before, for α smaller or equal to 1, gives

$$nn^\alpha\|q_\alpha(t)\|_2^2 \leq \|\nabla q_\alpha(t)\|_2^2 - n(n+1)\|q_\alpha(t)\|_2^2$$
$$\leq \|\nabla q(t)\|_2^2 - n(n+1)\|q(t)\|_2^2 \lesssim n^{-6s}.$$

Then

$$\begin{cases} \|q_\alpha(t)\|_2 \lesssim n^{-\frac{1}{2}-3s-\frac{\alpha}{2}}, \\ \|\nabla q_\alpha(t)\|_2 \lesssim n^{\frac{1}{2}-3s-\frac{\alpha}{2}}, \end{cases} \tag{15}$$

and by interpolation

$$\begin{cases} \|q_\alpha(t)\|_4 \lesssim n^{-3s-\frac{\alpha}{2}}, \\ \|q_\alpha(t)\|_6 \lesssim n^{\frac{1}{6}-3s-\frac{\alpha}{2}}. \end{cases}$$

Now we are able to estimate the $\mathbb{L}^p$ norms of q

$$\|q(t)\|_4 \leq \sum_{l=1}^{n^\alpha}|z_l(t)|\|h_{n+l}\|_4 + \|q_\alpha(t)\|_4 \leq \left(\sum_{j\geq 1} j|z_j|^2\right)^{\frac{1}{2}}\left(\sum_{j\geq 1}^{n^\alpha}\frac{\|h_{n+j}\|_4^2}{j}\right)^{\frac{1}{2}} + n^{-3s-\frac{\alpha}{2}}.$$

Sogge's estimates (2) on the norms of the h_{n+j}'s and the relation (13) imply

$$\|q(t)\|_4^2 \lesssim n^{-1-6s}n^{\frac{1}{4}}\log n + n^{-6s-\alpha} \leq n^{-\frac{3}{4}-6s}\log n$$

if α is chosen larger enough. Similarly,

$$\|q(t)\|_6 \leq \sum_{l=1}^{n^\alpha}|z_l(t)|\|h_{n+l}\|_6 + \|q_\alpha(t)\|_6 \leq \left(\sum_{j\geq 1}^{n^\alpha} j|z_j(t)|^2\right)^{\frac{1}{2}}\left(\sum_{j\geq 1}^{n^\alpha}\frac{\|h_{n+j}\|_6^2}{j}\right)^{\frac{1}{2}} + n^{\frac{1}{6}-3s-\frac{\alpha}{2}},$$

and

$$\|q(t)\|_6^3 \lesssim n^{-\frac{3}{2}-9s} n^{\frac{1}{2}} (\log n)^{\frac{3}{2}} + n^{\frac{1}{2}-9s-\frac{3\alpha}{2}} \leq n^{-1-9s} (\log n)^{\frac{3}{2}}$$

if $\alpha = 1$.

□

2.2. **Sharp estimates for $< |\psi|^2\psi, h_{n+2} >$.**

Lemma 2.4. *We have the sharp estimate*

$$< |\psi|^2\psi, h_{n+2} > \approx n^{\frac{1}{2}-3s}.$$

Proof. In polar coordinates

$$x_1 = \sin\theta\cos\phi \ , \ x_2 = \sin\theta\sin\phi \ , \ x_3 = \cos\theta,$$

the spherical harmonics h_{n+l} can be written in terms of the associated Legendre functions ([9])

$$h_{n+l}(\phi, \theta) = c_j e^{in\phi} P_{n+l}^n(\cos(\theta)),$$

where c_l is the coefficient of the $\mathbb{L}^2$ normalization

$$c_l = \sqrt{\frac{l!(n+l)}{2\pi(2n+l)!}}.$$

Since ψ is the restriction of $(x_1 + ix_2)^n$ to the sphere, we can calculate

$$< |\psi|^2\psi, h_{n+l} > = c_l \int_0^{2\pi} \int_0^{\pi} n^{\frac{3}{4}-3s} \sin^{3n}(\theta) e^{in\phi} \sin(\theta)\, e^{-in\phi} P_{n+l}^n(\cos(\theta)) d\theta d\phi$$
$$= 2\pi c_l n^{\frac{3}{4}-3s} \int_0^{\pi} P_{n+l}^n(\cos(\theta)) \sin^{3n+1}(\theta) d\theta.$$

A way to write the associated Legendre functions P_{n+l}^n is ([9])

$$P_{n+l}^n(\cos(\theta)) = (-1)^n \frac{(2n+l)!}{2^n n! l!} \sin^n(\theta) \left(\cos^l(\theta) - \frac{l(l-1)}{2(2n+2)} \cos^{l-2}(\theta) \sin^2(\theta) \right.$$
$$\left. + \frac{l(l-1)(l-3)(l-4)}{2 \cdot 4(2n+2)(2n+4)} \cos^{l-4}(\theta) \sin^4(\theta) - \ldots \right);$$

the sum ends when the power of the cosinus, decreasing each time by 2, becomes 1 or 0. In particular, this formula implies that for odd l

$$< |\psi|^2\psi, h_{n+l} > = \sum_{k=0}^{[\frac{l}{2}]} \widetilde{c_{l,k}} \int_0^{\pi} \cos^{l-2k}(\theta) \sin^{3n+1}(\theta) d\theta = 0.$$

Let us also remark that $< |\psi|^2\psi, h_{n+l} >$ is a real number.
For $l = 2$, Stirling's formula gives us the sharp estimate

$$< |\psi|^2\psi, h_{n+2} > \approx n^{\frac{5}{4}} n^{\frac{3}{4}-3s} \int_0^{\pi} \sin^{4n+1}(\theta) \left(1 - \frac{2n+3}{2n+2} \sin^2(\theta)\right) d\theta$$
$$\approx n^{2-3s} \frac{1}{\sqrt{4n+1}} \frac{2n}{(2n+2)(4n+3)} \approx n^{\frac{1}{2}-3s}.$$

□

Since
$$\|r\|_2 \leq \|\psi\|_6^3 \leq n^{\frac{1}{2}-3s},$$
then we also get the sharp estimate of r in $\mathbb{L}^2$
$$\|r\|_2 = \left(\sum_{l\geq 1} | < h_{n+l}, |\psi^2|\psi > |^2 \right)^{\frac{1}{2}} \approx n^{\frac{1}{2}-3s}.$$
Notice that we also get the equivalent
$$< A^{-1}r, r > \approx n^{-6s}. \tag{16}$$

2.3. **The description of $z(t)$.** The equation (11) verified by z can be developed
$$i\partial_t z = \frac{\kappa^2}{\|\psi\|_2^2}(z(|z|^2-1)\|\psi\|_4^4 + 2|z|^2|\psi|^2\overline{\psi}q + z^2|\psi|^2\psi\overline{q} + 2z|\psi|^2|q|^2 + \overline{z}(\overline{\psi})^2q^2 + \overline{\psi}|q|^2q).$$
On the one hand the identity (7) on z and the estimates (6) on q give
$$|z|^2 = 1 - \frac{\|q\|_2^2}{\|\psi\|_2^2} \lesssim 1 + n^{-1-4s}. \tag{17}$$
On the other hand q and ψ are orthogonal, so
$$\int |\psi|^2\overline{\psi}q = \omega \int \overline{\psi}q + \int \overline{r}q = < q, r > .$$
Then one has
$$i\partial_t z - \frac{\kappa^2}{\|\psi\|_2^2}(2 < q, r > + z^2 < \overline{q}, \overline{r} >) \lesssim \frac{\kappa^2}{\|\psi\|_2^2} \int |\psi|^4 n^{-1-4s} + |\psi^2 q^2| + |\psi q^3| + n^{-1-4s}|qr|.$$
By using the estimates (1) on ψ and (6), (14) on q, the terms on the right side are upper bounded as follows
$$\int |\psi|^4 n^{-1-4s} \lesssim n^{-\frac{1}{2}-8s},$$
$$\int |\psi^2 q^2| \leq \|\psi\|_\infty^2 \|q\|_2^2 \lesssim n^{-\frac{1}{2}-8s},$$
$$\int |\psi q^3| \lesssim \|\psi\|_2 \|q\|_6^3 \lesssim n^{-1-10s}(\log n)^{\frac{3}{2}},$$
and
$$\int |qr| n^{-1-4s} \lesssim \|\psi\|_6^3 \|q\|_2 n^{-1-4s} \lesssim n^{-1-10s}.$$
So, the equation of z becomes
$$i\partial_t z = \frac{\kappa^2}{\|\psi\|_2^2}(2 < q, r > + z^2 < \overline{q}, \overline{r} >) + O(n^{-\frac{1}{2}-6s}).$$
The equation of q is given by the projection of the equation (S) on the space spanned by the h_{n+j}'s
$$i\partial_t q - Aq = \kappa^2 \Pi \left(|z\psi + q|^2(z\psi + q) + \omega q \right),$$

where Π is the associated projector. One can write

$$< q, r >=< Aq, A^{-1}r >= - < \kappa^2 \Pi\left(|z\psi + q|^2(z\psi+q)+\omega q\right), A^{-1}r > + < i\partial_t q, A^{-1}r > .$$

The operator A^{-1} induces a decay of n^{-1} so the first term can be estimated as follows

$$\int |\psi^2 q A^{-1}r| \lesssim n^{-1}\|\psi\|_\infty^2 \|q\|_2\|r\|_2 \lesssim n^{-\frac{1}{2}-8s},$$

$$\int |q^3 A^{-1}r| \lesssim n^{-1}\|q\|_6^3\|r\|_2 \lesssim n^{-\frac{3}{2}-12s}(\log n)^{\frac{3}{2}},$$

$$\omega \int |qA^{-1}r| \lesssim \frac{\|\psi\|_4^4}{\|\psi\|_2^2} n^{-1}\|q\|_2\|r\|_2 \lesssim n^{-\frac{1}{2}-8s},$$

and

$$< |\psi|^2\psi, A^{-1}r >=< r, A^{-1}r > \approx n^{-6s}.$$

For the last term the equivalent is given by (16). His coefficient is $|z|^2 z$, so using the behavior (17) of z,

$$|z|^2 z < |\psi|^2\psi, A^{-1}r >= z < r, A^{-1}r > + O(n^{-1-10s}). \tag{18}$$

Therefore

$$< q, r >= -z\kappa^2 < r, A^{-1}r > + i\partial_t < q, A^{-1}r > + O(n^{-\frac{1}{2}-8s}).$$

Noticing that $< r, A^{-1}r >$ is a real number, and using (18), the equation of z can be written now

$$i\partial_t z = \frac{\kappa^2}{\|\psi\|_2^2}(-iz3\kappa^2 < r, A^{-1}r > + 2i\partial_t < q, A^{-1}r > - iz^2\partial_t < \overline{q}, A^{-1}\overline{r} >) + O(n^{-\frac{1}{2}-6s}).$$

The value of z at zero is 1 so the Duhamel formula implies

$$\left| z(t) - e^{-it\frac{3\kappa^4}{\|\psi\|_2^2}<r,A^{-1}r>}\right| \lesssim n^{2s}\left|\int_0^t e^{-i(t-s)\frac{3\kappa^4}{\|\psi\|_2^2}<r,A^{-1}r>}\partial_s < q, A^{-1}r > ds\right|$$

$$+ n^{2s}\left|\int_0^t e^{-i(t-s)\frac{3\kappa^4}{\|\psi\|_2^2}<r,A^{-1}r>} z^2(s)\partial_s < \overline{q}, A^{-1}\overline{r} > ds\right| + O(n^{-\frac{1}{2}-6s}).$$

By integration by parts in the first term

$$\int_0^t e^{-it\frac{3\kappa^4}{\|\psi\|_2^2}<r,A^{-1}r>}\partial_s < q, A^{-1}r > ds \lesssim (1 + n^{2s}n^{-1}\|r\|_2^2)n^{-1}\|q\|_2\|r\|_2$$

$$\lesssim (1+n^{-4s})n^{-1-6s} \lesssim n^{-1-6s}.$$

Let us notice that from (11), the derivative of z has the same upper bound as $z-1$, that is n^{-4s}. This fact, together with the behavior (17) of z, gives by integrations by parts the same upper bound for the second term as for the first one.

Therefore the description of the coefficient of ψ in the solution u is

$$z(t) = e^{-it\frac{3\kappa^4}{\|\psi\|_2^2}<r,A^{-1}r>} + O(n^{-\frac{1}{2}-6s}), \tag{19}$$

and the assertion $i)$ of the Theorem 3.1.1 is proved.

2.4. **The exact growth of $\|q\|_{H^s}$ and of $|z-1|$.** For $s>0$ the result (19) of the former section implies that

$$z(t)-1=-i\frac{3\kappa^4 t}{\|\psi\|_2^2}<A^{-1}r,r>+O(n^{-8s}),$$

and the equivalent (16) gives the exact growth

$$|z(t)-1|\approx tn^{-4s}.$$

The link (12) between the estimates of $z-1$ and q

$$|z(t)-1|\lesssim tn^{-s}\sup_{0<\tau<T}\|q(\tau)\|_{H^{\frac12}}\lesssim tn^{-4s},$$

implies that

$$\sup_{0<\tau<T}\|q(\tau)\|_{\mathbb{H}^{\frac12}}\approx n^{-3s}.$$

If $s=0$ then by (16)

$$\lim_{n\to\infty}\frac{3\kappa^4 t}{\|\psi\|_2^2}<A^{-1}r,r>\neq 0,$$

and by using the description (19) of z,

$$|z(t)-1|\approx t.$$

By the same arguments above

$$\sup_{0<t<T}\|q(t)\|_{\mathbb{H}^{\frac12}}\approx 1,$$

and the equivalents claimed in the begining of Theorem 3.1.1 are proved.

Remark 2.5. *As a consequence, fir $t>0$,*

$$\|(z(t)-1)\psi\|_{\mathbb{H}^s}\approx tn^{-4s}\gg tn^{-\frac12-2s}\approx\sup_{0<t<T}\|q(t)\|_{\mathbb{H}^{\frac12}}n^{-\frac12+s}\gtrsim\|q(t)\|_{\mathbb{H}^s}.$$

This shows that the main part in the remainder term in the evolution of ψ by the equation (S) remains parallel to ψ.

3. The ansatz of the solution

3.1. **The equations of the z_j's.** We recall the notations done in the introduction

$$\alpha_j=2nj+j^2+j-\kappa^2\omega\ ,\ k_{j,l}=\kappa^2<\overline{h_{n+l}}\psi^2,h_{n+j}>,$$

$$\mu_j=\sqrt{(3k_{j,j}+\alpha_j)(k_{j,j}+\alpha_j)}.$$

The equation of z_j is obtained by taking the scalar product of the equation (S) with the spherical harmonic h_{n+j}

$$\begin{cases} i\partial_t z_j-\alpha_j z_j=\kappa^2<|z\psi+q|^2(z\psi+q),h_{n+j}>,\\ z_j(0)=0.\end{cases}$$

Let α be a number smaller than 1. The equation can be written

$$i\partial_t z_j = \alpha_j z_j + \sum_{l=1}^{n^\alpha} (2z_l + \overline{z_l}) k_{j,l} + r_j,$$

where

$$r_j = \kappa^2 < |z\psi + q|^2 (z\psi + q), h_{n+j} > - \sum_{l=1}^{n^\alpha} (2z_l + \overline{z_l}) k_{j,l}.$$

Notice that r_j does not contain linear terms in z_l's. Consider now the system of equations of the real and imaginary parts of z_j

$$\begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix}' = \begin{pmatrix} 0 & \alpha_j + k_{j,j} \\ -\alpha_j - 3k_{j,j} & 0 \end{pmatrix} \begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix}$$

$$+ \sum_{l \neq j}^{n^\alpha} k_{j,l} \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} \Re z_l \\ \Im z_l \end{pmatrix} + \begin{pmatrix} \Im r_j \\ -\Re r_j \end{pmatrix}.$$

The eigenvalues of the first matrix on the righthandside are $\pm i\mu_j$. Notice that

$$\begin{pmatrix} 0 & \alpha_j + k_{j,j} \\ -\alpha_j - 3k_{j,j} & 0 \end{pmatrix} = B_j^{-1} A_j B_j,$$

where A_j is the diagonal matrix

$$A_j = \begin{pmatrix} -i\mu_j & 0 \\ 0 & i\mu_j \end{pmatrix},$$

and

$$B_j = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i\frac{\mu_j}{\alpha_j + 3k_{j,j}} \\ -i\frac{\mu_j}{\alpha_j + k_{j,j}} & 1 \end{pmatrix}.$$

Set

$$A_{j,l} = \frac{k_{j,l}}{2} B_j \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} B_l^{-1}$$

and

$$\begin{pmatrix} d_j \\ \widetilde{d_j} \end{pmatrix} = B_j \begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix}.$$

The system in the new variables is

$$\begin{pmatrix} d_j \\ \widetilde{d_j} \end{pmatrix}' = A_j \begin{pmatrix} d_j \\ \widetilde{d_j} \end{pmatrix} + \sum_{l \neq j}^{n^\alpha} A_{j,l} \begin{pmatrix} d_l \\ \widetilde{d_l} \end{pmatrix} - B_j \begin{pmatrix} \Im r_j \\ -\Re r_j \end{pmatrix}.$$

By performing a second change of variable

$$\begin{pmatrix} f_j \\ \widetilde{f_j} \end{pmatrix} = e^{-tA_j} \begin{pmatrix} d_j \\ \widetilde{d_j} \end{pmatrix},$$

the system becomes

$$\begin{pmatrix} f_j \\ \widetilde{f}_j \end{pmatrix}' = \sum_{l\neq j}^{n^\alpha} e^{-tA_j} A_{j,l} e^{tA_l} \begin{pmatrix} f_l \\ \widetilde{f}_l \end{pmatrix} - R'_{j,j}(t),$$

with

$$R_{j,l}(t) = \int_0^t e^{-\tau A_j} B_l \begin{pmatrix} \Im r_l \\ -\Re r_l \end{pmatrix} d\tau.$$

The integration in time gives

$$\begin{pmatrix} f_j(t) \\ \widetilde{f}_j(t) \end{pmatrix} = \sum_{l\neq j}^{n^\alpha} \int_0^t e^{-\tau A_j} A_{j,l} e^{\tau A_l} \begin{pmatrix} f_l(\tau) \\ \widetilde{f}_l(\tau) \end{pmatrix} d\tau - R_{j,j}(t).$$

Lemma 3.1. *For $j \neq l$ there are matrices $M_{j,l}(t)$ and $B_{j,l}$ verifying the relation*

$$M_{j,l}(t) = \int_0^t e^{-\tau A_j} A_{j,l} e^{\tau A_l} d\tau = e^{-tA_j} B_{j,l} e^{tA_l} - B_{j,l}$$

and the estimates

$$|M_{j,l}(t)| \approx |B_{j,l}| \lesssim \frac{n^{\frac{1}{2}-2s}}{n|j-l|}.$$

Proof. Finding $B_{j,l}$ is equivalent to solving the matrix equation

$$A_{j,l} = (B_{j,l}A_l - A_j B_{j,l}).$$

Let us denote

$$B_{j,l} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}.$$

In view of the expression of A_l, the equation becomes

$$A_{j,l} = (\mu_j - \mu_l) \begin{pmatrix} x & -y \\ z & -t \end{pmatrix}.$$

Thus the existence of $B_{j,l}$ is proved and estimates on it can be found as follows. Since $k_{j,l} \lesssim n^{\frac{1}{2}-2s}$, we have $\mu_j \approx \alpha_j$, so

$$|B_j| \approx 1 \quad , \quad |A_{j,l}| \lesssim n^{\frac{1}{2}-2s},$$

and consequently

$$|M_{j,l}(t)| \approx |B_{j,l}| \lesssim \frac{|A_{j,l}|}{|\mu_j - \mu_l|} \lesssim \frac{n^{\frac{1}{2}-2s}}{n|j-l|}.$$

□

Since $f_j(0) = \widetilde{f}_j(0) = 0$, after integrating by parts

$$\begin{pmatrix} f_j(t) \\ \widetilde{f}_j(t) \end{pmatrix} = \sum_{l\neq j}^{n^\alpha} M_{j,l}(t) \begin{pmatrix} f_j(t) \\ \widetilde{f}_j(t) \end{pmatrix} - \sum_{l\neq j}^{n^\alpha} \int_0^t M_{j,l}(\tau) \begin{pmatrix} f_l(\tau) \\ \widetilde{f}_l(\tau) \end{pmatrix}' d\tau - R_{j,j}(t).$$

Using the expression of f_l' and $\widetilde{f}_l'$ we obtain

$$(Rel_j)\ \begin{pmatrix} f_j(t) \\ \widetilde{f}_j(t) \end{pmatrix} = \sum_{l\neq j}^{n^\alpha} M_{j,l}(t) \begin{pmatrix} f_j(t) \\ \widetilde{f}_j(t) \end{pmatrix} - \sum_{l\neq j}^{n^\alpha} \int_0^t M_{j,l}(\tau) \sum_{k\neq l}^{n^\alpha} e^{-\tau A_l} A_{l,k} e^{\tau A_k} \begin{pmatrix} f_k(\tau) \\ \widetilde{f}_k(\tau) \end{pmatrix} d\tau$$

$$- \sum_{l\neq j}^{n^\alpha} \left(\int_0^t e^{-\tau A_j} B_{j,l} B_l \begin{pmatrix} \Im r_l \\ -\Re r_l \end{pmatrix} d\tau + B_{j,l} R_{l,l}(t) \right) - R_{j,j}(t).$$

Since

$$\begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix} = e^{tA_j} B_j^{-1} \begin{pmatrix} f_j(t) \\ \widetilde{f}_j(t) \end{pmatrix},$$

(Rel_j) is the searched relation between the z_j's.

3.2. **Estimates on the source terms $R_{j,l}$.**

Lemma 3.2. *Let $s \in]\frac{1}{12}, \frac{1}{4}[$, and let $\alpha \in]1-4s, 8s[$. Then we have the estimates*

$$(20)\qquad \begin{cases} |R_{2,2}| \approx n^{-\frac{1}{2}-3s}, \\ |R_{j,l}| \lesssim \frac{n^{-\frac{1}{2}-3s}}{j} \ \textit{if}\ j \ll n^{-\frac{1}{2}+2s+\frac{\alpha}{2}}, \\ |R_{j,l}| \lesssim n^{-5s-\frac{\alpha}{2}} \ \textit{for the other}\ j \ll n^\alpha. \end{cases}$$

Proof. Let us recall the expression of r_l

$r_l = \kappa^2 < |z|^2 z|\psi|^2\psi + 2(|z|^2 q - q + q_\alpha)|\psi|^2 + (z^2\overline{q} - \overline{q} + \overline{q_\alpha})\psi^2 + 2z\psi|q|^2 + \overline{z}\overline{\psi}q^2 + |q|^2 q, h_{n+l} >$.

Since $|B_l| \approx 1$ one can estimate

$$\left| R_{j,l} - < |\psi|^2\psi, h_{n+l} > \int_0^t e^{\pm i\tau\mu_j} d\tau \right| \lesssim \left| < |\psi|^2\psi, h_{n+l} > \int_0^t e^{\pm i\tau\mu_j} (|z(\tau)|^2 z(\tau) - 1) d\tau \right|$$

$$+ \int (|\psi|^2(|q||z^2 - 1| + |q_\alpha|) + |\psi q^2| + |q|^2 q)|h_{n+l}|.$$

By using Cauchy-Schwarz's inequality and (1)

$$| < |\psi|^2\psi, h_{n+l} > | \lesssim n^{\frac{1}{2}-3s},$$

with equivalence for $l = 2$, and cancellation for odd l, by Lemma 2.4. Therefore

$$(21)\qquad \left| < |\psi|^2\psi, h_{n+l} > \int_0^t e^{\pm i\tau\mu_j} d\tau \right| \lesssim \frac{n^{\frac{1}{2}-3s}}{nj},$$

also with equivalence for $l = 2$. Note that from (11) one has the same estimate for $(z-1)'$ as for $z - 1$. Integration by parts gives

$$\left| < |\psi|^2\psi, h_{n+l} > \int_0^t e^{\pm i\tau\mu_j} (|z(\tau)|^2 z(\tau) - 1) d\tau \right| \ll \frac{n^{\frac{1}{2}-3s}}{nj}.$$

By using the upper bounds (14) and (15) the other terms can be estimated as follows

$$\int |\psi^2 q(z^2-1)h_{n+l}| \leq \sup_{0<t<T} |z-1| \|\psi\|_\infty^2 \|q\|_2 \lesssim n^{-9s},$$

$$\int |\psi^2 q_\alpha h_{n+l}| \leq n^{\frac{1}{2}-2s} \|q_\alpha\|_2 \lesssim n^{-5s-\frac{\alpha}{2}},$$

$$\int |\psi q^2 h_{n+l}| \leq \|\psi\|_\infty \|q\|_4^2 \lesssim n^{-\frac{1}{2}-7s} \log n \ll n^{-5s-\frac{\alpha}{2}},$$

and

$$\int |q^3 h_{n+l}| \leq \|q\|_6^3 \lesssim n^{-1-9s} (\log n)^{\frac{3}{2}} \ll n^{-5s-\frac{\alpha}{2}}.$$

Therefore

$$|R_{j,l}| \lesssim \left| < |\psi|^2\psi, h_{n+l} > \int_0^t e^{\pm i\tau\mu_j} d\tau \right| + O(n^{-5s-\frac{\alpha}{2}}) + O(n^{-9s}).$$

In view of (21), in order to have the first term as dominant for small j, we choose

$$\alpha \in]1-4s, 8s[.$$

Indeed

$$n^{-5s-\frac{\alpha}{2}} \ll \frac{n^{\frac{1}{2}-3s}}{nj}$$

for all $j \ll n^{-\frac{1}{2}+2s+\frac{\alpha}{2}}$, and

$$n^{-9s} \ll n^{-5s-\frac{\alpha}{2}}$$

for all j. This condition on α implies the restriction $s \in]\frac{1}{12}, \frac{1}{4}[$. In conclusion, for α chosen in $]1-4s, 8s[$, we have the estimates (20). □

Remark 3.3. *The restriction $s \in]\frac{1}{12}, \frac{1}{4}[$ is due to the presence in the source terms $R_{j,l}$ of the linear terms in z_j's, that have only the decay $O(n^{-9s})$. These terms come from*

$$< z^2 \overline{q}\, \psi^2, h_{n+l} >,$$

and have variable coefficients. If we consider them in the linear part of the system of the z_j's, we are unable to obtain the decay estimates claimed in Theorem 3.1.1.

3.3. **Estimates on the z_j's.** Since $|e^{tA_j} B_j| \approx 1$

$$\left| \begin{pmatrix} f_j(t) \\ \tilde{f}_j(t) \end{pmatrix} \right| \approx \left| \begin{pmatrix} \Re z_j(t) \\ \Im z_j(t) \end{pmatrix} \right|.$$

The relation (Rel_j) gives

$$\left| \begin{pmatrix} f_j(t) \\ \tilde{f}_j(t) \end{pmatrix} + R_{j,j}(t) \right| \lesssim \sum_{l\neq j}^{n^\alpha} |M_{j,l}||z_j| - \sum_{l\neq j}^{n^\alpha} |M_{j,l}| \sum_{k\neq l}^{n^\alpha} |A_{l,k}||z_l| - \sum_{l\neq j}^{n^\alpha} |B_{j,l}|(|R_{j,l}| + |R_{l,l}|).$$

Then the estimate (13) on the z_l's, Lemma 3.3.1 and (20) imply that the term on the right is upper bounded by

$$n^{-\frac{1}{2}-2s} \log n (n^{-\frac{1}{2}-3s} + n^{\frac{1}{2}-2s} n^{-\frac{1}{2}-3s} \log n^{\frac{1}{2}} + (n^{-\frac{1}{2}-3s} + n^{-5s-\frac{\alpha}{2}})).$$

Therefore, since α is smaller than $1+4s$,

$$\left|\begin{pmatrix} f_j(t) \\ \tilde{f}_j(t) \end{pmatrix} + R_{j,j}(t)\right| \lesssim n^{-1-5s}\log n + n^{-\frac{1}{2}-7s}\log n \ll n^{-5s-\frac{\alpha}{2}}. \tag{22}$$

By using again (20) we get the behavior of $|f_j|$ and implicitly the one of $|z_j|$

$$\begin{cases} |z_2| \approx n^{-\frac{1}{2}-3s} \\ |z_j| \lesssim \frac{n^{-\frac{1}{2}-3s}}{j} \text{ if } j \ll n^{-\frac{1}{2}+2s+\frac{\alpha}{2}} \\ |z_j| \lesssim n^{-5s-\frac{\alpha}{2}} \text{ for the other } j \ll n^{\alpha} \end{cases}.$$

If α is taken to be $8s-2\epsilon$ with ϵ small and positive, these estimates become

$$\begin{cases} |z_2| \approx n^{-\frac{1}{2}-3s} \\ |z_j| \lesssim \frac{n^{-\frac{1}{2}-3s}}{j} \text{ if } j \ll n^{-\frac{1}{2}+6s-\epsilon} \\ |z_j| \lesssim n^{-9s+\epsilon} \text{ for the other } j \ll n^{8s-2\epsilon} \end{cases}.$$

3.4. **The ansatz.** As was proved in §3.2.2,

$$< |\psi|^2\psi, h_{n+j} >=< r, h_{n+j} >$$

is a real number independent of time. As a consequence, using (22) and the analysis of the source terms $R_{j,l}$ done in §3.3.2 for $\alpha = 8s-2\epsilon$, one can write for all $j \ll n^{8s-2\epsilon}$

$$e^{-tA_j}B_j\begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix} = O(n^{-9s+\epsilon}) - \int_0^t e^{-\tau A_j}d\tau B_j \begin{pmatrix} 0 \\ < r, h_{n+j} > \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} \Re z_j \\ \Im z_j \end{pmatrix} = O(n^{-9s+\epsilon}) - B_j^{-1}\int_0^t e^{\tau A_j}d\tau B_j \begin{pmatrix} 0 \\ < r, h_{n+j} > \end{pmatrix}.$$

Using the explicit form of A_j and B_j

$$z_j(t) = O(n^{-9s+\epsilon}) - i\frac{< r, h_{n+j} >}{2}\int_0^t e^{i\tau\mu_j}\left(\frac{\mu_j}{3k_{j,j}+\alpha_j}+1\right) + e^{-i\tau\mu_j}\left(\frac{\mu_j}{3k_{j,j}+\alpha_j}-1\right)d\tau =$$

$$= O(n^{-9s+\epsilon}) - i < r, h_{n+j} > \int_0^t \frac{\mu_j}{3k_{j,j}+\alpha_j}\cos\tau\mu_j + i\sin\tau\mu_j d\tau.$$

Let

$$\beta \in \left]0, 6s-\frac{1}{2}\right[$$

and q_β the part of q containing only the spherical harmonics of order greater than $n+n^\beta$. Hence $q(t,x)$ can be decomposed as follows

$$q(t,x) = -i\int_0^t e^{i\tau M} r\, d\tau + i\sum_{j=n^\beta..\infty} < r, h_{n+j} > \int_0^t e^{i\tau\mu_j}d\tau\, h_{n+j}(x) + \tilde{q}(t,x) + q_\beta(t,x),$$

where

$$\widetilde{q}(t,x)=\sum_{j=2..n^\beta}\left(O(n^{-9s+\epsilon})-i<r,h_{n+j}>\left(\frac{\mu_j}{3k_{j,j}+\alpha_j}-1\right)\int_0^t\cos\tau\mu_j d\tau\right)h_{n+j}(x).$$

As was proved in §3.2.2, the upper bound $n^{\frac12-3s}$ of $<r,h_{n+j}>$ is an equivalent for $j=2$ so the $\mathbb{H}^s$ norm of the principal part of q is

$$\left\|\int_0^t e^{i\tau M}r\,d\tau\right\|_{\mathbb{H}^s}\approx\left(\sum_{j=2..\infty}\frac{|<r,h_{n+j}>|^2}{(nj+j^2)^2}(n+j)^{2s}\right)^{\frac12}\approx n^{-\frac12-2s}.$$

Similarly

$$\left\|\sum_{j=n^\beta..\infty}|<r,h_{n+j}>|\int_0^t e^{i\tau\mu_j}d\tau\,h_{n+j}\right\|_{\mathbb{H}^s}\lesssim\sum_{j=n^\beta..\infty}\frac{<r,h_{n+j}>}{nj+j^2}(n+j)^s\ll n^{-\frac12-2s}.$$

Using the upper bounds (15) on the norms of q_β

$$\|q_\beta\|_{\mathbb{H}^s}\leq\|q_\beta\|_2^{1-s}\|q_\beta\|_{\mathbb{H}^1}^s\lesssim n^{-\frac12-2s-\frac{\beta}{2}}\ll n^{-\frac12-2s}.$$

Finally, one can estimate the $\mathbb{H}^s$ norm of $\widetilde{q}$ as follows

$$\|\widetilde{q}\|_{\mathbb{H}^s}\lesssim n^\beta n^{-9s+\epsilon}n^s+\|r\|_2\sum_{j=2..n^\beta}\left(\frac{1}{3k_{j,j}+\alpha_j}-\frac{1}{\mu_j}\right)(n+j)^s\lesssim$$

$$\lesssim n^{\beta-8s+\epsilon}+n^{\frac12-3s}\sum_{j=2..n^\beta}\frac{|k_{j,j}|}{\alpha_j^2}(n+j)^s.$$

The estimates

$$|k_{j,j}|\leq\|\psi\|_\infty^2=n^{\frac12-2s}\ll nj+j^2\approx\alpha_j$$

imply that

$$\|\widetilde{q}\|_{\mathbb{H}^s}\lesssim n^{\beta-8s+\epsilon}+n^{1-5s}\sum_{j=2..n^\beta}\frac{(n+j)^s}{(nj+j^2)^2}\lesssim$$

$$\lesssim n^{-\frac12-2s}(n^{\beta+\frac12-6s+\epsilon}+n^{-\frac12-2s})\ll n^{-\frac12-2s}.$$

Therefore

$$\|q+i\int_0^t e^{i\tau M}r\,d\tau\|_{\mathbb{H}^s}\ll n^{-\frac12-2s},$$

and the proof of Theorem 3.1.1 is complete.

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