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HECKE OPERATORS AND GENERALIZED BERNOULLI-EULER POLYNOMIALS

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Abstract

We introduce Hecke operators on the space of complex polynomials. We study their properties. We deduce that the generalized Bernoulli-Euler polynomials are eigenpolynomials for our Hecke operators. From this study, we obtain a new approach to the theory of Bernoulli and Euler polynomials.

1. Introduction and Main Results

Euler [1] defined the Bernoulli and Euler polynomials (resp., $B(x)$, $E(x)$) by

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$$\sum_{n\geq 0} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1}, \ |t| < 2\pi, \tag{1.1}$$

$$\sum_{n\geq 0} E_n(x)\frac{t^n}{n!} = \frac{2e^{xt}}{e^t + 1}, \ |t| < \pi. \tag{1.2}$$

Raabe [5] discovered their multiplication theorem

$$\sum_{k=0}^{m-1} B_n\left(\frac{x+k}{m}\right) = m^{1-n}B_n(x), \ \forall m \geq 1, \ \forall n \in \mathbb{N}, \tag{1.3}$$

$$\sum_{k=0}^{m-1} (-1)^k E_n\left(\frac{x+k}{m}\right) = m^{-n}E_n(x), \ \forall m \text{ odd} \geq 1, \ \forall n \in \mathbb{N}. \tag{1.4}$$

Now, we fix the following notations.

Let $a, N \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, ξ_N be a primitive root of unity of order N. We consider the functions $\chi_{a,N} : \mathbb{N} \to \mathbb{C}$ defined by

$$\chi_{a,N}(k) = \begin{cases} \xi_N^k, & N \geq 2 \\ \frac{1}{a}, & N = 1 \end{cases}, \ 0 \leq k \leq a-1.$$

We define on the space $\mathbb{C}[x]$ the following operators

$$T_{\chi_{a,N}}(P)(x) := \sum_{k=0}^{a-1} \chi_{a,N}(k) P\left(\frac{x+k}{a}\right), \tag{1.5}$$

$$T_N := \sum_{a \equiv 1 (\text{mod } N)} T_{\chi_{a,N}}. \tag{1.6}$$

It is easy to see, these are $\mathbb{C}$-linear operators acting on the space of polynomials $\mathbb{C}[x]$ and preserve degrees. We call them, respectively, "partial" and "total Hecke" operators on $\mathbb{C}[x]$.

In this paper, we establish the interesting arithmetical properties of our Hecke operators $T_{\chi_{a,N}}$ and T_N.

Our main results are:

Theorem 1. *Let* $a,\ N \in \mathbb{N}^*$ *such that* $a \equiv 1 (\bmod N)$. *Then,*

(1) *there exists unique sequence of monic polynomials* $(P_{n,N})_{n\in\mathbb{N}}$ *in* $\mathbb{Q}(\xi_N)[x]$, *with* $\deg P_{n,N} = n$ *such that*

$$T_{\chi_{a,N}}(P_{n,N})(x) = a^{-n} P_{n,N}(x). \tag{1.7}$$

(2) (*Eigenpolynomials*): *Polynomials* $P_{n,N}(x)$ *are eigenfunctions for the operators* T_N *with eigenvalues* $N^{-n}\zeta(-n, \frac{1}{N})$, *that is,*

$$T_N(P_{n,N})(x) = N^{-n}\zeta\left(-n, \frac{1}{N}\right) \cdot P_{n,N}(x),$$

where $\zeta(s, x) = \sum_{k\geq 0} \frac{1}{(x+k)^s}$ *is the Hurwitz zeta function.*

(3) (*Generating functions*): *Generating functions of* $P_{n,N}(x)$ *are*

$$\sum_{n=0}^{\infty} P_{n,N}(x)\frac{t^n}{n!} = \begin{cases} \dfrac{te^{tx}}{e^t - 1}, & N = 1, \\ \dfrac{(\xi_N - 1)e^{tx}}{\xi_N e^t - 1}, & N \geq 2. \end{cases}$$

The polynomials $P_{n,N}(x)$, we call them generalized Bernoulli-Euler polynomials.

Our Theorem 1 implies:

Corollary 2. *For* $N = 1, 2,$ *we obtain*

(1) $P_{n,1}(x) = B_n(x),\ P_{n,2}(x) = E_n(x),$

(2) *Raabe formulae* 1.3 *and* 1.4.

The identity 1.7 is a Raabe formula for the generalized Bernoulli-Euler polynomials $P_{n,N}$.

We now briefly review the content of the individual sections of our paper.

Section 2 contains various properties of our partial Hecke operators $T_{\chi_{a,N}}$. In Section 3, we show the Theorem 1.

Historical remark. It is well-known that the Hecke's operators theory is developed on the space of modular and cusps forms, see [2, 6]. In some sense, our paper can be viewed as an extension of this Hecke's operators theory to the space of complex polynomials.

2. Hecke Operators and their Properties

For this section, we fix the following notations.

Given $a, N \in \mathbb{N}^*$, ξ_N is a primitive root of unity of order N. Let $\chi_{a,N} : \mathbb{Z} \to \mathbb{C}$ given by

$$\chi_{a,N}(k) = \begin{cases} \xi_N^k, & N \geq 2 \\ \dfrac{1}{a}, & N = 1 \end{cases}, \ 0 \leq k \leq a-1.$$

We consider the $\mathbb{C}$-linear operator, acting on the complex space of polynomials $\mathbb{C}[x]$ as follows

$$T_{\chi_{a,N}} : \mathbb{C}[x] \to \mathbb{C}[x]$$

$$P(x) \to T_{\chi_a}(P)(x) = \sum_{k=0}^{a-1} \chi_{a,N}(k) \cdot P\left(\frac{x+k}{a}\right). \tag{2.1}$$

Our purpose is to give all monic polynomials satisfying the functional equation

$$T_{\chi_a}(P)(x) = a^{-n} P(x), \ P(x) \in \mathbb{C}[x], \text{ for any } a \equiv 1 (\text{mod } N). \tag{2.2}$$

Now, we have the following key lemma.

Lemma 1 (Hecke operators $T_{\chi_{a,N}}$). *For any* $a, N \in \mathbb{N}^*$ *such that* $a \equiv 1 (\text{mod } N)$, *we have the properties*

(i) $T_{\chi_{a,N}}$ *preserves the degree in* $\mathbb{C}[x]$.

(ii) *We have*

$$T_{\chi_{a,N}}(x^m) = \begin{cases} 1, & m = 0, \\ a^{-m}x^m + a^{-m}\sum_{\nu=0}^{m-1}\binom{m}{v}S_{m-\nu}(\chi_{a,N})\cdot x^v, & m \geq 1, \end{cases}$$

where $S_{m-\nu}(\chi_{a,N}) = \sum_{k=0}^{a-1}\chi_{a,N}(k)\cdot k^{m-v}$.

(iii) *For any* $m \in \mathbb{N}^*$, *let* $\beta_m = (1, x, x^2, \ldots, x^m)$ *be the canonical* $\mathbb{C}$*- basis of*

$$\mathbb{C}_m[x] := \{P(x) \in \mathbb{C}[x] : \textit{degree of } P(x) \leq m\}.$$

Then, the matrix $\mathbf{M}_{\beta_m}\left(T_{\chi_{a,N}}\right)$ *corresponding to the operator* $T_{\chi_{a,N}}$ (*restricted to* $\mathbb{C}_m[x]$) *in the basis* β_m *is given by*

$$\mathbf{M}_{\beta_m}\left(T_{\chi_{a,N}}\right) = \begin{pmatrix} S_0 & a^{-1}S_1 & a^{-2}S_2 & \cdots & a^{-m}S_m \\ 0 & a^{-1}S_0 & a^{-2}S_1 & \cdots & \binom{m}{1}S_{m-1} \\ 0 & 0 & a^{-2}S_0 & \cdots & \binom{m}{2}S_{m-2} \\ 0 & 0 & 0 & \cdots & \binom{m}{3}S_{m-3} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & a^{-m}S_0 \end{pmatrix}_{m\times m}, \tag{2.3}$$

where $S_l = \sum_{k=0}^{a-1}\chi_{a,N}(k)\cdot k^l$, $\forall 0 \leq l \leq m-1$.

(iv) *Let* $a, b \geq 1$, *such that* $a \equiv b \equiv 1 (\text{mod } N)$, *then*

$$T_{\chi_{a,N}} T_{\chi_{b,N}} = T_{\chi_{b,N}} T_{\chi_{a,N}}.$$

Then, there are polynomials $P_{n,N}$ *with degree* n *in* x *satisfying the functional equation* (2.2). *Furthermore, for a given integer* n, *there is only one monic polynomial* $P_{n,N}$ *of degree* n *satisfying* (2.2).

Proof of Lemma 1. If there exist any $(P_{n,N})_{n\in\mathbb{N}}$ such that (2.2) holds, then (i) is obvious since $\sum_{k=0}^{a-1} \chi_{a,N}(k) = 1$.

Since $\sum_{k=0}^{a-1} \chi_{a,N}(k) = 1$, $T_{\chi_{a,N}}(1) = 1$ for $m = 0$. Also by induction in (2.2), we obtain

$$T_{\chi_{a,N}}(x^m) = a^{-m} \sum_{v=0}^{m} \left\{ \binom{m}{v} \sum_{k=0}^{a-1} \chi_{a,N}(k) \cdot k^{m-v} \right\} x^v, \qquad m \geq 1,$$

and therefore (ii) is satisfied.

If we write operator $T_{\chi_{a,N}}$ for the canonical basis $\beta_m = (1, x, x^2, \ldots, x^m)$, $m \in \mathbb{N}^*$, then from (ii), $\mathbf{M}_{\beta_m}(T_{\chi_{a,N}})$ is given by (2.3).

For $a = 1$, the operator $T_{\chi_{a,N}}$ is diagonal and for all $a \geq 2$, $\mathbf{M}_{\beta_m}(T_{\chi_{a,N}})$ is a diagonalizable matrix and has $m+1$ distinct eigenvalues $S_0, a^{-1}S_0, a^{-2}S_0, \ldots, a^{-m}S_0$. Then, we know that there exist any $(P_{n,N})_{n\in\mathbb{N}}$ such that (2.2) holds.

Since $\mathbf{M}_{\beta_m}(T_{\chi_{a,N}})$ is a diagonalizable matrix for all $a \geq 1$, there exist $\tilde{\beta}_m$ is a $\mathbb{C}$-basis of $\mathbb{C}_m[x]$ such that

$$\mathbf{M}_{\widetilde{\beta}_m}\left(T_{\chi_{a,N}}\right) = \begin{pmatrix} S_0 & 0 & 0 & \cdots & 0 \\ 0 & a^{-1}S_0 & 0 & \cdots & 0 \\ 0 & 0 & a^{-2}S_0 & \cdots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdots & a^{-m}S_0 \end{pmatrix}$$

is a diagonal matrix.

Thanks to linear algebra, it is well-known that: two $n \times n$ diagonalizable matrices A and B (in $\mathbb{C}$) are simultaneously diagonalizable (i.e., diagonal in the same basis), if and only if $AB = BA$. In fact, the linear operators $T_{\chi_{a,N}}$ and $T_{\chi_{b,N}}$ are diagonal in the canonical basis of $\mathbb{C}[x]$, then are simultaneously diagonalizable, if and only if $T_{\chi_{a,N}} T_{\chi_{b,N}} = T_{\chi_{b,N}} T_{\chi_{a,N}}$.

Now, to prove the existence of $(P_{n,N})_{n\in\mathbb{N}}$ in $\mathbb{C}[x]$, we must show that $T_{\chi_{a,N}} T_{\chi_{b,N}} = T_{\chi_{b,N}} T_{\chi_{a,N}}$ for all $a \equiv 1 (\text{mod } N)$.

From now, we assume $a \equiv b \equiv 1 (\text{mod } N)$.

For $N = 1$, it is obvious that $T_{\chi_{a,N}} T_{\chi_{b,N}} = T_{\chi_{b,N}} T_{\chi_{a,N}}$.

For $N \geq 2$, we can write

$$\chi_{a,N}(k_1).\chi_{b,N}(k_2) = \xi_N^{k_1}\xi_N^{k_2} = \xi_N^{k_1 b + k_2} = \xi_N^{k_1 + k_2 a}.$$

Then, $\chi_{a,N}(k_1).\chi_{b,N}(k_2) = \chi_{ab,N}(k_1 b + k_2) = \chi_{ab,N}(k_1 + k_2 a)$ and for any $P \in \mathbb{C}[x]$,

$$T_{\chi_{a,N}} T_{\chi_{b,N}}(P)(x) = \sum_{\substack{0 \leq k_1 \leq a-1 \\ 0 \leq k_2 \leq b-1}} \chi_{a,N}(k_1).\chi_{b,N}(k_2) \cdot P\left(\frac{x + k_1 b + k_2}{ab}\right)$$

$$= \sum_{k=0}^{ab-1} \chi_{ab,N}(k) \cdot P\left(\frac{x+k}{ab}\right), \quad k = k_1 b + k_2$$

$$= T_{\chi_{ab,N}}(P) = T_{\chi_{ba,N}}(P) = T_{\chi_{a,N}} T_{\chi_{b,N}}(P).$$

Therefore, for fixed $N \in \mathbb{N}^*$, we deduce the existence of polynomials $P_{n,N} \in \mathbb{C}[x]$ with degree n such that

$$T_{\chi_{a,N}}(P_{n,N})(x) = a^{-n} P_{n,N}(x), \ \forall n \in \mathbb{N}, \ \forall a \equiv 1 (\text{mod } N).$$

To show $P_{n,N} \in \mathbb{Q}(\xi_N)[x]$, define any monic polynomial $(P_{n,N})_{n \in \mathbb{N}}$ by

$$P_{n,N}(x) = \sum_{v=0}^{n} b_v x^{n-v}, \ b_0 = 1,$$

and $P_{n,N} \in \mathbb{C}[x]$. Then,

$$a^{-n} P_{n,N}(x) = T_{\chi_{a,N}}(P_{n,N})(x) = \sum_{k=0}^{a-1} \chi_{a,N}(k) \cdot P_{n,N}\left(\frac{x+k}{a}\right), \tag{2.4}$$

and it follows that

$$(a^r - 1) b_r = -\sum_{v=0}^{r-1} \binom{n-v}{r-v} \cdot b_v \cdot a^v \cdot S_{r-v}(\chi_{a,N}).$$

Since $S_{r-v}(\chi_a) = \sum_{k=0}^{a-1} \chi_{a,N}(k) \cdot k^{r-v}$, we obtain $S_{r-v}(\chi_{a,N}) \in \mathbb{Z}[\xi_N]$ and therefore, $b_r \in \mathbb{Q}(\xi_N)$.

Finally, we must now to observe the uniqueness of $(P_{n,N})_{n \in \mathbb{N}}$.

If possible, let $P_{n,N}(x)$ and $Q_{n,N}(x)$ be two different monic polynomials of degree n satisfying (2.4).

Suppose that $P_{n,N}(x) - Q_{n,N}(x) = \Delta_d(x) = A_0 x^d + A_1 x^{d-1} + \ldots$, where $d < n$ and $A_0 \neq 0$. Subtracting in (2.4), we get

$$\sum_{k=0}^{a-1} \chi_{a,N}(k) \cdot \Delta_d\left(\frac{x+k}{a}\right) = a^{-n}(A_0 x^d + A_1 x^{d-1} + \ldots).$$

Identifying the coefficients of x^d on both sides, we have $A_0 = a^{-n} A_0$, but this contradicts our stipulations that $A_0 \neq 0, d < n$, and $a \geq 2$. Hence, it follows (iv).

Therefore, we obtain our Lemma 1.

3. Proofs of Main Results

Let $a \in \mathbb{N}$, $N \in \mathbb{N}^*$. We recall that the Hecke operator T_N is given by

$$T_N = \sum_{\substack{a \equiv 1 (\mathrm{mod}\, N) \\ a \geq 0}} T_{\chi_{a,N}}, \ \forall \chi_{a,N},$$

i.e.,

$$T_N(P)(x) = \sum_{\substack{a \equiv 1 (\mathrm{mod}\, N) \\ a \geq 0}} \sum_{k=0}^{a-1} \chi_{a,N}(k) \cdot P\left(\frac{x+k}{a}\right).$$

For every $n \geq 0$ and $N \geq 1$, from the Lemma 1, there are unique monic polynomials $P_{n,N}$ in $\mathbb{Q}(\xi_N)[x]$ with deg $P_{n,N} = n$ such that

$$T_{\chi_{a,N}}(P_{n,N})(x) = a^{-n} P_{n,N}(x), \ \forall a \equiv 1 (\mathrm{mod}\, N).$$

It implies the part (1) of the Theorem 1. ■

Now, we prove the part (2) of the Theorem 1.

It is easy to see that

$$T_N(P_{n,N})(x) = \sum_{\substack{a \equiv 1 (\mathrm{mod}\, N) \\ a \geq 0}} T_{\chi_{a,N}}(P_{n,N})(x)$$

$$= \left(\sum_{\substack{a \equiv 1 (\mathrm{mod}\, N) \\ a \geq 0}} a^{-n} \right) P_{n,N}(x),$$

and we write $\alpha = 1 + kN$,

$$T_N(P_{n,N})(x) = \sum_{k\geq 0}(1+kN)^{-n}P_{n,N}(x)$$

$$= N^{-n}\sum_{k\geq 0}\left(k+\frac{1}{N}\right)^{-n}P_{n,N}(x)$$

$$= N^{-n}\zeta(-n,\frac{1}{N})P_{n,N}(x). \qquad \blacksquare$$

We prove now, the generating function of $P_{n,N}(x)$ is given by

$$\sum_{n=0}^{\infty}P_{n,N}(x)\frac{t^n}{n!} = \begin{cases}\dfrac{te^{xt}}{e^t-1}, & N=1,\\ \dfrac{(\xi_N-1)e^{xt}}{\xi_N e^t-1}, & N\geq 2.\end{cases} \tag{3.1}$$

For that, we introduce the function

$$F_N(x,t) = \begin{cases}\dfrac{te^{xt}}{e^t-1}, & N=1,\\ \dfrac{(\xi_N-1)e^{xt}}{\xi_N e^t-1}, & N\geq 2.\end{cases}$$

Let us expand the function $F_N(x,t)$ in powers of x and t and collect the coefficients as a polynomial $\Psi_{n,N}(x)$ of degree n in x:

$$F_N(x,t) = \sum_{n=0}^{\infty}\Psi_{n,N}(x)\frac{t^n}{n!}. \tag{3.2}$$

Suppose $\Psi_{n,N}(x) = A_0^{(n,N)}x^n + A_1^{(n,N)}x^{n-1} + \ldots + A_n^{(n,N)}$.

If we replace x by $\frac{1}{y}$ and t by $t\cdot y$ in (3.2), we get

$$F_N(\frac{1}{y}, t\cdot y) = \begin{cases}\dfrac{t\cdot y\cdot e^t}{e^{ty}-1}, & N=1\\ \dfrac{(\xi_N-1)e^t}{\xi_N e^{ty}-1}, & N\geq 2\end{cases}$$

$$= \sum_{n=0}^{\infty} y^n . \Psi_{n,N}(x) \frac{t^n}{n!}.$$

Letting y tends to zero, we obtain $e^t = \sum_{n=0}^{\infty} A_0^{(n,N)} \frac{t^n}{n!}$.

Hence, for all $n \in \mathbb{N}$, we have $A_0^{(n,N)} = 1$, and hence, $\Psi_{n,N}(x)$ is monic.

If in (3.2), we replace x by $x + \frac{k}{a}$ and multiply by ξ_N^k and sum over k, we get

$$\sum_{k=0}^{a-1} \xi_N^k \sum_{n=0}^{\infty} \Psi_{n,N}\left(x + \frac{k}{a}\right) \frac{t^n}{n!} = \sum_{k=0}^{a-1} F_N\left(x + \frac{k}{a}\right) \cdot \xi_N^k$$

$$= \begin{cases} \dfrac{\frac{t}{a} e^{xt}}{e^{\frac{t}{a}} - 1}, & N = 1, \\ \dfrac{(\xi_N - 1)e^{xt}}{\xi_N e^t - 1} \cdot \dfrac{e^t . \xi_N^a - 1}{e^{\frac{t}{a}} \xi_N - 1}, & N \geq 2, \end{cases}$$

and for $a \equiv 1 (\text{mod } N)$, we write $a = kN + 1, k \in \mathbb{Z}$, it follows that

$$\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{a-1} \xi_N^k \cdot \Psi_{n,N}\left(x + \frac{k}{a}\right) \right\} \frac{t^n}{n!} = \begin{cases} \dfrac{\frac{t}{a} e^{xt}}{e^{\frac{t}{a}} - 1}, & N = 1, \\ \dfrac{(\xi_N - 1)e^{xt}}{e^{\frac{t}{a}} \xi_N - 1}, & N \geq 2. \end{cases} \tag{3.3}$$

If, instead, we replace in (3.2) x by ax and t by $\frac{t}{a}$, we obtain

$$F_N\left(ax, \frac{t}{a}\right) = \sum_{n=0}^{\infty} \frac{1}{a^n} \cdot \Psi_{n,N}(ax) \cdot \frac{t^n}{n!} = \begin{cases} \dfrac{\frac{t}{a} e^{xt}}{e^{\frac{t}{a}} - 1}, & N = 1, \\ \dfrac{(\xi_N - 1)e^{xt}}{e^{\frac{t}{a}} \xi_N - 1}, & N \geq 2. \end{cases} \tag{3.4}$$

Identifying the coefficients of $\frac{t^n}{n!}$ in (3.3) and (3.4), we conclude that $\Psi_{n,N}(x)$ satisfies the functional equation (2.2)

$$\sum_{k=0}^{a-1} \chi_{a,N}(k) \cdot \Psi_{n,N}\left(x + \frac{k}{a}\right) = a^{-n} \cdot \Psi_{n,N}(x).$$

Since $\Psi_{n,N}(x)$ is monic, by Theorem 1, we have $\Psi_{n,N}(x) = P_{n,N}(x)$ and 3.1 is proved. ■

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