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On the $\left|\overline{N}, p_n\right|_k$ Summability of Factoried Fourier Series

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Abstract. In this paper we deal with a main theorem on the local property of $\left|\overline{N}, p_n\right|_k$, $k \geq 1$ summability of factored Fourier series, which generalizes some known results.

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1. Introduction:

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \text{ as } n \to \infty \ \ (P_{-i} = p_{-i} = 0 \ , \ i \geq 1) \tag{1.1}$$

The sequence -to- sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu \tag{1.2}$$

defines the sequence (t_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n) generated by the sequence of coefficients (p_n).

The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty \tag{1.3}$$

In the Special Cases:

1. If $P_n = 1$ for all values of n, the $\left|\overline{N}, p_n\right|_k$ summability is the same as summability. Also, if we take $k = 1$ and $P_n = \frac{1}{n+1}$, then the summability $\left|\overline{N}, p_n\right|_k$ is equivalent to the summability $|R, \log n, 1|$.

   Let θ_n be any sequence of positive constants. The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n, \theta_n\right|_k$, $k \geq 1$ if $\sum\limits_{n=1}^{\infty} \theta_n^{k-1} |\Delta\sigma_{n-1}|^k < \infty$.

   If we take $\theta_n = \frac{P_n}{p_n}$,then $\left|\overline{N}, p_n, \theta_n\right|_k$ summability reduces to $\left|\overline{N}, p_n\right|_k$ summability. Also, if we take $\theta_n = n$ and $n = 1$ for all values of n, then we get the $[C, 1]_k$ Summability.
2. In the special case in which $P_n = \frac{1}{n+1}$ and therefore $P_n \approx \log n$ as $n \to \infty$, t_n reduces to the familiar harmonic mean $\{S_n\}$ and if it is denoted by t'_n , that the series $\sum a_n$ or the sequence $\{S_n\}$ is said to be summable by harmonic means to the sum S if $t'_n = S$ as $n \to \infty$.

   If the method of the summability (N, p_n) is super imposed on Cesaro means of order one, another method of summability $(N, p_n)C_1$ is obtained. For $P_n = \frac{1}{n+1}$ this method reduces tothe summability $\left(N, \frac{1}{n+1}\right) C_1$.
3. If we take $\theta_n = \frac{P_n}{p_n}$, then the $\left|\overline{N}, p_n, \theta_n\right|_k$ summability reduces to the $\left|\overline{N}, p_n\right|_k$ summability. Also if we take $\theta_n = n$ and $P_n = 1$ for all values of n, then we get $|C, 1|_k$ summability.

   – If we take $\theta_n = n$, then $\left|\overline{N}, p_n, \theta_n\right|_k$ summability reduces to $\left|\overline{N}, p_n\right|_k$ summability.

Let $f(t)$ be periodic function with period 2π and integrable (L) over $(-\pi, \pi)$ without any loss of generality we may assume that the constant

term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{1.5}$$

and

$$f(t) \approx \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n \tag{1.6}$$

It is well known that the convergence of the Fourier series at $t = x$ is a local property of the generating function f (i.e. it depends only on the behavior of f in an arbitrarily small neighbourhood of), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of the generating function f.

2. Known results

Mohanty [4] has demonstrated that the $|R, \log n, 1|$ summability of the factored Fourier series

$$\sum \frac{A_n(t)}{\log(n+1)} \tag{2.1}$$

at $t = x$, is a local property of the generating function of f , whereas the $|C, 1|$ summability of this series is not. Matsumoto [3] improved this result by replacing the series (2.1) by

$$\sum \frac{A_n(t)}{\{\log\log(n+1)\}^{\delta}} \ , \ \delta > 1 \tag{2.2}$$

Generalizing the above result by Bhatt [1] proved the following theorem.

Theorem 1. *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $|R, \log n, 1|$ of the series $\sum A_n(t)\lambda_n \log n$ at a point can be ensured by a local property.*

Theorem 2. *Let $k \geq 1$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, then the summability $\left|\overline{N}, p_n\right|_k$ of the series $\sum A_n(t)p_n\lambda_n$ at a point is a local property of the generating function f .*

3. Main Results

The aim of this paper is to generalize above theorems for $|N, p_n|_k$ summability methods, where $k \geq 1$ and $\delta \geq 0$. Now we shall prove the following theorems.

Theorem 3. *Let $k \geq 1$ and $0 \leq \delta < 1$. If the sequence (S_n) is bounded and the sequence (λ_n) and (P_n) satisfy the following conditions.*

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |\lambda_n|^k = 0(1) \text{ as } m \to \infty \tag{1}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta\lambda_n| = 0(1) \text{ as } m \to \infty \tag{2}$$

and

$$\sum_{n=\nu+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = 0\left\{\left(\frac{P_\nu}{p_\nu}\right)^{\delta k} \frac{1}{P_\nu}\right\}$$

Then the series $\sum a_n\lambda_n$ is summable $\left|\overline{N}, P_n, \delta\right|_k$

Theorem 4. *Let $k \geq 1$ and $0 \leq \delta k \leq 1$. The summability $\left|\overline{N}, P_n, \delta\right|_k$ of the series $\sum A_n(t)p_n\lambda_n$ at a point is a local property of the generating function if the conditions (1) and (2) are satisfied.*

We need the following lemmas for the proof of our theorems.

Lemma 5. *If λ_n is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, where p_n is sequence of positive numbers such that $P_n \to \infty$ as $n \to \infty$, then $P_n\lambda_n = 0(1)$ as $n \to \infty$ and $\sum P_n\Delta\lambda_n < \infty$.*

Proof: Since (λ_n) is non-increasing, we have that

$$P_m\lambda_m = \lambda_m \sum_{n=0}^{m} p_n = 0(1) \sum_{n=0}^{m} p_n\lambda_n = 0(1) \text{ as } m \to \infty$$

Applying the Abel transform to the sum $\sum\limits_{n=0}^{m} p_n\lambda_n$, we get that

$$\sum_{n=0}^{m} P_n\Delta\lambda_n = \sum_{n=0}^{m} p_n\lambda_n - P_m\lambda_{m+1}$$

since $\lambda_n \geq \lambda_{n+1}$, n we obtain

$$\sum_{n=0}^{m} P_n\Delta\lambda_n \leq P_m\lambda_m + \sum_{n=0}^{m} p_n\lambda_n = 0(1) + 0(1) = 0(1) \text{ as } m \to \infty$$

Lemma 6. *Let $k \geq 1$ and $sn = 0(1)$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum\limits_{n=0}^{m} p_n\lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $p_n \to \infty$ as $n \to \infty$, then the series $\sum a_nP_n\lambda_n$ is summable $\left|\overline{N}, p_n\right|_k$.*

Proof: Let (T_n) be the sequence of $\left(\overline{N}, p_n\right)$ means of the series $\sum a_n P_n \lambda_n$. Then, by definition,

$$T_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \sum_{r=0}^{\nu} a_r \lambda_r P_r = \frac{1}{P_n} \sum_{\nu=0}^{n} \left(P_n - P_{\nu-1}\right) a_\nu \lambda_\nu P_\nu$$

Then , for $n \geq 1$, we have

$$\begin{aligned} T_n - T_{n-1} &= \tfrac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu P_\nu s_\nu \Delta\lambda_\nu - \tfrac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu P_\nu s_\nu \lambda_\nu - \tfrac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu P_{\nu+1} s_\nu \lambda_{\nu+1} + s_n p_n \lambda_n \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \text{ say.} \end{aligned}$$

By Minkowskis inequality for $k > 1$, to complete the proof of lemma 6, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k < \infty \; r = 1, 2, 3, 4 \tag{3.1}$$

Now, applying Holders inequality with indices k and k' , where $\frac{1}{k} + \frac{1}{k'} = 1$ and $k > 1$,we get

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k \leq \sum_{n=2}^{m+1} \frac{P_n}{P_n P_{n-1}} \left\{ \sum_{\nu=1}^{n-1} |s_\nu|^k P_\nu P_\nu \Delta\lambda_\nu \right\} \left\{ \frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu P_\nu \Delta\lambda_\nu \right\}^{k-1}$$

Since

$$\sum_{\nu=1}^{n-1} P_\nu P_\nu \Delta\lambda_\nu \leq P_{n-1} \sum_{\nu=1}^{n-1} P_\nu \Delta\lambda_\nu$$

It follows by lemma 5 that

$$\frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu P_\nu \Delta\lambda_\nu \leq \sum_{\nu=1}^{n-1} P_\nu \Delta\lambda_\nu = 0(1) \text{ as } n \to \infty$$

Therefore

$$\begin{aligned} \sum_{n=2}^{m+1} \left(\tfrac{P_n}{p_n}\right)^{k-1} |T_{n,1}|^k &= 0(1) \sum_{n=2}^{m+1} \tfrac{P_n}{P_n P_{n-1}} \left\{ \sum_{\nu=1}^{n-1} |s_\nu|^k P_\nu P_\nu \Delta\lambda_\nu \right\} \\ &= 0(1) \sum_{\nu=1}^{m} |s_\nu|^k P_\nu P_\nu \Delta\lambda_\nu \sum_{n=\nu+1}^{m+1} \tfrac{P_n}{P_n P_{n-1}} \\ &= 0(1) \sum_{\nu=1}^{m} P_\nu \Delta\lambda_\nu = 0(1) \text{ as } m \to \infty \end{aligned}$$

by virtue of the hypotheses of theorem 2 and lemma 5. Again

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\tfrac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k &= 0(1)\sum_{n=2}^{m+1}\tfrac{P_n}{P_nP_{n-1}}\left\{\sum_{\nu=1}^{n-1}|s_\nu|^k p_\nu(P_\nu\lambda_\nu)^k\right\}\left\{\tfrac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}P_\nu\right\}^{k-1}\\
&= 0(1)\sum_{n=2}^{m+1}\tfrac{P_n}{P_nP_{n-1}}\left\{\sum_{\nu=1}^{n-1}|s_\nu|^k p_\nu(P_\nu\lambda_\nu)^k\right\}\\
&= 0(1)\sum_{n=1}^{m}|s_\nu|^k(P_\nu\lambda_\nu)^k p_\nu\left\{\sum_{n=\nu+1}^{m+1}\tfrac{P_n}{P_nP_{n-1}}\right\}\\
&= 0(1)\sum_{n=1}^{m}|s_\nu|^k(P_\nu\lambda_\nu)^k\tfrac{p_\nu}{P_\nu}\\
&= 0(1)\sum_{n=1}^{m}|s_\nu|^k(P_\nu\lambda_\nu)^{k-1}p_\nu\lambda_\nu\\
&= 0(1)\sum_{n=1}^{m}p_\nu\lambda_\nu = 0(1)\text{ as } m\to\infty
\end{aligned}$$

the fact that $P_\nu < P_{\nu+1}$, similarly we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k = 0(1)\sum_{\nu=1}^{m}p_{\nu+1}\lambda_{\nu+1} = 0(1)\text{ as } m\to\infty$$

Finally, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\tfrac{P_n}{p_n}\right)^{k-1}|T_{n,4}|^k &= \sum_{n=1}^{m}|s_n|^k(P_n\lambda_n)^{k-1}p_n\lambda_n\\
&= 0(1)\sum_{n=1}^{m}p_n\lambda_n = 0(1)\text{ as } m\to\infty
\end{aligned}$$

by virtue of the hypotheses of the theorem and lemma 5. Therefore, we get that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}|^k = 0(1)\text{ as } m\to\infty\ ,\ r=1,2,3,4$$

This completes the proof of lemma 6.

In the particular case if we take $p_n = 1$ for all values of n in Lemma 3.3, then we get the following corollary.

Corollary 7. *Let $k \geq 1$ and $s_n = 0(1)$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum\lambda_n$ is convergent, then the series $\sum na_n\lambda_n$ is summable $|C,1|_k$.*

Proof of theorem 2: Since the behavior of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behavior of the function in the immediate neighbourhood of this point only, hence the truth of theorem 2 is a consequence of lemma 6. If we take $p_n = 1$ for all values of n in this theorem, then we get a new local property result conserning the $|C,1|_k$ summability.

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