Reciprocity formulae for general multiple Dedekind-Rademacher sums and enumeration of lattice points

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Abstract

In this paper we introduce and investigate generalized multiple Dedekind-Rademacher sums. Our multiple Dedekind-Rademacher sums are written in terms of values of Bernoulli functions. The first aim of this paper is to give and prove a reciprocity law for our sums. The second aim is an application. We state that our sums have a connection to multiple Radermacher sums and we prove that these sums appear in lattice-point enumeration functions.

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1 Introduction and statement of main results

The multiple Dedekind-Rademacher sums that we introduce in this work are a generalization for the Dedekind-Rademacher sums studied by Hall, Lewis and Zagier in [4] (from dimension $n = 3$ to arbitrary dimension n).

It should be noted that these sums which are expressed in terms of values of Bernoulli functions are different from those introduced by Zagier [10] and from Dedekind cotangent sums studied by Beck[1], which are expressed in term of cotangent function.

We recall that the Bernoulli polynomials are defined by the generating function

$$\frac{ze^{zx}}{e^z-1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{z^n}{n!}\,, \qquad |z| < 2\pi\,. \tag{1.0.1}$$

and that the Bernoulli functions are defined by

$$\overline{B}_n(x) := \begin{cases} 0 & \text{if } x \in \mathbb{Z}, n = 1 \\ B_n(\{x\}) & \text{Otherwise} \end{cases}. \tag{1.0.2}$$

For each pair (a, b) of coprime integers, the classical Dedekind sum $S(a, b)$ is defined by

$$S(a,b) = \sum_{t=0}^{a-1} \overline{B}_1\left(\frac{bt}{a}\right)\overline{B}_1\left(\frac{t}{a}\right) \tag{1.0.3}$$

Dedekind [2] introduced this sum in connection with the transformation formula for the Dedekind η-function. But the story of Dedekind sum started with a paper of L.Kronecker [5] written in 1885, seven years before the widely quoted paper of R.Dedekind [2] on Riemann's Work. In that paper, Kronecker obtains a Dedekind sum by calculating the logarithm of the quadratic residue symbol using the Gauss lemma. Rademacher introduced the homogeneous sum $S(a,b,c)$ given by

$$S(a,b,c) := \sum_{t=0}^{a-1} \overline{B}_1\left(\frac{bt}{a}\right)\overline{B}_1\left(\frac{ct}{a}\right) \tag{1.0.4}$$

That is the Rademacher generalizaton of the classical Dedekind sum.
The most fundamental and important theorems for any of the generalized Dedekind sums are the reciprocity laws.
Along this paper we use the notation: $\mathbb{N}^*$ is the set of all integers $n > 0$.
The famous reciprocity laws for the classical Dedekind and Dedekind-Rademacher sums are

Theorem 1.0.1 (Dedekind-Rademacher)

i) **Dedekind:** *If $a, b \in \mathbb{N}^*$ are relatively prime then*

$$S(a,b) + S(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) . \tag{1.0.5}$$

ii) **Rademacher:** *If $a, b, c \in \mathbb{N}^*$ are pairwise coprime then*

$$S(a,b,c) + S(b,c,a) + S(c,a,b) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{c}{ab} + \frac{b}{ac}\right) . \tag{1.0.6}$$

The standard reference for ordinary Dedekind sums is Rademacher and Grosswald [3].
Let us now introduce the main object of study, the multiple Dedekind-Rademacher sums as follows:
For fixed integers $n \geqslant 2$, we consider $r_1, \ldots, r_n$ be positive integers and $a_1, \ldots, a_n$ be positive integers pairwise coprime and we set for any $1 \leqslant k \leqslant n$

$$\overrightarrow{A_k} = (a_1, .., \check{a}_k, ..., a_n) \ , \overrightarrow{R_k} = (r_1, .., \check{r}_k, ..., r_n) \in \mathbb{N}^{n-1}.$$

We define multiple Dedekind-Rademacher sums by the formula:

$$S_n(\overrightarrow{A_k}, \overrightarrow{R_k}) := \sum_{t=0}^{a_k-1} \prod_{1 \leqslant j \neq k \leqslant n} \overline{B}_{r_j}\left(\frac{a_j t}{a_k}\right). \tag{1.0.7}$$

It's easy to see for $n = 3, k = 3, \overrightarrow{A_3} = (\check{a}, b, c)$ and $\overrightarrow{R_3} = (1, 1)$, that

$$S(a,b,c) = S_3(\overrightarrow{A_3}, \overrightarrow{R_3}).$$

Our aim here is to prove that these sums satisfy certain reciprocity relations under cyclic permutations of $(a_1, .., a_k, ..., a_n)$. We state the reciprocity law in terms of the generating function defined by

$$\mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) := \sum_{\overrightarrow{R_k} \in \mathbb{N}^{n-1}} \frac{S_n(\overrightarrow{A_k}, \overrightarrow{R_k})}{r_1! \ldots \check{r}_k! \ldots r_n!} \prod_{1 \leqslant j \neq k \leqslant n} \left(\frac{2\pi i \varphi_j}{a_j}\right)^{r_j - 1} \tag{1.0.8}$$

where $\overrightarrow{\Phi_k} = (\phi_1, \ldots, \check{\phi}_k, \ldots, \phi_n)$ is a $n-1$-uples of reals.
In this paper we prove the following generalized result

Theorem 1.0.2 *Let n be integer $\geqslant 2$, let $a_1, ..., a_n$ be positive integers and pairwise coprime and let $\varphi_1, ..., \varphi_n$ n-real variables in the interval $]-1,1[$ with sum zero. Then*

$$\sum_{k=1}^{n} \mathfrak{S}_n\left(\overrightarrow{A_k}, \overrightarrow{\Phi_k}\right) = -\frac{1}{(2i)^{n-1}} \frac{\sin\left(\pi \sum_{j=1}^{n} \frac{\varphi_j}{a_j}\right)}{\prod_{j=1}^{n} \sin\left(\pi \frac{\varphi_j}{a_j}\right)} + \frac{1}{(2i)^{n-1}} \sum_{k=1}^{n} \prod_{1 \leqslant j \neq k \leqslant n} \cot\left(\pi \frac{\varphi_j}{a_j}\right). \tag{1.0.9}$$

By our result we recover the well known theorem of Hall, Wilson and Zagier in [4], which is stated for $n = 3$. Precisely, we have the following interesting relations deduced directly from the last theorem using the expansion of $\sin(\sum_{j=1}^{n} U_j)$ for $n = 2, 3, 4, 5$.

Corollary 1.0.3 *Under the hypothesis of theorem 1.0.2, we have the simple identities*

n	2	3	4	5
$\sum_{k=1}^{n} \mathfrak{S}_n\left(\overrightarrow{A_k}, \overrightarrow{\Phi_k}\right)$	0	$-\frac{1}{4}$	$\frac{i}{8} \sum_{j=1}^{4} \cot\left(\pi \frac{\varphi_j}{a_j}\right)$	$-\frac{1}{2^4} \sum_{1 \leqslant i < j \leqslant 5} \cot\left(\pi \frac{\varphi_i}{a_i}\right) \cot\left(\pi \frac{\varphi_j}{a_j}\right)$

Proof of theorem 1.0.1

We show how to obtain the theorem 1.0.1 from our main result theorem 1.0.2.
We put $a_1 = a, a_2 = b, a_3 = c$ and we define

$$S_{r_1,r_2}(a,b,c) := S_3\left(\overrightarrow{A_3}, \overrightarrow{R_3}\right), S_{r_2,r_3}(b,c,a) := S_3\left(\overrightarrow{A_1}, \overrightarrow{R_1}\right), S_{r_3,r_1}(c,a,b) := S_3\left(\overrightarrow{A_2}, \overrightarrow{R_2}\right).$$

Then we can write

$$\mathfrak{S}_3\left(\overrightarrow{A_3}, \overrightarrow{\Phi_3}\right) = \sum_{r_1,r_2 \geqslant 0} \frac{S_{r_1,r_2}(a,b,c)}{r_1! r_2!} \left(\frac{2\pi i \varphi_1}{a}\right)^{r_1-1} \left(\frac{2\pi i \varphi_2}{b}\right)^{r_2-1}$$

$$\mathfrak{S}_3\left(\overrightarrow{A_1}, \overrightarrow{\Phi_1}\right) = \sum_{r_2,r_3 \geqslant 0} \frac{S_{r_2,r_3}(b,c,a)}{r_2! r_3!} \left(\frac{2\pi i \varphi_2}{b}\right)^{r_2-1} \left(\frac{2\pi i \varphi_3}{c}\right)^{r_3-1}$$

$$\mathfrak{S}_3\left(\overrightarrow{A_2}, \overrightarrow{\Phi_2}\right) = \sum_{r_3,r_1 \geqslant 0} \frac{S_{r_3,r_1}(c,a,b)}{r_3! r_1!} \left(\frac{2\pi i \varphi_3}{c}\right)^{r_3-1} \left(\frac{2\pi i \varphi_1}{a}\right)^{r_1-1}$$

The theorem 1.0.2 shows that

$$\mathfrak{S}_3\left(\overrightarrow{A_1}, \overrightarrow{\Phi_1}\right) + \mathfrak{S}_3\left(\overrightarrow{A_2}, \overrightarrow{\Phi_2}\right) + \mathfrak{S}_3\left(\overrightarrow{A_3}, \overrightarrow{\Phi_3}\right) = -\frac{1}{4}. \tag{1.0.10}$$

By writing

$$\begin{aligned}
\mathfrak{S}_3\left(\overrightarrow{A_k}, \overrightarrow{\Phi_k}\right) = & \Big(\text{ part of total degree in terms of } \varphi_1, \varphi_2, \varphi_3 \text{ equal to } -2\Big) \\
& + \Big(\text{ part of total degree in terms of } \varphi_1, \varphi_2, \varphi_3 \text{ equal to } -1\Big) \\
& + \Big(\text{ part of total degree in terms of } \varphi_1, \varphi_2, \varphi_3 \text{ equal to } 0\Big) \\
& + \Big(\text{ Analytic part of total degree } \geqslant 1 \text{ in terms of } \varphi_1, \varphi_2, \varphi_3\Big).
\end{aligned}$$

In fact, for each $k = 1, 2, 3$ we compute the homogeneous part of degree zero in φ_1, φ_2 and φ_3 in the left of 1.0.10, we obtain that the **part of total degree** $l = 0$ **in terms of** $\varphi_1, \varphi_2, \varphi_3$ of

$$\mathfrak{S}_3\left(\overrightarrow{A_1}, \overrightarrow{\Phi_1}\right) + \mathfrak{S}_3\left(\overrightarrow{A_2}, \overrightarrow{\Phi_2}\right) + \mathfrak{S}_3\left(\overrightarrow{A_3}, \overrightarrow{\Phi_3}\right)$$

is equal to $-\frac{1}{4}$.
Precisely, for

$$\mathfrak{S}_3\left(\overrightarrow{A_3}, \overrightarrow{\Phi_3}\right) \text{ we obtain } S_{1,1}(a,b,c) + \frac{S_{0,2}(a,b,c)}{0!2!}\frac{a}{b} \times \frac{\varphi_2}{\varphi_1} + \frac{S_{2,0}(a,b,c)}{2!0!}\frac{b}{a} \times \frac{\varphi_1}{\varphi_2}$$

for

$$\mathfrak{S}_3\left(\overrightarrow{A_2}, \overrightarrow{\Phi_2}\right) \text{ we obtain } S_{1,1}(c,a,b) + \frac{S_{0,2}(c,a,b)}{0!2!}\frac{c}{a} \times \frac{\varphi_1}{\varphi_3} + \frac{S_{2,0}(c,a,b)}{2!0!}\frac{a}{c} \times \frac{\varphi_3}{\varphi_1}$$

and for

$$\mathfrak{S}_3\left(\overrightarrow{A_1}, \overrightarrow{\Phi_1}\right) \text{ we obtain } S_{1,1}(b,c,a) + \frac{S_{0,2}(b,c,a)}{0!2!}\frac{b}{c} \times \frac{\varphi_3}{\varphi_2} + \frac{S_{2,0}(b,c,a)}{2!0!}\frac{c}{b} \times \frac{\varphi_2}{\varphi_3}.$$

Now by using, the additive distribution formula 2.1.11 and the definition (3.0.18), we obtain

$$S_{1,1}(a,b,c) + \frac{S_{0,2}(a,b,c)}{0!2!}\frac{a}{b} \times \frac{\varphi_2}{\varphi_1} + \frac{S_{2,0}(a,b,c)}{2!0!}\frac{b}{a} \times \frac{\varphi_1}{\varphi_2} = S(a,b,c) + \frac{1}{2}\left(\frac{a}{bc} \times \frac{\varphi_2}{\varphi_1} + \frac{b}{ac} \times \frac{\varphi_1}{\varphi_2}\right)\overline{B}_2(0)$$

$$S_{1,1}(c,a,b) + \frac{S_{0,2}(c,a,b)}{0!2!}\frac{c}{a} \times \frac{\varphi_1}{\varphi_3} + \frac{S_{2,0}(c,a,b)}{2!0!}\frac{a}{c} \times \frac{\varphi_3}{\varphi_1} = S(c,a,b) + \frac{1}{2}\left(\frac{c}{ab} \times \frac{\varphi_1}{\varphi_3} + \frac{a}{bc} \times \frac{\varphi_3}{\varphi_1}\right)\overline{B}_2(0)$$

$$S_{1,1}(b,c,a) + \frac{S_{0,2}(b,c,a)}{0!2!}\frac{b}{c} \times \frac{\varphi_3}{\varphi_2} + \frac{S_{2,0}(b,c,a)}{2!0!}\frac{c}{b} \times \frac{\varphi_2}{\varphi_3} = S(b,c,a) + \frac{1}{2}\left(\frac{b}{ac} \times \frac{\varphi_3}{\varphi_2} + \frac{c}{ab} \times \frac{\varphi_2}{\varphi_3}\right)\overline{B}_2(0).$$

Finally, the homogeneous part of degree zero of

$$\mathfrak{S}_3\left(\overrightarrow{A_1}, \overrightarrow{\Phi_1}\right) + \mathfrak{S}_3\left(\overrightarrow{A_2}, \overrightarrow{\Phi_2}\right) + \mathfrak{S}_3\left(\overrightarrow{A_3}, \overrightarrow{\Phi_3}\right)$$

is

$$S(a,b,c) + S(b,c,a) + S(c,a,b) - \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right)$$

since $\varphi_1 + \varphi_2 + \varphi_3 = 0$ and $\overline{B}_2(0) = \frac{1}{6}$. This leads to the result

$$S(a,b,c) + S(b,c,a) + S(c,a,b) - \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right) = -\frac{1}{4}$$

by comparing both sides of (1.0.10). $\square$

Remark 1.0.4

In general, for any integer $N \geqslant 0$, by comparing the coefficients of degree $N-(n-1)$ of both sides of 1.0.9 we get relations for non-trivial Dedekind-Rademacher sums $S_n\left(\overrightarrow{A_k}, \overrightarrow{R_k}\right), 1 \leqslant k \leqslant n$, with $\sum_{1 \leqslant j \neq k \leqslant n} r_j = N$, and $r_j \geqslant 1$, let us express all the sums $S_n\left(\overrightarrow{A_k}, \overrightarrow{R_k}\right)$ as linear combinations of those.

2 Bernoulli functions and proof of main results

We use throughout the notation : $e(z) = e^{2\pi i z} \quad (z \in \mathbb{C})$.

2.1 Bernoulli functions: An overview

We will need the additive distribution formula satisfied by Bernoulli functions, which is proved by Raabe in [6]

$$\forall a \in \mathbb{N}^*, \quad \forall x \in \mathbb{R}, \quad \sum_{t=0}^{a-1} \overline{B}_m\left(x + \frac{t}{a}\right) = a^{1-m}\overline{B}_m(ax). \tag{2.1.11}$$

We recall the difference and addition formulas satisfied by Bernoulli polynomials

$$B_m(X+1) - B_m(X) = mX^{m-1} = \sum_{k=0}^{m-1} \binom{m}{k} B_k(X), \quad (m \in \mathbb{N}) \tag{2.1.12}$$

$$B_m(X+Y) = \sum_{k=0}^{m} \binom{m}{k} X^{m-k} B_k(Y), \quad (m \in \mathbb{N}). \tag{2.1.13}$$

Also we need the Fourier expansion formula of $\overline{B}_m(x)$ [4]

$$\overline{B}_m(x) = -\frac{m!}{(2\pi i)^m} \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{e^{2\pi i k x}}{k^m}, \quad (m \in \mathbb{N}^*,\ x \in \mathbb{R}). \tag{2.1.14}$$

Now we consider for all $\phi \in \mathbb{R} \setminus \mathbb{Z}$, the function

$$F(z,\phi) = \begin{cases} \dfrac{1}{2i}\cot(\pi\phi) & \text{if } z \in \mathbb{Z}, \\ \dfrac{e(\{z\}\phi)}{e(\phi)-1} & \text{if } z \notin \mathbb{Z}. \end{cases} \tag{2.1.15}$$

By the definition (1.0.1) of $B_m(x)$, we have for all $\phi \in \mathbb{R} \setminus \mathbb{Z}$,

$$F(z,\phi) = \sum_{m=0}^{+\infty} \frac{\overline{B}_m(z)}{m!}(2\pi i\phi)^{m-1}, \quad (z \in \mathbb{R}). \tag{2.1.16}$$

2.2 Proof of theorem 1.0.2

We begin to prove the following lemma

Lemma 2.2.1

$$\sum_{k=1}^{n} \mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) = \sum_{\substack{(t_1,\dots,t_n)\in\mathbb{N}^n \\ 0\leqslant t_i \leqslant a_i - 1}} \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\neq k}}^{n} F\left(\frac{t_k}{a_k} - \frac{t_j}{a_j}, \phi_j\right).$$

Proof : Using the distribution property (2.1.11) of $\overline{B}_m$, we rewrite $\mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k})$ as

$$\mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) \quad = \quad \sum_{\overrightarrow{R_k} \in \mathbb{N}^{n-1}} S_n(\overrightarrow{A_k}, \overrightarrow{R_k}) \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{r_j!}\left(\frac{2\pi i \phi_j}{a_j}\right)^{r_j - 1}$$

$$
\begin{aligned}
&= \sum_{\substack{0\leqslant t_i\leqslant a_i-1\\ 1\leqslant i\leqslant n}} \sum_{\overrightarrow{R_k}\in\mathbb{N}^{n-1}} \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{r_j!}\overline{B}_{r_j}\left(\frac{t_k}{a_k}-\frac{t_j}{a_j}\right)(2\pi i\phi_j)^{r_j-1}\\
&= \sum_{\substack{0\leqslant t_i\leqslant a_i-1\\ 1\leqslant i\leqslant n}} \prod_{\substack{j=1\\ j\neq k}}^{n} \sum_{r_j=0}^{\infty} \frac{1}{r_j!}\overline{B}_{r_j}\left(\frac{t_k}{a_k}-\frac{t_j}{a_j}\right)(2\pi i\phi_j)^{r_j-1}.
\end{aligned}
$$

Using Laurent expansion of F (2.1.16), we obtain thus

$$
\mathfrak{S}_n(\overrightarrow{A_k},\overrightarrow{\Phi_k}) = \sum_{\substack{0\leqslant t_i\leqslant a_i-1\\ 1\leqslant i\leqslant n}} \prod_{\substack{j=1\\ j\neq k}}^{n} F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right)
$$

from which the lemma follows. □

Lemma 2.2.2 *Let $(u_1,\ldots,u_n)\in\mathbb{R}^n$ such that for all distinct $k,j\in\{1,\ldots,n\}$*

$$
u_k-u_j\notin\mathbb{Z}.
$$

Then we have

$$
\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n} F\left(u_k-u_j,\phi_j\right)=0.
$$

Proof :

The Fourier expansions (2.1.14) and (2.1.16) give the identity

$$
\begin{aligned}
F(z,\phi) &= \frac{1}{2\pi i}\sum_{k\in\mathbb{Z}}\frac{e(kz)}{\phi-k}\\
&= \frac{1}{2\pi i}\sum_{\lambda=\phi \pmod{\mathbb{Z}}}\frac{e((\phi-\lambda)z)}{\lambda}
\end{aligned}
$$

where the last sum, obtained by writing $\lambda=\phi-k$, is to be interpreted as a Cauchy principal value (sum over $|\lambda|<L$ and let $L\to\infty$) for convergence.
Then

$$
\prod_{\substack{j=1\\ j\neq k}}^{n} F\left(u_k-u_j,\phi_j\right)=\frac{1}{(2\pi i)^{n-1}}\sideset{}{^*}\sum_{(\lambda_1,\ldots,\check{\lambda}_k,\ldots,\lambda_n)}\frac{e\left((u_k-u_j)(\phi_j-\lambda_j)\right)}{\lambda_j}.
$$

where the sum $\sum^*$ is over all $(\lambda_1,\ldots,\check{\lambda}_k,\ldots,\lambda_n)\equiv(\phi_1,\ldots,\check{\phi}_k,\ldots,\phi_n) \pmod{\mathbb{Z}^{n-1}}$.
Now since $\sum_{j=1}^{n}\phi_j=0$, then by setting $\lambda_n=-\sum_{j=1}^{n-1}\lambda_j$, one has

$$
\begin{aligned}
\sum_{\substack{j=1\\ j\neq k}}^{n}(u_k-u_j)(\phi_j-\lambda_j) &= \sum_{j=1}^{n}(u_k-u_j)(\phi_j-\lambda_j)\\
&= -\sum_{j=1}^{n}u_j(\phi_j-\lambda_j)
\end{aligned}
$$

which is independent of k.
We observe also

$$\begin{aligned}\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{\lambda_j} &= \sum_{k=1}^{n}\lambda_k\prod_{j=1}^{n}\frac{1}{\lambda_j}\\ &= 0.\end{aligned}$$

We conclude

$$\begin{aligned}\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}F\left(u_k-u_j,\phi_j\right) &= \frac{1}{(2\pi i)^{n-1}}\sum_{k=1}^{n}\sum_{(\lambda_1,\ldots,\check{\lambda_k},\ldots,\lambda_n)}^{*}\frac{e\left(-\sum_{j=1}^{n}u_j(\phi_j-\lambda_j)\right)}{\prod_{\substack{j=1\\ j\neq k}}^{n}\lambda_j}\\ &= 0.\end{aligned}$$

This proves the lemma 2.2.2.□

In order to prove the formula 1.0.9, we compute the sum

$$\sum_{\substack{(t_1,\ldots,t_n)\in\mathbb{N}^n\\ 0\leqslant t_i\leqslant a_i-1}}\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right). \tag{2.2.17}$$

Thanks to the lemma (2.2.2), we know that the contribution in the sum (2.2.17) coming from the $\frac{t_k}{a_k}-\frac{t_j}{a_j}\notin\mathbb{Z}$ for all distinct $k,j\in\{1,\ldots,n\}$ and all $t_k,t_j\in\mathbb{Z}$, is zero.
Let $(a_1,\ldots,a_n)$ and $(t_1,\ldots,t_n)$ be fixed. We set

$$\sum=\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right).$$

Our aim here is to calculate this quantity $\sum$.
We assume first that $(t_1,\ldots,t_n)$ is of the form :

$$(t_1,\ldots,t_k)=(0,\ldots,0,t_{j_0+1},\ldots,t_n)\in\prod_{j=1}^{j_0}\{0\}\times\prod_{j=j_0+1}^{n}\,]0,a_{j-1}]\cap\mathbb{N}.$$

Throughout, we admit that every sum (resp. product) over the empty set is zero (resp. equal to 1).
We will observe easily that the general case follows without difficulty.
For simplicity, in our compution we can assume that:

$$\frac{t_j}{a_j}\leqslant\frac{t_{j+1}}{a_{j+1}}\qquad(j_0<j<n).$$

We decompose the sum $\sum$ in two sums $\sum_1+\sum_2$ as $k\leqslant j_0$ or $k>j_0$.
Let us put $h_j=t_j/a_j$ $(j=1,\ldots,n)$.

On the one hand, we have

$$\begin{aligned}
\sum{}_1 &:= \sum_{k=1}^{j_0} \prod_{\substack{j=1\\ j\neq k}}^{n} F\left(h_k - h_j, \phi_j\right) \\
&= \sum_{k=1}^{j_0} \prod_{\substack{j=1\\ j\neq k}}^{n} F\left(-h_j, \phi_j\right) \\
&= \sum_{k=1}^{j_0} \prod_{\substack{j=1\\ j\neq k}}^{j_0} F(0,\phi_j) \prod_{j>j_0} F\left(-h_j, \phi_j\right) \\
&= \sum_{k=1}^{j_0} \frac{1}{F(0,\phi_k)} \prod_{j\leqslant j_0} F(0,\phi_j) \prod_{j>j_0} F\left(-h_j, \phi_j\right) \\
&= \prod_{j\leqslant j_0} F(0,\phi_j) \sum_{j\leqslant j_0} \frac{1}{F(0,\phi_j)} \prod_{j>j_0} \frac{e\left(\{-h_j\}\phi_j\right)}{e(\phi_j)-1} \\
&= \prod_{j\leqslant j_0} F(0,\phi_j) \sum_{j\leqslant j_0} \frac{1}{F(0,\phi_j)} \prod_{j>j_0} \frac{e\left((1-h_j)\,\phi_j\right)}{e(\phi_j)-1} \\
&= \prod_{j\leqslant j_0} \frac{1}{2}\frac{e(\phi_j)+1}{e(\phi_j)-1} \sum_{j\leqslant j_0} \frac{1}{F(0,\phi_j)} \frac{e\Big(\sum_{j>j_0}\phi_j\Big) \times e\left(-\sum_{j>j_0} h_j\phi_j\right)}{\prod_{j>j_0}\left(e(\phi_j)-1\right)}.
\end{aligned}$$

So after rearrangement

$$\begin{aligned}
\sum{}_1 &= \frac{e\left(-\sum_{j=1}^{n} h_j\phi_j\right)}{\prod_{j=1}^{n}(e(\phi_j)-1)} e\left(\sum_{j>j_0}\phi_j\right) \prod_{j\leqslant j_0} \frac{1}{2}\Big(e(\phi_j)+1\Big) \sum_{j\leqslant j_0} \frac{1}{F(0,\phi_j)} \\
&= \frac{e\left(-\sum_{j=1}^{n} h_j\phi_j\right)}{\prod_{j=1}^{n}(e(\phi_j)-1)} \prod_{j\leqslant j_0} \frac{1}{2}\Big(1+e(-\phi_j)\Big) \sum_{j\leqslant j_0} \frac{1}{F(0,\phi_j)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum{}_2 &= \sum_{k>j_0} \prod_{\substack{j=1\\ j\neq k}}^{n} F\left(h_k - h_j, \phi_j\right) \\
&= \sum_{k>j_0} \frac{e\left(\sum_{\substack{j\neq k\\ 1\leqslant j\leqslant n}} \{h_k - h_j\}\phi_j\right)}{\prod_{j\neq k}\Big(e(\phi_j)-1\Big)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\prod_{j=1}^{n}\Big(e(\phi_j)-1\Big)}\sum_{k>j_0}\Big(e(\phi_k)-1\Big)e\left(\sum_{j\leqslant k}(h_k-h_j)\,\phi_j\right)e\left(\sum_{j>k}(1-h_j+h_k)\,\phi_j\right)\\
&= \frac{1}{\prod_{j=1}^{n}(e(\phi_j)-1)}\sum_{k>j_0}\Big(e(\phi_k)-1\Big)e\left(-\sum_{j\leqslant n}h_j\phi_j\right)e\left(\sum_{j>k}\phi_j\right)\\
&= \frac{e\left(-\sum_{j=1}^{n}h_j\phi_j\right)}{\prod_{j=1}^{n}\Big(e(\phi_j)-1\Big)}\sum_{k>j_0}\Big(e(\phi_k)-1\Big)e\Big(\sum_{j>k}\phi_j\Big).
\end{aligned}
$$

Thus

$$
\sum{}_2 = \frac{e\left(-\sum_{j=1}^{n}h_j\phi_j\right)}{\prod_{j=1}^{n}\Big(e(\phi_j)-1\Big)}\Big(e(\sum_{j>j_0}\phi_j)-1\Big).
$$

We have therefore shown that

$$
\sum = \sum{}_1 + \sum{}_2 \;=\; \frac{e\left(-\sum_{j=1}^{n}h_j\phi_j\right)}{\prod_{j=1}^{n}(e(\phi_j)-1)}\left(\prod_{j\leqslant j_0}\frac{1}{2}(e(-\phi_j)+1)\sum_{j\leqslant j_0}\frac{1}{F(0,\phi_j)}+e(\sum_{j>j_0}\phi_j)-1\right).
$$

For every $\overrightarrow{t}=(t_1,\ldots,t_n)$, we set $J\left(\overrightarrow{t}\right)=\{j : t_j=0\}$.
By permutation, we can consider the previous case and so

$$
\sum = \frac{e\left(-\sum_{j=1}^{n}\frac{t_j}{a_j}\phi_j\right)}{\prod_{j=1}^{n}(e(\phi_j)-1)}\left(e(-\sum_{j\in J}\phi_j)-1+\prod_{j\in J}\frac{1}{2}(e(-\phi_j)+1)\sum_{j\in J}\frac{1}{F(0,\phi_j)}\right).
$$

We consider now the sum of $\sum$ over all $\overrightarrow{t}\in\prod_{j=1}^{n}[0,a_j-1]\cap\mathbb{N}$.

Let us write

$$
\sum_{\overrightarrow{t}}\sum \;=\; \sum_{J\subset[\![1,n]\!]}\;\sum_{\substack{\overrightarrow{t}\\ J\left(\overrightarrow{t}\right)=J}}\;\sum.
$$

We see first that

$$\begin{aligned}
\sum_{{}_J(\overrightarrow{t})=J} e\left(-\sum_{j=1}^{n}\frac{t_j}{a_j}\phi_j\right) &= \prod_{j\notin J}\sum_{t_j=1}^{a_j-1} e\left(-\frac{\phi_j}{a_j}t_j\right)\\
&= \prod_{j\notin J}\frac{e\left(-\frac{\phi_j}{a_j}\right)-e(-\phi_j)}{1-e\left(-\frac{\phi_j}{a_j}\right)}\\
&= \prod_{j=1}^{n}\frac{e\left(-\frac{\phi_j}{a_j}\right)-e\left(-\phi_j\right)}{1-e\left(-\frac{\phi_j}{a_j}\right)}\prod_{j\in J}\frac{1-e\left(-\frac{\phi_j}{a_j}\right)}{e\left(-\frac{\phi_j}{a_j}\right)-e\left(-\phi_j\right)}.
\end{aligned}$$

In other hand, we have

$$\begin{aligned}
\sum_J\prod_{j\in J}\frac{1-e\left(-\frac{\phi_j}{a_j}\right)}{e\left(-\frac{\phi_j}{a_j}\right)-e\left(-\phi_j\right)}e\left(-\phi_j\right) &= \sum_J\prod_{j\in J}\frac{1-e\left(-\frac{\phi_j}{a_j}\right)}{e\left((1-\frac{1}{a_j})\phi_j\right)-1}\\
&= \prod_{i=1}^{n}\left(1+\frac{1-e(-\frac{\phi_i}{a_i})}{e((1-\frac{1}{a_i})\phi_i)-1}\right)\\
&= \prod_{i=1}^{n}\frac{e\left(\phi_i\right)-1}{e\left(\phi_i\right)-e\left(\frac{\phi_i}{a_i}\right)}.
\end{aligned}$$

It remains to give a formula for

$$S=\sum_J\prod_{j\in J}\frac{1}{2}(e(-\phi_j)+1)\frac{1-e\left(-\frac{\phi_j}{a_j}\right)}{e\left(-\frac{\phi_j}{a_j}\right)-e\left(-\phi_j\right)}\sum_{j\in J}\frac{1}{F(0,\phi_j)}.$$

By putting

$$\alpha_j=\frac{1}{2}\Big(1+e(-\phi_j)\Big)\times\frac{1-e\left(-\frac{\phi_j}{a_j}\right)}{e\left(-\frac{\phi_j}{a_j}\right)-e\left(-\phi_j\right)}$$

and

$$\beta_j=\frac{1}{F\left(0,\phi_j\right)}=2\frac{e(\phi_j)-1}{e(\phi_j)+1},$$

we may write

$$S=\sum_J\prod_{j\in J}\alpha_j\sum_{j\in J}\beta_j.$$

We set $\beta=\sum_{j\in J}\beta_j$. It is clear that

$$S=H'(1),$$

where

$$H(y) = \sum_J\left(\prod_{j\in J}\alpha_j\right)y^{\beta}$$

$$
\begin{aligned}
&= \sum_{J} \prod_{j \in J} \alpha_j y^{\beta_j} \\
&= \prod_{i=1}^{n} \left(1 + \alpha_i y^{\beta_i}\right).
\end{aligned}
$$

Hence

$$
S = \prod_{i=1}^{n} (1 + \alpha_i) \sum_{i=1}^{n} \frac{\alpha_i \beta_i}{1 + \alpha_i}.
$$

Since

$$
1 + \alpha_j = \frac{1}{2} \frac{A_j}{e\left(-\frac{\phi_j}{a_j}\right) - e\left(-\phi_j\right)}
$$

with

$$
\begin{aligned}
A_j &= 2e\left(-\frac{\phi_j}{a_j}\right) - 2e\left(-\phi_j\right) + \Big(1 + e(-\phi_j)\Big)\left(1 - e\left(-\frac{\phi_j}{a_j}\right)\right) \\
&= e\left(-\frac{\phi_j}{a_j}\right) - e\left(-\phi_j\right) + 1 + e\Big(-(1 + \frac{1}{a_j})\phi_j)\Big) \\
&= \left(1 + e\left(-\frac{\phi_j}{a_j}\right)\right)\Big(1 - e\left(-\phi_j\right)\Big),
\end{aligned}
$$

it follows that

$$
1 + \alpha_j = \frac{1}{2} \frac{\left(1 + e\left(-\frac{\phi_j}{a_j}\right)\right)\Big(1 - e\left(-\phi_j\right)\Big)}{e\left(-\frac{\phi_j}{a_j}\right) - e\left(-\phi_j\right)}.
$$

On the other hand, we have

$$
\begin{aligned}
\alpha_j \beta_j &= \frac{1}{2}\Big(1 + e(-\phi_j)\Big) \frac{1 - e\left(-\frac{\phi_j}{a_j}\right)}{e\left(-\frac{\phi_j}{a_j}\right) - e(-\phi_j)} \frac{2\left(e(\phi_j) - 1\right)}{e(\phi_j) + 1} \\
&= \Big(1 - e(-\phi_j)\Big) \frac{1 - e\left(-\frac{\phi_j}{a_j}\right)}{e\left(-\frac{\phi_j}{a_j}\right) - e(-\phi_j)}.
\end{aligned}
$$

Hence, we obtain the following expression :

$$
\frac{\alpha_j \beta_j}{1 + \alpha_j} = \frac{1 - e\left(-\frac{\phi_j}{a_j}\right)}{1 + e\left(-\frac{\phi_j}{a_j}\right)}
$$

and consequently,

$$
S = \prod_{j=1}^{n} \frac{1}{2} \frac{\left(1 + e\left(-\frac{\phi_j}{a_j}\right)\right)\left(1 - e(-\phi_j)\right)}{e\left(-\frac{\phi_j}{a_j}\right) - e(-\phi_j)} \sum_{j=1}^{n} 2 \frac{1 - e\left(-\frac{\phi_j}{a_j}\right)}{1 + e\left(-\frac{\phi_j}{a_j}\right)}.
$$

Finally, by using lemma (2.2.2)

$$\begin{aligned}\sum_{\substack{0\leqslant t_j\leqslant a_j-1\\1\leqslant j\leqslant n}}\sum_{k=1}^{n}\prod_{\substack{j=1\\j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right) &= \prod_{j=1}^{n}\frac{e\left(-\frac{\phi_j}{a_j}\right)-e(-\phi_j)}{1-e\left(-\frac{\phi_j}{a_j}\right)}\prod_{j=1}^{n}\frac{1}{e(\phi_j)-e\left(\frac{\phi_j}{a_j}\right)}\\ &- \prod_{j=1}^{n}\frac{1}{1-e\left(-\frac{\phi_j}{a_j}\right)}+\prod_{j=1}^{n}\frac{1}{2}\frac{\left(1+e\left(-\frac{\phi_j}{a_j}\right)\right)}{1-e\left(-\frac{\phi_j}{a_j}\right)}\sum_{j=1}^{n}2\frac{1-e\left(-\frac{\phi_j}{a_j}\right)}{1+e\left(-\frac{\phi_j}{a_j}\right)},\end{aligned}$$

which also can be rewritten as follows

$$\sum_{\substack{0\leqslant t_j\leqslant a_j-1\\1\leqslant j\leqslant n}}\sum_{k=1}^{n}\prod_{\substack{j=1\\j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right) = \frac{1}{2^{n-1}}\sum_{k=1}^{n}\prod_{1\leqslant j\neq k\leqslant n}\frac{e\left(\frac{\phi_j}{a_j}\right)+1}{e\left(\frac{\phi_j}{a_j}\right)-1}-\frac{e\left(\sum_{j=1}^{n}\frac{\phi_j}{a_j}\right)-1}{\prod_{j=1}^{n}e\left(\frac{\phi_j}{a_j}\right)-1}.$$

Then, we obtain our formula in terms of sin and cot

$$\sum_{\substack{0\leqslant t_j\leqslant a_j-1\\1\leqslant j\leqslant n}}\sum_{k=1}^{n}\prod_{\substack{j=1\\j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right)=-\frac{1}{(2i)^{n-1}}\frac{\sin\left(\pi\sum_{j=1}^{n}\frac{\varphi_j}{a_j}\right)}{\prod_{j=1}^{n}\sin\left(\pi\frac{\varphi_j}{a_j}\right)}+\frac{1}{(2i)^{n-1}}\sum_{k=1}^{n}\prod_{1\leqslant j\neq k\leqslant n}\cot\left(\pi\frac{\varphi_j}{a_j}\right).$$

Now, by using lemma (2.2.1), we obtain

$$\sum_{k=1}^{n}\mathfrak{S}_n\left(\overrightarrow{A_k},\overrightarrow{\Phi_k}\right)=\sum_{k=1}^{n}\prod_{\substack{j=1\\j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right).$$

This finish the proof of our theorem 1.0.2. □

3 Connection to multiple Rademacher sums

Let $a_1,...,a_n$ be positive integers pairwise coprime . Inspired by Rademacher's work [7] we define multiple Rademacher sums by

$$B(a_1,\ldots,a_n)=\sum_{i=1}^{n}\sum_{t_i=0}^{a_i-1}\frac{t_i}{a_i}\prod_{\substack{j=1\\j\neq i}}^{n}\left[\frac{t_ia_j}{a_i}\right]. \tag{3.0.18}$$

We have a connection between $B(a_1,..,a_n)$ and our multiple Dedekind-Rademacher sums $S_n(\overrightarrow{A_k},\overrightarrow{R_k})$ given by the following result

Theorem 3.0.3 *For any $a_1,...,a_n$, positive integers pairwise coprime, we have*

$$\begin{aligned}B(a_1,\ldots,a_n) &= A_n\sum_{i=1}^{n}\frac{1}{a_i}\sum_{J\subset\{1,\ldots,n\}\setminus\{i\}}F(J)\sum_{k=0}^{|\overline{J}|}\binom{|\overline{J}|+1}{k}\sum_{L\subset J}2^{|L|}S_k(a_i;(a_l)_{l\in L})\\ &-\frac{A_n}{2}\sum_{i=1}^{n}\frac{1}{a_i}\sum_{J\subset\{1,\ldots,n\}\setminus\{i\}}F(J)\left(|\overline{J}|+1\right)\end{aligned}$$

where

$$A_n = \prod_{j=1}^{n} a_j, \quad F(J) = \frac{1}{2^{|J|}} . \frac{1}{|\overline{J}|+1} \prod_{j\in J} \left(\frac{-1}{a_j}\right),$$

and

$$\overline{J} = \{1,\dots,n\} \setminus J \quad , \quad S_k\Big(a_i;(a_l)_{l\in L}\Big) = \sum_{t_i=0}^{a_i-1} \overline{B}_k\left(\frac{t_i}{a_i}\right) \prod_{l\in L} \overline{B}_1\left(\frac{t_i a_l}{a_i}\right).$$

Proof : Let us put $u_i = \frac{t_i}{a_i}, \quad i = 1,\dots,n$. We have

$$\begin{aligned} u_i \prod_{\substack{j=1\\ j\neq i}}^{n} [u_i a_j] &= u_i \prod_{\substack{j=1\\ j\neq i}}^{n} a_j . \prod_{\substack{j=1\\ j\neq i}}^{n} \left(u_i - \frac{1}{a_j}\{u_i a_j\}\right) \\ &= \frac{A_n}{a_i} . \prod_{j=1}^{n} \left(u_i - \frac{1}{a_j}\{u_i a_j\}\right) \\ &= \frac{A_n}{a_i} \sum_{J\subset\{1,\dots,n\}} {u_i}^{|\overline{J}|} \prod_{j\in J} \left(\frac{-1}{a_j}\right) \{u_i a_j\}. \end{aligned}$$

Remark that, if $i \in J$ the product

$$\prod_{j\in J} \left(\frac{-1}{a_j}\right) \{u_i a_j\} = 0$$

so

$$u_i \prod_{\substack{j=1\\ j\neq i}}^{n} [u_i a_j] = \frac{A_n}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} {u_i}^{|\overline{J}|} \prod_{j\in J} \left(\frac{-1}{a_j}\right) \{u_i a_j\}.$$

It follows that

$$B(a_1,\dots,a_n) = A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} \prod_{j\in J} \left(\frac{-1}{a_j}\right) \sum_{t_i=0}^{a_i-1} {u_i}^{|\overline{J}|} \prod_{j\in J} \{u_i a_j\}. \tag{3.0.19}$$

Now, we set $m = |\overline{J}|+1$, $u = \{u_i\}$ and $v = \{u_i a_j\}$.
Then, we use the difference formula 2.1.12 and the identity

$$\{v\} = \overline{B}_1(v) + \frac{1}{2}, \qquad v \in \mathbb{R}\setminus\mathbb{Z}$$

we deduce that, for all $t_i = 1,\cdots,a_i-1$ and all $J \subset \{1,\dots,n\}\setminus\{i\}$, we have

$$\begin{aligned} {u_i}^{|\overline{J}|} &= \frac{1}{|\overline{J}|+1} \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \overline{B}_k(u_i) \\ \prod_{j\in J} \{u_i a_j\} &= \prod_{j\in J} \left(\overline{B}_1(u_i a_j) + \frac{1}{2}\right) \\ &= \sum_{L\subset J} \left(\frac{1}{2}\right)^{|J|-|L|} \prod_{l\in L} \overline{B}_1(u_i a_l). \end{aligned}$$

By substituting these equalities in (3.0.19), we obtain

$$\begin{aligned}
B(a_1,\dots,a_n) &= A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} \frac{1}{|\overline{J}|+1} \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \prod_{j\in J} \frac{-1}{a_j} \\
&\quad \sum_{L\subset J} \left(\frac{1}{2}\right)^{|J|-|L|} \sum_{t_i=1}^{a_i-1} \overline{B}_k(u_i) \prod_{l\in L} \overline{B}_1(u_i a_l) \\
&= A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} \frac{1}{|\overline{J}|+1} \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \prod_{j\in J} \frac{-1}{a_j} \\
&\quad \sum_{L\subset J} \left(\frac{1}{2}\right)^{|J|-|L|} \left(\sum_{t_i=0}^{a_i-1} \overline{B}_k(u_i) \prod_{l\in L} \overline{B}_1(u_i a_l) - \overline{B}_k(0) \prod_{l\in L} \overline{B}_1(0) \right) \\
&= A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} F(J) \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \\
&\quad \sum_{L\subset J} 2^{|L|} \left(\sum_{t_i=0}^{a_i-1} \overline{B}_k(u_i) \prod_{l\in L} \overline{B}_1(u_i a_l) - \overline{B}_k(0) \prod_{l\in L} \overline{B}_1(0) \right).
\end{aligned}$$

We can rewrite this equality as follows

$$\begin{aligned}
B(a_1,\dots,a_n) &= A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} F(J) \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \sum_{L\subset J} 2^{|L|} S_k\left(a_i;(a_l)_{l\in L}\right) \\
&\quad - A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} F(J) \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \overline{B}_k(0).
\end{aligned}$$

Indeed

$$\begin{aligned}
\sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \overline{B}_k(0) &= \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} B_k(0) - \binom{|\overline{J}|+1}{1} B_1(0) \\
&= \left(|\overline{J}|+1\right) 0^{|\overline{J}|} - B_1(0)\left(|\overline{J}|+1\right) \\
&= \frac{|\overline{J}|+1}{2}, \quad \text{since } |\overline{J}| \geqslant 1.
\end{aligned}$$

Finally, we obtain our desired formula. □

4 Connection to the enumeration of lattice points in pyramid in dimension $n+1$.

Let $a_1,...,a_n$ be positive integers pairwise coprime. Let P denote the integral point $(0,...,0,A_n)$ in the Euclid space $\mathbb{R}^{n+1}$ where $A_n = a_1...a_n$. For $I \subset [\![1,n]\!]$ let $P_I = \left(x_1^{(I)},...,x_n^{(I)},0\right)$, where $, x_i^{(I)} = a_i$ or 0 according to wheter $i \in I$ or not. Denote by $N(a_1,...,a_n)$ the number of lattice points $(x_1,...,x_n,x_{n+1})$ with $0 < x_{n+1} < A_n$ in the pyramid with vertices P and $P_I, I \subset [\![1,n]\!]$. Sun in [9] proves the following relation

$$N(a_1,...,a_n) = A_n\left(\prod_{i=1}^{n}(a_i-1) - B(a_1,\dots,a_n)\right). \tag{4.0.20}$$

By combining our result in theorem 3.0.3 with the relation (4.0.20) it's obvious to see that our multiple Dedekind-Rademacher appear in lattice-point enumeration functions. More precisely, we obtain the result

Theorem 4.0.4 *Let $a_1,...,a_n$ be positive integers and pairwise coprime. Then*

$$\begin{aligned} N(a_1,\dots,a_n) &= A_n\prod_{i=1}^{n}(a_i-1) - A_n\sum_{i=1}^{n}\frac{1}{a_i}\sum_{J\subset\{1,\dots,n\}\setminus\{i\}} F(J)\sum_{k=0}^{|\overline{J}|}\binom{|\overline{J}|+1}{k}\sum_{L\subset J}2^{|L|}S_k(a_i;(a_l)_{l\in L}) \\ &\quad + \frac{A_n}{2}\sum_{i=1}^{n}\frac{1}{a_i}\sum_{J\subset\{1,\dots,n\}\setminus\{i\}} F(J)\left(|\overline{J}|+1\right). \end{aligned}$$

Corollary 4.0.5 *(Explicit formulas)*
Let a_1, a_2 be positive integers and coprime. Then

$$\begin{aligned} N(a_1) &= \frac{a_1(a_1-1)}{2}, \\ N(a_1,a_2) &= a_1a_2\left(\frac{1}{3}a_1a_2 - \frac{(a_1+a_2)}{4} + \frac{1}{4}\right) - \frac{a_1^2+a_2^2-1}{12}. \end{aligned}$$

Proof :

i) For $n=1$, thanks to theorem 3.0.3 we have

$$\begin{aligned} B(a_1) &= F(\emptyset)\sum_{k=0}^{1}\binom{2}{k}\sum_{t_1=0}^{a_1-1}\overline{B}_k\left(\frac{t_1}{a_1}\right) - \frac{1}{2}\times F(\emptyset)\times 2 \\ &= \frac{1}{2}\sum_{k=0}^{1}\binom{2}{k}\times a_1^{1-k}\overline{B}_k(0) - \frac{1}{2}. \end{aligned}$$

Then,

$$B(a_1) = \frac{a_1-1}{2}.$$

Using (4.0.20) we obtain

$$N(a_1) = \frac{a_1(a_1-1)}{2}.$$

ii) For $n=2$, from our theorem 3.0.3 we obtain

$$B(a_1,a_2) =$$

$$\begin{aligned} &a_2\left(F(\emptyset)\sum_{k=0}^{2}\binom{3}{k}\sum_{t_1=0}^{a_1-1}\overline{B}_k\left(\frac{t_1}{a_1}\right) + F(\{2\})\left(\sum_{k=0}^{1}\binom{2}{k}\sum_{t_1=0}^{a_1-1}\overline{B}_k\left(\frac{t_1}{a_1}\right) + 2\sum_{k=0}^{1}\binom{2}{k}S_k(a_1,a_2)\right)\right) + \\ &a_1\left(F(\emptyset)\sum_{k=0}^{2}\binom{3}{k}\sum_{t_2=0}^{a_2-1}\overline{B}_k\left(\frac{t_2}{a_2}\right) + F(\{1\})\left(\sum_{k=0}^{1}\binom{2}{k}\sum_{t_2=0}^{a_2-1}\overline{B}_k\left(\frac{t_2}{a_2}\right) + 2\sum_{k=0}^{1}\binom{2}{k}S_k(a_2,a_1)\right)\right) \\ &\quad -\tfrac{a_1a_2}{2}\left(\tfrac{1}{a_1}\times F(\emptyset)\times 3 + \tfrac{1}{a_1}\times F(\{2\})\times 2 + \tfrac{1}{a_2}\times F(\emptyset)\times 3 + \tfrac{1}{a_2}\times F(\{1\})\times 2\right). \end{aligned}$$

We remark, that $F(\emptyset)=\frac{1}{3}$, $F(\{1\})=-\frac{1}{4a_1}, F(\{2\})=-\frac{1}{4a_2}$. By using Raabe formula 2.1.11, we obtain

$$\begin{aligned}
&B(a_1,a_2)=\\
&a_1a_2\left(\frac{1}{3a_1}\sum_{k=0}^{2}\left(\begin{array}{c}3\\k\end{array}\right)a_1^{1-k}\overline{B}_k(0)-\frac{1}{4a_1a_2}\left(\sum_{k=0}^{1}\left(\begin{array}{c}2\\k\end{array}\right)a_1^{1-k}\overline{B}_k(0)+2\times\overline{B}_1(0)+4S(a_1,a_2)\right)\right)+\\
&a_1a_2\left(\frac{1}{3a_2}\sum_{k=0}^{2}\left(\begin{array}{c}3\\k\end{array}\right)a_2^{1-k}\overline{B}_k(0)-\frac{1}{4a_1a_2}\left(\sum_{k=0}^{1}\left(\begin{array}{c}2\\k\end{array}\right)a_2^{1-k}\overline{B}_k(0)+2\times\overline{B}_1(0)+4S(a_2,a_1)\right)\right)\\
&-\frac{1}{2}(a_1+a_2)+\frac{1}{2}.
\end{aligned}$$

We rewrite this formula as follows

$$\begin{aligned}
&B(a_1,a_2)=-\tfrac{1}{2}(a_1+a_2)+\tfrac{1}{2}\\
&a_1a_2\left(\tfrac{1}{3a_1}\left(\sum_{k=0}^{3}\left(\begin{array}{c}3\\k\end{array}\right)a_1^{1-k}B_k(0)-3B_1(0)\right)-\tfrac{1}{4a_1a_2}\left(\sum_{k=0}^{2}\left(\begin{array}{c}2\\k\end{array}\right)a_1^{1-k}B_k(0)-2B_1(0)+4S(a_1,a_2)\right)\right)+\\
&a_1a_2\left(\tfrac{1}{3a_1}\left(\sum_{k=0}^{3}\left(\begin{array}{c}3\\k\end{array}\right)a_1^{1-k}B_k(0)-3B_1(0)\right)-\tfrac{1}{4a_1a_2}\left(\sum_{k=0}^{2}\left(\begin{array}{c}2\\k\end{array}\right)a_1^{1-k}B_k(0)-2B_1(0)+4S(a_1,a_2)\right)\right).
\end{aligned}$$

We use addition formula 2.1.13, then

$$\begin{aligned}
&B(a_1,a_2)=-\frac{1}{2}(a_1+a_2)+\frac{1}{2}\\
&a_1a_2\left(\frac{1}{3a_1}\left(\frac{B_3(a_1)}{a_1^2}+\frac{3}{2}\right)-\frac{1}{4a_1a_2}\left(\frac{B_2(a_1)-B_2(0)}{a_1}+1+4S(a_1,a_2)\right)\right)+\\
&a_1a_2\left(\frac{1}{3a_2}\left(\frac{B_3(a_2)}{a_1^2}+\frac{3}{2}\right)-\frac{1}{4a_1a_2}\left(\frac{B_2(a_2)-B_2(0)}{a_2}+1+4S(a_2,a_1)\right)\right).
\end{aligned}$$

We deduce from $B_1(x)=x-\frac{1}{2}$ $B_2(x)=x^2-x+\frac{1}{6}$ $B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x$ that

$$B(a_1,a_2)=-\frac{1}{2}(a_1+a_2)+\frac{1}{2}+\frac{1}{6}\left(\frac{a_2}{a_1}+\frac{a_1}{a_2}\right)-\frac{1}{4}(a_1+a_2)-\left(S(a_1,a_2)+S(a_2,a_1)\right).$$

By using our main theorem 1.0.2 , we obtain

$$B(a_1,a_2)\quad=\tfrac{2}{3}a_1a_2-\tfrac{3(a_1+a_2)}{4}+\tfrac{3}{4}+\tfrac{a_1^2+a_2^2-1}{12a_1a_2}.$$

From (4.0.20) we obtain our desired result

$$N(a_1,a_2)=a_1a_2\left(\frac{1}{3}a_1a_2-\frac{(a_1+a_2)}{4}+\frac{1}{4}\right)-\frac{a_1^2+a_2^2-1}{12}.\ \square$$

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