Modular properties of elliptic Bernoulli and Euler functions

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Abstract

Bernoulli and Euler numbers play an important and quite mysterious role in mathematics and in various places like analysis, number theory, differential geometry and algebraic topology. In the first part of this paper, we present old, recent and new results of Bernoulli(resp. Euler) numbers, polynomials and functions. We also discuss the relationship which exists between Bernoulli and Euler objects. In the second part of this paper we introduce their elliptic analogues. This part is completely new. Our method and results are quite similar to the classical one except the use of elliptic functions and Jacobi forms. We introduce elliptic Bernoulli and Euler numbers and functions and we study their properties. In fact we show the Raabe formulae satisfied by these functions.
Our new results are the theorems 1.4.2, 2.3.1, 2.4.1, 2.4.2.

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1 Bernoulli and Euler polynomials: A modern overview

We start by fixing notation and recalling some stantard facts about Bernoulli and Euler numbers and polynomials.

1.1 Notations and definitions

Definition 1.1.1.

i) The Bernoulli polynomials are given by their generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad |t|<\pi. \tag{1.1.1}$$

ii) The Bernoulli numbers are given by

$$B_n = B_n(0). \tag{1.1.2}$$

iii) The Euler polynomials are given by their generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi. \tag{1.1.3}$$

iv) There are two types of Euler numbers are given by

$$E_n = 2^n E_n\left(\frac{1}{2}\right) \tag{1.1.4}$$

$$E_n^* = E_n(0) \tag{1.1.5}$$

The first few Bernoulli polynomials are

$$B_0(x) = 1 \tag{1.1.6}$$

$$B_1(x) = x - 1/2 \tag{1.1.7}$$

$$B_2(x) = x^2 - x + 1/6 \tag{1.1.8}$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \tag{1.1.9}$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30} \tag{1.1.10}$$

$$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x \tag{1.1.11}$$

$$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}. \tag{1.1.12}$$

The first few Euler polynomials are

$$E_0(x) = 1 \tag{1.1.13}$$

$$E_1(x) = x - 1/2 \tag{1.1.14}$$

$$E_2(x) = x^2 - x \tag{1.1.15}$$

$$E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4} \tag{1.1.16}$$

$$E_4(x) = x^4 - 2x^3 + x \tag{1.1.17}$$

$$E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2} \tag{1.1.18}$$

$$E_6(x) = x^6 - 3x^5 + 5x^3 - 3x. \tag{1.1.19}$$

The Bernoulli and Euler numbers and polynomials are connected by the following relations

Lemma 1.1.2. The main formulas connecting the Eulerian numbers (resp. polynomials) to the Bernoulli numbers (resp.polynomials) are:

i)

$$B_n = -\frac{n}{2(2^n-1)}E_{n-1}^* \text{ and } B_n \in \mathbb{Q}, \quad \forall n \geqslant 1$$

ii)

$$E_n^* = \frac{(-1)^n}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}E_k \text{ and } 2^nE_n^* \in \mathbb{Z}, \quad \forall n \geqslant 0.$$

iii)

$$E_n = \sum_{k=0}^{n}\binom{n}{k}E_k^* \text{ and } E_n \in \mathbb{Z}, \quad \forall n \geqslant 0.$$

iv)

$$\begin{aligned} E_n(x) &= \frac{2^{n+1}}{n}\left[B_{n+1}\left(\frac{x+1}{2}\right) - B_{n+1}\left(\frac{x}{2}\right)\right] \\ &= \frac{2}{n+1}\left[B_{n+1}(x) - 2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right], \quad \forall n \geqslant 1 \end{aligned}$$

v)

$$E_n(x) = 2\binom{n+2}{2}^{-1}\sum_{k=0}^{n}\binom{n+2}{k}\left[\left(2^{n+2-k}-1\right)B_{n+2-k}B_k(x)\right], \quad \forall n \geqslant 0$$

Proof : For the proofs of i)–v) we refer to [1]. □

1.2 Old and recent results: Analytic view

We quote, from [1], in the following theorem the main first analytical properties of the Bernoulli numbers and polynomails

Theorem 1.2.1. Bernoulli numbers and polynomials have the following properties

i) $B_n(1-x) = (-1)^nB_n(x), \quad n \geq 0,$

ii) $(-1)^nB_n(-x) = B_n(x) + nx^{n-1} \quad n \geq 0$

iii) $B_n^{'}(x) = nB_{n-1}(x), \quad n \geq 0$

iv) $B_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k}, \quad n \geq 0$

v) $B_m(x+1) - B_m(x) = mx^{m-1}, m \geqslant 1$

vi) $\zeta(1-n) = -\frac{B_n}{n}, \quad n \geqslant 1$

vii) (**Euler Theorem**): $\zeta(2n) = -\frac{(2\pi i)^{2n}}{2}\frac{B_{2n}}{(2n)!}, \quad n \geq 0$

viii)

$$\zeta(-n, X) = -\frac{B_{n+1}(X)}{n+1}, \quad n \in \mathbb{N}$$

ix) (Raabe Formulae)[22]

$$B_n(ax) = a^{n-1}\sum_{k=0}^{a-1}B_n\left(x + \frac{k}{a}\right)$$

x)

$$B_m(x) = -\frac{m!}{(2\pi i)^m} \sum_{k\neq 0} \frac{e^{2\pi ikx}}{k^m}, \forall m \geqslant 1, \quad 0 \leqslant x < 1.$$

xi)

$$B_m(x) = -\frac{m!}{(2\pi i)^m} \left[Li_m(e^{2\pi ix}) + (-1)^m Li_m(e^{-2\pi ix}) \right],$$

where $Li_n(x)$ is the polylogarithm function given by

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

Similarly, from [1], we quote in the following theorem the main first analytical properties of the Euler numbers and polynomials

Theorem 1.2.2. Euler numbers and polynomials have the following properties

i) $E_n(1-x) = (-1)^n E_n(x) \quad n \geq 0$

ii) $(-1)^n E_n(-x) = -E_n(x) + 2x^n, \quad n \geq 0$

iii) $E_n'(x) = nE_{n-1}(x)$

iv) $E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}$

v) $E_n(x+1) + E_n(x) = 2x^n, \quad n \geqslant 0$

vi) $\zeta(2n) = -\frac{(2\pi i)^{2n}}{4(2n-1)!} \frac{E_{2n-1}^*}{(1-4^n)}, \qquad n \geq 1$

vii) $\zeta_*(-n) = E_n^*$, where $\zeta_*(s) = 2\sum_{n\geqslant 1} \frac{(-1)^n}{n^s}$,

viii) $\zeta_*(-n, X) = E_n^*(X)$, where $\zeta_*(s, X) = 2\sum_{n\geqslant 0} \frac{(-1)^n}{(n+X)^s}$,

ix) $E_n(ax) = a^n \sum_{k=0}^{a-1} (-1)^k E_n\left(x + \frac{k}{a}\right) \qquad$ if a is odd

x) $E_n(ax) = \frac{-2}{n+1} a^n \sum_{k=0}^{a-1} (-1)^k B_{n+1}\left(x + \frac{k}{a}\right) \qquad$ if a is even

Sketch of proofs: The main idea, for the proofs of these properties in theorems 1.2.1 and 1.2.2, is based on the use of the generating functions. We give the proof of the property **ix)** in theorem 1.2.1 . For the rest the idea is the same.

$$B_n(ax) = a^{n-1} \sum_{k=0}^{a-1} B_n\left(x + \frac{k}{a}\right)$$

We compute the generating functions of $B_n(ax)$ and $a^{n-1}\sum_{k=0}^{a-1} B_n\left(x+\frac{k}{a}\right)$. Then, by using the identity (1.1.1), we obtain

$$\sum_{n\geqslant 0} B_n(ax)\frac{t^n}{n!} = \frac{te^{axt}}{e^t-1}$$

and

$$\begin{aligned}
\sum_{n\geqslant 0} a^{n-1}\sum_{k=0}^{a-1} B_n\left(x+\frac{k}{a}\right)\frac{t^n}{n!} &= \frac{1}{a}\sum_{k=0}^{a-1}\sum_{n\geqslant 0} B_n\left(x+\frac{k}{a}\right)\frac{(at)^n}{n!} \\
&= \frac{1}{a}\sum_{k=0}^{a-1}\frac{(at)e^{\left(x+\frac{k}{a}\right)at}}{e^{at}-1} \\
&= \frac{te^{axt}}{e^{at}-1}\sum_{k=0}^{a-1} e^{kt} \\
&= \frac{te^{axt}}{e^t-1}
\end{aligned}$$

Hence, we obtain the desired property. □

1.3 Old and recent results: Arithmetical view

Now, we quote in the following theorem the main first arithmetical properties of the Bernoulli numbers and polynomails

Theorem 1.3.1. We have the famous results

i) **Faulhaber's formula [14]**:

$$\sum_{k=0}^{x} k^p = \frac{B_{p+1}(x+1)-B_{p+1}(0)}{p+1}, \quad x\in\mathbb{N}$$

ii) **Von Staudt-Clausen theorem [13, 26]**:

$$B_n + \sum_{\substack{p-1|n \\ p \text{ prime}}} \frac{1}{p} \in \mathbb{Z}.$$

Then,

$$Denominator(B_n) = \prod_{\substack{p-1|n \\ p \text{ prime}}} p.$$

iii) **Adams theorem [2]**: $\forall p$ prime, $n\geqslant 1$, $p-1 \not| 2n$ and $p^e|2n$ for some $e\geqslant 1$. Then, $p^e|Numerator(B_{2n})$.

iv) **Kummer congruence [19]**: If p prime $\geqslant 5$ and $p-1 \not| 2m$. Then,

$$\frac{B_{2m+(p-1)}}{2m+(p-1)} \equiv \frac{B_{2m}}{2m} \pmod{p}.$$

Similarly, we quote in the following theorem the main first arithmetical properties of the Euler numbers and polynomials

Theorem 1.3.2. We have the results

i) $\displaystyle\sum_{k=0}^{m}(-1)^k k^p = \frac{(-1)^m}{2}\frac{E_{p+1}(m+1)+(-1)^{m+1}E_{p+1}(1)}{m+1}, \quad m \in \mathbb{N}$,

ii) $2^n E_n^*$ **and** E_n **are in** $\mathbb{Z}$. (**Von Staudt-Clausen Analogue**)

iii) $E_{2n+1} = 0, E_{2n+2}^* = 0$

iv) E_{2n} are **odd integers** which alternate in sign.

v) If p prime $\geqslant 5$ and $p-1 \not| 2n$. Then,

$$E_{2n+(p-1)} \equiv E_{2n} \pmod{p}, \quad \forall n \in \mathbb{N},$$

(**Kummer's analogue**)

vi) $\forall p$ odd prime, $n \geqslant 1$, if $(p-1)p^e \not| 2n$ and $p^e | 2n$ for some $e \geqslant 1$. Then,

$$E_{2n} \equiv 0 \textbf{ or } 2 \pmod{p^{e+1}}$$

according as $p \equiv 1$ or $3 \pmod 4$. (**Adams analogue**)

Proof : The proofs are elementary and similar to those in theorem 1.3.1. □

1.4 Bernoulli and Euler functions: Raabe formulas

We state and prove the Raabe formulas for Bernoulli and Euler functions.

Definition 1.4.1. The mth Bernoulli function is defined by

$$\overline{B}_m(x) = -\frac{m!}{(2\pi i)^m}\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{e^{2\pi i k x}}{k^m}, \forall m \geqslant 1, \quad x \in \mathbb{R} \tag{1.4.20}$$

The first kind of mth Euler function is defined by

$$\overline{E}_m(x) = 2\frac{m!}{(2\pi i)^{m+1}}\sum_{k\in\mathbb{Z}}\frac{e^{2\pi i(k+\frac{1}{2})x}}{(k+\frac{1}{2})^{m+1}}, \quad \forall m \geqslant 1, \quad 0 \leqslant x < 1 \tag{1.4.21}$$

$$\overline{E}_m(x+1) = \overline{E}_m(x), \quad x \in \mathbb{R} \tag{1.4.22}$$

The second kind of mth Euler function is defined by

$$\overline{E}_m^*(x) = (-1)^{[x]}\overline{E}_m(x) \tag{1.4.23}$$

Theorem 1.4.2. (**First main result**)
These functions satisfied the so-called "distribution property" (Raabe's formulas)

i) $\displaystyle\overline{B}_n(ax) = a^{n-1}\sum_{k=0}^{a-1}\overline{B}_n\left(x+\frac{k}{a}\right)$

ii) $\overline{E}_n(ax) = a^n \sum_{k=0}^{a-1} (-1)^{k+[x+\frac{k}{a}]+[ax]} \overline{E}_n \left(x + \frac{k}{a}\right)$ if a is odd.

iii) $\overline{E}_n^*(ax) = a^n \sum_{k=0}^{a-1} (-1)^k \overline{E}_n^* \left(x + \frac{k}{a}\right)$ if a is odd.

Proof : These formulas follows easly from 1.4.20, 1.4.21 and 1.4.23 . For example, for the equality i), we have

$$\begin{aligned} a^{n-1} \sum_{l=0}^{a-1} \overline{B}_n \left(x + \frac{l}{a}\right) &= a^{n-1} \sum_{l=0}^{a-1} -\frac{n!}{(2\pi i)^n} \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{e^{2\pi i k\left(x+\frac{l}{a}\right)}}{k^n} \\ &= a^{n-1} \sum_{k \in \mathbb{Z}\setminus\{0\}} -\frac{n!}{(2\pi i)^n} \frac{e^{2\pi i k x}}{k^n} \sum_{l=0}^{a-1} e^{2\pi i k \frac{l}{a}} \end{aligned}$$

Now, we use

$$\sum_{l=0}^{a-1} e^{2\pi i k \frac{l}{a}} = \begin{cases} a & \text{if } k \in a\mathbb{Z}, \\ 0 & \text{Otherwise} \end{cases}$$

Then

$$\begin{aligned} a^{n-1} \sum_{l=0}^{a-1} \overline{B}_n \left(x + \frac{l}{a}\right) &= \sum_{k \in \mathbb{Z}\setminus\{0\}} -\frac{n!}{(2\pi i)^n} \frac{e^{2\pi i k a x}}{k^n} \\ &= \overline{B}_n (ax) \end{aligned}$$

Hence, we obtain the desired property.
The proofs of equalities ii) and iii) are similars to i).

□

2 Elliptic Bernoulli and Euler functions:

In this section, we review the definition of the elliptic Lerch series. We define the elliptic Lerch function following Weil ([28] VIII §12) as follows.

Definition 2.0.3. Let $L = \mathbb{Z}$ **or** $L = \mathbb{Z}\tau + \mathbb{Z}$, where $\mathbf{Im}(\tau) > 0$. We set

$$E_{\mathbb{Z}}(\omega, X) = \omega X$$

$$E_L(\omega, X) = X_1\omega_2 - X_2\omega_1, \quad L = \mathbb{Z}\tau + \mathbb{Z}$$

if $X = X_1\tau + X_2, \omega = \omega_1\tau + \omega_2, (X_i, \omega_i) \in \mathbb{R}^2, i = 1, 2.$

2.1 Elliptic Lerch series

Definition 2.1.1. Let m be an integer, and let z, $\varphi \in \mathbb{C}$. We introduce the elliptic Lerch series

$$L_m(s, z, \varphi) = \sum_{\omega \in L\setminus\{-z\}} \frac{(\overline{z} + \overline{\omega})^m}{|z + \omega|^{2s}} e\left(E_L(\omega, \varphi)\right)$$

Proposition 2.1.2. [28] Let m be an integer.

i) The function $L_m(s,z,\varphi)$ converges for $\mathrm{Re}(s) > \frac{m}{2}+1$,

ii) The function $L_m(s,z,\varphi)$for s continues meromorphically to a function on the whole s plane, with a simple pole only at $s=1$ if $m=0$ and $\varphi \in L$.

iii) The function $L_m(s,z,\varphi)$ satisfies the functional equation

$$\Gamma(s)L_m(s,z,\varphi) = A^{m+1-2s}\Gamma(m+1-s)L_m(m+1-s,\varphi,z)e\left(E_L(z,\varphi)\right).$$

2.2 Main objects

The main objects which we study here are

Definition 2.2.1. The **elliptic Bernoulli functions** are defined by

$$\begin{aligned}\overline{B}_m(X,L) &= -\frac{\Gamma(m)}{(2\pi i)^m}L_m(s=m,z=0,X)\\ &= -\frac{m!}{(2\pi i)^m}\sum_{\omega\in L\setminus\{0\}}\frac{e\left(E_L(\omega,X)\right)}{\omega^m},\end{aligned}$$

and The **elliptic Euler functions** are given by

$$\begin{aligned}\overline{E}_m(X,L) &= 2\frac{\Gamma(m)}{(2\pi i)^{m+1}}L_{m+1}(s=m+1,z=\frac{1}{2},X)\\ &= 2\frac{m!}{(2\pi i)^{m+1}}\sum_{\omega\in L}\frac{e\left(E_L(\omega,X)\right)}{\left(\omega+\frac{1}{2}\right)^{m+1}}.\end{aligned}$$

Examples: For $L=\mathbb{Z}$ we obtain the classical Bernoulli and Euler functions. In fact,

i) For $L=\mathbb{Z}$:

$$\begin{aligned}\overline{B}_m(X,L) &= -\frac{m!}{(2\pi i)^m}\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{e\left(kX\right)}{k^m}\\ &= \overline{B}_m(X)\end{aligned}$$

ii) $L=\mathbb{Z}$: Let $0\leqslant X<1$. Then,

$$\begin{aligned}\overline{E}_m(X,L)e\left(\frac{X}{2}\right) &= 2\frac{m!}{(2\pi i)^{m+1}}\sum_{k\in\mathbb{Z}}\frac{e\left(\left(k+\frac{1}{2}\right)X\right)}{\left(k+\frac{1}{2}\right)^{m+1}}\\ &= \overline{E}_m(X)\end{aligned}$$

2.3 Jacobi forms

Our elliptic Bernoulli and Euler functions are connected to Jacobi forms in three variables. Precisely, we have

Theorem 2.3.1. (**Second main result**)

i) The generating function of $\overline{B}_m(\varphi, L)$ is the following Jacobi form

$$D_L(z;\varphi) = e\Big(z\varphi_1\Big)\frac{\theta'_\tau(0)\theta_\tau\left(z+\varphi\right)}{\theta_\tau\left(z\right)\theta_\tau\left(\varphi\right)}$$

ii) The generating function of $\overline{E}_m(\varphi, L)$ is the following Jacobi form

$$F_L(z;\varphi) = -\frac{1}{\pi i}e\Big((z-\frac{1}{2})\varphi_1\Big)\frac{\theta'_\tau(0)\theta_\tau\left(z+\varphi-\frac{1}{2}\right)}{\theta_\tau\left(z-\frac{1}{2}\right)\theta_\tau\left(\varphi\right)}$$

where

$$\theta_\tau(z) = \sum_{n\in\mathbb{Z}} e\left(\frac{1}{2}(n+\frac{1}{2})^2\tau + (n+\frac{1}{2})(z+\frac{1}{2})\right)$$

Proof : The generating function of $\overline{B}_m(\varphi, L)$ is

$$\sum_{m\geqslant 0}\overline{B}_m(\varphi, L)\frac{(2\pi i)^m}{m!}z^{m-1}$$

Now, we are going to prove a). First, we prove the equlaity

$$D_L(z;\varphi) = \sum_{\omega\in L}^{(e)}\frac{e(-E_L(\omega,\varphi))}{z+\omega}. \tag{2.3.1}$$

Where $\sum_{\omega\in L}^{(e)}$ is the Eisenstein summation equal to

$$\sum_{\omega\in L}^{(e)} = \lim_{M,N\to\infty}\sum_{m=-M}^{m=M}\sum_{m=-N}^{m=N}, \text{ Where } \omega = m\tau + n.$$

The functions $z \to D_L(z;\varphi)$ and $F_\varphi : z \to \sum_{\omega\in L}^{(e)}\frac{e(-E_L(\omega,\varphi))}{z+\omega}$ are meromorphic functions with only simple poles in $\omega \in L$. Moreover,

$$\begin{aligned} D_L(z+\rho;\varphi) &= e(E_L(\rho,\varphi))D_L(z;\varphi) \\ F_\varphi(z+\rho) &= e(E_L(\rho,\varphi))F_\varphi(z)\forall\rho\in L. \end{aligned}$$

and finally, they have the same residue at $z = \omega, \omega \in L$:

$$\mathrm{Res}\Big(F_\varphi(z)dz, z=\omega\Big) = \mathrm{Res}\Big(D_L(z;\varphi)dz, z=\omega\Big) = e\Big(E_L(\omega,\varphi)\Big)$$

We introduce the quotient function $z \to \frac{F_\varphi(z)}{D_L(z;\varphi)}$. It is meromorphic, periodic with periods the lattice L and modulo L it has only one pole (a simple one) at $z = -\varphi$. Then, from Liouville's Theorem, the functions $z \to D_L(z;\varphi)$ and $F_\varphi(z)$ are proportional. But, we know that they have the same residue at $z = \omega, \omega \in L$. Thus we get our desired equality 2.3.1.
We use the Laurent expansion of $z \to \frac{1}{z+\omega}$. For $\omega \in L\backslash\{0\}$ we have

$$\frac{1}{z+\omega} = \sum_{k\geqslant 0}(-1)^k\left(\frac{z}{\omega}\right)^k$$

For our convenance we use $-\omega$ instead ω. Then,

$$D_{L_\tau}(z;\varphi) = \frac{1}{z} + \sum_{m\geqslant 1}\left(\sum_{\substack{\omega\in L\\ \omega\neq 0}}^{(e)} \frac{e(E_L(\omega,\varphi))}{\omega^m}\right) z^{m-1}$$

Now, we know that

$$\overline{B}_m(\varphi, L) = -\frac{m!}{(2\pi i)^m}\sum_{\omega\in L\setminus\{0\}} \frac{e\left(E_L(\omega,\varphi)\right)}{\omega^m},$$

Then, we conclude that the Jacobi form $D_{L_\tau}(z;\varphi)$ is the generating function of $\overline{B}_m(\varphi, L)$.

Similarly, we can prove the property b). In fact, the generating function of $\overline{E}_m(\varphi, L)$ is defined by

$$\sum_{m\geqslant 0}\overline{E}_m(\varphi, L)\frac{(2\pi i z)^m}{m!}$$

We remark that

$$F_L(z;\varphi) = -\frac{1}{\pi i}D_L(z-\frac{1}{2};\varphi)$$

Then,

$$F_L(z;\varphi) = -\frac{1}{\pi i}\sum_{\omega\in L}^{(e)} \frac{e(-E_L(\omega,\varphi))}{z+\omega-\frac{1}{2}} \tag{2.3.2}$$

wich is equal to

$$F_L(z;\varphi) = -\frac{1}{\pi i}\sum_{m\geqslant 0}\left(\sum_{\omega\in L}^{(e)} \frac{e(E_L(\omega,\varphi))}{\left(\omega+\frac{1}{2}\right)^{m+1}}\right) z^m$$

Now,

$$\overline{E}_m(X, L) = 2\frac{m!}{(2\pi i)^{m+1}}\sum_{\omega\in L} \frac{e\left(E_L(\omega, X)\right)}{\left(\omega+\frac{1}{2}\right)^{m+1}}$$

Hence

$$F_L(z;\varphi) = \sum_{m\geqslant 0}\overline{E}_m(\varphi, L)\frac{(2\pi i z)^m}{m!}$$

Thus, it's give us our desired result. □

For more details concerning the Jacobi form $D_L(z;\varphi)$, see [3, 5, 7, 8, 9, 11, 12].

2.4 Elliptic Raabe formulas

The aim of this section, is to study the distribution formulas (ellitpic Raabe formulas) satisfied by elliptic Bernoulli and elliptic Euler functions.

Theorem 2.4.1. (**Third main result**)

i)

$$\overline{B}_m(\lambda\varphi, \lambda L) = \lambda^{-m}\overline{B}_m(\varphi, L), \forall \lambda \in \mathbb{C}\backslash\{0\}.$$

ii)

$$\overline{B}_m\left(\frac{\varphi}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^m\overline{B}_m(\varphi;\tau), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}).$$

iii)

$$\overline{B}_m(\varphi+\rho;\tau) = \overline{B}_m(\varphi;\tau), \forall \rho \in \mathbb{Z}\tau + \mathbb{Z}$$

iv)

$$\overline{B}_m(-\varphi;\tau) = (-1)^{m-1}\overline{B}_m(\varphi;\tau)$$

v) For L, Λ lattices such that : $L \subset \Lambda$. Then,

$$\sum_{\bar{t}\in\Lambda/L} \overline{B}_m\big(\varphi+t; L\big) = [\Lambda : L]^{1-m}\overline{B}_m\big(\varphi;\Lambda\big), m \geqslant 1,$$

Proof : We give the proof of property (e). We quote from [11] the relation

$$\sum_{\bar{t}\in\Lambda/L} D_L\big(lz;\varphi+t\big) = D_\Lambda\big(z;\varphi\big) \tag{2.4.3}$$

which is valid for all complex lattices L and Λ s.t: $L \subset \Lambda$, $[\Lambda : L] = l$ Now, we use the Laurent expansion the functions $z \to D_L\big(lz;\varphi+t\big)$ and $z \to D_\Lambda(z;\varphi)$. We compare their coeficients, then we obtain our distribution formulae

$$\sum_{\bar{t}\in\Lambda/L} \overline{B}_m\big(\varphi+t; L\big) = [\Lambda : L]^{1-m}\overline{B}_m\big(\varphi;\Lambda\big).$$

We prove the modularity property (b). We know that, the Jacobi form D_L is a modular form for $\mathrm{SL}_2(\mathbb{Z})$, with index 0 and weight 1 i.e

$$D_{\frac{a\tau+b}{c\tau+d}}\left(\frac{z}{c\tau+d}; \frac{\varphi}{c\tau+d}\right) = (c\tau+d)D_\tau(z;\varphi), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

where

$$D_\tau(z;\varphi) := D_{L_\tau}(z;\varphi), L_\tau = \mathbb{Z}\tau + \mathbb{Z}, \tau \in \mathcal{H}.$$

It's obvious to deduce from this result the modularity of $\overline{B}_m\big(\varphi; L\big)$.
Also The periodicity (c), symmetry (d) and homogeneity (a) of $\overline{B}_m\big(\varphi; L\big)$ coming from the periodicity and homogeneity of the Jacobi form D_L. □

Theorem 2.4.2. (**Fourth main result**)

i) For L, Λ lattices such that : $L \subset \Lambda$ with $[\Lambda : L]$ odd. Then, for $m \geqslant 1$

$$\overline{E}_m\big(\varphi;\Lambda\big) = [\Lambda : L]^m \sum_{\bar{t}\in\Lambda/L} e\left(\frac{1}{2}E_\Lambda(1,\varphi+t) - \frac{1}{2}E_L(1,\varphi+t)\right)\overline{E}_m\big(\varphi+t; L\big).$$

ii) For L, Λ lattices such that : $L \subset \Lambda$ with $[\Lambda : L]$ even. Then, for $m \geqslant 1$

$$\overline{E}_m(\varphi;\Lambda) = -2\frac{[\Lambda : L]^{-m}}{m+1} \sum_{\bar{t}\in\Lambda/L} e\left(\frac{1}{2}E_\Lambda(1,\varphi+t)\right)\overline{B}_m(\varphi+t;L).$$

Proof :
The proofs of this theorem are similars to the proofs of theorem 2.4.1. In fact, in place of the Jacobi form D_L we consider the new one $F_L(z;\varphi)$.

□

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