Multiple polylogarithms and multi-poly-Bernoulli polynomials *

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Abstract. In this paper we introduce special generalized Bernoulli polynomials which generalize poly-Bernoulli polynomials and numbers. We call them *multi-poly-Bernoulli polynomials* and *numbers*. We prove a collection of important and fundamental identities satisfied by our multi-poly-Bernoulli polynomials and numbers.

Key words: multiple polylogarithms, zeta function, multi-poly-Bernoulli numbers and polynomials.

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1 Introduction and known results

Let us briefly review poly-Bernoulli polynomials. For details, we refer to [2], [7]. For an integer $k \in \mathbb{Z}$, put

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},$$

which is the k-th polylogarithm if $k \geq 1$, and a rational function if $k \leq 0$. One knows that $\mathrm{Li}_1(z) = -\log(1-z)$. The formal power series $\mathrm{Li}_k(z)$ can be used to define poly-Bernoulli polynomials. The polynomials $B_n^{(k)}(x)$ $(n = 0, 1, 2, \ldots)$ are said to be *poly-Bernoulli polynomials* if they satisfy

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}.$$

For any $n \geq 0$, we have

$$(-1)^n B_n^{(1)}(-x) = B_n(x),$$

the classical Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

We proved in [3] the following formulae.

Theorem 1.1 (Explicit formula) *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$B_n^{(k)}(x) = \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (x-j)^n. \tag{1}$$

Theorem 1.2 (Recurrence formula 1) *For $k \in \mathbb{Z}$ and $n \geq 2$,*

$$B_n^{(k)}(x) = \frac{1}{n+1} \left\{ B_n^{(k-1)}(x) + xB_0^{(k)}(x) - \sum_{m=1}^{n-1} \left(\binom{n}{m-1} - \binom{n}{m} x \right) B_m^{(k)}(x) \right\}, \tag{2}$$

and

$$\begin{aligned} B_0^{(k)}(x) &= 1, \\ B_1^{(k)}(x) &= \frac{1}{2} \left(B_1^{(k-1)}(x) + xB_0^{(k)}(x) \right). \end{aligned}$$

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Theorem 1.3 (Recurrence formula 2) *For all $k \geq 0$, $n \geq 0$,*

$$B_n^{(k)}(x) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} B_{n-m}^{(k-1)} \sum_{l=0}^{m} \frac{(-1)^l}{n-l+1} \binom{m}{l} B_l(x).$$

Theorem 1.4 (Appell sequence) *For $k \in \mathbb{Z}$, $n \geq 0$,*

$$\frac{d}{dx} B_{n+1}^{(k)}(x) = (n+1) B_n^{(k)}(x). \tag{3}$$

Theorem 1.5 (Addition formula) *For $k \in \mathbb{Z}$, $n \geq 0$,*

$$B_n^{(k)}(x+y) = \sum_{m=0}^{n} \binom{n}{m} B_m^{(k)}(x) y^{n-m}. \tag{4}$$

For $m, n \geq 0$, set

$$C_n^{(-m)}(x,y) = \sum_{k=0}^{m} \binom{m}{k} B_n^{(-k)}(x) y^{m-k}.$$

Then we have the following result:

Theorem 1.6 (Symmetric formula)

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n^{(-m)}(x,y) \frac{t^n}{n!} \frac{u^m}{m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_m^{(-n)}(x,y) \frac{t^n}{n!} \frac{u^m}{m!} \tag{5}$$

$$= \frac{e^{xt+yu+t+u}}{e^t + e^y - e^{t+u}}. \tag{6}$$

Theorem 1.7 (Duality) *For $m, n \geq 0$, we have*

$$C_n^{(-m)}(x,y) = C_m^{(-n)}(y,x). \tag{7}$$

Theorem 1.8 (Inversion formula) *For $m, n \geq 0$,*

$$B_n^{(-m)}(x) = \sum_{k=0}^{m} C_n^{(-k)}(x,y)(-y)^{m-k}. \tag{8}$$

Theorem 1.9 (Closed formula) *For $m, n \geq 0$,*

$$C_n^{(-m)}(x,y) = \sum_{j=0}^{\infty} (j!)^2 \left(\sum_{a=0}^{n} (x+1)^{n-a} \binom{n}{a} \left\{ \begin{array}{c} a \\ j \end{array} \right\} \right) \left(\sum_{b=0}^{m} (y+1)^{m-a} \binom{m}{b} \left\{ \begin{array}{c} b \\ j \end{array} \right\} \right), \tag{9}$$

where $\left\{ \begin{array}{c} n \\ m \end{array} \right\}$ are the Stirling numbers of the second kind.

We now introduce certain zeta functions in terms of the Laplace-Mellin integral. Let $k \in \mathbb{Z}$. Define

$$Z_k(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-xt} t^{s-1} dt.$$

It is defined for $\mathrm{Re}(s) > 0$ and $x > 0$ if $k \geq 1$, and for $\mathrm{Re}(s) > 0$ and $x > |k| + 1$ if $k \leq 0$. In particular, $Z_k(s,1)$ is called the *Arakawa-Kaneko zeta function* defined in [1].

The next result was independently proved by Coppo-Candelpergher [4] and the authors.

Theorem 1.10 (Interpolation formula) *The function $s \mapsto Z_k(s,x)$ is analytically continued to an entire function on the complex s-plane and for $n \geq 0$ and $x > 0$,*

$$Z_k(-n,x) = (-1)^n B_n^{(k)}(-x) \tag{10}$$

is satisfied. In addition, this zeta function can be rewritten as follows: For $k \in \mathbb{Z}$, we have

$$Z_k(s,x) = \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{1}{(x+j)^s}. \tag{11}$$

We investigated in [5], [6] generalized poly-Bernoulli numbers, which are called *multi-poly-Bernoulli numbers*, according to a suggestion mentioned in [1], [8]. In this paper, we introduce generalized poly-Bernoulli polynomials, which are called *multi-poly-Bernoulli polynomials* in our paper. The constant terms of these polynomials are *multi-poly-Bernoulli numbers*. We prove a collection of formulae which generalize the above ones. Specializing $x = 0$, our some results reduces to the results about multi-poly-Bernoulli numbers proved in [5], [6].

2 Multi-poly-Bernoulli polynomials and numbers

Firstly, we recall a generalization of $\mathrm{Li}_k(z)$. For $k_1, \ldots, k_r \in \mathbb{Z}$, define the *multiple polylogarithm* by

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{\substack{m_1,\ldots,m_r \in \mathbb{Z} \\ 0<m_1<\cdots<m_r}} \frac{z^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

The following result will be used in the next section.

Lemma 2.1

$$\frac{d}{dz}\mathrm{Li}_{k_1,\ldots,k_r}(z) = \begin{cases} \frac{1}{z}\mathrm{Li}_{k_1,\ldots,k_{r-1},k_r-1}(z) & (k_r > 1) \\ \frac{1}{1-z}\mathrm{Li}_{k_1,\ldots,k_{r-1}}(z) & (k_r = 1) \end{cases}.$$

Next, using multiple polylogarithms, let us introduce a generalization of poly-Bernoulli polynomials.

Definition 2.2 *The multi-poly-Bernoulli polynomials* $B_n^{(k_1,\ldots,k_r)}(x)$ $(n = 0, 1, 2, \ldots)$ are defined for each integer $k_1, \ldots, k_r$ by the generating series

$$\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_r)}(x) \frac{t^n}{n!}. \tag{12}$$

We call $B_n^{(k_1,\ldots,k_r)} := B_n^{(k_1,\ldots,k_r)}(0)$ $(n = 0, 1, 2, \ldots)$ *multi-poly-Bernoulli numbers*, which were investigated in [5] and [6]. The multi-poly-Bernoulli numbers satisfy the following recurrence formulae ([6]).

Theorem 2.3 (Recurrence formula 1) (1) *If* $k_r > 1$ *and* $n \geq 2$*, then*

$$B_n^{(k_1,\ldots,k_r)} = \frac{1}{n+r}\left[B_n^{(k_1,\ldots,k_r-1)}(x) - \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{(k_1,\ldots,k_r)}\right].$$

(2) *If* $k_r = 1$ *and* $n \geq 2$*, then*

$$B_n^{(k_1,\ldots,k_{r-1},1)} = \frac{1}{n+r}\left[B_n^{(k_1,\ldots,k_{r-1})} - \sum_{m=0}^{n-1} (-1)^{n-m}\left\{r\binom{n}{m} + \binom{n}{m-1}\right\} B_m^{(k_1,\ldots,k_{r-1},1)}\right].$$

Theorem 2.4 (Recurrence formula 2) (1) *If* $k_r > 1$*, then for any* $n \geq 0$*,*

$$\begin{aligned}
&B_n^{(k_1,\ldots,k_r)} \\
&= (-1)^{r-1}\frac{n!}{(n+r-1)!} \\
&\times \sum_{m=0}^{n}\left\{\sum_{p=0}^{n-m} \sum_{\substack{j_1+\cdots+j_{r-1} \\ =n-m+r-1-p}} \frac{(-1)^{n-m+r-1-p}(n-m+r-1-p)!}{j_1!\cdots j_{r-1}!}\binom{n-m+r-1}{p} B_p^{(k_1,\ldots,k_{r-1},k_r-1)}\right\} \\
&\times (-1)^m \binom{n+r-1}{m} \sum_{l=0}^{m} \frac{(-1)^l}{n-l+r}\binom{m}{l} \sum_{\substack{i_1+\cdots+i_r \\ =l}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{l!}{i_1!\cdots i_r!}.
\end{aligned}$$

(2) *If* $k_r = 1$*, then for any* $n \geq 0$*,*

$$\begin{aligned}
&B_n^{(k_1,\ldots,k_r)} \\
&= (-1)^{r-1}\frac{n!}{(n+r-1)!} \\
&\times \sum_{m=0}^{n}\left\{\sum_{p=0}^{n-m} \sum_{\substack{j_1+\cdots+j_{r-1} \\ =n-m+r-1-p}} \frac{(-1)^{n-m+r-1-p}(n-m+r-1-p)!}{j_1!\cdots j_r!}\binom{n-m+r-1}{p} B_p^{(k_1,\ldots,k_{r-1})}\right\} \\
&\times \binom{n+r-1}{m} \frac{1}{n-m+r} \sum_{\substack{i_1+\cdots i_r \\ =m}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{m!}{i_1!\cdots i_r!}.
\end{aligned}$$

Remark 2.5 If $k_1 = \cdots = k_r = 1$, then the above defining equation becomes

$$\frac{1}{r!}\left(\frac{-t}{e^{-t}-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(1,\ldots,1)}(x)\frac{t^n}{n!},$$

which gives the definition of higher order Bernoulli polynomials.

3 Some formulae

In this section, we will prove some fundamental identities which generalize ones mentioned in the previous sections.

Theorem 3.1 (Explicit formula)

$$B_n^{(k_1,\ldots,k_r)}(x) = \sum_{0<m_1<\cdots<m_r\le n+r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \sum_{j=0}^{m_r-r}(-1)^j\binom{m_r-r}{j}(x-j)^n. \tag{13}$$

Proof. Since

$$\begin{aligned}\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} &= \sum_{0<m_1<\cdots<m_r}\frac{(1-e^{-t})^{m_r-r}}{m_1^{k_1}\cdots m_r^{k_r}}\\ &= \sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\sum_{j=0}^{m_r-r}(-1)^j\binom{m_r-r}{j}e^{-jt},\end{aligned}$$

we have

$$\begin{aligned}\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r}e^{xt} &= \sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\sum_{j=0}^{m_r-r}(-1)^j\binom{m_r-r}{j}e^{(x-j)t}\\ &= \sum_{0<m_1<\cdots<m_r}\sum_{n=0}^{\infty}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\sum_{j=0}^{m_r-r}(-1)^j\binom{m_r-r}{j}(x-j)^n\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{0<m_1<\cdots<m_r\le n+r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\sum_{j=0}^{m_r-r}(-1)^j\binom{m_r-r}{j}(x-j)^n\frac{t^n}{n!}.\end{aligned}$$

This proves the theorem. □

Theorem 3.2 (Recurrence formula 1) (1) *If $k_r > 1$ and $n \ge 2$, then*

$$B_n^{(k_1,\ldots,k_r)}(x) = \frac{1}{n+r}\left[B_n^{(k_1,\ldots,k_r-1)}(x) + xB_0^{(k_1,\ldots,k_r)}(x) - \sum_{m=1}^{n-1}\left\{\binom{n}{m-1} - x\binom{n}{m}\right\}B_m^{(k_1,\ldots,k_r)}(x)\right].$$

(2) *If $k_r = 1$ and $n \ge 2$, then*

$$B_n^{(k_1,\ldots,k_{r-1},1)}(x) = \frac{1}{n+r}\left[B_n^{(k_1,\ldots,k_{r-1})}(x) - \sum_{m=0}^{n-1}(-1)^{n-m}\left\{(x+r)\binom{n}{m} + \binom{n}{m-1}\right\}B_m^{(k_1,\ldots,k_{r-1},1)}(x)\right].$$

Proof. Put $z = 1-e^{-t}$.
(1) The identity (12) leads

$$\mathrm{Li}_{k_1,\ldots,k_r}(z)e^{x\mathrm{Li}_1(z)} = z^r\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x)\frac{\mathrm{Li}_1(z)^n}{n!}. \tag{14}$$

Differentiation by z, and division by z^{r-1} gives

$$\begin{aligned}&\frac{\mathrm{Li}_{k_1,\ldots,k_r-1}(z)}{z^r}e^{x\mathrm{Li}_1(z)} + \frac{xz}{1-z}\cdot\frac{\mathrm{Li}_{k_1,\ldots,k_r}(z)}{z^r}e^{x\mathrm{Li}_1(z)}\\ &\qquad = r\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x)\frac{\mathrm{Li}_1(z)^n}{n!} + \frac{z}{1-z}\sum_{n=1}^{\infty}B_n^{(k_1,\ldots,k_r)}\frac{\mathrm{Li}_1(z)^{n-1}}{(n-1)!}.\end{aligned}$$

Using $z/(1-z)=e^t-1$,

$$\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r-1)}(x)\frac{t^n}{n!} - x\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!} - r\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!} + \sum_{n=0}^{\infty} B_{n+1}^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!}$$
$$= e^t\sum_{n=0}^{\infty} B_{n+1}^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!} - e^t\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!}.$$

Comparing both sides of the above identity,

$$B_n^{(k_1,\dots,k_r-1)}(x) - xB_n^{(k_1,\dots,k_r)} - rB_n^{(k_1,\dots,k_r)} + B_{n+1}^{(k_1,\dots,k_r)} = \sum_{m=0}^{n}\binom{n}{m}B_{m+1}^{(k_1,\dots,k_r)}(x) - x\sum_{m=0}^{n}\binom{n}{m}B_m^{(k_1,\dots,k_r)}(x).$$

From this, we have

$$B_n^{(k_1,\dots,k_r-1)}(x) - (n+r)B_n^{(k_1,\dots,k_r)}(x) = \sum_{m=1}^{n-1}\left\{\binom{n}{m-1} - x\binom{n}{m}\right\}B_m^{(k_1,\dots,k_r)}(x) - xB_0^{(k_1,\dots,k_r)}(x).$$

(2) Differentiating (14) by z, and then dividing by z^{r-1}, we obtain

$$\frac{1}{1-z}\cdot\frac{\mathrm{Li}_{k_1,\dots,k_{r-1}}(z)}{z^{r-1}}e^{x\mathrm{Li}_1(z)} + \frac{xz}{1-z}\cdot\frac{\mathrm{Li}_{k_1,\dots,k_r}(z)}{z^r}e^{x\mathrm{Li}_1(z)}$$
$$= r\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(x)\frac{\mathrm{Li}_1(z)^n}{n!} + \frac{z}{1-z}\sum_{n=0}^{\infty} B_{n+1}^{(k_1,\dots k_r)}(x)\frac{\mathrm{Li}_1(z)^n}{n!}.$$

Using $z/(1-z)=e^t-1$ and $1/(1-z)=e^t$,

$$\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_{r-1})}(x)\frac{t^n}{n!} + x\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_{n+1}^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!}$$
$$= e^{-t}\sum_{n=0}^{\infty} xB_n^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!} + e^{-t}\sum_{n=0}^{\infty} rB_n^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!} - e^{-t}\sum_{n=0}^{\infty} rB_{n+1}^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!}.$$

Comparing both sides of this identity, we get

$$B_n^{(k_1,\dots,k_{r-1})}(x) + xB_n^{(k_1,\dots,k_r)}(x) - B_{n+1}^{(k_1,\dots,k_r)}(x)$$
$$= \sum_{m=0}^{n}(-1)^{n-m}x\binom{n}{m}B_m^{(k_1,\dots,k_r)}(x) + \sum_{m=0}^{n}(-1)^{n-m}r\binom{n}{m}B_m^{(k_1,\dots,k_r)}(x) - \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}B_{m+1}^{(k_1,\dots,k_r)}(x)$$
$$= xB_n^{(k_1,\dots,k_r)}(x) - B_{n+1}^{(k_1,\dots,k_r)}(x) + (n+r)B_n^{(k_1,\dots,k_r)}(x)$$
$$+ \sum_{m=0}^{n-1}(-1)^{n-m}\left\{(x+r)\binom{n}{m} + \binom{n}{m-1}\right\}B_m^{(k_1,\dots,k_r)}(x).$$

This leads our formula. □

Theorem 3.3 (Recurrence formula 2) (1) *If $k_r>1$, then for any $n\geq 0$,*

$$B_n^{(k_1,\dots,k_r)}(rx)$$
$$= (-1)^{r-1}\frac{n!}{(n+r-1)!}$$
$$\times\sum_{m=0}^{n}\left\{\sum_{p=0}^{n-m}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n-m+r-1-p}}\frac{(-1)^{n-m+r-1-p}(n-m+r-1-p)!}{j_1!\cdots j_{r-1}!}\binom{n-m+r-1}{p}B_p^{(k_1,\dots,k_{r-1},k_r-1)}\right\}$$
$$\times(-1)^m\binom{n+r-1}{m}\sum_{l=0}^{m}\frac{(-1)^l}{n-l+r}\binom{m}{l}\sum_{\substack{i_1+\cdots+i_r\\=l}}B_{i_1}^{(1)}(x)\cdots B_{i_r}^{(1)}(x)\frac{l!}{i_1!\cdots i_r!}.$$

(2) *If* $k_r = 1$, *then for any* $n \geq 0$,

$$
\begin{aligned}
&B_n^{(k_1,\dots,k_r)}(rx) \\
&= (-1)^{r-1}\frac{n!}{(n+r-1)!} \\
&\times \sum_{m=0}^{n}\left\{\sum_{p=0}^{n-m}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n-m+r-1-p}}\frac{(-1)^{n-m+r-1-p}(n-m+r-1-p)!}{j_1!\cdots j_r!}\binom{n-m+r-1}{p}B_p^{(k_1,\dots,k_r)}\right\} \\
&\times\binom{n+r-1}{m}\frac{1}{n-m+r}\sum_{\substack{i_1+\cdots i_r\\=m}}B_{i_1}^{(1)}(x)\cdots B_{i_r}^{(1)}(x)\frac{m!}{i_1!\cdots i_r!}.
\end{aligned}
$$

Proof. (1) By Lemma 2.1, we get

$$
\mathrm{Li}_{k_1,\dots k_r}(1-e^{-t}) = \int_0^t \frac{e^{-s}}{1-e^{-s}}\mathrm{Li}_{k_1,\dots k_{r-1},k_r-1}(1-e^{-s})ds.
$$

Using this, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}B_n^{(k_1,\dots,k_r)}(rx)\frac{t^n}{n!} \\
&= \frac{e^{rxt}}{(1-e^{-t})^r}\int_0^t\frac{e^{-s}}{1-e^{-s}}\mathrm{Li}_{k_1,\dots,k_{r-1},k_r-1}(1-e^{-s})ds \\
&= \left(\frac{e^{xt}}{1-e^{-t}}\right)^r\int_0^t e^{-s}(1-e^{-s})^{r-1}\frac{\mathrm{Li}_{k_1,\dots,k_{r-1},k_r-1}(1-e^{-s})}{(1-e^{-s})^r}ds \\
&= \left(\sum_{n=0}^{\infty}B_n^{(1)}(x)\frac{t^{n-1}}{n!}\right)^r\int_0^t\sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\left(-\sum_{n=1}^{\infty}\frac{(-s)^n}{n!}\right)^{r-1}\sum_{n=0}^{\infty}B_n^{(k_1,\dots,k_{r-1},k_r-1)}\frac{s^n}{n!}ds.
\end{aligned}
$$

Rewriting the last expression as in the proof of Theorem 7 (1) in [6] leads the result.

(2) By Lemma 2.1,

$$
\mathrm{Li}_{k_1,\dots,k_{r-1},1}(1-e^{-t}) = \int_0^t \mathrm{Li}_{k_1,\dots,k_{r-1}}(1-e^{-s})ds.
$$

From this identity, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}B_n^{(k_1,\dots,k_{r-1},1)}(rx)\frac{t^n}{n!} \\
&= \left(\frac{e^{xt}}{1-e^{-t}}\right)^r\int_0^t(1-e^{-s})^{r-1}\frac{\mathrm{Li}_{k_1,\dots,k_{r-1}}(1-e^{-s})}{(1-e^{-s})^{r-1}}ds \\
&= \left(\sum_{n=0}^{\infty}B_n^{(1)}(x)\frac{t^{n-1}}{n!}\right)^r\int_0^t\left(-\sum_{n=1}^{\infty}\frac{(-s)^n}{n!}\right)^{r-1}\sum_{n=0}^{\infty}B_n^{(k_1,\dots,k_{r-1})}\frac{s^n}{n!}ds.
\end{aligned}
$$

Rewriting this as in the proof of Theorem 7 (2) in [6], we obtain the result. □

Theorem 3.4 (Appell sequence) *For any* $n \geq 0$,

$$
\frac{d}{dx}B_{n+1}^{(k_1,\dots,k_r)}(x) = (n+1)B_n^{(k_1,\dots,k_r)}(x). \tag{15}
$$

Proof. Differentiation of both sides of (12) by x gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{d}{dx}B_n^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!} &= \frac{t\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(1-e^{-t})^r}e^{xt} \\
&= \sum_{n=1}^{\infty}nB_{n-1}^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!},
\end{aligned}
$$

which yields the result. □

Theorem 3.5 (Addition formula) *For any $n \geq 0$,*

$$B_n^{(k_1,\ldots,k_r)}(x+y) = \sum_{m=0}^{n} \binom{n}{m} B_m^{(k_1,\ldots,k_r)}(x) y^{n-m}. \tag{16}$$

In particular, we have

$$B_n^{(k_1,\ldots,k_r)}(x) = \sum_{m=0}^{n} \binom{n}{m} B_m^{(k_1,\ldots,k_r)} x^{n-m}.$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_r)}(x+y)\frac{t^n}{n!} &= \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} e^{(x+y)t} \\
&= \left(\sum_{m=0}^{\infty} B_m^{(k_1,\ldots,k_r)}(x)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\frac{y^l t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} B_m^{(k_1,\ldots,k_r)}(x) y^{n-m}\right)\frac{t^n}{n!}.
\end{aligned}$$

This proves the theorem. □

4 Special multi-poly-Bernoulli polynomials

In this section we consider multi-poly-Bernoulli polynomials of special type. For a positive integer r, we define

$$B[r]_n^{(k)}(x) := B_n^{(\overbrace{0,\ldots,0}^{r-1},k)}(x).$$

It is easy to see $B[1]_n^{(k)}(x) = B_n^{(k)}(x)$. Also, we easily verify that $B[r]_n^{(k)}(0) = B[r]_n^{(k)}$, which is defined by

$$B[r]_n^{(k)} := B_n^{(\overbrace{0,\ldots,0}^{r-1},k)}.$$

One sees in [5] that the generating function of these numbers is written as

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B[r]_n^{(-k)}\frac{t^n}{n!}\frac{u^k}{k!} = \left(\frac{e^{t+u}}{e^t+e^u-e^{t+u}}\right)^r. \tag{17}$$

From this, for $n, k \geq 0$, we have

$$B[r]_n^{(-k)} = B[r]_k^{(-n)}. \tag{18}$$

For $m, n \geq 0$, set

$$C[r]_n^{(-m)}(x,y) = \sum_{k=0}^{m}\binom{m}{k} B[r]_n^{(-k)}(x) y^{m-k}.$$

Then we have

Theorem 4.1 (Symmetric formula)

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C[r]_n^{(-m)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C[r]_m^{(-n)}(y,x)\frac{t^n}{n!}\frac{u^m}{m!} \\
&= e^{xt+yu}\left(\frac{e^{t+u}}{e^t+e^u-e^{t+u}}\right)^r.
\end{aligned}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C[r]_n^{(-m)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} B[r]_n^{(-k)}(x) y^l \frac{t^n}{n!}\frac{u^k}{k!}\frac{u^l}{l!} \\
&= e^{yu}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B[r]_n^{(-k)}(x)\frac{t^n}{n!}\frac{u^k}{k!} \\
&\qquad \text{(by definition of } B[r]_n^{(-k)}(x))
\end{aligned}$$

$$= e^{yu}\sum_{k=0}^{\infty}\left(e^{xt}\sum_{n=0}^{\infty}B[r]_n^{(-k)}\frac{t^n}{n!}\right)\frac{u^k}{k!}$$
$$= e^{xt+yu}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}B[r]_n^{(-k)}\frac{t^n}{n!}\frac{u^k}{k!}.$$

Using (17), the last identity yields the claim. □

As a corollary, we have the following result:

Theorem 4.2 (Duality) *For all $n, m \geq 0$,*

$$C[r]_n^{(-m)}(x,y) = C[r]_m^{(-n)}(y,x). \tag{19}$$

In particular, for $x = y = 0$ we obtain (18), the duality property of special multi-poly-Bernoulli numbers. Furthermore, for $r = 1$ we have

$$B_n^{(-m)} = B_m^{(-n)},$$

the duality property of poly-Bernoulli numbers.

Theorem 4.3 (Inversion formula) *For $m, n \geq 0$, we have*

$$B[r]_n^{(-m)}(x) = \sum_{k=0}^{m}\binom{m}{k}C[r]_n^{(-k)}(x,y)(-y)^{m-k}. \tag{20}$$

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}C[r]_n^{(-k)}(x,y)(-y)^{m-k}\right)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$(j = m-k)$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j+k=m}C[r]_n^{(-k)}(x,y)\frac{(-uy)^j}{j!}\frac{t^n}{n!}\frac{u^k}{k!}$$
$$= e^{-yu}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C[r]_n^{(-k)}(x,y)\frac{t^n}{n!}\frac{u^k}{k!}$$
$$\text{(by Theorem 4.1)}$$
$$= e^{xt}\left(\frac{e^{t+u}}{e^t+e^u-e^{t+u}}\right)^r.$$

One sees that right hand side is the generating series of $B[r]_n^{(-m)}(x)$ $(n, m \geq 0)$. □

Theorem 4.4 (Closed formula) *For $m, n \geq 0$, we have*

$$C[r]_n^{(-m)}(x,y)$$
$$= \sum_{\substack{n=n_1+\cdots n_r\\ n_1,\ldots,n_r\geq 0}}\sum_{\substack{m=m_1+\cdots m_r\\ m_1,\ldots,m_r\geq 0}}\frac{n!m!}{n_1!\cdots n_r!m_1!\cdots m_r!}$$
$$\times\sum_{j_1=0}^{\min(n_1,m_1)}\cdots\sum_{j_r=0}^{\min(n_r,m_r)}(j_1!\cdots j_r!)^2\sum_{i=1}^{r}\left(\sum_{a_i=0}^{n_i}(\frac{x}{r}+1)^{n_i-a_i}\binom{n_i}{a_i}\left\{{a_i \atop j_i}\right\}\right)\left(\sum_{b_i=0}^{m_i}(\frac{y}{r}+1)^{m_i-b_i}\binom{m_i}{b_i}\left\{{b_i \atop j_i}\right\}\right).$$

Proof. We have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C[r]_n^{(-m)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} = e^{xt+yu}\left(\frac{e^{t+u}}{e^t+e^u-e^{t+u}}\right)^r$$
$$= \frac{e^{(x+r)t+(y+r)u}}{(1-(e^t-1)(e^u-1))^r}$$
$$= e^{(x+r)t+(y+r)u}\left(\sum_{j=0}^{\infty}(e^t-1)^j(e^u-1)^j\right)^r$$
$$= \left(\sum_{j=0}^{\infty}e^{(x/r+1)t}(e^t-1)^j e^{(y/r+1)u}(e^u-1)^j\right)^r.$$

Here making use of

$$\sum_{n=0}^{\infty} \left\{ \begin{array}{c} n \\ k \end{array} \right\} \frac{u^n}{n!} = \frac{(e^u-1)^k}{k},$$

the right hand side of the last expression becomes

$$\left[\sum_{j=0}^{\infty} \left(j! \sum_{n=0}^{\infty} \frac{(\frac{x}{r}+1)^n t^n}{n!} \sum_{m=0}^{\infty} \left\{ \begin{array}{c} m \\ j \end{array} \right\} \frac{t^m}{m!} \right) \left(j! \sum_{n=0}^{\infty} \frac{(\frac{y}{r}+1)^n u^n}{n!} \sum_{m=0}^{\infty} \left\{ \begin{array}{c} m \\ j \end{array} \right\} \frac{u^m}{m!} \right) \right]^r$$

$$= \left[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n}{n!} \frac{u^m}{m!} \sum_{j=0}^{\min(n,m)} (j!)^2 \left(\sum_{a=0}^{n} \left(\frac{x}{r}+1 \right)^{n-a} \binom{n}{a} \left\{ \begin{array}{c} a \\ j \end{array} \right\} \right) \left(\sum_{b=0}^{m} \left(\frac{y}{r}+1 \right)^{m-b} \binom{m}{b} \left\{ \begin{array}{c} b \\ j \end{array} \right\} \right) \right]^r.$$

This gives the result. □

5 Generalized Arakawa-Kaneko zeta functions

Let $k_1, k_2, \ldots, k_r \in \mathbb{Z}$. We consider

$$Z_{k_1,k_2,\ldots,k_r}(s,x) := \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{\mathrm{Li}_{k_1,k_2,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} e^{-xt} t^{s-1} dt,$$

the Laplace-Mellin integral. When all k_i are positive, this function is defined for $\mathrm{Re}(s) > 0$ and $x > 0$. We call it the *generalized Arakawa-Kaneko zeta function*.

Theorem 5.1 (Interpolation formula) *The function $s \mapsto Z_{k_1,\ldots,k_r}(s,x)$ is analytically continued to the whole complex s-plane and for $n \geq 0$ and $x > 0$,*

$$Z_{k_1,\ldots,k_r}(-n,x) = (-1)^n B_n^{(k_1,\ldots,k_r)}(-x) \tag{21}$$

is satisfied. Moreover, $Z_{k_1,\ldots,k_r}(s,x)$ is written as follows:

$$Z_{k_1,\ldots,k_r}(s,x) = \sum_{0 \leq m_1 < \cdots < m_r} \frac{1}{(m_1+1)^{k_1} \cdots (m_r+1)^{k_r}} \sum_{j=0}^{m_r+1-r} (-1)^j \binom{m_r+1-r}{j} \frac{1}{(x+j)^s}. \tag{22}$$

Proof. We split up $Z_{k_1,\ldots k_r}(s,x)$ as the sum of two integrals:

$$\begin{aligned} Z_{k_1,\ldots,k_r}(s,x) \quad := \quad & \frac{1}{\Gamma(s)} \int_0^1 \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} e^{-xt} t^{s-1} dt \\ & + \frac{1}{\Gamma(s)} \int_1^{\infty} \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} e^{-xt} t^{s-1} dt. \end{aligned}$$

The second integral converges absolutely for any $s \in \mathbb{C}$ and $x > 0$ and cancels at negative integers because $1/\Gamma(s)$ so does. If $\mathrm{Re}(s) > 0$, then the first integral is expressed as

$$\frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{B_n^{(k_1,\ldots,k_r)}(-x)}{n!} \cdot \frac{1}{n+s}.$$

From this, for a nonnegative integer n,

$$\begin{aligned} Z_{k_1,\ldots,k_r}(-n,x) \quad &= \quad \left(\lim_{s \to -n} \frac{1}{\Gamma(s)(n+s)} \right) \frac{B_n^{(k_1,\ldots,k_r)}(-x)}{n!} \\ &= \quad (-1)^n B_n^{(k_1,\ldots,k_r)}(-x). \end{aligned}$$

As for the latter part, we calculate

$$\begin{aligned}
Z_{k_1,\dots,k_r}(s,x) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-xt}\sum_{0<m_1<\cdots<m_r}\frac{(1-e^{-t})^{m_r-r}}{m_1^{k_1}\cdots m_r^{k_r}}dt \\
&= \frac{1}{\Gamma(s)}\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\int_0^\infty (1-e^{-t})^{m_r-r}t^{s-1}e^{-xt}dt \\
&= \frac{1}{\Gamma(s)}\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\int_0^\infty \left(\sum_{j=0}^{m_r-r}\binom{m_r-r}{j}(-1)^j e^{-jt}\right)t^{s-1}e^{-xt}dt \\
&= \frac{1}{\Gamma(s)}\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\sum_{j=0}^{m_r-r}(-1)^j\binom{m_r-r}{j}\int_0^\infty t^{s-1}e^{-(x+j)t}dt \\
&= \sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\sum_{j=0}^{m_r-r}(-1)^j\binom{m_r-r}{j}\frac{1}{(x+j)^s} \\
&= \sum_{0\le m_1<\cdots<m_r}\frac{1}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\sum_{j=0}^{m_r+1-r}(-1)^j\binom{m_r+1-r}{j}\frac{1}{(x+j)^s}.
\end{aligned}$$

□

Remark 5.2 In the case $x=r$, we have

$$\begin{aligned}
Z_{k_1,\dots,k_r}(s,r) &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(e^t-1)^r}t^{s-1}dt \\
&=: \zeta_n(k_1,\dots,k_r;s),
\end{aligned}$$

which are investigated in [9]. The above theorem is an extension of a result in [9].

Theorem 5.3 (Difference formula) *We have*

$$\begin{aligned}
&Z_{k_1,\dots,k_r}(s,x+1)-Z_{k_1,\dots,k_r}(s,x) \\
&\qquad = \sum_{0\le m_1<\cdots<m_r}\frac{1}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\sum_{j=0}^{m_r-r+2}(-1)^{j+1}\binom{m_r-r+2}{j}\frac{1}{(x+j)^s}.
\end{aligned}$$

Proof.

$$\begin{aligned}
\text{L.H.S.} &= -\frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(1-e^{-t})^{r-1}}e^{-xt}t^{s-1}dt \\
&= -\frac{1}{\Gamma(s)}\sum_{0<m_1<\cdots<m_r}\int_0^\infty \frac{(1-e^{-t})^{m_r-r+1}}{m_1^{k_1}\cdots m_r^{k_r}}e^{-xt}t^{s-1}dt \\
&= -\frac{1}{\Gamma(s)}\sum_{0\le m_1<\cdots<m_r}\int_0^\infty \frac{(1-e^{-t})^{m_r-r+2}}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}e^{-xt}t^{s-1}dt \\
&= \sum_{0\le m_1<\cdots<m_r}\frac{1}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\sum_{j=0}^{m_r-r+2}(-1)^{j+1}\binom{m_r-r+2}{j}\frac{1}{\Gamma(s)}\int_0^\infty e^{(x+j)t}t^{s-1}dt,
\end{aligned}$$

which yields the result. □

Theorem 5.4 (Raabe type formulae) (1) *If $s\neq 1$, then*

$$\begin{aligned}
&\int_0^1 Z_{k_1,\dots,k_r}(s,x+w)dw \\
&\qquad = \frac{1}{s-1}\sum_{0\le m_1<\cdots<m_r}\frac{1}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\sum_{j=0}^{m_r-r+2}(-1)^j\binom{m_r-r+2}{j}\frac{1}{(x+j)^{s-1}}.
\end{aligned}$$

(2)

$$\int_0^1 B_n^{(k_1,\dots,k_r)}(x-w)dw$$
$$= \frac{1}{n+1}\sum_{0\le m_1<\cdots<m_r}\frac{1}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\sum_{j=0}^{m_r-r+2}(-1)^j\binom{m_r-r+2}{j}(x+j)^{n+1}.$$

Proof. (1)

$$\begin{aligned}\text{L.H.S.} &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(1-e^{-t})^r}e^{-xt}t^{s-1}\int_0^1 e^{-wt}dwdt\\ &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(1-e^{-t})^r}e^{-xt}t^{s-2}(1-e^{-t})dt\\ &= \frac{\Gamma(s-1)}{\Gamma(s)}\left(Z_{k_1,\dots,k_r}(s-1,x)-Z_{k_1,\dots,k_r}(s-1,x+1)\right).\end{aligned}$$

Applying the last theorem to this expression, we get the result.

(2) By $B_n^{(k_1,\dots,k_r)}(x)=(-1)^n Z_{k_1,\dots,k_r}(-n,-x)$, the left hand side becomes

$$(-1)^n\int_0^1 Z_{k_1,\dots,k_r}(-n,-x+w)dw.$$

From the above result (1), we complete the proof. □

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