NEW MEANS AND ITS PROPERTIES

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Abstract. The object of this paper is to introduce three new means on the basis of proportions and its dual forms, study certain properties, monotonicities and new inequalities involving them. Further, we defined weighted three new means and its dual form, its properties are stated and deduced the ten Neo-Pythagorean means, weighted Oscillatory mean, weighted r^{th} Oscillatory mean, also some familiar means and various other means.

1. Introduction

Quite recently, V.Lokesha and etal. [2, 8] defined the Oscillatory mean, the rth Oscillatory mean and its dual forms, studied the properties and monotonicities and obtained inequalities involving Power mean, Identric mean and Logarithmic mean. Further obtained the best possible values for these inequalities. In [10], G.Toader and S.Toader explored inequalities among Greek means. Authors obtained the relation between Greek means and various other means in [4], this motivates us to define the three new means and its dual forms and also their respective weighted means and its dual forms on the basis of proportions.

This article devoted to study the properties, monotonic results and new inequality of new means. Further deduced the expressions for weighted Oscillatory mean, r^{th} Oscillatory mean and its dual forms, related results.

For positive numbers a and b, Logarithmic mean, Heron mean, Power mean and Identric mean are as follows.

$$L = L(a,b) = \begin{cases} \frac{a-b}{\ln a - \ln b} & , \quad a \neq b \\ a & , \quad a = b \end{cases} \tag{1.1}$$

$$H_e = H_e(a,b) = \frac{a + \sqrt{ab} + b}{3} \tag{1.2}$$

$$M_r = M_r(a,b) = \begin{cases} \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}} & , \quad r \neq 0 \\ \sqrt{ab} & , \quad r = 0 \end{cases} \tag{1.3}$$

and

$$I = I(a,b) = \begin{cases} e^{\left(\frac{a \ln a - b \ln b}{a-b} - 1\right)} & , \quad a \neq b \\ a & , \quad a = b \end{cases} \tag{1.4}$$

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All means defined above have been studied extensively by many researchers. Many remarkable inequalities and identities have been established. For more details the interested reader is referred to [1, 5, 6, 7, 9].

We recall some Definitions which are essential for the paper.

Definition 1.1. [8] *For a,b > 0 and $\alpha \in (0,1)$, Oscillatory mean and its dual form are as follows;*

$$O(a,b;\alpha) = \alpha G(a,b) + (1-\alpha)A(a,b) \tag{1.5}$$

$$O^{(d)}(a,b;\alpha) = G(a,b)^{\alpha} A(a,b)^{1-\alpha} \tag{1.6}$$

Definition 1.2. [2] *For a,b > 0 and $\alpha \in (0,1)$, r^{th} Oscillatory mean and its dual form are as follows;*

$$O(a,b;\alpha,r) = \alpha M_r(a,b) + (1-\alpha)A(a,b) \tag{1.7}$$

$$O^{(d)}(a,b;\alpha,r) = M_r(a,b)^{\alpha} A(a,b)^{1-\alpha} \tag{1.8}$$

Definition 1.3. [10] *For a,b > 0, a mean $M(a,b)$ of a and b is defined as the function $M(a,b) : R_+^2 \to R_+$, which has the property that*

$$a \wedge b \leq M(a,b) \leq a \vee b \tag{1.9}$$

where $a \wedge b = \ Min\ (a,b)$ and $a \vee b = \ Max\ (a,b)$

For positive numbers a,m,b>0, then m is said to be mean of (a,b) is defined on the basis of proportions. The following table shows ten Greek means of which four are un-named.

Sl.No.	Proportion	Name of Mean	Notation
1	$\frac{a-m}{m-b} = \frac{a}{a}$	Arithmetic Mean	$F_1 = A = \frac{a+b}{2}$
2	$\frac{a-m}{m-b} = \frac{m}{b} = \frac{a}{m}$	Geometric Mean	$F_2 = G = \sqrt{ab}$
3	$\frac{a-m}{m-b} = \frac{a}{b}$	Harmonic Mean	$F_3 = H = \frac{2ab}{a+b}$
4	$\frac{a-m}{m-b} = \frac{b}{a}$	Contra harmonic Mean	$F_4 = C = \frac{a^2+b^2}{a+b}$
5	$\frac{a-m}{m-b} = \frac{b}{m}$	first kind of contra geometric Mean	$F_5 = \frac{a-b+\sqrt{(a-b)^2+4b^2}}{2}$
6	$\frac{a-m}{m-b} = \frac{m}{a}$	second kind of contra geometric Mean	$F_6 = \frac{b-a+\sqrt{(a-b)^2+4a^2}}{2}$
7	$\frac{a-m}{a-b} = \frac{b}{a}$	un named Mean	$F_7 = \frac{a^2-ab+b^2}{a}$
8	$\frac{a-m}{a-b} = \frac{m}{a}$	un named Mean	$F_8 = \frac{a^2}{2a-b}$
9	$\frac{m-b}{a-b} = \frac{b}{a}$	un named Mean	$F_9 = \frac{b(2a-b)}{a}$
10	$\frac{m-b}{a-b} = \frac{b}{m}$	un named Mean	$F_{10} = \frac{b+\sqrt{b(4a-3b)}}{2}$

Table 1

2. Definitions

In this section, we introduced the new means, weighted new means and their dual forms.

Let $M(a,b)$ and $N(a,b)$ are means of $(a\ ,\ b)$. $M(a,b)$ and $N(a,b)$ are any one of the well known means or one of the ten Greek means such that $M(a,b) \neq N(a,b)$.

Definition 2.1. *For $a > m > b > 0$, choose $\alpha = \frac{a}{b}$ or $\frac{b}{m}$ or $\frac{m}{a}$, then $\alpha \in (0,1)$.*

In table 1, the proportion 9 motivate us to consider the following proportion as,

$$\alpha = \frac{m - N(a,b)}{M(a,b) - N(a,b)} \tag{2.1}$$

On rearranging and replace m by R, that is

$$R = R(a,b;\alpha) = \alpha M(a,b) + (1-\alpha)N(a,b) \tag{2.2}$$

In the proportion (2.1), replace a by lna, b by lnb, m by lnm, $M(a,b)$ by $lnM(a,b)$ and $N(a,b)$ by $lnN(a,b)$, on simplifying and replacing m by $R^{(d)}$, that is

$$R^{(d)} = R^{(d)}(a,b;\alpha) = M(a,b)^{\alpha} N(a,b)^{1-\alpha} \tag{2.3}$$

we call $R^{(d)}(a,b;\alpha)$ is the dual form of $R(a,b;\alpha)$.

In table 1, the proportion 8 motivate us to consider the following proportion as,

$$\alpha = \frac{M(a,b) - m}{M(a,b) - N(a,b)} \tag{2.4}$$

The mean $S(a,b;\alpha)$ and its dual $S^{(d)}(a,b;\alpha)$ as below.

$$S = S(a,b;\alpha) = \alpha N(a,b) + (1-\alpha)M(a,b) \tag{2.5}$$

and

$$S^{(d)} = S^{(d)}(a,b;\alpha) = N(a,b)^{\alpha} M(a,b)^{1-\alpha} \tag{2.6}$$

Also in table 1, the proportion 6 motivate us to consider the following proportion as,

$$\alpha = \frac{M(a,b) - m}{m - N(a,b)} \tag{2.7}$$

The mean $T(a,b;\alpha)$ and its dual $T^{(d)}(a,b;\alpha)$ as below.

$$T = T(a,b;\alpha) = \frac{M(a,b) + \alpha N(a,b)}{1+\alpha} \tag{2.8}$$

and

$$T^{(d)} = T^{(d)}(a,b;\alpha) = M(a,b)^{\frac{1}{1+\alpha}} N(a,b)^{\frac{\alpha}{1+\alpha}} \tag{2.9}$$

respectively $R(a,b;\alpha)$, $S(a,b;\alpha)$, $T(a,b;\alpha)$ are the three new means and their dual forms are $R^{(d)}(a,b;\alpha)$, $S^{(d)}(a,b;\alpha)$, $T^{(d)}(a,b;\alpha)$.

Now we define weighted means and their dual forms as follows.

Definition 2.2. *For $a > m > b > 0$, choose $\alpha = \frac{a}{b}$ or $\frac{b}{m}$ or $\frac{m}{a}$, then $\alpha \in (0,1)$. If $\lambda \in (0,1)$, then*

$$R = R(a,b;\alpha,\lambda) = \frac{(1-\lambda)\alpha M(a,b) + (\lambda - (1-\lambda)\alpha)N(a,b)}{\lambda} \tag{2.10}$$

and

$$R^{(d)} = R^{(d)}(a,b;\alpha,\lambda) = M(a,b)^{(\frac{1-\lambda}{\lambda})\alpha} N(a,b)^{\frac{\lambda-(1-\lambda)\alpha}{\lambda}} \tag{2.11}$$

$$S = S(a,b;\alpha,\lambda) = \frac{(1-\lambda)\alpha N(a,b) + (\lambda - (1-\lambda)\alpha)M(a,b)}{\lambda} \tag{2.12}$$

and

$$S^{(d)} = S^{(d)}(a,b;\alpha,\lambda) = N(a,b)^{(\frac{1-\lambda}{\lambda})\alpha} M(a,b)^{\frac{\lambda-(1-\lambda)\alpha}{\lambda}} \tag{2.13}$$

$$T = T(a,b;\alpha,\lambda) = \frac{\lambda}{\lambda+(1-\lambda)} M(a,b) + \frac{(1-\lambda)\alpha}{\lambda+(1-\lambda)\alpha} N(a,b) \tag{2.14}$$

and

$$T^{(d)} = T^{(d)}(a,b;\alpha,\lambda) = M(a,b)^{\frac{\lambda}{\lambda+(1-\lambda)}} N(a,b)^{\frac{(1-\lambda)\alpha}{\lambda+(1-\lambda)\alpha}} \tag{2.15}$$

eqs (2.10)-(2.15) are respectively the three weighted means $R(a,b;\alpha,\lambda)$, $S(a,b;\alpha,\lambda)$, $T(a,b;\alpha,\lambda)$ and their dual forms are $R^{(d)}(a,b;\alpha,\lambda)$, $S^{(d)}(a,b;\alpha,\lambda)$, $T^{(d)}(a,b;\alpha,\lambda)$.

Remark 1: If $\lambda = \frac{1}{2}$ in eqs (2.10)-(2.15) of Definition 2.2, the weighted means becomes eqs (2.1)-(2.9).

3. Properties and Monotonic results

In this section using Definition 1.3, we verify that eqs (2.2),(2.3),(2.5),(2.6) (2.8),(2.9) and (2.10)-(2.15)are all means.

Theorem 3.1. *For $a > b > 0$ and $\alpha, \lambda \in (0,1)$, then $R(a,b;\alpha,\lambda)$ and its dual $R^{(d)}(a,b;\alpha,\lambda)$ are means.*

Proof. Let $M(a,b)$ and $N(a,b)$ are the means of (a,b) and for $a > b > 0$, from Definition 1.3 of mean, we have $b \le M(a,b) \le a$ and $b \le N(a,b) \le a$.

$$b(1-\lambda)\alpha \le (1-\lambda)\alpha M(a,b) \le (1-\lambda)\alpha a \tag{3.1}$$

and

$$b(\lambda - (1-\lambda)\alpha) \le (\lambda - (1-\lambda)\alpha)N(a,b) \le (\lambda - (1-\lambda)\alpha)a \tag{3.2}$$

From eqs (3.1) and (3.2), we obtain

$$b \le \frac{[(1-\lambda)\alpha M(a,b) + (\lambda - (1-\lambda)\alpha)N(a,b)]}{\lambda} \le a \tag{3.3}$$

This proves that,

$$b \le R(a,b;\alpha,\lambda) \le a \tag{3.4}$$

Thus we verified from Definition 1.3 of mean that $R(a,b;\alpha,\lambda)$ is a mean.

Again, for duals we have with usual meaning

$$b^{\frac{\alpha(1-\lambda)}{\lambda}} \leq M(a,b)^{\frac{\alpha(1-\lambda)}{\lambda}} \leq a^{\frac{\alpha(1-\lambda)}{\lambda}} \tag{3.5}$$

and

$$b^{\frac{\lambda-(1-\lambda)\alpha}{\lambda}} \leq N(a,b)^{\frac{\lambda-(1-\lambda)\alpha}{\lambda}} \leq a^{\frac{\lambda-(1-\lambda)\alpha}{\lambda}} \tag{3.6}$$

From eqs (3.5) and (3.6), we obtain

$$b \leq M(a,b)^{\frac{\alpha(1-\lambda)}{\lambda}} N(a,b)^{\frac{\lambda-(1-\lambda)\alpha}{\lambda}} \leq a \tag{3.7}$$

This proves that,

$$b \leq R^{(d)}(a,b;\alpha,\lambda) \leq a \tag{3.8}$$

Thus we verified from Definition 1.3 of mean that $R^{(d)}(a,b;\alpha,\lambda)$ is a mean. □

We have observed the following;

Note 1: Similarly, we can verify $R(a,b;\alpha)$, $R^{(d)}(a,b;\alpha)$, $S(a,b;\alpha)$, $T(a,b;\alpha)$, $S(a,b;\alpha,\lambda)$, $T(a,b;\alpha,\lambda)$ and their duals $S^{(d)}(a,b;\alpha)$, $T^{(d)}(a,b;\alpha)$, $S^{(d)}(a,b;\alpha,\lambda)$, $T^{(d)}(a,b;\alpha,\lambda)$ are means.

Now we state the following characteristic properties of the means $R(a,b;\alpha)$ and its dual $R^{(d)}(a,b;\alpha,\lambda)$. All these properties are also holds for the means $S(a,b;\alpha)$, $T(a,b;\alpha)$, $S(a,b;\alpha,\lambda)$, $T(a,b;\alpha,\lambda)$ and their dual forms $S^{(d)}(a,b;\alpha)$, $T^{(d)}(a,b;\alpha)$, $S^{(d)}(a,b;\alpha,\lambda)$, $T^{(d)}(a,b;\alpha,\lambda)$.

Proposition 3.1. *For $a > b > 0$ and $\alpha,\lambda \in (0,1)$, we have*

(1) *If $M(a,b)$ and $N(a,b)$ are symmetric, then*
$R(a,b;\alpha,\lambda)$=$R(b,a;\alpha,\lambda)$ *and* $R^{(d)}(a,b;\alpha,\lambda)$=$R^{(d)}(b,a;\alpha,\lambda)$.
(2) *If $M(a,b)$ and $N(a,b)$ are homogeneous, then*
$R(at,bt;\alpha,\lambda)$=$tR(a,b;\alpha,\lambda)$ *and* $R^{(d)}(at,bt;\alpha,\lambda)$=$tR^{(d)}(a,b;\alpha,\lambda)$, *if* $t > 0$.
(3) $R(a,b;\alpha,\lambda)$=$R^{(d)}(a,b;\alpha,\lambda) \Leftrightarrow a = b$.
(4) $R(a,b;\frac{1}{2},\frac{1}{2})$=$\frac{M(a,b)+N(a,b)}{2}$.
(5) $R(a,b;\alpha,\frac{1}{2})$=$R(a,b;\alpha)$ *and* $R^{(d)}(a,b;\alpha,\frac{1}{2})$=$R^{(d)}(a,b;\alpha)$.
(6) $R^{(d)}(a,b;\frac{1}{2},\frac{1}{2})$=$[M(a,b)N(a,b)]^{1/2}$.
(7) $min\{a,b\} \leq R^{(d)}(a,b;\alpha,\lambda) \leq R(a,b;\alpha,\lambda) \leq max\{a,b\}$.

Note 2: Some Identities

(1) $R(a,b;\alpha) + S(a,b;\alpha)$=$M(a,b) + N(a,b)$=$2A(M,N)$.
(2) $R^{(d)}(a,b;\alpha).S^{(d)}(a,b;\alpha)$=$M(a,b)N(a,b)$=$G^2(M,N)$.
(3) $R(a,b;\alpha) + S(a,b;\alpha) + 2T(a,b;\alpha)$=$2M(a,b) + N(a,b)$=$4A(M,N)$.
(4) For $\alpha = \frac{1}{2}$, $R(a,b;\alpha)R^{(d)}(a,b;\alpha) + S(a,b;\alpha)S^{(d)}(a,b;\alpha)$=$2G(M,N)A(M,N)$.
(5) For $\alpha = \frac{1}{2}$,$R^{(d)}(a,b;\alpha).T^{(d)}(a,b;\alpha) + S^{(d)}(a,b;\alpha).T^{(d)}(a,b;\alpha)$=$(MN)^{\frac{5}{6}}.\{M^{\frac{1}{3}} + N^{\frac{1}{3}}\}$.
(6) For $\alpha = \frac{1}{2}$,$R^{(d)}(a,b;\alpha).T^{(d)}(a,b;\alpha).S^{(d)}(a,b;\alpha).T^{(d)}(a,b;\alpha)$=$(MN)^{\frac{5}{3}}$.

Theorem 3.2. *For $a > b > 0$ and $\alpha, \lambda \in (0,1)$, then*

$$R^{(d)}(a,b;\alpha,\lambda) \leq R(a,b;\alpha,\lambda) \tag{3.9}$$

and equality holds when $a = b$.

Proof. The proof of the Theorem 3.2 is obvious from well known power mean inequality. □

Theorem 3.3. *For $a > b > 0$ and $\alpha, \lambda \in (0,1)$, then $R^{(d)}(a,b;\alpha,\lambda)$ and $R(a,b;\alpha,\lambda)$ are monotonically increasing with α, if $M(a,b) > N(a,b)$ and monotonically decreasing with α, if $M(a,b) < N(a,b)$, for all λ.*

Proof. We recall eqs (2.10)and (2.11)

$$R(a,b;\alpha,\lambda) = \frac{(1-\lambda)\alpha M(a,b) + (\lambda - (1-\lambda)\alpha)N(a,b)}{\lambda} \tag{3.10}$$

and

$$R^{(d)}(a,b;\alpha,\lambda) = M(a,b)^{(\frac{1-\lambda}{\lambda})\alpha} N(a,b)^{\frac{\lambda-(1-\lambda)\alpha}{\lambda}} \tag{3.11}$$

This result can be verified by the studying the sign of the partial derivatives of the corresponding means, in case of $R(a,b;\alpha,\lambda)$ we have

$$\frac{\partial R(a,b;\alpha,\lambda)}{\partial \alpha} = \frac{1-\lambda}{\lambda}\{M(a,b) - N(a,b)\} \tag{3.12}$$

it is positive if and only if $M(a,b) - N(a,b) > 0$ and in case of $R^{(d)}(a,b;\alpha,\lambda)$ we have,

$$\frac{\partial R^{(d)}(a,b;\alpha,\lambda)}{\partial \alpha} = \frac{1-\lambda}{\lambda}\{lnM(a,b) - lnN(a,b)\} \tag{3.13}$$

it is positive if and only if $lnM(a,b) - lnN(a,b) > 0$, that is equivalently we say $M(a,b) - N(a,b) > 0$. This completes the proof of Theorem 3.3. □

Theorem 3.4. *For $a > b > 0$ and $\alpha, \lambda \in (0,1)$, then $R^{(d)}(a,b;\alpha,\lambda)$ and $R(a,b;\alpha,\lambda)$ are monotonically increasing with λ, if $M(a,b) < N(a,b)$ and monotonically decreasing with λ, if $M(a,b) > N(a,b)$, for all α.*

Proof. This result can be verified by the studying the sign of the partial derivatives of the corresponding means (2.10)and (2.11), in case of $R(a,b;\alpha,\lambda)$ we have

$$\frac{\partial R(a,b;\alpha,\lambda)}{\partial \lambda} = \frac{\alpha}{\lambda^2}\{N(a,b) - M(a,b)\} \tag{3.14}$$

it is positive if and only if $N(a,b) - M(a,b) > 0$ and in case of $R^{(d)}(a,b;\alpha,\lambda)$ we have,

$$\frac{\partial R^{(d)}(a,b;\alpha,\lambda)}{\partial \lambda} = \frac{\alpha}{\lambda^2}\{lnN(a,b) - lnM(a,b)\} \tag{3.15}$$

it is positive if and only if $lnN(a,b) - lnM(a,b) > 0$, that is equivalently $N(a,b) - M(a,b) > 0$. This completes the proof of Theorem 3.4. □

Note 3: For the mean $S(a,b;\alpha,\lambda)$ and its dual $S^{(d)}(a,b;\alpha)$ are monotonically increasing with α, if $M(a,b) < N(a,b)$ and monotonically decreasing with α, if $M(a,b) > N(a,b)$, for all λ.

Note 4: For the mean $S(a,b;\alpha,\lambda)$ and its dual $S^{(d)}(a,b;\alpha)$ are monotonically increasing with λ, if $M(a,b) > N(a,b)$ and monotonically decreasing with λ, if $M(a,b) < N(a,b)$, for all α.

Theorem 3.5. *For $a > b > 0$ and $\alpha \in (0,1)$, then $T^{(d)}(a,b;\alpha,\lambda)$ and $T(a,b;\alpha,\lambda)$ are monotonically increasing with α, if $M(a,b) < N(a,b)$ and monotonically decreasing with α, if $M(a,b) > N(a,b)$, for fixed λ.*

Proof. We recall eqs (2.14)and (2.15)

$$T(a,b;\alpha,\lambda) = \frac{\lambda}{\lambda+(1-\lambda)}M(a,b) + \frac{(1-\lambda)\alpha}{\lambda+(1-\lambda)\alpha}N(a,b) \tag{3.16}$$

and

$$T^{(d)}(a,b;\alpha,\lambda) = M(a,b)^{\frac{\lambda}{\lambda+(1-\lambda)}} N(a,b)^{\frac{(1-\lambda)\alpha}{\lambda+(1-\lambda)\alpha}} \tag{3.17}$$

The result can be verified by the study of the sign of the partial derivatives of the corresponding means, in case of $R(a,b;\alpha,\lambda)$ we have

$$\frac{\partial T(a,b;\alpha,\lambda)}{\partial \alpha} = \frac{\lambda(1-\lambda)}{(\lambda+(1-\lambda)\alpha)^2}\{N(a,b) - M(a,b)\} \tag{3.18}$$

it is positive if and only if $N(a,b) - M(a,b) > 0$ and in case of $R^{(d)}(a,b;\alpha,\lambda)$ we have,

$$\frac{\partial T^{(d)}(a,b;\alpha,\lambda)}{\partial \alpha} = \frac{\lambda(1-\lambda)}{(\lambda+(1-\lambda)\alpha)^2}\{lnN(a,b) - lnM(a,b)\} \tag{3.19}$$

it is positive if and only if $lnN(a,b) - lnM(a,b) > 0$, equivalently we say $N(a,b) - M(a,b) > 0$. This completes the proof of Theorem 3.5. □

Theorem 3.6. *For $a > b > 0$ and $\alpha \in (0,1)$, then $T^{(d)}(a,b;\alpha,\lambda)$ and $T(a,b;\alpha,\lambda)$ are monotonically increasing with λ, if $M(a,b) > N(a,b)$ and monotonically decreasing with λ, if $M(a,b) < N(a,b)$, for fixed α.*

Proof. This result can be verified by the studying the sign of the partial derivatives of the corresponding means (2.14)and (2.15), in case of $R(a,b;\alpha,\lambda)$ we have

$$\frac{\partial T(a,b;\alpha,\lambda)}{\partial \lambda} = \frac{\alpha}{(\lambda+(1-\lambda)\alpha)^2}\{M(a,b) - N(a,b)\} \tag{3.20}$$

it is positive if and only if $M(a,b) - N(a,b) > 0$ and in case of $R^{(d)}(a,b;\alpha,\lambda)$ we have,

$$\frac{\partial T^{(d)}(a,b;\alpha,\lambda)}{\partial \lambda} = \frac{\alpha}{(\lambda+(1-\lambda)\alpha)^2}\{lnM(a,b) - lnN(a,b)\} \tag{3.21}$$

it is positive if and only if $lnM(a,b) - lnN(a,b) > 0$, equivalently we say $M(a,b) - N(a,b) > 0$. This completes the proof of Theorem 3.6. □

Note 5: As $\alpha \to \infty$, $T^{(d)}(a,b;\alpha,\lambda) = N(a,b) = T(a,b;\alpha,\lambda)$.

4. New inequality

In this section, we sharpened the old inequality and established the general inequality.

In [10], authors stated and established the inequalities among Greek means such as,

$$F_3(a,b) \leq F_2(a,b) \leq F_1(a,b) \leq F_6(a,b) \leq F_5(a,b) \leq F_4(a,b) \tag{4.1}$$

I n view of the above inequality, we have by selecting $F_3(a,b)$= min $\{M(a,b),\ N(a,b)\}$ and $F_4(a,b)$= max $\{M(a,b),\ N(a,b)\}$, then the following inequality holds for a,$b > 0$, $\alpha \in (0,1)$.

$$F_3(a,b) \leq R^{(d)}(a,b;\alpha) \leq R(a,b,\alpha) \leq F_4(a,b) \tag{4.2}$$

Example 1: If $\alpha = \frac{1}{2}$, in the eqn (4.2) we have the following inequality.

$$AH \leq GM_2 \leq \frac{G^2+M^2}{2} \leq AC. \tag{4.3}$$

Theorem 4.1. *For a,b > 0, α_1,$\alpha_2 \in (0,1)$ and β_i is a scalar so that $\alpha_2 \leq \frac{1-\beta_i}{2} \leq \alpha_1$, Then*

$$R^{(d)}(a,b;\alpha_1) \leq F_i(a,b;\beta_i) \leq R(a,b;\alpha_2) \tag{4.4}$$

Further more, α_2=$\frac{1-\beta_i}{2}$= $\alpha_1$is the best possible for Theorem 4.1. where i=1,2,5 & 6.

Proof. Recall eqs (2.2) and (2.3), $R(a,b;\alpha) = \alpha M(a,b) + (1-\alpha)N(a,b)$ and its dual $R^{(d)}(a,b;\alpha) = M(a,b)^{\alpha}N(a,b)^{1-\alpha}$.

Let $F_3(a,b)$= $M(a,b)$ and $F_4(a,b)$= $N(a,b)$, by applying Taylor's theorem and setting $a = x = t+1$ and $b = 1$ in (2.2)and(2.3), we have

$$R(a,b;\alpha_2) = 1 + \frac{t}{2} + \frac{1-\alpha_2}{8}t^2 + ... \tag{4.5}$$

$$R(a,b;\alpha_1) = 1 + \frac{t}{2} + \frac{1-\alpha_1}{8}t^2 + ... \tag{4.6}$$

and for arbitrary mean $F_i(a,b;\beta_i)$, we have

$$F_i(a,b;\beta_i) = 1 + \frac{t}{2} + \frac{\beta_i}{8}t^2 + ... \tag{4.7}$$

with simple manipulation, $\frac{1-\alpha_1}{8} \leq \frac{\beta_i}{8} \leq \frac{1-\alpha_2}{8}$ is holds for some β_i.

on rearranging, $\alpha_2 \leq \frac{1-\beta_i}{2} \leq \alpha_1$

Further more $\alpha_2 = \frac{1-\beta_i}{2} = \alpha_1$ is the best possible for Theorem 4.1. □

The following table gives the best possible value of β_i.

Sl.No.	1	2	3	4	5	6
Mean F_i	F_3	F_2	F_1	F_5	F_6	F_4
Value of β_i	-2	-1	0	1	1	2

Table 2

5. Some Deductions and Inequalities

In this section, we deduced the well known means. In [10], the authors defined weighted Greek means using the proportions, with this motivation and by using the proportion (2.1), we establishing the expressions for weighted Oscillatory mean, weighted r^{th} Oscillatory mean and its dual forms.

Let $N(a,b)$=$A(a,b)$=$F_1(a,b)$ and $M(a,b)$=$M_r(a,b)$=$F_2(a,b)$, then eqs (2.10) and (2.11) of Definition 2.1 as;

$$R = R(a,b;\alpha,\lambda) = \frac{(1-\lambda)\alpha M_r(a,b) + (\lambda-(1-\lambda)\alpha)A(a,b)}{\lambda} \tag{5.1}$$

and

$$R^{(d)} = R^{(d)}(a,b;\alpha,\lambda) = M_r(a,b)^{(\frac{1-\lambda}{\lambda})\alpha} A(a,b)^{\frac{\lambda-(1-\lambda)\alpha}{\lambda}} \tag{5.2}$$

are respectively called weighted r^{th} Oscillatory mean and its dual, are denoted by $O_\lambda(a,b;\alpha,r)$ and$O^d_\lambda(a,b;\alpha,r)$.

when $r=0$, $M_r(a,b)=G(a,b)$. Then the equations (5.1) and (5.2) takes in the form;

$$R = R(a,b;\alpha,\lambda) = \frac{(1-\lambda)\alpha G(a,b) + (\lambda-(1-\lambda)\alpha)A(a,b)}{\lambda} \tag{5.3}$$

and

$$R^{(d)} = R^{(d)}(a,b;\alpha,\lambda) = G(a,b)^{(\frac{1-\lambda}{\lambda})\alpha} A(a,b)^{\frac{\lambda-(1-\lambda)\alpha}{\lambda}} \tag{5.4}$$

are respectively called weighted Oscillatory mean and its dual, these are denoted by $O_\lambda(a,b;\alpha)$ and $O^d_\lambda(a,b;\alpha)$.

According to equations (5.1)-(5.4), the following characteristic properties for weighted r^{th} oscillatory mean $O_\lambda(a,b;\alpha,r)$ and its dual $O^d_\lambda(a,b;\alpha,r)$ are highlighted. Further when r=0, the characteristic properties are true for weighted oscillatory mean $O_\lambda(a,b;\alpha)$ and its dual $O^{(d)}_\lambda(a,b;\alpha)$.

Proposition 5.1. *Let $\alpha,\lambda\in(0,1)$ and $r\in[0,1]$. Then*

(1) $O_\lambda(a,b;\alpha,r)$=$O_\lambda(b,a;\alpha,r)$ *and* $O^{(d)}_\lambda(a,b;\alpha,r)$=$O^{(d)}_\lambda(b,a;\alpha,r)$.
(2) $O_\lambda(at,bt;\alpha,r)$=$tO_\lambda(a,b;\alpha,r)$ *and* $O^{(d)}_\lambda(at,bt;\alpha,r)$=$tO^{(d)}_\lambda(a,b;\alpha,r)$, $t>0$.
(3) $O_\lambda(a,b;\alpha,r)$=$O^{(d)}_\lambda(a,b;\alpha,r)$=$a \Leftrightarrow a=b$.
(4) $O_\lambda(a,b;0,r)$=$O^{(d)}_\lambda(a,b;0,r)$=$A(a,b)$.
(5) $O_\lambda(a,b;0,0)$=$O^{(d)}_\lambda(a,b;0,0)$=$A(a,b)$.
(6) $O_\lambda(a,b;\alpha,0)$=$O_\lambda(a,b;\alpha)$ *and* $O^{(d)}_\lambda(a,b;\alpha,0)$=$O^{(d)}_\lambda(a,b;\alpha)$.
(7) $O_\lambda(a,b;1,1)$=$O^{(d)}_\lambda(a,b;1,1)$=$A(a,b)$
(8) $O_{1/2}(a,b;\alpha,r)$=$O(a,b;\alpha,r)$ *and* $O^{(d)}_{1/2}(a,b;\alpha,r)$=$O^{(d)}(a,b;\alpha,r)$.

(9) $O_{1/2}(a,b;\alpha,0)$=$O(a,b;\alpha)$ *and* $O^{(d)}_{1/2}(a,b;\alpha,0)$=$O^{(d)}(a,b;\alpha)$.
(10) $O_{1/2}(a,b;\frac{1}{3},0)$=$H_e(a,b)$.

The proof of the following propositions are verified easily for details see[8, 2].

Proposition 5.2. *For $\alpha,\lambda\in(0,1)$ and $r\in(0,1)$. Then $O_\lambda(a,b;\alpha)$ and $O^{(d)}_\lambda(b,a;\alpha)$ both are monotonically decreasing with respect to α, if b>a>0.*

Proposition 5.3. *For $\alpha,\lambda\in(0,1)$ and $r\in(0,1)$, then*

$$O^{(d)}_\lambda(a,b;\alpha)\le O_\lambda(a,b;\alpha) \tag{5.5}$$

with the equality holds if and only if a=b or α =1or0.

Proposition 5.4. *For a,b > 0, α_1,$\alpha_2\in(0,1)$ and r is a fixed number so that $\alpha_1\le 1-r\le\alpha_2$, then*

$$O^{(d)}(a,b;\alpha_2)\le M_r(a,b)\le O(a,b;\alpha_1) \tag{5.6}$$

Further more, $\alpha_1=1-r=\alpha_2$ is the best possible for Proposition 5.4.

Proposition 5.5. *For a,b > 0, α_1,$\alpha_2\in(0,1)$ so that $\alpha_1\le\frac{2}{3}\le\alpha_2$, then*

$$O^{(d)}(a,b;\alpha_2)\le L(a,b)\le O(a,b;\alpha_1) \tag{5.7}$$

Further more, $\alpha_1=\frac{2}{3}=\alpha_2$ is the best possible for Proposition 5.5.

Proposition 5.6. *For a,b > 0, α_1,$\alpha_2\in(0,1)$ so that $\alpha_1\le\frac{1}{3}\le\alpha_2$, then*

$$O^{(d)}(a,b;\alpha_2)\le I(a,b)\le O(a,b;\alpha_1) \tag{5.8}$$

Further more, $\alpha_1=\frac{1}{3}=\alpha_2$ is the best possible for Proposition 5.6.

Proposition 5.7. *For a,b > 0, α_1,α_2,$\lambda\in(0,1)$ and r is a fixed number, so that $\alpha_1\le\frac{\lambda}{1-\lambda}(1-r)\le\alpha_2$, then*

$$O^{(d)}_\lambda(a,b;\alpha_2)\le M_r(a,b)\le O_\lambda(a,b;\alpha_1) \tag{5.9}$$

Further more, $\alpha_1=\frac{\lambda}{1-\lambda}(1-r)=\alpha_2$ is the best possible for Proposition 5.7.

Proposition 5.8. *For a,b > 0 and α_1,α_2,$\lambda\in(0,1)$, so that $\alpha_1\le\frac{\lambda}{1-\lambda}\frac{2}{3}\le\alpha_2$, then*

$$O^{(d)}_\lambda(a,b;\alpha_2)\le L(a,b)\le O_\lambda(a,b;\alpha_1) \tag{5.10}$$

Further more, $\alpha_1=\frac{\lambda}{1-\lambda}\frac{2}{3}=\alpha_2$ is the best possible for Proposition 5.8.

Proposition 5.9. *For a,b > 0 and α_1,α_2,$\lambda\in(0,1)$, so that $\alpha_1\le\frac{\lambda}{1-\lambda}\frac{1}{3}\le\alpha_2$, then*

$$O^{(d)}_\lambda(a,b;\alpha_2)\le I(a,b)\le O_\lambda(a,b;\alpha_1) \tag{5.11}$$

Further more, $\alpha_1=\frac{\lambda}{1-\lambda}\frac{1}{3}=\alpha_2$ is the best possible for Proposition 5.9.

Finally, we deduce the ten Neo Pythagorean means(Greek means) and other familiar means and new means are given in the following tables 3-6.

For $M(a,b)=a$ and $N(a,b)=b$, we have

Sl.No.	$\alpha \in (0,1)$	$M(a,b)$	$N(a,b)$	$R^{(d)}(a,b;\alpha)$=m	$R(a,b;\alpha)$=m
1	$\frac{b}{a}$	a	b	$a^{\frac{b}{a}}b^{1-\frac{b}{a}}$	$F_9(a,b)$
2	$\frac{b}{m}$	a	b	$a^{\frac{b}{m}}b^{1-\frac{b}{m}}$	$F_{10}(a,b)$
3	$\frac{m}{a}$	a	b	$a^{\frac{m}{a}}b^{1-\frac{m}{a}}$	a

Table 3

Sl.No.	$\alpha \in (0,1)$	$M(a,b)$	$N(a,b)$	$S^{(d)}(a,b;\alpha)$=m	$S(a,b;\alpha)$=m
1	$\frac{b}{a}$	a	b	$b^{\frac{b}{a}}a^{1-\frac{b}{a}}$	$F_7(a,b)$
2	$\frac{b}{m}$	a	b	$b^{\frac{b}{m}}a^{1-\frac{b}{m}}$	$\frac{a+\sqrt{a^2+4(b^2-ab)}}{2}$
3	$\frac{m}{a}$	a	b	$b^{\frac{m}{a}}a^{1-\frac{m}{a}}$	$F_8(a,b)$

Table 4

Sl.No.	$\alpha \in (0,1)$	$M(a,b)$	$N(a,b)$	$T^{(d)}(a,b;\alpha)$=m	$T(a,b;\alpha)$=m
1	$\frac{b}{a}$	a	b	$a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}$	$F_4(a,b)$
2	$\frac{b}{m}$	a	b	$a^{\frac{b}{b+m}}b^{\frac{m}{b+m}}$	$F_5(a,b)$
3	$\frac{m}{a}$	a	b	$a^{\frac{b}{a+b}}b^{\frac{a}{a+b}}$	$F_6(a,b)$

Table 5

Sl.No.	$\alpha \geq 1$	$M(a,b)$	$N(a,b)$	$T^{(d)}(a,b;\alpha)$=m	$T(a,b;\alpha)$=m
1	$\frac{a}{a}=\frac{b}{b}=\frac{m}{m}=1$	a	b	$a^{\frac{1}{2}}b^{\frac{1}{2}}$	$F_1(a,b)$
2	$\frac{m}{b},\frac{a}{m}$	a	b	$a^{\frac{b}{b+m}}b^{\frac{m}{b+m}}, a^{\frac{m}{a+m}}b^{\frac{a}{a+m}}$	$F_2(a,b)$
3	$\frac{a}{b}$	a	b	$a^{\frac{b}{a+b}}b^{\frac{a}{a+b}}$	$F_3(a,b)$

Table 6

By studying the particular values of $M(a,b)$ and $N(a,b)$, we notice out of 24 means, five means having extreme values(max or min), three means are transcendental means the mean values are evaluated by numerical methods namely Newton-Raphson method and others have usual mean values which are given in the following tables 7-10.

Sl.No.	$\alpha \in (0,1)$	$M(a,b)$	$N(a,b)$	$R^{(d)}(a,b;\alpha)$=m	$R(a,b;\alpha)$=m
1	$\frac{1}{2}$	$a=2$	$b=1$	$a^{\frac{b}{a}}b^{1-\frac{b}{a}}=1.4142$	$F_9(a,b)$=1.5
2	$\frac{1}{m}$	$a=2$	$b=1$	$a^{\frac{b}{m}}b^{1-\frac{b}{m}}=1.55965$	$F_{10}(a,b)$=1.618
3	$\frac{m}{2}$	$a=2$	$b=1$	$a^{\frac{m}{a}}b^{1-\frac{m}{a}}=2$	$a=2$

Table 7

Sl.No.	$\alpha \in (0,1)$	$M(a,b)$	$N(a,b)$	$S^{(d)}(a,b;\alpha)$=m	$S(a,b;\alpha)$=m
1	$\frac{1}{2}$	$a=2$	$b=1$	$b^{\frac{b}{a}}a^{1-\frac{b}{a}}=1.4142$	$F_7(a,b)$=1.5
2	$\frac{1}{m}$	$a=2$	$b=1$	$b^{\frac{b}{m}}a^{1-\frac{b}{m}}=1$	$\frac{a+\sqrt{a^2+4(b^2-ab)}}{2}=1$
3	$\frac{m}{2}$	$a=2$	$b=1$	$b^{\frac{m}{a}}a^{1-\frac{m}{a}}1.3333$	$F_8(a,b)=1$

Table 8

Sl.No.	$\alpha \in (0,1)$	$M(a,b)$	$N(a,b)$	$T^{(d)}(a,b;\alpha)$=m	$T(a,b;\alpha)$=m
1	$\frac{1}{2}$	$a=2$	$b=1$	$a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}=1.5874$	$F_4(a,b)=1.6666$
2	$\frac{1}{m}$	$a=2$	$b=1$	$a^{\frac{b}{b+m}}b^{\frac{m}{b+m}}=1.3438$	$F_5(a,b)=1.618$
3	$\frac{m}{2}$	$a=2$	$b=1$	$a^{\frac{b}{a+b}}b^{\frac{a}{a+b}}=1.2599$	$F_6(a,b)=1.4365$

Table 9

Sl.No.	$\alpha \geq 1$	$M(a,b)$	$N(a,b)$	$T^{(d)}(a,b;\alpha)$=m	$T(a,b;\alpha)$=m
1	$\frac{2}{2}=\frac{1}{1}=\frac{m}{m}=1$	$a=2$	$b=1$	$a^{\frac{1}{2}}b^{\frac{1}{2}}=1.4142$	$F_1(a,b)=1.5$
2	$\frac{m}{1},\frac{2}{m}$	$a=2$	$b=1$	$a^{\frac{b}{b+m}}b^{\frac{m}{b+m}}, a^{\frac{m}{a+m}}b^{\frac{a}{a+m}}=1.3438$	$F_2(a,b)=1.4142$
3	$\frac{2}{1}$	$a=2$	$b=1$	$a^{\frac{b}{a+b}}b^{\frac{a}{a+b}}=1.2599$	$F_3(a,b)=1.3333$

Table 10

From Tables 7-10, it is easy to analyze the deviation of mean values of various means with each other.

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