A STUDY ON THE p-ADIC q-INTEGRAL REPRESENTATION ON $\mathbb{Z}_p$ ASSOCIATED WITH THE WEIGHTED q-BERNSTEIN AND q-BERNOULLI POLYNOMIALS

T. KIM, A. BAYAD, AND Y.-H. KIM

Abstract In this paper, we investigate some interesting properties of the weighted q-Bernstein polynomials related to the weighted q-Bernoulli numbers and polynomials by using p-adic q-integral on $\mathbb{Z}_p$.

1. Introduction and preliminaries

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let q be regarded as either a complex number $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1-q|_p < 1$. In this paper, we define the q-number as $[x]_q = \dfrac{1-q^x}{1-q}$ (see [1-13]).

Let $C[0,1]$ be the set of continuous functions on $[0,1]$. For $\alpha \in \mathbb{N}$ and $n, k \in \mathbb{Z}_+$, the weighted q-Bernstein operator of order n for $f \in C[0,1]$ is defined by

$$\mathbb{B}_{n,q}^{(\alpha)}(f|x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x]_{q^\alpha}^k[1-x]_{q^{-\alpha}}^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}^{(\alpha)}(x,q).$$

Here $B_{k,n}^{(\alpha)}(x,q)$ is called the weighted q-Bernstein polynomials of degree n (see [2,5,6]).

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$, which is called the bosonic q-integral on $\mathbb{Z}_p$, is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \quad \text{(see [10])}. \tag{1}$$

The Carlitz's q-Bernoulli numbers are defined by

$$\beta_{0,q} = 1, \quad q(q\beta+1)^k - \beta_{k,q} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases}$$

2010 Mathematics Subject Classification : 11B68, 41A30.

Key words and phrases : q-Bernstein polynomial, Bernoulli numbers and polynomials, p-adic q-integral.

with the usual convention about replacing β^k by $\beta_{k,q}$ (see [3,9,10]). In [3], Carlitz also defined the expansion of Carlitz's q-Bernoulli numbers as follows:

$$\beta_{0,q}^h = \frac{h}{[h]_q}, \quad q^h(q\beta^h+1)^n - \beta_{n,q}^h = \begin{cases} 1 & \text{if } n=1, \\ 0 & \text{if } n>1, \end{cases}$$

with the usual convention about replacing $(\beta^h)^n$ by $\beta_{n,q}^h$.

The weighted q-Bernoulli numbers are constructed in previous paper [6] as follows: for $\alpha \in \mathbb{N}$,

$$\tilde{\beta}_{0,q}^{(\alpha)} = 1, \quad q(q^\alpha \tilde{\beta}^{(\alpha)}+1)^n - \tilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q} & \text{if } n=1, \\ 0 & \text{if } n>1, \end{cases}$$

with the usual convention about replacing $(\tilde{\beta}^{(\alpha)})^n$ by $\tilde{\beta}_{n,q}^{(\alpha)}$. Let $f_n(x) = f(x+n)$. By the definition (1) of p-adic q-integral on $\mathbb{Z}_p$, we easily get

$$\begin{aligned} qI_q(f_1) &= q \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x+1)q^x, \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x + \lim_{N\to\infty} \frac{f(p^N)q^{p^N} - f(0)}{[p^N]_q} \\ &= \int_{\mathbb{Z}_p} f(x)d\mu_q(x) + (q-1)f(0) + \frac{q-1}{\log q} f'(0), \end{aligned}$$

Continuing this process, we obtain easily the relation

$$q^n \int_{\mathbb{Z}_p} f_n(x)d\mu_q(x) - \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} q^l f'(l),$$

where $n \in \mathbb{N}$ and $f'(l) = \frac{df(l)}{dx}$ (see [6]).

Then by (1), applying to the function $x \to [x]_{q^\alpha}^n$, we can see that

$$\tilde{\beta}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_q(x) = -\frac{n\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{m\alpha+m}[m]_{q^\alpha}^{n-1} + (1-q)\sum_{m=0}^{\infty} q^m [m]_{q^\alpha}^n.$$

The weighted q-Bernoulli polynomials are also defined by the generating function as follows:

$$\begin{aligned} F_q^{(\alpha)}(t,x) &= -t\frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{m\alpha+m} e^{[m+x]_{q^\alpha} t} + (1-q)\sum_{m=0}^{\infty} q^m e^{[m+x]_{q^\alpha} t} \\ &= \sum_{n=0}^{\infty} \tilde{\beta}_{n,q}^{(\alpha)}(x)\frac{t^n}{n!}, \quad \text{(see [6])}. \end{aligned}$$

Thus, we note that

$$\begin{aligned} \tilde{\beta}_{n,q}^{(\alpha)}(x) &= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(\alpha)} \\ &= -\frac{n\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{m\alpha+m}[m+x]_{q^\alpha}^{n-1} + (1-q)\sum_{m=0}^{\infty} q^m [m+x]_{q^\alpha}^n. \end{aligned}$$

From (1) and the previous equalities, we obtain the Witt's formula for the weighted q-Bernoulli polynomials as follows:

$$\tilde{\beta}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n d\mu_q(y) = \sum_{l=0}^{n} \binom{n}{l} q^{\alpha l x} [x]_{q^\alpha}^{n-l} \int_{\mathbb{Z}_p} [y]_{q^\alpha}^l d\mu_q(y).$$

By using (1) and the weighted q-Bernoulli polynomials, we easily get

$$q^n \tilde{\beta}_{m,q}^{(\alpha)}(n) - \tilde{\beta}_{m,q}^{(\alpha)} = (q-1) \sum_{l=0}^{n-1} q^l [l]_{q^\alpha}^m + \frac{m\alpha}{[\alpha]_q} \sum_{l=0}^{n-1} q^{\alpha l + l} [l]_{q^\alpha}^{m-1},$$

where $n, \alpha \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ (see [6]).

In this paper, we consider the weighted q-Bernstein polynomials to express the bosonic q-integral on $\mathbb{Z}_p$ and investigate some properties of the weighted q-Bernstein polynomials associated with the weighted q-Bernoulli polynomials by using the expression of p-adic q-integral on $\mathbb{Z}_p$ of those polynomials.

2. Weighted q-Bernstein polynomials and q-Bernoulli polynomials

In this section, we assume that $\alpha \in \mathbb{N}$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$.

Now we consider the p-adic weighted q-Bernstein operator as follows:

$$\mathbb{B}_{n,q}^{(\alpha)}(f|x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}^{(\alpha)}(x,q). \tag{2}$$

The p-adic q-Bernstein polynomials with weight α of degree n are given by

$$B_{k,n}^{(\alpha)}(x,q) = \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k}, \tag{3}$$

where $x \in \mathbb{Z}_p$, $\alpha \in \mathbb{N}$ and $n, k \in \mathbb{Z}_+$ (see [6,7]). Note that $B_{k,n}^{(\alpha)}(x,q) = B_{n-k,n}^{(\alpha)}(1-x, \frac{1}{q})$. That is, the weighted q-Bernstein polynomials are symmetric.

From the definition of the weighted q-Bernoulli polynomials, we have

$$\tilde{\beta}_{n,q^{-1}}^{(\alpha)}(1-x) = (-1)^n q^{\alpha n} \tilde{\beta}_{n,q}^{(\alpha)}(x). \tag{4}$$

By the definition of p-adic q-integral on $\mathbb{Z}_p$, we get

$$\begin{aligned} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n d\mu_q(x) &= q^{\alpha n}(-1)^n \int_{\mathbb{Z}_p} [-1+x]_{q^\alpha}^n d\mu_q(x) \\ &= \int_{\mathbb{Z}_p} (1-[x]_{q^\alpha})^n d\mu_q(x). \end{aligned} \tag{5}$$

From (4) and (5), we have

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n d\mu_q(x) = \sum_{l=0}^{n} \binom{n}{l} (-1)^l \tilde{\beta}_{l,q}^{(\alpha)} = q^{\alpha n}(-1)^n \tilde{\beta}_{n,q}^{(\alpha)}(-1) = \tilde{\beta}_{n,q}^{(\alpha)}(2). \tag{6}$$

Therefore, we obtain the following lemma.

Lemma 1. *For $n \in \mathbb{Z}_+$, we have*

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n d\mu_q(x) = \sum_{l=0}^{n} \binom{n}{l} (-1)^l \tilde{\beta}_{l,q}^{(\alpha)} = q^{\alpha n}(-1)^n \tilde{\beta}_{n,q}^{(\alpha)}(-1) = \tilde{\beta}_{n,q}^{(\alpha)}(2)$$

and

$$\tilde{\beta}^{(\alpha)}_{n,q^{-1}}(1-x) = (-1)^n q^{\alpha n} \tilde{\beta}^{(\alpha)}_{n,q}(x).$$

By (3), (4) and (5), we get

$$q^2 \tilde{\beta}^{(\alpha)}_{n,q}(2) = n\frac{\alpha}{[\alpha]_q} q^{1+\alpha} + q^2 - q + \tilde{\beta}^{(\alpha)}_{n,q}, \quad \text{if } n > 1.$$

Thus, we have

$$\tilde{\beta}^{(\alpha)}_{n,q}(2) = \frac{1}{q^2}\tilde{\beta}^{(\alpha)}_{n,q} + \frac{n\alpha}{[\alpha]_q} q^{\alpha-1} + 1 - \frac{1}{q}, \quad \text{if } n > 1. \tag{7}$$

Therefore, by (7), we obtain the following proposition.

Proposition 2. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\tilde{\beta}^{(\alpha)}_{n,q}(2) = \frac{1}{q^2}\tilde{\beta}^{(\alpha)}_{n,q} + \frac{n\alpha}{[\alpha]_q} q^{\alpha-1} + 1 - \frac{1}{q}.$$

By using Proposition 2 and Lemma 1, we obtain the following corollary.

Corollary 3. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\int_{\mathbb{Z}_p} [1-x]^n_{q^{-\alpha}} d\mu_q(x) = q^2 \tilde{\beta}^{(\alpha)}_{n,q^{-1}} + \frac{n\alpha}{[\alpha]_q} + 1 - q \tag{8}$$

and

$$\begin{aligned} \int_{\mathbb{Z}_p} [1-x]^n_{q^{-\alpha}} d\mu_q(x) &= \frac{n\alpha}{[\alpha]_q} + 1 - q + q^2 \int_{\mathbb{Z}_p} [x]^n_{q^{-\alpha}} d\mu_{q^{-1}}(x) \\ &= \int_{\mathbb{Z}_p} (1 - [x]_{q^\alpha})^n d\mu_q(x). \end{aligned} \tag{9}$$

Taking the bosonic q-integral on $\mathbb{Z}_p$ for one weighted q-Bernstein polynomials in (2), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} B^{(\alpha)}_{k,n}(x,q) d\mu_q(x) &= \binom{n}{k} \int_{\mathbb{Z}_p} [x]^k_{q^\alpha} [1-x]^{n-k}_{q^{-\alpha}} d\mu_q(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]^{k+l}_{q^\alpha} d\mu_q(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \tilde{\beta}^{(\alpha)}_{k+l,q}. \end{aligned} \tag{10}$$

By the symmetry of q-Bernstein polynomials, we get

$$\begin{aligned} \int_{\mathbb{Z}_p} B^{(\alpha)}_{k,n}(x,q) d\mu_q(x) &= \int_{\mathbb{Z}_p} B^{(\alpha)}_{n-k,n}(1-x, \frac{1}{q}) d\mu_q(x) \\ &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} [1-x]^{n-l}_{q^{-\alpha}} d\mu_q(x). \end{aligned} \tag{11}$$

For $n > k+1$, by (9) and (11), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)d\mu_q(x) &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\left(\frac{n\alpha}{[\alpha]_q}+1-q+q^2\int_{\mathbb{Z}_p}[x]_{q^{-\alpha}}^{n-l}d\mu_{q^{-1}}(x)\right)\\ &= \begin{cases} \frac{n\alpha}{[\alpha]_q}+1-q+q^2\tilde{\beta}_{n,q^{-1}}^{(\alpha)} & \text{if } k=0,\\ \binom{n}{k}q^2\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\tilde{\beta}_{n-l,q^{-1}}^{(\alpha)} & \text{if } k>0.\end{cases}\end{aligned} \tag{12}$$

By comparing the coefficients on the both sides of (10) and (12), we obtain the following theorem.

Theorem 4. *For $n,k\in\mathbb{Z}_+$ with $n>k+1$, we have*

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\tilde{\beta}_{k+l,q}^{(\alpha)} = q^2\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\tilde{\beta}_{n-l,q^{-1}}^{(\alpha)}, \quad \text{if } k\neq 0. \tag{13}$$

In particular, when $k=0$, we have

$$\frac{n\alpha}{[\alpha]_q}+1-q+q^2\tilde{\beta}_{n,q^{-1}}^{(\alpha)} = \sum_{l=0}^{n}\binom{n}{l}(-1)^l\tilde{\beta}_{l,q}^{(\alpha)}. \tag{14}$$

Let $m,n,k\in\mathbb{Z}_+$ with $m+n>2k+1$. Then we see that

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)B_{k,m}^{(\alpha)}(x,q)d\mu_q(x)\\ &= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[x]_{q^{\alpha}}^{2k}[1-x]_{q^{-\alpha}}^{n+m-2k}d\mu_q(x)\\ &= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n+m-l}d\mu_q(x).\\ &= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(\frac{n\alpha}{[\alpha]_q}+1-q+q^2\int_{\mathbb{Z}_p}[x]_{q^{-\alpha}}^{n+m-l}d\mu_{q^{-1}}(x)\right)\\ &= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(\frac{n\alpha}{[\alpha]_q}+1-q+q^2\tilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)}\right).\end{aligned} \tag{15}$$

Therefore, by (15), we obtain the following theorem.

Theorem 5. *For $m,n,k\in\mathbb{Z}_+$ with $m+n>2k+1$, we have*

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)B_{k,m}^{(\alpha)}(x,q)d\mu_q(x)\\ &= \begin{cases} \frac{n\alpha}{[\alpha]_q}+1-q+q^2\tilde{\beta}_{n+m,q^{-1}}^{(\alpha)} & \text{if } k=0,\\ \binom{n}{k}\binom{m}{k}q^2\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\tilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)} & \text{if } k\neq 0.\end{cases}\end{aligned} \tag{16}$$

For $m,n,k\in\mathbb{Z}_+$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)B_{k,m}^{(\alpha)}(x,q)d\mu_q(x)\\
&=\binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[x]_{q^\alpha}^{2k}[1-x]_{q^{-\alpha}}^{n+m-2k}d\mu_q(x)\\
&=\binom{n}{k}\binom{m}{k}\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^l\int_{\mathbb{Z}_p}[x]_{q^\alpha}^{2k+l}d\mu_q(x)\\
&=\binom{n}{k}\binom{m}{k}\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^l\tilde{\beta}_{l+2k,q}^{(\alpha)}.
\end{aligned}\tag{17}$$

Therefore, by (16) and (17), we obtain the following theorem.

Theorem 6. *For $m,n,k\in\mathbb{Z}_+$ with $m+n>2k+1$, we have*

$$\frac{n\alpha}{[\alpha]_q}+1-q+q^2\tilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)}=\sum_{l=0}^{n+m}\binom{n+m}{l}(-1)^l\tilde{\beta}_{l,q}^{(\alpha)}.$$

Furthermore, for $k\neq 0$, we have

$$\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^l\tilde{\beta}_{l+2k,q}^{(\alpha)}=q^2\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\tilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)}.$$

By the induction hypothesis, we obtain the following theorem.

Theorem 7. *For $s\in\mathbb{N}$ and $k,n_1,\dots,n_s\in\mathbb{Z}_+$ with $n_1+n_2+\cdots+n_s>sk+1$, we have*

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{s}B_{k,n_i}^{(\alpha)}(x,q)\right)d\mu_q(x)\\
&=\begin{cases}\frac{n\alpha}{[\alpha]_q}+1-q+q^2\tilde{\beta}_{n_1+\cdots+n_s,q^{-1}}^{(\alpha)} & \text{if } k=0,\\ \left(\prod_{i=1}^{s}\binom{n_i}{k}\right)\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\tilde{\beta}_{n_1+\cdots+n_s-l,q^{-1}}^{(\alpha)} & \text{if } k\neq 0.\end{cases}
\end{aligned}$$

For $s\in\mathbb{N}$, let $k,n_1,\dots,n_s\in\mathbb{Z}_+$ with $n_1+n_2+\cdots+n_s>sk+1$. Then we show that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{s}B_{k,n_i}^{(\alpha)}(x,q)\right)d\mu_q(x)\\
&=\left(\prod_{i=1}^{s}\binom{n_i}{k}\right)\sum_{l=0}^{n_1+\cdots+n_s-sk}\binom{n_1+\cdots+n_s-sk}{l}(-1)^l\tilde{\beta}_{l+sk,q}^{(\alpha)}.
\end{aligned}\tag{18}$$

Therefore, by Theorem 7 and (18), we obtain the following theorem.

Theorem 8. *For $s\in\mathbb{N}$, let $k,n_1,\dots,n_s\in\mathbb{Z}_+$ with $n_1+n_2+\cdots+n_s>sk+1$. Then we see that for $k=0$*

$$\sum_{l=0}^{n_1+\cdots+n_s}\binom{n_1+\cdots+n_s}{l}(-1)^l\tilde{\beta}_{l,q}^{(\alpha)}=\frac{n\alpha}{[\alpha]_q}+1-q+q^2\tilde{\beta}_{n_1+\cdots+n_s,q^{-1}}^{(\alpha)}.$$

For $k \neq 0$, we have

$$\sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \tilde{\beta}^{(\alpha)}_{n_1+\cdots+n_s-l,q^{-1}} = \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l \tilde{\beta}^{(\alpha)}_{l+sk,q}.$$

References

[1] M. Acikgoz, Y. Simsek, *On multiple interpolation function of the Nörlund-type q-Euler polynomials*, Abst. Appl. Anal. **2009** (2009), Article ID 382574, 14 pages.

[2] A. Bayad, J. Choi, T. Kim, Y.-H. Kim, L.C. Jang, *q-extension of Bernstein polynomials with weighted* (α, β), J. Comput. Anal. Appl. (submitted)

[3] L. Carlitz, *Expansion of q-Bernoulli numbers*, Duke Math. J. **25** (1958), 355-364.

[4] A. S. Hegazi, M. Mansour *A note on q-Bernoulli numbers and polynomials*, J. Nonlinear Math. Phys. **13** (2006), 9-18.

[5] L. C. Jang, W.-J. Kim and Y. Simsek, *A study on the p-adic integral representation on $\mathbb{Z}_p$ associated with Bernstein and Bernoulli polynomials, Advan. Diff. Eqn.*, 2010(2010), Article ID 163217, 6 pages.

[6] T. Kim, *On the weighted q-Bernoulli numbers and polynomials*, arXiv 1011.5305.

[7] T. Kim, *A note on q-Bernstein polynomials*, Russ. J. Math. Phys. (accepted).

[8] T. Kim, *Barnes-type multiple q-zeta functions and q-Euler polynomials*, J. Physics A:Math. Theor. **43** (2010), 255201, 11pp.

[9] T. Kim, *q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients*, Russ. J. Math. Phys. **15** (2008), 51–57.

[10] T. Kim, *On a q-analogue of the p-adic log gamma functions and related integrals*, J. Number Theory **76** (1999), 320-329.

[11] B. A. Kupershmidt, *Reflection symmetries of q-Bernoulli polynomials*, J. Nonlinear Math. Phys. **12** (2005), 412-422.

[12] H. Ozden, I. N. Cangul, Y. Simsek, *Remarks on the q-Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. **18** (2009), 41-48.

[13] S.-H. Rim, J.-H. Jin, E.-J. Moon, S.-J. Lee, *On multiple interpolation function of the q-Genocchi polynomials*, J. Inequl. Appl. 2010(2010), Article ID 351419, 13 pages.

Taekyun Kim. Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea,
E-mail address: tkkim@kw.ac.kr

Abdelmejid Bayad. Département de mathématiques, Université d'Evry Val d'Essonne, Bd. F. Mitterrand, 91025 Evry Cedex, France,
E-mail address: abayad@maths.univ-evry.fr

Young-Hee Kim. Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea,
E-mail address: yhkim@kw.ac.kr