*J. Appl. Math. & Informatics Vol.* **29***(2011), No. 1 - 2, pp. 511 - 516*
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# ON THE SYMMETRY PROPERTIES OF THE GENERALIZED HIGHER-ORDER EULER POLYNOMIALS†

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Abstract. In this paper we prove a generalized symmetry relation between the generalized Euler polynomials and the generalized higher-order (attached to Dirichlet character) Euler polynomials. Indeed, we prove a relation between the power sum polynomials and the generalized higher-order Euler polynomials..

AMS Mathematics Subject Classification : 11B68, 11S80
*Key words and phrases* : Euler polynomials, Euler numbers, symmetry

## 1. Introduction

Let $d$ be a fixed positive integer and let $\chi$ be the Dirichlet's character with conductor $d$. Then the generalized Euler numbers and polynomials attached to $\chi$ are defined as

$$\begin{aligned}\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1} &= \sum_{n=0}^{\infty}E_{n,\chi}\frac{t^n}{n!},\\ \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1}e^{xt} &= \sum_{n=0}^{\infty}E_{n,\chi}(x)\frac{t^n}{n!},\ \text{for } |t|<\frac{\pi}{d},\ \text{(see [1-4])}. \qquad (1)\end{aligned}$$

For a real or complex parameter $\alpha$, we define the generalized higher-order Euler numbers and polynomials, of order $\alpha$, attached to $\chi$ as follows:

$$\begin{aligned}\left(\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1}\right)^{\alpha} &= \sum_{n=0}^{\infty}E_{n,\chi}^{(\alpha)}\frac{t^n}{n!},\ \text{and}\\ \left(\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1}\right)^{\alpha}e^{xt} &= \sum_{n=0}^{\infty}E_{n,\chi}^{(\alpha)}(x)\frac{t^n}{n!},\ \text{where } |t|<\frac{\pi}{d}. \qquad (2)\end{aligned}$$

Received October 6, 2010. Revised October 30, 2010. Accepted November 5, 2010.
*Corresponding author. †The present research has been conducted by the Research Grant of Kwangwoon University in 2010
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The main purpose of this paper is to prove an identity of symmetry for the generalized higher-order Euler polynomials. It turn out that the recurrence relation and multiplication theorem for the generalized Euler polynomials attached to $\chi$.

For the basic definitions and properties of the Euler polynomials, see [1-6].

## 2. Symmetry Identities related to the generalized higher-order Euler polynomials

**Theorem 1.** *Let $\chi$ be the Dirichlet's character with conductor $d \in \mathbb{N}^*$, and $a, b$ be the natural numbers with the same parity. For $n, m \in \mathbb{N}$, we have*

$$\begin{aligned} & \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} E_{n-k,\chi}^{(m-1)}(aY) \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j)(-1)^{i+j} E_{k,\chi}^{(m)} \left( bX + \frac{bd}{a} i + \frac{b}{a} j \right) \\ = & \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} E_{n-k,\chi}^{(m-1)}(bY) \sum_{j=0}^{d-1} \sum_{i=0}^{b-1} \chi(j)(-1)^{i+j} E_{k,\chi}^{(m)} \left( aX + \frac{ad}{b} i + \frac{a}{b} j \right). \end{aligned} \tag{3}$$

*Proof.* The generating function of the left hand side of (3) is

$$\begin{aligned} & \sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} E_{n-k,\chi}^{(m-1)}(aY) \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j)(-1)^{i+j} E_{k,\chi}^{(m)} \left( bX + \frac{bd}{a} i + \frac{b}{a} j \right) \frac{t^n}{n!} \\ = & \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j)(-1)^{i+j} \sum_{n \geq 0} \sum_{k=0}^{n} E_{n-k,\chi}^{(m-1)}(aY) \frac{(bt)^{n-k}}{(n-k)!} E_{k,\chi}^{(m)} \left( bX + \frac{bd}{a} i + \frac{b}{a} j \right) \frac{(at)^k}{k!} \\ = & \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j)(-1)^{i+j} \left( \sum_{k \geq 0} E_{k,\chi}^{(m-1)}(aY) \frac{(bt)^k}{(k!} \right) \\ & \times \left( \sum_{k \geq 0} E_{k,\chi}^{(m)} \left( bX + \frac{bd}{a} i + \frac{b}{a} j \right) \frac{(at)^k}{(k!} \right) \\ = & \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j)(-1)^{i+j} \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jbt}}{e^{dbt} + 1} \right)^{m-1} e^{abYt} \\ & \times \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jat}}{e^{dbt} + 1} \right)^{m} e^{\left( bX + \frac{bd}{a} i + \frac{b}{a} j \right) at}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} & \sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} E_{n-k,\chi}^{(m-1)}(aY) \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j)(-1)^{i+j} E_{k,\chi}^{(m)} \left( bX + \frac{bd}{a} i + \frac{b}{a} j \right) \frac{t^n}{n!} \\ = & \frac{1}{2} \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jat}}{e^{dat} + 1} \right)^{m} e^{abXt} \left( 1 - (-1)^a e^{dabt} \right) \end{aligned}$$

$$\times \left( \frac{2\sum_{j=0}^{d-1}(-1)^j \chi(j) e^{jbt}}{e^{dbt}+1} \right)^m e^{abYt},$$

and in the same way we obtain the generating function of the right hand side of (3) is

$$\begin{aligned} & \sum_{n\geq 0}\sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} E_{n-k,\chi}^{(m-1)}(bY) \sum_{j=0}^{d-1}\sum_{i=0}^{b-1} \chi(j)(-1)^{i+j} E_{k,\chi}^{(m)}\left(aX + \frac{ad}{b}i + \frac{a}{b}j\right) \frac{t^n}{n!} \\ = \; & \frac{1}{2}\left( \frac{2\sum_{j=0}^{d-1}(-1)^j \chi(j) e^{jat}}{e^{dat}+1} \right)^m e^{abXt}\left(1-(-1)^b e^{dabt}\right) \\ & \times \left( \frac{2\sum_{j=0}^{d-1}(-1)^j \chi(j) e^{jbt}}{e^{dbt}+1} \right)^m e^{abYt}. \end{aligned}$$

Since $a \equiv b \pmod 2$ then we have the same generating functions. Thus, the coefficients of these generating functions are the same. Hence, we obtain our desired Theorem 1. □

**Remark**

1) Our theorem 1 is valid for arbitrary conductor $d$ of the Dirichlet character $\chi$. $a$ and $b$ must only have the same parity.

2) Let $m = 1$ in (3). Then we have

$$\begin{aligned} & a^n \sum_{j=0}^{d-1}\sum_{i=0}^{a-1} \chi(j)(-1)^{i+j} E_{n,\chi}\left(bX + \frac{bd}{a}i + \frac{b}{a}j\right) \\ = \; & b^n \sum_{j=0}^{d-1}\sum_{i=0}^{b-1} \chi(j)(-1)^{i+j} E_{n,\chi}\left(aX + \frac{ad}{b}i + \frac{a}{b}j\right). \end{aligned}$$

3) Let $m = 1$ in (3) and $\chi = 1$. Then we have

$$a^n \sum_{i=0}^{a-1}(-1)^{i+j} E_n\left(bX + \frac{b}{a}i\right) = b^n \sum_{i=0}^{b-1}(-1)^i E_n\left(aX + \frac{a}{b}i\right),$$

for all $a$ and $b$ with same parity. For instance,

$$\sum_{i=0}^{a-1}(-1)^i E_n\left(X + \frac{i}{a}\right) = a^{-n} E_n(aX), \text{ if } a \text{ is odd}, \quad (\text{here we take } b = 1).$$

And

$$\sum_{i=0}^{a-1}(-1)^i E_n\left(X + \frac{i}{a}\right) = a^{-n} 2^n \left(E_n(aX) - E_n\left(aX + \frac{a}{2}\right)\right), \text{ if } a \text{ is even},$$

(here we take $b = 2$).

Now, for the rest of this paper, we set

$$\tilde{T}_{k,\chi}(n) = \sum_{i=0}^{d-1}\sum_{j=0}^{n-1}(-1)^{i+j}\chi(j)(i+jd)^k.$$

By (1), we can also easily see that

$$(-1)^{n-1}E_{k,\chi}(nd) + E_{k,\chi} = 2\tilde{T}_{k,\chi}(n).$$

Note that if $\chi_0$ is the trivial Dirichlet's character, then

$$\tilde{T}_k(n) := \tilde{T}_{k,\chi_0}(n) = \sum_{j=1}^{n}(-1)^{i-1}i^k = \frac{1}{2}\Big((-1)^{n-1}E_k(n) + E_k\Big).$$

**Theorem 2.** *Let $\chi$ be the Dirichlet's character with an arbitrary conductor $d \in \mathbb{N}^*$. For $a, b \in \mathbb{N}$ with same parity we have*

$$\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j}b^j a^{n-j}E^{(m)}_{n-j,\chi}(bX)\sum_{k=0}^{j}\binom{j}{k}\tilde{T}_{k,\chi}(a)E^{(m-1)}_{j-k,\chi}(aY)\\
=\ &\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}E^{(m)}_{n-j,\chi}(aX)\sum_{k=0}^{j}\binom{j}{k}\tilde{T}_{k,\chi}(b)E^{(m-1)}_{j-k,\chi}(bY). \qquad (4)
\end{aligned}$$

*Proof.* For $a, b \in \mathbb{N}$ with $a \equiv b \pmod 2$, let us consider the following functional equation:

$$\begin{aligned}
F_{d,m}(a,b;X,Y) &= \frac{1}{2}\left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jat}}{e^{dat}+1}\right)^m e^{abXt}\left(1-(-1)^a e^{dabt}\right)\\
&\quad\times\left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jbt}}{e^{dbt}+1}\right)^m e^{abYt} \qquad (5)\\
&= \left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jat}}{e^{dat}+1}\right)^m e^{abXt}\left(\frac{1-(-1)^a e^{dabt}}{e^{bdt}+1}\right)\\
&\quad\times\left(\sum_{j=0}^{d-1}\chi(j)(-1)^j e^{jbt}\right)\times\left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jbt}}{e^{dbt}+1}\right)^{m-1} e^{abYt}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\left(\frac{1-(-1)^a e^{abdt}}{e^{bdt}+1}\right)\left(\sum_{i=0}^{d-1}\chi(i)e^{ibt}(-1)^i\right)\\
=\ &\left(\sum_{l=0}^{a-1}e^{ldbt}(-1)^l\right)\left(\sum_{i=0}^{d-1}\chi(i)(-1)^i e^{ibt}\right)\\
=\ &\sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}e^{(id+j)bt}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}(id+j)^k\right)\frac{b^k t^k}{k!}\\
&= \sum_{k=0}^{\infty}\tilde{T}_{k,\chi}(a)\frac{b^k t^k}{k!}.
\end{aligned}
\tag{6}
$$

By (5) and (6), we have

$$
\begin{aligned}
&F_{d,m}(a,b;X,Y)\\
&= \left(\sum_{i=0}^{\infty}E_{i,\chi}^{(m)}(bX)\frac{a^i t^i}{i!}\right)\left(\sum_{l=0}^{\infty}\tilde{T}_{l,\chi}(a)\frac{b^l t^l}{l!}\right)\left(\sum_{k=0}^{\infty}E_{k,\chi}^{(m-1)}(aY)\frac{b^k t^k}{k!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}b^j a^{n-j}E_{n-j,\chi}^{(m)}(bX)\sum_{k=0}^{j}\tilde{T}_{k,\chi}(a)\binom{j}{k}E_{j-k,\chi}^{(m-1)}(aY)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{7}
$$

By the symmetry of $F_{d,m}(a,b;X,Y)$ in $a$ and $b$ (because $a \equiv b \pmod 2$), we also see that

$$
F_{d,m}(a,b;X,Y) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}E_{n-j,\chi}^{(m)}(aX)\sum_{k=0}^{j}\binom{j}{k}\tilde{T}_{k,\chi}(b)E_{j-k,\chi}^{(m-1)}(bY)\right)\frac{t^n}{n!}. \tag{8}
$$

By comparing the coefficients on the both sides of (7) and (8), we obtain the Theorem 2. □

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