# On values of Jacobi forms and elliptic Dedekind Sums

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November 22, 2008

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# 1 Introduction

In this paper, we are interesting by the study of special values of The Jacobi forms in **two variables** $D_L(z;\varphi)$. We introduce shifted elliptic Dedekind sums in terms of values of these Jacobi forms and we state and prove their Dedekind reciprocity Laws. Furthermore, in this study, we show how to use our techniques to obtain a closed new reciprocity law for the so-called **Shifted-elliptic Dedekind-Sczech Sums** 1.0.1. It allows us to obtain a strong generalization of Sczech's result [23].

In a concise way, we introduce the following main object of our study

$$\frac{1}{a_k}\sum_{t\in L/a_kL}\prod_{1\leqslant j\neq k\leqslant n} D_L\left(\frac{\varphi_j}{\bar{a}_j};a_j\frac{z_k+t}{a_k}-z_j\right) \tag{1.0.1}$$

We called them the **Shifted-elliptic Dedekind Sums**. We state and prove Dedekind reciprocity laws for these sums.

Let $E_1(z;L)$ be the first Eisenstein serie. As an application of our study, we show how to use our techniques to obtain a closed new reciprocity law for the following "**Shifted-elliptic Dedekind-Sczech Sums**"1.0.2. Precisely, we deduce reciprocity laws for

$$\frac{1}{a_k}\sum_{t\in L/a_kL}\prod_{1\leqslant j\neq k\leqslant 3} E_1\left(a_j\frac{z_k+t}{a_k}-z_j;L\right) \tag{1.0.2}$$

Then we get a strong generalization to Sczech's result [23].

Basically, it's well known that the classical Dedekind sums appears in various areas. In analytic and algebraic number theory, topology, differential geometry, algebraic and combinatorial geometry, algorithmic complexity ... etc. See [2, 14, 15, 16, 20, 24].

From our main result we can get several applications in the following areas: Study the Eisenstein Cohomology of the groups $\mathrm{SL}_2(O_K)$ where $K$ is an imaginary quadratic field, we establish connection between special values of Hecke $L$-functions at $s=1$ associated to some quadratic forms, closely related to quadratic residue symbols (i.e we can connect Legendre symbol ) to elliptic Dedekind sums. Finally we give recent result concerning the density of elliptic Dedekind sums.

In this paper several questions which are formulated are only partially solved and thus they remain still opened.

## 2 Special properties of Jacobi forms $D_L(z;\varphi)$

### 2.1 Definitions and notations

Let $\tau \in \mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ =upper half plane. We use the following notations. For all $\varphi \in \mathbb{C}$, we can write $\varphi = \varphi_1\tau + \varphi_2, (\varphi_1, \varphi_2) \in \mathbb{R}^2$, because $\{\tau, 1\}$ is an $\mathbb{R}$-basis of $\mathbb{C}$. We put

$$e(z) = e^{2\pi i z}, q_\tau = e(\tau). \tag{2.1.3}$$

We recall the Jacobi's Theta function

$$\theta_\tau(z) = \sum_{n\in\mathbb{Z}} e\left(\frac{1}{2}(n+\frac{1}{2})^2\tau + (n+\frac{1}{2})(z+\frac{1}{2})\right) \tag{2.1.4}$$

By the well known Jacobi Triple product formula, we can rewrite Theta function as follows

$$\theta_\tau(z) = iq_\tau^{1/8}\left(e(z/2) - e(-z/2)\right)\prod_{n=1}^{\infty}\left(1-q_\tau^n\right)\left(1-q_\tau^n e(z)\right)\left(1-q_\tau^n e(-z)\right) \tag{2.1.5}$$

In this paper we need to define the following $\mathbb{R}$-alternating bilinear form

$$E_L(z,\varphi) = \frac{\bar{z}\varphi - z\bar{\varphi}}{2iIm(\tau)} \tag{2.1.6}$$

is a symplectic form on $\mathbb{C}$ associated to the complex lattice $L = \mathbb{Z}\tau + \mathbb{Z}$.

### 2.2 Eisenstein series and Jacobi forms of two variables $D_L(z,\varphi)$.

We fix a complex Lattice $L$. Eisenstein series, associated to Lattice $L$, are defined by

$$E_k(z,L) = \lim_{s\to 0+}\sum_{w\in L}^{(e)}(w+z)^{-k}|w+z|^{-s}, k = 1, ... \tag{2.2.7}$$

the sum being over $\omega \in L$ if $z \notin L$ and $\omega \in L\backslash\{-z\}$ if $z \in L$. Where $\sum_{\omega\in L}^{(e)}$ is the Eisenstein summation equal to

$$\sum_{\omega\in L}^{(e)} = \lim_{M,N\to\infty}\sum_{m=-M}^{m=M}\sum_{n=-N}^{n=N}, \text{ Where } \omega = m\tau + n. \tag{2.2.8}$$

For $\varphi \in \mathbb{C}$, we write

$$\varphi = \varphi_1\tau + \varphi_2, (\varphi_1, \varphi_2) \in \mathbb{R}^2.$$

We associate to $L$ a Jacobi form of two variables

$$D_L(z;\varphi) = \frac{1}{\omega_2}e\left(z\varphi_1\right)\frac{\theta'_\tau(0)\theta_\tau\left(z+\varphi\right)}{\theta_\tau\left(z\right)\theta_\tau\left(\varphi\right)} \tag{2.2.9}$$

## 2.3 Properties of $D_L(z;\varphi)$.

We quote from [6] the following fondamental properties of $D_L(z;\varphi)$.

**Theorem 2.3.1** *(***Properties of** $D_L(z;\varphi)$*)*

*i)* $D_L$ *is meromorphic in the first variable* $z$*, and only real analytic on the second variable* $\varphi$*.*

*ii)* **Homogeneity:** $D_L$ *is homogenous of degre* $-1$ *i.e*

$$D_{\lambda L}(\lambda z;\lambda\varphi) = \lambda^{-1} D_L(z;\varphi), \forall \lambda \in \mathbb{C}\backslash\{0\}.$$

*iii)* **Symmetry:** *we have the following symmetry*

$$D_L(-z;-\varphi) = -D_L(z;\varphi).$$

*iv) (* **Periodicity** *of* $D_L(z;\varphi)$*):*

$$\begin{cases} D_L(z;\varphi+\rho) = D_L(z;\varphi) \\ D_L(z+\rho;\varphi) = e(E_L(\rho,\varphi))D_L(z;\varphi) \end{cases} \quad ,\forall \rho \in L$$

*v) (***Functional Equation***):*

$$D_L(z;\varphi)e(-E_L(z,\varphi)) = D_L(\varphi;z).$$

*vi) (***Modularity***):* $D_L$ *is a Jacobi modular form for* $SL_2(\mathbb{Z})$*, with index 0 and weight 1 i.e*

$$D_{\frac{a\tau+b}{c\tau+d}}\left(\frac{z}{c\tau+d};\frac{\varphi}{c\tau+d}\right) = (c\tau+d)D_\tau(z;\varphi), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

*where*

$$D_\tau(z;\varphi) := D_{L_\tau}(z;\varphi), L_\tau = \mathbb{Z}\tau + \mathbb{Z}, \tau \in \mathcal{H}.$$

*vii) The Jacobi form* $D_L(z,\varphi)$ *have the following* **Laurent expansion**

$$D_\tau(z;\varphi) = \sum_{m\geqslant 0} \bar{B}_m(\varphi;L)\frac{(2\pi i)^m}{m!}z^{m-1}$$

*Where*

$$\begin{cases} \bar{B}_0(\varphi;L) = 1, \\ \bar{B}_1(\varphi;L).2\pi i = E_1(\varphi;L), \\ \bar{B}_2(\varphi;L)\frac{(2\pi i)^2}{2!} = \frac{1}{2}E_1(\varphi,L)^2 - \frac{1}{2}\wp_L(\varphi), \end{cases}$$

$$\bar{B}_m(\varphi;L) = \begin{cases} -\frac{m!}{(2\pi i)^m}\sum_{\substack{\omega\in L \\ \omega\neq 0}}^{(e)} \frac{e(E_L(\omega,\varphi))}{\omega^m} & \text{If } m \geqslant 1 \\ 1 & \text{If } m = 0 \end{cases}$$

**Eisenstein-Kronecker series of Weight** $m$**.**

*viii)* *(***at Cusp** $\infty$*)*
*For* $z \in \mathbb{R}\backslash\mathbb{Z}, \varphi = \varphi_1\tau + \varphi_2$ *with* $\varphi_1 \notin \mathbb{Z}$*. Then*

$$\frac{1}{2\pi i} \lim_{\mathbf{Im}(\tau)\to\infty} D_\tau(z,\varphi) = \frac{e(z\{\varphi_1\})}{e(z)-1}$$

*ix)* **( Distribution Formulas for** $D_L(z;\varphi)$**) :**
*For* $L$*,* $\Lambda$ *complex lattices such that :* $L \subset \Lambda$*,* $[\Lambda : L] = l$*. We have:*

$$\sum_{\bar{t}\in\Lambda/L} D_L(lz;\varphi+t) = D_\Lambda(z;\varphi).$$

In the next section we study the properties of $\bar{B}_m(\varphi, L)$ coefficents of Laurent expansion of $D_L(z,\varphi)$ in the first variable $z$.

# 3 Elliptic Bernoulli functions $\bar{B}_m(\varphi, L)$ and their fondamental properties.

In the following theorem we precise the most important properties of elliptic Bernoulli numbers and functions.

**Theorem 3.0.2** *(***properties of** $\bar{B}_m(\varphi, L)$*)*

*i)* *(***Homogeneity***) For each* $m \in \mathbb{N}^*$*,* $\bar{B}_m(\varphi, L_\tau)$ *is homogenous of degre* $-m$ *i.e*

$$\bar{B}_m(\lambda\varphi, \lambda L) = \lambda^{-m}\bar{B}_m(\varphi, L), \forall\lambda \in \mathbb{C}\backslash\{0\}.$$

*ii)* *(***Periodicity***):*

$$\bar{B}_m(\varphi+\rho; L) = \bar{B}_m(\varphi; L), \forall\rho \in \mathbb{L} \tag{3.0.10}$$

*iii)* *(***Symmetry***):*

$$\bar{B}_m(-\varphi; L) = (-1)^{m-1}\bar{B}_m(\varphi; L)$$

*iv)* *(***Modularity***): We let* $\bar{B}_m(\varphi,\tau) := \bar{B}_m(\varphi, L_\tau)$*.*
$\bar{B}_m(\varphi,\tau)$ *is a modular form for* $SL_2(\mathbb{Z})$*, with index 0 and weight* $m$ *i.e*

$$\bar{B}_m\left(\frac{\varphi}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^m\bar{B}_m(\varphi;\tau),$$

*v)* **Bernoulli functions:** *For all* $\varphi \in \mathbb{C}\backslash\mathbb{Z}\tau+\mathbb{Z}$*, we have*

$$\mathbf{Re}\left(\lim_{\mathrm{Im}(\tau)\to\infty} \overline{B_m}\Big(\varphi; \mathbb{Z}\tau+\mathbb{Z}\Big)\right) = B_m(\{\varphi_1\})$$

*vi)* **Elliptic Raabe Formula**

*For $L$, $\Lambda$ complex lattices such that : $L \subset \Lambda$. Then, we have*

$$\sum_{\bar{t}\in\Lambda/L} \bar{B}_m(\varphi+t;L) = [\Lambda:L]^{1-m}\bar{B}_m(\varphi;\Lambda), \forall \quad \varphi\in\mathbb{C}\backslash\Lambda \text{ and } \forall \quad m\geqslant 1$$

*and*

$$\sum_{\bar{t}\in\Lambda/L\backslash\{\bar{0}\}} d_m(t;L) = E_m(0;L) - [\Lambda:L]^{1-m}E_m(0;\Lambda), m\geqslant 1$$

*In particular, for $\Lambda = \frac{1}{\omega}L$ we have*

$$\sum_{\bar{t}\in\Lambda/L\backslash\{0\}} d_m(t;L) = \left(1-\frac{|\omega|^2}{\bar{\omega}^m}\right)E_m(0,L), \forall m\geqslant 2$$

*where*

$$d_m(\varphi;L) = \bar{B}_m(\varphi;L)\frac{(2\pi i)^m}{m!}.$$

*Proof:*

This theorem 3.0.2 is a direct consequence of the theorem 2.3.1. The idea consists in extracting the coefficients of Laurent series of the Jacobi forms in the results of the previous theorem 2.3.1.

# 4 Dedekind Sums.

This paragraph consists of two parts. In the first one, we regroup together the significant results on the classical Dedekind sums. In the second part of this section, we clarify the problems connected to the elliptic Dedekind sums (or special values of Jacobi forms). We formulate the problems similar to the classical case and give some new results.

## 4.1 Classical Dedekind sums: An overview

The Bernoulli numbers are defined by the serie

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n\frac{z^n}{n!}, \qquad |z|<2\pi,$$

The Bernoulli polynomials are similarly defined by the generating function

$$\frac{ze^{zx}}{e^z-1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{z^n}{n!}, \qquad |z|<2\pi.$$

To recall the famous **Dedekind reciprocity Law** we define:

$$\bar{B}_n(x) := \begin{cases} 0 & \text{if } x \in \mathbb{Z}, n = 1 \\ B_n(\{x\}) & \text{Otherwise} \end{cases} .$$

and

$$\bar{B}_1(x) = \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{Otherwise} \end{cases} .$$

Let $a, b \in \mathbb{N}^*$. **The classical Dedekind sums is**

$$s(a,b) := \sum_{k \bmod b} \bar{B}_1\left(\frac{ka}{b}\right) \bar{B}_1\left(\frac{k}{b}\right) ;$$

By **discrete Fourier series** of the sawtooth function (see, for example, [**?**, p. 14]) we obtain

$$\bar{B}_1\left(\frac{n}{p}\right) = \left(\left(\frac{n}{p}\right)\right) = \frac{i}{2p} \sum_{k=1}^{p-1} \cot\left(\frac{\pi k}{p}\right) e^{2\pi i k n/p} , \tag{4.1.11}$$

Now, we can rewrite $s(a,b)$ **in terms of cotangents:**

$$s(a,b) = \frac{1}{4b} \sum_{k=1}^{b-1} \cot \frac{\pi k a}{b} \cot \frac{\pi k}{b} .$$

Then, we obtain two representations of $s(a,b)$ and provides us various generalizations of the Dedekind sums in the litterature.
In the next theorem, we state the Dedekind reciprocity law

**Theorem 4.1.1 (Dedekind-1880)**
*Let $a, b \in \mathbb{N}^*$ coprime. then*

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) .$$

Since $s(-a,-b) = s(a,b)$, we can define $s(\frac{a}{b}) = s(a,b)$ for an irreductible fraction $\frac{a}{b}$ and get a function on the rational field $\mathbb{Q}$. Hickerson in [13] has proved the following.

**Theorem 4.1.2** *(**Hickerson-1977**) The set $\left\{\left(\frac{a}{b}, s(\frac{a}{b})\right) : \frac{a}{b} \in \mathbb{Q}\right\}$ is dense in the plane.*

Now, we state the problems for **generalized classical Dedekind sums:**
Let $a, a_1, \ldots, a_n$ be positive integers, $m_1, \ldots, m_n$ nonnegative integers, and $z_1, \ldots, z_n$ complex numbers. **Study the arithmetic properties of the following two Shifted classical Dedekind Sums**

$$\boxed{\textbf{Sums of first Kind :} \sum_{k \bmod a} \prod_{j=1}^{n} \bar{B}_{m_j}\left(\frac{k a_j}{a} + z_j\right)}$$

$$\boxed{\textbf{Sums of second Kind:} \quad \sum_{k \bmod a} \prod_{j=1}^{n} \cot^{(m_j)} \pi \left( \frac{ka_j}{a} + z_j \right)}$$

We have here two kinds of generalized classical Dedekind sums.
In general, these representations are differents, but there are the same in the case stutied by Dedekind himself ( corresponding to $n = 2$).
Essentially, the problem considered here is to give a generalized Dedekind reciprocity laws for these two sums and get their applications in differents areas in mathematics.
Several mathematicians studied the Dedekind reciprocity Laws for these two sums. In particular: Apostol [1], Beck [7], Berndt [8], Carlitz [9], Dieter [10], Meyer [22], Rademacher [21], Halbritter [11], Hall- Wilson-Zagier [12, 25] and many others mathematicians.

On the other hand the above density problem remains still opened concerning the study of two differents kinds of Dedekind two sums.
Now, I am going to clarify the wellknown results on the Dedekind reciprocity laws of them. The above reciprocity law problem is solved:

i) partially for the sums of the first kind only for $n = 2$ and arbitrary $m_1, m_2$. See [12]

ii) On the other hand for the sums of the second kind is solved completely for arbitrary $n$, $m_1, ..., m_d$. See [7].

These results are formulated in the following two theorems.

We set

$$S(\overrightarrow{A_k}, \overrightarrow{X_k}, \overrightarrow{R_k}; \tau) := \sum_{t=0}^{a_k-1} \prod_{1 \leqslant j \neq k \leqslant n} \bar{B}_{r_j} \left( a_j \frac{x_k + t}{a_k} - x_j \right)$$

Now, we consider the generating function of these sums

$$\mathfrak{S}(\overrightarrow{A_k}, \overrightarrow{X_k}, \overrightarrow{\Phi_k}) := \sum_{\overrightarrow{R_k} \in \mathbb{N}^{n-1}} \frac{S(\overrightarrow{A_k}, \overrightarrow{X_k}, \overrightarrow{R_k}; \tau)}{r_1! \ldots \check{r}_j! \ldots r_n!} \prod_{1 \leqslant j \neq k \leqslant n} \left( \frac{\varphi_j}{a_j} \right)^{r_j - 1}$$

Now, for $n = 3$ we can formulate the main theorem in [12] as follows

**Theorem 4.1.3 (Hall-Wilson-Zagier-1995)** *Let $a_1, a_2, a_3$ be three positive integers which have no common factor,$\overrightarrow{A} = (a_1, a_2, a_3)$,$x_1, x_2, x_3$ three real numbers, and $\varphi_1, \varphi_2, \varphi_3$ three variables with sum zero. Then*

$$\sum_{k=1}^{3} \mathfrak{S}(\overrightarrow{A_k}, \overrightarrow{X_k}, \overrightarrow{\Phi_k}) = \begin{cases} -\frac{1}{4} & \textit{If } \overrightarrow{X} \in \mathbb{R}\overrightarrow{A} + \mathbb{Z}^3 \\ 0 & \textit{otherwise.} \end{cases}$$

*Where $\overrightarrow{A} = (a_1, a_2, a_3), \overrightarrow{X} = (x_1, x_2, x_3)$*

In order to state the next theorem, we denote by $\cot^{(m)}$ the $m$'th derivative of the cotangent function and let $a_1, \ldots, a_n \in \mathbb{N}$ , $m_1, \ldots, m_n \in \mathbb{N}$, $x_1, \ldots, x_n$ are real numbers. Beck in his paper [7] study the following sums

$$\mathfrak{C}\left(\begin{array}{c|ccccc} a_k & a_1 & \cdots & \widehat{a_k} & \cdots & a_n \\ m_k & m_1+l_1 & \cdots & \widehat{m_k+l_k} & \cdots & m_n+l_n \\ x_k & x_1 & \cdots & \widehat{x_k} & \cdots & x_n \end{array}\right) := \frac{1}{{a_k}^{m_k+1}} \sum_{t \bmod a_k} \prod_{1 \leqslant j \neq k \leqslant n} \cot^{(m_j)} \pi \left(a_j \frac{t+x_k}{a_k} - x_j\right)$$

We quote from [7] the result

**Theorem 4.1.4** *(**Beck-2003**)*
*Let $a_1, \ldots, a_n \in \mathbb{N}$ , $m_1, \ldots, m_n \in \mathbb{N}$, $x_1, \ldots, x_n$ are real numbers. Assuming that*

$$a_j x_k - a_k x_j \notin \mathbb{Z}a_j + \mathbb{Z}a_k, \ \forall\ 1 \ \leqslant j \neq k \leqslant n.$$

*Then, we have the following Dedekind reciprocity Law*

$$\sum_{k=1}^{n} (-1)^{m_k} m_k! \sum_{\substack{l_1, \ldots, \widehat{l_k}, \ldots, l_n \geqslant 0 \\ l_1 + \ldots + \widehat{l_k} + \ldots + l_n = m_k}} \frac{a_1^{l_1} \cdots \widehat{a_k^{l_k}} \cdots a_n^{l_n}}{l_1! \cdots \widehat{l_k!} \cdots l_n!} \mathfrak{C}\left(\begin{array}{c|ccccc} a_k & a_1 & \cdots & \widehat{a_k} & \cdots & a_n \\ m_k & m_1+l_1 & \cdots & \widehat{m_k+l_k} & \cdots & m_n+l_n \\ x_k & x_1 & \cdots & \widehat{x_k} & \cdots & x_n \end{array}\right)$$
$$= \begin{cases} (-1)^{\frac{n-1}{2}} & \textit{if all } m_k = 0 \textit{ and } n \textit{ odd} \\ 0 & \textit{otherwise.} \end{cases}$$

## 4.2 Elliptic Dedekind sums: Problems and results.

Now, we precise the **elliptics problem** for the multiple elliptic Dedekind sums.

Let $L = \mathbb{Z}\tau + \mathbb{Z}$ be a complex lattice we denote by $O_L := \{x \in \mathbb{C} : xL \subset L\}$. Let $a_1, \ldots, a_n$ be elements in $O_L$ , the $a_i$ being pairwise coprime , $z_1, \ldots, z_n \in \mathbb{C}$ and $\varphi_1, \cdots, \varphi_n$ be complex variables.
We study the arithmetical properties of the following elliptic Dedekind sums

$$\boxed{\textbf{Elliptic sums of first Kind:} \sum_{\bar{k} \in L/aL} \prod_{j=1}^{n} \bar{B}_{m_j}\left(a_j \frac{k+z_0}{a} + z_j; L\right)}$$

$$\boxed{\textbf{Elliptic sums of second Kind:} \sum_{\bar{k} \in L/aL} \prod_{j=1}^{n} D_L^{(m_j)}\left(\varphi_j; a_j \frac{k+z_0}{a} + z_j\right)}$$

In my paper [6], recently i proved the Dedekind reciprocity law for the elliptic Dedekind sums of the first kind.
In this paper we study the reciprocity law for the shifted-Dedekind elliptic sums in terms of special values of Jacobi forms (i.e **elliptic sums of second kind**) for $m_j = 1, \forall j = 1, ..., n$.

In the last section of this paper, we treat the other interesting questions connected to these sums.

Now, we state the following reciprocity laws

**Theorem 4.2.1**

*Let $a_1, \ldots, a_n$ be elements in $O_L$, $z_1, \ldots, z_n$ complex numbers and $\varphi_1, ..., \varphi_n$ complex variables such that $\sum_{j=1}^{n} \varphi_j = 0$.*
*We obtain the following reciprocity laws*

*i) If we assume that*

$$a_j z_k - a_k z_j \notin a_j L + a_k L, \ \forall \ , 1 \leqslant j \neq k \leqslant n,$$

*Then*

$$\sum_{k=1}^{n} \frac{1}{a_k} \sum_{t \in L/a_k L} \prod_{1 \leqslant j \neq k \leqslant n} D_L \left( \frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k + t}{a_k} - z_j \right) = 0$$

*ii) More general, for fixed $1 \leqslant l \leqslant n$ such that: $z_1 = \cdots = z_l = 0$*
*we assume that*

$$a_j z_k - a_k z_j \notin a_j L + a_k L$$

*for all $1 \leqslant j \neq k \leqslant n$ such that*

$$\{j, k\} \not\subset \{1, \cdots, l\}$$

*then we obtain*

$$\sum_{k=1}^{n} \frac{1}{a_k} \sum_{t}^{\prime} \prod_{1 \leqslant j \neq k \leqslant n} D_L \left( \frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k + t}{a_k} - z_j \right) = -Res \left( \prod_{1 \leqslant j \leqslant n} D_L \left( \frac{\varphi_j}{\bar{a}_j}; a_j z - z_j \right); z = 0 \right)$$

*where*

$$\sum_{t}^{\prime} = \begin{cases} \sum\limits_{\substack{t \in L/a_k L \\ t \neq 0}} & \text{if } 1 \leqslant k \leqslant l \\ \sum\limits_{t \in L/a_k L} & \text{Otherwise} \end{cases}$$

We will establish in the following that our theorem 4.2.1 contains and generalized the Sczech's result [23].

**Theorem 4.2.2** *Let $a_1, a_2, a_3$ be three elements in $O_L$ being pairwise coprime, $z_1, z_2, z_3$ complex numbers.*

*Then, we have*

$$\sum_{k=1}^{n=3} \frac{1}{a_k} \sum_{t mod(a_k L)} \prod_{1 \leqslant j \neq k \leqslant n} E_1\left(a_j \frac{z_k + t}{a_k} - z_j; L\right)$$

$$\begin{cases} = \frac{\bar{a}_1}{\bar{a}_2\bar{a}_3} d_2(a_2 z_3 - a_3 z_2; L) + \frac{\bar{a}_2}{\bar{a}_1\bar{a}_3} d_2(a_1 z_3 - a_3 z_1; L) + \frac{\bar{a}_3}{\bar{a}_2\bar{a}_1} d_2(a_2 z_1 - a_1 z_2; L) \ \textit{If} \\ a_j z_k - a_k z_j \notin L, \forall 1 \leqslant j \neq k \leqslant n, \\ \\ = \frac{\bar{a}_1}{\bar{a}_2\bar{a}_3} d_2(a_2 z_3; L) + \frac{\bar{a}_2}{\bar{a}_1\bar{a}_3} d_2(a_1 z_3; L) - \frac{\bar{a}_3}{\bar{a}_2\bar{a}_1} G_2(L) - \frac{a_3}{a_1 a_2} E_2(z_3; L) \ \textit{If} \\ z_1 = z_2 = 0 \ \textit{and} \ a_j z_3 \notin a_j L + a_3 L, \forall j = 1, 2, 3 \\ \\ = G_2(L) I\left(\frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2}\right) \ \textit{If} \ z_1 = z_2 = z_3 = 0. \end{cases}$$

*where $I(z) = z - \bar{z}$ and $d_j(\varphi; L) = \frac{(2\pi i)^j}{j!} B_j(\varphi; L)$, $\forall\ j \in \mathbb{N}$ and $G_{2k}(L) = \sum_{\omega \in L \setminus \{o\}} \frac{1}{|\omega|^{2k}}, \forall\ k \geqslant 2$ are the classical Eisenstein series ( or numbers).*

**Remark 4.2.3**

*The result in the last case of our corollary 4.2.2, represent an Homogenization of the main Theorem of Sczech in [23]. More precisely, for $a_1 = a, a_2 = c$ coprime and $a_3 = 1$, we get exactly the Sczech's result in [23].*

# 5 Proofs of Theorems 4.2.1 and 4.2.2

We consider the function

$$F(z, \overrightarrow{\Phi}, \overrightarrow{A}) = \prod_{j=1}^{n} D_L\left(a_j z - z_j; \frac{\varphi_j}{\bar{a}_j}\right)$$

Where

$$\overrightarrow{A} = (a_1 \ldots, a_n), \overrightarrow{\Phi} = (\varphi_1, \ldots, \varphi_n)$$

Our hypothesis is:

$$\sum_{j=1}^{n} \varphi_j = 0 \tag{5.0.12}$$

and we assume that

$$a_j z_k - a_k z_j \notin a_j L + a_k L, \ \forall \ j \neq k \tag{5.0.13}$$

**Otherwise** if there exists $2 \leqslant l \leqslant n$ such that:

$$z_1 = \cdots = z_l = 0,$$

we assume that

$$a_j z_k - a_k z_j \notin a_j L + a_k L, \ \forall \ 1 \leqslant j \neq k \leqslant n \ \text{ with } \{j, k\} \not\subset \{1, \cdots, l\}. \tag{5.0.14}$$

Now, we have the following interesting properties of $F$

**Proposition 5.0.4** *The function*

$$F : z \to F(z, \overrightarrow{\Phi}, \overrightarrow{A})$$

*i) $F$ is meromorphic, with poles only at*

$$z = \frac{z_k + t}{a_i}, t \in L$$

*ii) Under the conditions 5.0.12 and 5.0.13 all the poles of $F$ have order 1.*

*iii) Under the conditions 5.0.12 and 5.0.14 all the poles, except for $z = 0$ modulo $L$, of $F$ have order 1, otherwise the order is $l$ for the pole at $z = 0$ modulo $L$.*

*iv) $F$ is periodic with periods the lattice $L$ i.e*

$$F(z + \rho, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M}) = F(z, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M}), \forall \rho \in L$$

Then, to prove the theorem 4.2.1 we use Liouville residue theorem for the elliptic function $F$ with periods the Lattice $L$, as follows:
For the proof of our this theorem we have the first hypothesis

$$\sum_{j=1}^{n} \varphi_j = 0$$

Now, we have to consider two different cases

**Case 1:** We assume that

$$a_j z_k - a_k z_j \notin a_j L + a_k L, \ \forall\ 1 \geqslant j \neq k \geqslant n. \tag{5.0.15}$$

Then, by Liouville's theorem we obtain

$$\sum_{k=1}^{n} \sum_{t \in L/a_k L} \text{Res}\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A})dz; z = \frac{z_k + t}{a_k}\right) = 0$$

$$\sum_{k=1}^{n} \frac{1}{a_k} \sum_{t \in L/a_k L} e\left(E_L(t, \frac{\varphi_k}{\bar{a}_k})\right) \prod_{1 \leqslant j \neq k \leqslant n} D_L\left(a_j \frac{z_k + t}{a_k} - z_j; \frac{\varphi_j}{\bar{a}_j}\right) = 0$$

Now by using the functional equation of $D_L(z, \varphi)$ and the condition 5.0.12 we obtain

$$\sum_{k=1}^{n} \frac{1}{a_k} \sum_{t \in L/a_k L} \prod_{1 \leqslant j \neq k \leqslant n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k + t}{a_k} - z_j\right) = 0$$

This proves the first part of Theorem 4.2.1.

**Case 2:** Now we fix $1 \leqslant l \leqslant n$ such that: $z_1 = \cdots = z_l = 0$ and we assume that

$$a_j z_k - a_k z_j \notin a_j L + a_k L, \ \forall\ 1 \geqslant j \neq k \geqslant n.$$

The proof is similar as below, except at the pole at $z = 0$ modulo $L$ because it has order $l$, of the function $z \to F(z, \overrightarrow{\Phi}, \overrightarrow{A})$. Precisely, in the first case we have no pole at $z = 0$, but here we have a pole at $z = 0$ of order equal to $l$. Then again by using Liouville residue theorem and functional equation of $D_L(z; \varphi)$ we obtain the desired remaing part of Theorem 4.2.1 $\square$

*Proof of Theorem 4.2.2:*
Here we must consider three different cases

**Case 1:** We assume that

$$a_j z_k - a_k z_j \notin a_j L + a_k L, \ \forall\ 1 \leqslant j \neq k \leqslant 3$$

We use the following Laurent expansion of Jacobi forms

$$D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k + t}{a_k} - z_j\right) = \frac{\bar{a}_j}{\varphi_j} + d_1\left(a_j \frac{z_k + t}{a_k} - z_j; L\right) + d_2\left(a_j \frac{z_k + t}{a_k} - z_j; L\right) \frac{\varphi_j}{\bar{a}_j} + ..., \text{ for } j = 1, 2, 3. \tag{5.0.16}$$

Now, we compute the product

$$\prod_{1 \leqslant j \neq k \leqslant 3} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k + t}{a_k} - z_j\right)$$

as a serie in terms of variables $\varphi_1, \varphi_2$ and $\varphi_3$.
Then, we obtain

$$\sum_{k=1}^{n=3} \frac{1}{a_k} \sum_{t \in L/a_k L} \prod_{1 \leqslant j \neq k \leqslant n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k + t}{a_k} - z_j\right) = \tag{5.0.17}$$

$$\sum_{k=1}^{n=3} \frac{1}{a_k} \sum_{t \bmod (a_k L)} \prod_{1 \leqslant j \neq k \leqslant n} E_1\left(a_j \frac{z_k + t}{a_k} - z_j; L\right) +$$

$$\frac{\bar{a}_1}{\bar{a}_2 \bar{a}_3} \frac{\varphi_2 + \varphi_3}{\varphi_1} d_2(a_2 z_3 - a_3 z_2; L) + \frac{\bar{a}_2}{\bar{a}_1 \bar{a}_3} \frac{\varphi_1 + \varphi_3}{\varphi_2} d_2(a_1 z_3 - a_3 z_1; L) + \frac{\bar{a}_3}{\bar{a}_1 \bar{a}_2} \frac{\varphi_1 + \varphi_2}{\varphi_3} d_2(a_1 z_2 - a_2 z_1; L)$$

+monomial terms of total degre greater than 1 in terms of $\varphi_j, \bar{\varphi}_k, j, k = 1, 2, 3$.

Now, by using the first part of Theorem 4.2.1 we obtain that coefficient, of our serie in 5.0.19, is equal to zero. Now, the first coefficient here is zero implies that

$$\sum_{k=1}^{n=3} \frac{1}{a_k} \sum_{t \bmod (a_k L)} \prod_{1 \leqslant j \neq k \leqslant n} E_1\left(a_j \frac{z_k + t}{a_k} - z_j; L\right) =$$

$$\frac{\bar{a}_1}{\bar{a}_2 \bar{a}_3} d_2(a_2 z_3 - a_3 z_2; L) + \frac{\bar{a}_2}{\bar{a}_1 \bar{a}_3} d_2(a_1 z_3 - a_3 z_1; L) + \frac{\bar{a}_3}{\bar{a}_2 \bar{a}_1} d_2(a_2 z_1 - a_1 z_2; L)$$

This represent the case 1 of our second theorem 4.2.2.

**Case 2:** We assume $z_1 = z_2 = 0$ and $a_j z_3 \notin a_j L + a_3 L, \forall j = 1, 2, 3$. Now, we compute the residue of $z \to F(z, \overrightarrow{\Phi}, \overrightarrow{A})$ at $z = 0$ is a pole of order 2 where

$$F(z, \overrightarrow{\Phi}, \overrightarrow{A}) = \prod_{j=1}^{n=3} D_L\left(a_j z - z_j; \frac{\varphi_j}{\bar{a}_j}\right)$$

and

$$\overrightarrow{A} = (a_1, a_2, a_3), \overrightarrow{\Phi} = (\varphi_1, \varphi_2, \varphi_3)$$

After mutatis mutandis, a laborious, computation we obtain

$$\operatorname{Res}\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A}) dz; z = 0\right) = \tag{5.0.18}$$

$$D_L\left(-z_3; \frac{\varphi_3}{\bar{a}_3}\right) \left(\frac{d_1\left(\frac{\varphi_2}{\bar{a}_2}; L\right)}{a_1} + \frac{d_1\left(\frac{\varphi_1}{\bar{a}_1}; L\right)}{a_2}\right) + \frac{a_3}{a_1 a_2} D'_L\left(-z_3; \frac{\varphi_3}{\bar{a}_3}\right)$$

Then

$$\sum_{k=1}^{3}\frac{1}{a_k}\sum_{t}^{\prime}\prod_{1\leqslant j\neq k\leqslant n=3}D_L\left(\frac{\varphi_j}{\bar{a}_j};a_j\frac{z_k+t}{a_k}-z_j\right)=$$

$$-D_L\left(\frac{\varphi_3}{\bar{a}_3};-z_3\right)\left(\frac{d_1\left(\frac{\varphi_2}{\bar{a}_2};L\right)}{a_1}+\frac{d_1\left(\frac{\varphi_1}{\bar{a}_1};L\right)}{a_2}\right)-\frac{a_3}{a_1a_2}D_L'\left(-z_3;\frac{\varphi_3}{\bar{a}_3}\right)e\left(E_L\left(z_3,\frac{\varphi_3}{\bar{a}_3}\right)\right)$$

where

$$\sum_{t}^{\prime}=\begin{cases}\displaystyle\sum_{\substack{t\in L/a_kL\\ t\neq 0}} & \text{if } 1\leqslant k\leqslant l\\ \displaystyle\sum_{t\in L/a_kL} & \text{Otherwise}\end{cases}$$

Next, we use the Laurent expansion 5.0.16 of Jacobi forms to compute the product

$$\prod_{1\leqslant j\neq k\leqslant 3}D_L\left(\frac{\varphi_j}{\bar{a}_j};a_j\frac{z_k+t}{a_k}-z_j\right)$$

as a serie in terms of variables $\varphi_1,\varphi_2$ and $\varphi_3$.
Then, the term corresponding to the part of total degre zero of

$$\sum_{k=1}^{n=3}\frac{1}{a_k}\sum_{t\in L/a_kL}^{\prime}\prod_{1\leqslant j\neq k\leqslant n=3}D_L\left(\frac{\varphi_j}{\bar{a}_j};a_j\frac{z_k+t}{a_k}-z_j\right)\tag{5.0.19}$$

Hence, this term is exactly the qunatity

$$\sum_{k=1}^{n=3}\frac{1}{a_k}\sum_{t\bmod(a_kL)}\prod_{1\leqslant j\neq k\leqslant n}E_1\left(a_j\frac{z_k+t}{a_k}-z_j;L\right)-\frac{\bar{a}_1}{\bar{a}_2\bar{a}_3}d_2(a_2z_3;L)-\frac{\bar{a}_2}{\bar{a}_1\bar{a}_3}d_2(a_1z_3;L)$$

$$-\frac{\bar{a}_2}{a_1\bar{a}_3}\frac{\varphi_3}{\varphi_2}d_2(z_3;L)-\frac{\bar{a}_1}{a_2\bar{a}_3}\frac{\varphi_3}{\varphi_1}d_2(z_3;L)+\frac{\bar{a}_3}{a_1\bar{a}_2}\frac{\varphi_2}{\varphi_3}\left(1-\frac{a_1}{\bar{a}_1}\right)G_2(L)+\frac{\bar{a}_3}{a_2\bar{a}_1}\frac{\varphi_1}{\varphi_3}\left(1-\frac{a_2}{\bar{a}_2}\right)G_2(L)$$

Now, to compute the term corresponding to the part of total degre zero of

$$-D_L\left(\frac{\varphi_3}{\bar{a}_3};-z_3\right)\left(\frac{d_1\left(\frac{\varphi_2}{\bar{a}_2};L\right)}{a_1}+\frac{d_1\left(\frac{\varphi_1}{\bar{a}_1};L\right)}{a_2}\right)-\frac{a_3}{a_1a_2}D_L'\left(-z_3;\frac{\varphi_3}{\bar{a}_3}\right)e\left(E_L\left(z_3,\frac{\varphi_3}{\bar{a}_3}\right)\right)$$

we use Laurent expansion 5.0.16 and

$$D'_L(z;\varphi) = -E_2(z;L) - \frac{\pi}{a(L)}\frac{\bar{\varphi}}{\varphi} + \text{ positive power of } \varphi, \bar{\varphi}.$$

and the functional equation of $D_L(z;\varphi)$. Therefore, the term corresponding to the part of total degre zero of

$$-D_L\left(-z_3;\frac{\varphi_3}{\bar{a}_3}\right)\left(\frac{d_1\left(\frac{\varphi_2}{\bar{a}_2};L\right)}{a_1}+\frac{d_1\left(\frac{\varphi_1}{\bar{a}_1};L\right)}{a_2}\right)-\frac{a_3}{a_1a_2}D'_L\left(-z_3;\frac{\varphi_3}{\bar{a}_3}\right)e\left(E_L\left(z_3,\frac{\varphi_3}{\bar{a}_3}\right)\right)$$

is exactly equal to

$$-\frac{\bar{a}_2}{a_1\bar{a}_3}\frac{\varphi_3}{\varphi_2}d_2(z_3;L)-\frac{\bar{a}_1}{a_2\bar{a}_3}\frac{\varphi_3}{\varphi_1}d_2(z_3;L)+\frac{\bar{a}_3}{a_1\bar{a}_2}\frac{\varphi_2}{\varphi_3}G_2(L)+\frac{\bar{a}_3}{a_2\bar{a}_1}\frac{\varphi_1}{\varphi_3}G_2(L)+\frac{a_3}{a_2a_1}E_2(z_3;L)$$

Finally, we apply our theorem 4.2.1, we obtain

$$\sum_{k=1}^{n=3}\frac{1}{a_k}\sum_{t\bmod(a_kL)}\prod_{1\leqslant j\neq k\leqslant n}E_1\left(a_j\frac{z_k+t}{a_k}-z_j;L\right)=$$

$$=\frac{\bar{a}_1}{\bar{a}_2\bar{a}_3}d_2(a_2z_3;L)+\frac{\bar{a}_2}{\bar{a}_1\bar{a}_3}d_2(a_1z_3;L)-\frac{\bar{a}_3}{\bar{a}_2\bar{a}_1}G_2(L)-\frac{a_3}{a_1a_2}E_2(z_3;L).$$

This proves the second case of our theorem 4.2.2.

**Case 3:** we assume $z_1 = z_2 = z_3 = 0$
Now, we compute the residue of $z \to F(z, \overrightarrow{\Phi}, \overrightarrow{A})$ at $z = 0$ is a pole of order 3 where again

$$F(z, \overrightarrow{\Phi}, \overrightarrow{A}) = \prod_{j=1}^{n=3} D_L\left(a_j z; \frac{\varphi_j}{\bar{a}_j}\right)$$

and

$$\overrightarrow{A} = (a_1, a_2, a_3), \overrightarrow{\Phi} = (\varphi_1, \varphi_2, \varphi_3)$$

Our computation of this residue is

$$\text{Res}\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A})dz; z = 0\right) = \tag{5.0.20}$$

$$\frac{a_3}{a_1a_2}d_2\left(\frac{\varphi_3}{\bar{a}_3};L\right)+\frac{a_2}{a_1a_3}d_2\left(\frac{\varphi_2}{\bar{a}_2};L\right)+\frac{a_1}{a_1a_3}d_2\left(\frac{\varphi_1}{\bar{a}_1};L\right)+$$

$$\frac{1}{a_1}d_1\left(\frac{\varphi_2}{\bar{a}_2};L\right)d_1\left(\frac{\varphi_3}{\bar{a}_3};L\right)+\frac{1}{a_2}d_1\left(\frac{\varphi_1}{\bar{a}_1};L\right)d_1\left(\frac{\varphi_3}{\bar{a}_3};L\right)+\frac{1}{a_3}d_1\left(\frac{\varphi_2}{\bar{a}_2};L\right)d_1\left(\frac{\varphi_1}{\bar{a}_1};L\right).$$

Again, by using the following expansions near zero

$$d_1(z,L) = \frac{1}{z} - G_2(L)z - \frac{\pi}{a(L)}\bar{z} + O(z^2).$$

and

$$d_2(z,L) = -G_2(L) - \frac{\pi}{a(L)}\frac{\bar{z}}{z} + \frac{\pi G_2(L)}{a(L)}\bar{z}z + o(z).$$

We obtain, the part of total degre zero ( in terms of $\varphi_1, \varphi_2, \varphi_3$ and $\bar{\varphi_1}, \bar{\varphi_2}, \bar{\varphi_3}$) for the following quantity

$$-\mathrm{Res}\left(\prod_{j=1}^{n=3} D_L\left(a_j z; \frac{\varphi_j}{\bar{a}_j}\right)\right) \tag{5.0.21}$$

is equal to

$$G_2(L)\left(\frac{a_1}{a_2a_3} + \frac{a_2}{a_1a_3} + \frac{a_3}{a_1a_2}\right) + \frac{\pi}{a(L)}\left(\frac{\bar{a}_3}{a_1a_2}\frac{\bar{\varphi_3}}{\varphi_3} + \frac{\bar{a}_2}{a_1a_3}\frac{\bar{\varphi_2}}{\varphi_2} + \frac{\bar{a}_1}{a_2a_3}\frac{\bar{\varphi_1}}{\varphi_1}\right) +$$

$$G_2(L)\left(\frac{\bar{a}_2}{a_1\bar{a}_3}\frac{\varphi_3}{\varphi_2} + \frac{\bar{a}_3}{a_1\bar{a}_2}\frac{\varphi_2}{\varphi_3} + \frac{\bar{a}_1}{a_2\bar{a}_3}\frac{\varphi_3}{\varphi_1} + \frac{\bar{a}_3}{a_2\bar{a}_1}\frac{\varphi_1}{\varphi_3} + \frac{\bar{a}_1}{a_3\bar{a}_2}\frac{\varphi_2}{\varphi_1} + \frac{\bar{a}_2}{a_3\bar{a}_1}\frac{\varphi_1}{\varphi_2}\right) +$$

$$\frac{\pi}{a(L)}\left(\frac{\bar{a}_2}{a_1a_3}\frac{\bar{\varphi_3}}{\varphi_2} + \frac{\bar{a}_3}{a_1a_2}\frac{\bar{\varphi_2}}{\varphi_3} + \frac{\bar{a}_1}{a_2a_3}\frac{\bar{\varphi_3}}{\varphi_1} + \frac{\bar{a}_3}{a_2a_1}\frac{\bar{\varphi_1}}{\varphi_3} + \frac{\bar{a}_1}{a_3a_2}\frac{\bar{\varphi_2}}{\varphi_1} + \frac{\bar{a}_2}{a_3a_1}\frac{\bar{\varphi_1}}{\varphi_2}\right)$$

Now using that

$$\varphi_1 + \varphi_2 + \varphi_3 = 0$$

we obtain that the part of total degre zero term of

$$-\mathrm{Res}\left(\prod_{j=1}^{n=3} D_L\left(a_j z; \frac{\varphi_j}{\bar{a}_j}\right)\right) \tag{5.0.22}$$

is exactly

$$G_2(L)\left(\frac{a_1}{a_2a_3} + \frac{a_2}{a_1a_3} + \frac{a_3}{a_1a_2} + \frac{\bar{a}_2}{a_1\bar{a}_3}\frac{\varphi_3}{\varphi_2} + \frac{\bar{a}_3}{a_1\bar{a}_2}\frac{\varphi_2}{\varphi_3} + \frac{\bar{a}_1}{a_2\bar{a}_3}\frac{\varphi_3}{\varphi_1} + \frac{\bar{a}_3}{a_2\bar{a}_1}\frac{\varphi_1}{\varphi_3} + \frac{\bar{a}_1}{a_3\bar{a}_2}\frac{\varphi_2}{\varphi_1} + \frac{\bar{a}_2}{a_3\bar{a}_1}\frac{\varphi_1}{\varphi_2}\right)$$

To finish our computation, we use again the Laurent expansion 5.0.16 of the Jacobi form $D_L(z;\varphi)$. Hence, we obtain the part of total term of degre zero of

$$\sum_{k=1}^{n}\frac{1}{a_k}\sum_{t}{}' \prod_{1\leqslant j\neq k\leqslant n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j\frac{z_k+t}{a_k} - z_j\right) \tag{5.0.23}$$

it's correspond to the following quantity

$$\sum_{k=1}^{n}\frac{1}{a_k}\sum_{t\in L/a_kL\setminus\{\bar{0}\}}\prod_{1\leqslant j\neq k\leqslant n}d_1\left(a_j\frac{t}{a_k};L\right)+$$

$$G_2(L)\left(\frac{\bar{a}_2}{a_1\bar{a}_3}\frac{\varphi_3}{\varphi_2}+\frac{\bar{a}_3}{a_1\bar{a}_2}\frac{\varphi_2}{\varphi_3}+\frac{\bar{a}_1}{a_2\bar{a}_3}\frac{\varphi_3}{\varphi_1}+\frac{\bar{a}_3}{a_2\bar{a}_1}\frac{\varphi_1}{\varphi_3}+\frac{\bar{a}_1}{a_3\bar{a}_2}\frac{\varphi_2}{\varphi_1}+\frac{\bar{a}_2}{a_3\bar{a}_1}\frac{\varphi_1}{\varphi_2}\right)$$

$$-G_2(L)\left(\frac{\bar{a}_2}{\bar{a}_1\bar{a}_3}\frac{\varphi_3}{\varphi_2}+\frac{\bar{a}_3}{\bar{a}_1\bar{a}_2}\frac{\varphi_2}{\varphi_3}+\frac{\bar{a}_1}{\bar{a}_2\bar{a}_3}\frac{\varphi_3}{\varphi_1}+\frac{\bar{a}_3}{\bar{a}_2\bar{a}_1}\frac{\varphi_1}{\varphi_3}+\frac{\bar{a}_1}{\bar{a}_3\bar{a}_2}\frac{\varphi_2}{\varphi_1}+\frac{\bar{a}_2}{\bar{a}_3\bar{a}_1}\frac{\varphi_1}{\varphi_2}\right)$$

Finally, we now by theorem 4.2.1, that the quantities 5.0.22 and 5.0.22 are the same. Then, there parts of total degre zero are also the same. The above compuation implies that

$$\sum_{k=1}^{n=3}\frac{1}{a_k}\sum_{t\mathrm{mod}(a_kL)}\prod_{1\leqslant j\neq k\leqslant n}E_1\left(a_j\frac{t}{a_k};L\right)=$$

$$G_2(L)\left(\frac{a_1}{a_2a_3}+\frac{a_2}{a_1a_3}+\frac{a_3}{a_1a_2}-\frac{\bar{a}_1}{\bar{a}_2\bar{a}_3}-\frac{\bar{a}_2}{\bar{a}_1\bar{a}_3}-\frac{\bar{a}_3}{\bar{a}_1\bar{a}_2}\right)$$

Thus we obtain our theorem 4.2.2. □

# 6 Applications and questions

In this section we give some known applications of Dedekind elliptic sums and we precise some open questions related to them.

## 6.1 Applications

The following applications concerns only the case when $n=2$ ( the number of elements $a_1,...,a_n$). Our references for this section are [17, 19, 23]. In fact, in these papers [17, 19, 23] H. Ito and R. Sczech studied a simplest form of our elliptic Dedekind sums and give some intersting applications.

Let $K=\mathbb{Q},\mathbb{Q}(\sqrt{D})$, $D$ negative integer.
Put

$$D(a_1,a_2):=\frac{1}{a_2}\sum_{t\mathrm{mod}(a_2L)}E_1\left(a_1\frac{t}{a_2};L\right)E_1\left(\frac{t}{a_2};L\right)$$

Let $A=\begin{pmatrix}a_1 & a_3\\ a_2 & a_4\end{pmatrix}$ in $\mathrm{SL}_2(O_K)$ We define the function

$$\Phi:A\mapsto\begin{cases}E_2(0)\frac{a_1+a_4}{a_2}-D(a_1,a_2) & \textbf{If } a_2\neq 0\\ E_2(0)\frac{a_3}{a_4} & \textbf{Otherwise}\end{cases}$$

In 1984 R. Sczech in his paper [23] proved the following result

**Theorem 6.1.1 (Eisenstein Cohomology)** *$\Phi$ is a group morphism: $(SL_2(O_K), \times) \to (\mathbb{C}, +)$. Non trivial iff $K \neq \mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$.*

Then, we obtain a cocycle in $H^1(\mathrm{SL}_2(O_K), \mathbb{C})$ it represented Eisenstein cohomology class of $\mathrm{SL}_2(O_K)$ For $A = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}$ in $\mathrm{SL}_2(O_K)$ with $a_1 + a_4 \neq 0, \pm 1, \pm 2$. Then $a_2 X^2 + (a_4 - a_1)X - a_2 = 0$ has two distincts solutions $\alpha, \alpha'$ in $\mathbb{C}$. We define the quadratic form

$$Q(m, n) = (m\alpha + n)(m\alpha' + n)$$

satisfies

$$Q((m, n)A) = Q(m, n)$$

Then $A$ operates on $L = O_K$ Define the Hecke $L$-function

$$L(s, A) = \sum_{(m,n)}{}' \frac{\overline{Q(m, n)}}{Q(m, n)}, \mathrm{Re}(s) > \frac{3}{2}.$$

In 1987 H. Ito proved the result [17]

**Theorem 6.1.2 (Values of Hecke L-function)** *Let $\epsilon = a_3\alpha + a_4$. Then*

$$\Phi(A) = sgn\,(log|\epsilon|)\,.(\alpha - \alpha').L(1, A)$$

Now, we set $\Psi(A) = \frac{\Phi(A)}{\sqrt{D}E_2(0)}$

In 1990 H. Ito proved in [18] the following result

**Theorem 6.1.3 (Quadratic residues symbols)** *There exists explicit group morphism*

$$\chi : (SL_2(O_K), \times) \to (Z/8\mathbb{Z}, +)$$

*such that*

$$\Psi(A) = \chi(A) mod(8O_K)$$

Then we get cohomology class in $H^1\left(\Gamma(8), \mathbb{Z}/2\mathbb{Z}\right)$.

In 2004 H. Ito again proved the following result [19]

**Theorem 6.1.4 (Density of elliptic Dedekind sums)** *If $K$ is euclidian $\neq \mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$. Then the set*

$$\left\{ \left( \frac{a_1}{a_2}, \frac{D(a_1, a_2)}{\sqrt{D}E_2(0)} \right) : \frac{a_1}{a_2} \in K \right\}$$

*is dense in the space $\mathbb{C} \times \mathbb{R}$.*

### 6.2 Open questions.

The following questions are not solved yet:

i) Find the similar theorems as 6.1.1, 6.1.2, 6.1.3 and 6.1.4 for multiple elliptic Dedekind sums?

ii) Similar question for the classical multiple Dedekind sums ?

iii) Generalized these results to quantum modular forms?

iv) Give geometric or topological interpretation of these results?

## References


[1] T. M Apostol, *Theorems on generalized Dedekind Sums*, Pacific. J. Math, **2** (1952) 1–9.

[2] M.F. Atiyah, *The Logarithm of the Dedekind $\eta$-Function*, Math. Ann, **278**, (1987), 335-380.

[3] A. Bayad, *Sommes de Dedekind elliptiques et formes de Jacobi*, Ann. Instit. Fourier, Vol. **51**, Fasc. 1, 2001, 29-42.

[4] A. Bayad, *Sommes elliptiques multiples d'Apostol-Dedekind-Zagier*, C.R.A.S Paris, Ser. I **339**, fascicule 7, Série I, 2004, 457-462.

[5] A. Bayad, *Applications aux sommes elliptiques multiples d'Apostol-Dedekind-Zagier*, C.R.A.S Paris, Ser. I **339**, fascicule 8, Série I, 2004, 529-532.

[6] A. Bayad, *Jacobi forms in two variables: Multiple elliptic Dedekind sums, The Kummer-von Staudt Clausen Congruences for elliptic Bernoulli functions and values of Hecke L-functions,* Submitted for publication.

[7] M. Beck, Dedekind cotangent sums, Acta Arithmetica **109**, no. 2 (2003), 109-130.

[8] B. C. Berndt, Reciprocity theorems for Dedekind sums and generalizations, *Adv. in Math.* **23**, no. 3 (1977), 285–316.

[9] L.Carlitz, *Arithmetics properties of generalized Bernoulli numbers*, J. Reine Angew. Math; **202** (1959), 164-182.

[10] U. Dieter, Cotangent sums, a further generalization of Dedekind sums, *J. Number Th.* **18** (1984), 289–305.

[11] U. Halbritter, *Some new reciprocity formulas for generalized Dedekind sums*, Results in Mathematics 8, (1985), 21-46.

[12] R. R. Hall, J. C. Wilson, D. Zagier, Reciprocity formulae for general Dedekind-Rademacher sums, *Acta Arith.* **73**, no. 4 (1995), 389–396.

[13] D.Hickerson, *Continued fractions and density results for Dedekind sums*, J. reine und angew Math. 290, ( 1977), 113–116.

[14] F. Hirzebruch, *The signature theorem: reminiscences and recreation* , Prospects in Mathematics. Ann. of Math.Studies 70, 3-31, Princeton University Press, Princeton, 1971.

[15] F. Hirzebruch, T. Berger and R. Jung, *Manifolds and Modular forms* , Aspects of Math.E. 20,Vieweg (1992).

[16] F. Hirzebruch and D. Zagier, *The Atiyah-Singer Theorem and Elementary Number Theory*, Math. Lecture Series 3, Publish or Perish Inc, 1974.

[17] H. Ito, *A function on the upper half space which is analogous to the imaginary part of* $log\eta(z)$, J. reine angew. Math. 373, (1986), 148 – 165.

[18] H. Ito, *Dedekind sums and quadratic residues symbols*, Nagoya Math. J, Vol. 118, (1990), 35 – 43.

[19] H. Ito, *A density result for elliptic Dedekind sums*, Acta. arith. 112, (2004), 199 – 2008.

[20] R. Kirby, P. Melvin, *Dedekind sums, $\mu$-invariants and the signature cocycle*, Math. Annalen **299** (1994), 231-267.

[21] H. Rademacher, Some remarks on certain generalized Dedekind sums, *Acta Arith.* **9** (1964), 97–105.

[22] C. Meyer, *Über einige Anwendungen Dedekindscher Summen*, J.Reine angew. Math. **198**, (1957), 143-203.

[23] R.Sczech, *Dedekindsummen mit elliptischen Funktionen*, Invent.math, **76**, (1984), 523-551.

[24] U. Weselmann, *EisensteinKohomologie und Dedekindsummen für $GL_2$ über imaginärquadratischen Zahlenkörpern*, J. reine. angew. Math. 389, (1988), 90–121.

[25] D. Zagier, *Higher order Dedekind sums*, Math.Ann, **202**, 1973, 149-172.