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On the $\left|\overline{N}, p_n, \delta\right|_k$ Summability of Factored Fourier Series

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Abstract. In this paper we deal with a main theorem on the local property of $\left|\overline{N}, p_n, \delta\right|_k$, $k \geq 1$ summability of factored Fourier series, which generalizes a theorem of A.Ganesh,G.Balasubramanian and A. Bayad [1] $\left|\overline{N}, p_n\right|_k$ summability factors, is proved.

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1 Introduction and main result

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \quad \text{as} \quad n \to \infty \quad \text{and} \quad (P_{-i} = p_{-i} = 0 \ , \ i \geq 1) \tag{1.1}$$

The sequence -to- sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu \tag{1.2}$$

defines the sequence (t_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n) generated by the sequence of coefficients (p_n).

The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty, \tag{1.3}$$

and is said to be summable $\left|\overline{N}, p_n, \delta\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |t_n - t_{n-1}|^k < \infty, \tag{1.4}$$

In the special case when $\delta = 0$ (resp. $p_n = 1$ for values of n) ,$\left|\overline{N}, p_n, \delta\right|_k$, $k \geq 1$ summability is the same as $\left|\overline{N}, p_n\right|_k$ (resp. $|C, p_n, \delta|_k$) summability. We have the following result

Theorem 1 ([1]). *Let (p_n) be a sequence of positive numbers such that*

$$p_n = O(np_n) \quad \text{as } n \to \infty \tag{1}$$

If (X_n) be a positive non-decreasing sequence and suppose that there exists (λ_n) and (β_n) is a sequence of positive numbers such that

$$|\Delta \lambda_n| \leq \beta_n. \tag{2}$$

$$\beta_n \to 0 \quad \text{as } n \to \infty; \tag{3}$$

$$\sum_{n=1}^{\infty} n|\Delta \beta_n| X_n \leq \infty \tag{4}$$

$$|\lambda_n| X_n = O(1) \quad as\ n \to \infty \tag{5}$$

If

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m)\ as\ m \to \infty \tag{6}$$

Where

$$t_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \tag{7}$$

Then the sequence $\sum a_n \lambda_n$ *is summable* $\left|\overline{N}, p_n\right|_k$, $k \geq 1$.

The aim of this paper is to generalize theorem 1 for $\left|\overline{N}, p_n, \delta\right|_k$, $k \geq 1$, summability.
Now, we shall prove the following theorem.

Theorem 2 (main result). *Let* (X_n) *be a positive non-decreasing sequence and the sequence* (λ_n) *and* (β_n) *be such that conditions (2)-(5) of theorem 1 are satisfied. If* (p_n) *is a sequence such that condition (1) of theorem 1 is satisfied and*

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left\{\left(\frac{p_v}{P_v}\right)^{\delta k} \frac{1}{P_v}\right\} \tag{8}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |t_n|^k = O(X_m)\ as\ m \to \infty, \tag{9}$$

Where (t_n) *is sa in* (7), *then the series* $\sum a_n \lambda_n$ *is summable* $\left|\overline{N}, p_n, \delta\right|_k$ *for* $k \geq 1$ *and* $0 \leq \delta < \frac{1}{k}$.

Now, we need to prove the following lemma.

Lemma 3. *If the conditions* $(2) - (5)$ *on* (X_n), (λ_n) *and* (β_n) *are satisfied , then*

$$n\beta_n X_n = O(1)\ as\ m \to \infty, \tag{10}$$

$$\sum_{n=1}^{m} \beta_n X_n < \infty. \tag{11}$$

2 Proof of the theorem 2

Proof: Let (T_n) be the sequence of $\left(\overline{N}, p_n\right)$ means of the series $\sum a_n P_n \lambda_n$. Then, by definition,

$$T_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \sum_{r=0}^{\nu} a_r \lambda_r = \frac{1}{P_n}\sum_{\nu=0}^{n} \left(P_n - P_{\nu-1}\right) a_\nu \lambda_\nu$$

Then , for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu-1} a_\nu \lambda_\nu = \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \frac{P_{\nu-1}\lambda_\nu}{\nu} \nu a_\nu$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= \tfrac{(n+1)}{nP_n} p_n t_n \lambda_n - \tfrac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} p_n t_n \lambda_n \tfrac{\nu+1}{\nu} + \tfrac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu \Delta t_\nu \lambda_\nu \tfrac{\nu+1}{\nu} + \tfrac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} p_n t_n \lambda_{\nu+1} \tfrac{1}{\nu} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \text{ say.} \end{aligned}$$

Since $\left|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}\right|^k \leq 4^k \left(\left|T_{n,1}\right|^k + \left|T_{n,2}\right|^k + \left|T_{n,3}\right|^k + \left|T_{n,4}\right|^k\right)$, to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \left|T_{n,r}\right|^k < \infty \qquad r = 1, 2, 3, 4 \tag{3.1}$$

Since $\lambda_n = O(1/X_n) = O(1)$, by (5), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\tfrac{P_n}{p_n}\right)^{\delta k + k - 1} \left|T_{n,1}\right|^k &= O(1) \sum_{n=1}^{\infty} \left|\lambda_n\right|^{k-1} \left|\lambda_n\right| \left(\tfrac{P_n}{p_n}\right)^{\delta k - 1} \left|t_n\right|^k \\ &= O(1) \sum_{n=1}^{\infty} \left|\lambda_n\right| \left(\tfrac{P_n}{p_n}\right)^{\delta k - 1} \left|t_n\right|^k \end{aligned}$$

$$\begin{aligned} &= O(1) \sum_{n=1}^{m} \Delta\left|\lambda_n\right| \sum_{\nu=1}^{n} \left(\tfrac{P_\nu}{p_\nu}\right)^{\delta k - 1} \left|t_\nu\right|^k + O(1)\left|\lambda_m\right| \sum_{n=1}^{\infty} \left(\tfrac{P_n}{p_n}\right)^{\delta k - 1} \left|t_n\right|^k \\ &= O(1) \sum_{n=1}^{m} \left|\Delta\lambda_n\right| X_n + O(1)\left|\lambda_m\right| X_m \\ &= O(1) \sum_{n=1}^{m} \beta_n X_n + O(1)\left|\lambda_m\right| X_m \\ &= O(1) \text{ as } m \to \infty \end{aligned}$$

Now, applying Holder's inequality with indices k and k' , where $\frac{1}{k} + \frac{1}{k'} = 1$,

as in $T_{n,1}$ we get

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\tfrac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\tfrac{P_n}{P_n}\right)^{\delta k-1}\tfrac{1}{P_{n-1}}\left\{\sum_{\nu=1}^{n-1}|t_\nu|^k p_\nu|\lambda_\nu|^k\right\}\left\{\tfrac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}P_\nu\right\}^{k-1}\\
&= O(1)\sum_{\nu=1}^{m}|t_\nu|^k p_\nu|\lambda_\nu|^{k-1}|\lambda_\nu|\sum_{n=\nu+1}^{m+1}\left(\tfrac{P_n}{p_n}\right)^{\delta k-1}\tfrac{1}{P_{n-1}}\\
&= O(1)\sum_{\nu=1}^{m}|t_\nu|^k\left(\tfrac{P_n}{p_n}\right)^{\delta k-1}|\lambda_\nu|\\
&= O(1)\text{ as } m\to\infty
\end{aligned}$$

Using the fact that

$$p_\nu = 0(\nu p_\nu)$$

by (1) and $n\beta_n = O\left(\frac{1}{X_n}\right) = O(1)$, by (10), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\tfrac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\left(\tfrac{P_n}{P_n}\right)^{\delta k-1}\tfrac{1}{P_{n-1}}\left\{\sum_{\nu=1}^{n-1}|t_\nu|^k p_\nu(\nu\beta_\nu)^k\right\}\left\{\tfrac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}P_\nu\right\}^{k-1}\\
&= O(1)\sum_{\nu=1}^{m}|t_\nu|^k p_\nu(\nu\beta_\nu)^{k-1}(\nu\beta_\nu)\sum_{n=\nu+1}^{m+1}\left(\tfrac{P_n}{p_n}\right)^{\delta k-1}\tfrac{1}{P_{n-1}}\\
&= O(1)\sum_{\nu=1}^{m}|t_\nu|^k\left(\tfrac{P_n}{p_n}\right)^{\delta k-1}(\nu\beta_\nu)\\
&= O(1)\sum_{\nu=1}^{m}|t_\nu|^k\left(\tfrac{P_n}{p_n}\right)^{\delta k-1}\Delta(\nu\beta_\nu)+O(1)m\beta_m\sum_{\nu=1}^{m}|t_\nu|^k\left(\tfrac{P_n}{p_n}\right)^{\delta k-1}\\
&= O(1)\sum_{\nu=1}^{m}\nu X_\nu|\Delta(\beta_\nu)|+O(1)\sum_{\nu=1}^{m}(\nu+1)X_\nu+O(1)m\beta_m X_m\\
&= O(1)\text{ as } m\to\infty
\end{aligned}$$

by (2), (4), (8), (9), (10) and (11)

Finally, using the fact that $p_\nu = 0(\nu p_\nu)$, by (1), as in $T_{n,1}$ and $T_{n,2}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\tfrac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,2}|^k &= O(1)\sum_{\nu=1}^{m}|t_\nu|^k\left(\tfrac{P_n}{p_n}\right)^{\delta k-1}|\lambda_{\nu+1}|\\
&= O(1)\text{ as } m\to\infty
\end{aligned}$$

There fore, we get

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,r}|^k<\infty \qquad \text{as } m\to\infty,\ \text{for } r=1,2,3,4 \tag{3.1}$$

In the particular case if we take $p_n = 1$ for all values of n in this theorem, then we get the concerning the $|C,1,\delta|_k$, is summablity methods.

References

[1] A.Ganesh, G.Balasubramanian, A. Bayad, *On thesummability of Factored Fourier series*, International Mathematical Forum, **Volume 5 (2010), No.22**, pp. 1067–1073.

[2] A.Ganesh, G.Balasubramanian, *Localization of Factored Fourier Series*, Global Journal of Pure and Applied Mathematics, **Volume 5 No. 1 (2009)** , pp.69–74.

[3] A. Ganesh and G. Balasubramanian, *Novel Complex Valued Neural Networks*, International Journal of Computational and Applied Mathematics (IJCAM), **Volume 4, No.2 (2009)**, pp. 165–171.

[4] A.Ganesh, G.Balasubramanian, *On Signal Processing by using Fourier Analysis of Time Series*, Global Journal of Pure and Applied Mathematics, **Volume 6 , No.1 (2010)**, pp.31–44.

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