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Arithmetical Properties of Elliptic Bernoulli and Euler Numbers

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Abstract

We introduce **elliptic analogues** to the Bernoulli (resp. Euler) numbers and functions. The first aim of this paper is to state and prove that our elliptic Bernoulli and Euler functions satisfied Raabe's formulas (cf. Theorems 3.1.1, 3.2.1).
We define two kinds of **elliptic Dedekind-Rademacher sums**, in terms of values of our elliptic Bernoulli (resp. Euler) functions. The second aim of this paper is to state and prove a **Dedekind reciprocity Law** for our elliptic Dedekind-Rademacher sums (cf.Theorems 4.3.1, 4.4.1).
Our results recover the well known classical results.

1 Bernoulli and Euler polynomials and functions : An overview

Bernoulli and Euler numbers play an important and quite mysterious role in mathematics and in various places like analysis, number theory, differential geometry and algebraic topology.

1.1 Bernoulli and Euler polynomials

We start by fixing notation and recalling some stantard facts about Bernoulli and Euler numbers and polynomials.

Definition 1.1.1

i) The Bernoulli polynomials are given by their generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad |t|<\pi. \tag{1.1.1}$$

ii) The Bernoulli numbers are given by

$$B_n=B_n(0). \tag{1.1.2}$$

iii) The Euler polynomials are given by their generating function

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad |t|<\pi. \tag{1.1.3}$$

iv) There are two types of Euler numbers are given by

$$E_n = 2^nE_n\left(\frac{1}{2}\right) \tag{1.1.4}$$

$$E_n^* = E_n(0) \tag{1.1.5}$$

Bernoulli numbers and polynomials have the following properties

i) (**Raabe Formulae 1**)

$$B_n(ax)=a^{n-1}\sum_{k=0}^{a-1}B_n\left(x+\frac{k}{a}\right),\quad \forall\ a\in\mathbb{N}^*$$

ii) (**Fourier expansion**)

$$B_m(x)=-\frac{m!}{(2\pi i)^m}\sum_{k\neq 0}\frac{e^{2\pi ikx}}{k^m},\forall m\geqslant 1,\quad 0\leqslant x<1.$$

iii) (**Raabe Formulae 2**)

$$E_n(ax)=a^n\sum_{k=0}^{a-1}(-1)^kE_n\left(x+\frac{k}{a}\right)\qquad \text{if } a \text{ is odd integer}$$

iv) (**Raabe Formulae 3**)

$$E_n(ax)=\frac{-2}{n+1}a^n\sum_{k=0}^{a-1}(-1)^kB_{n+1}\left(x+\frac{k}{a}\right)\qquad \text{if } a \text{ is even integer}$$

1.2 Bernoulli and Euler functions

Definition 1.2.1 *The mth Bernoulli function is defined by*

$$\overline{B}_m(x) = -\frac{m!}{(2\pi i)^m} \sum_{k\in\mathbb{Z}\setminus\{0\}}' \frac{e^{2\pi ikx}}{k^m}, \forall m \geqslant 1, \quad x \in \mathbb{R} \tag{1.2.6}$$

The first kind of mth Euler function is defined by

$$\overline{E}_m(x) \quad = 2\frac{m!}{(2\pi i)^{m+1}} \sum_{k\in\mathbb{Z}} \frac{e^{2\pi i(k+\frac{1}{2})x}}{(k+\frac{1}{2})^{m+1}}, \quad \forall m \geqslant 1, \quad 0 \leqslant x < 1 \tag{1.2.7}$$

$$\overline{E}_m(x+1) \quad = \overline{E}_m(x), \quad x \in \mathbb{R} \tag{1.2.8}$$

The second kind of mth Euler function is defined by

$$\overline{E}^*_m(x) = (-1)^{[x]}\overline{E}_m(x) \tag{1.2.9}$$

Here in the definition (1.2.6) the sum $\sum'$ means that in the non-absolutely convergent case $m = 1$, the sum to be interpreted as a Cauchy principal value. Then, by the definition (1.2.6) of $\overline{B}_m(x)$, we have

$$\bar{B}_n(x) = \begin{cases} 0, & \text{If } n = 1, x \in \mathbb{Z} \\ B_n(\{x\}) & \text{Otherwise} \end{cases}$$

Equivalently,

$$\sum_{n=0}^{\infty} \frac{\bar{B}_n(x)}{n!} t^n = \begin{cases} \frac{t}{2}\frac{e^t+1}{e^t-1}, & \text{If } x \in \mathbb{Z} \\ \frac{te^{t\{x\}}}{e^t-1} & \text{Otherwise} \end{cases} \tag{1.2.10}$$

These functions satisfied the so-called "distribution property" (Raabe's formulas)

i)

$$\overline{B}_n(ax) = a^{n-1} \sum_{k=0}^{a-1} \overline{B}_n\left(x + \frac{k}{a}\right), \quad \forall a \in \mathbb{N}^* \tag{1.2.11}$$

ii)

$$\overline{E}_n(ax) = a^n \sum_{k=0}^{a-1} (-1)^{k+[x+\frac{k}{a}]+[ax]} \overline{E}_n\left(x + \frac{k}{a}\right) \text{ if } a \text{ is odd integer} \tag{1.2.12}$$

iii)

$$\overline{E}^*_n(ax) = a^n \sum_{k=0}^{a-1} (-1)^k \overline{E}^*_n\left(x + \frac{k}{a}\right) \text{ if } a \text{ is odd integer.} \tag{1.2.13}$$

For more details about these properties see [1, 2, 36]

2 Elliptic Bernoulli and Euler functions:

We start this section by fixing notations and reviewing the definition of the elliptic Lerch series.

2.1 Elliptic Lerch series

We define the elliptic Lerch function following Weil ([42] VIII §12) as follows.

Definition 2.1.1 *(***The form** $E_L(z,\varphi)$ *Let* $L=\mathbb{Z}$ **or** $L=\mathbb{Z}\tau+\mathbb{Z}$, *where* $\mathbf{Im}(\tau)>0$. *For* $X=X_1\tau+X_2,\ \omega=\omega_1\tau+\omega_2,\ (X_i,\omega_i)\in\mathbb{R}^2,\ i=1,2.$
The form $E_L(z,\varphi)$ *is given by*

$$E_L(\omega,X)=\begin{cases}\omega X & \text{If } L=\mathbb{Z}\\ X_1\omega_2-X_2\omega_1 & \text{If } L=\mathbb{Z}\tau+\mathbb{Z}\end{cases} \tag{2.1.14}$$

Remark that $E_L(z,\varphi)$ *is the symplectic form on* $\mathbb{C}$ *associated to the oriented complex lattice* L. *Note that for two complex lattices* $L\subset\Lambda$ *we have*

$$E_\Lambda=[\Lambda:L]E_L$$

where $[\Lambda:L]$ *indicates the number of elements of* Λ/L.

Definition 2.1.2 *Let* m *be an integer,* L *a complex lattice and let* $z,\ \varphi\in\mathbb{C}$. *We introduce the elliptic Lerch series*

$$L_m(s,z,\varphi;L)=\sum_{\omega\in L\setminus\{-z\}}\frac{(\overline{z}+\overline{\omega})^m}{|z+\omega|^{2s}}e\left(E_L(\omega,\varphi)\right)$$

We quote from [42] the following result

Proposition 2.1.3 *Let* m *be an integer.*

i) The function $L_m(s,z,\varphi;L)$ *converges for* $\Re(s)>\frac{m}{2}+1$,

ii) The function $L_m(s,z,\varphi;L)$ *for* s *continues meromorphically to a function on the whole* s *plane, with a simple pole only at* $s=1$ *if* $m=0$ *and* $\varphi\in L$.

iii) The function $L_m(s,z,\varphi;L)$ *satisfies the functional equation*

$$\Gamma(s)L_m(s,z,\varphi,L)=A^{m+1-2s}\Gamma(m+1-s)L_m(m+1-s,\varphi,z;L)e\left(E_L(z,\varphi)\right).$$

2.2 Main objects

In this section we consider $L = \mathbb{Z}\tau + \mathbb{Z}$, $\mathrm{Im}(\tau) > 0$. The case $L = \mathbb{Z}$ it corresponds to the classical case.
Now, we introduce the main objects of the study, are defined by

Definition 2.2.1

i) The **elliptic Bernoulli functions** *are defined by*

$$\begin{aligned}\overline{B}_m(X,L) &= -\frac{\Gamma(m)}{(2\pi i)^m} L_m(s=m, z=0, X; L) \\ &= -\frac{m!}{(2\pi i)^m} \sideset{}{'}\sum_{\omega\in L\setminus\{0\}} \frac{e\left(E_L(\omega, X)\right)}{\omega^m}, \ \forall m \geqslant 1\end{aligned}$$

Here the sum $\sideset{}{'}\sum$ *means that in the non-absolutely convergent case* $m = 1$*, the sum to be interpreted as a Cauchy principal value.*

ii) The **elliptic Euler functions** *are given by*

$$\begin{aligned}\overline{E}_m(X,L) &= 2\frac{\Gamma(m)}{(2\pi i)^{m+1}} L_{m+1}(s=m+1, z=\frac{1}{2}, X; L) \\ &= 2\frac{m!}{(2\pi i)^{m+1}} \sum_{\omega\in L} \frac{e\left(E_L(\omega, X)\right)}{\left(\omega+\frac{1}{2}\right)^{m+1}}, \ \forall m \geqslant 1.\end{aligned}$$

Examples:

i) For $L = \mathbb{Z}$:

$$\begin{aligned}\overline{B}_m(X,L) &= -\frac{m!}{(2\pi i)^m} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e\left(kX\right)}{k^m} \\ &= \overline{B}_m(X)\end{aligned}$$

ii) $L = \mathbb{Z}$: Let $0 \leqslant X < 1$. Then,

$$\begin{aligned}\overline{E}_m(X,L)e\left(\frac{X}{2}\right) &= 2\frac{m!}{(2\pi i)^{m+1}} \sum_{k\in\mathbb{Z}} \frac{e\left(\left(k+\frac{1}{2}\right)X\right)}{\left(k+\frac{1}{2}\right)^{m+1}} \\ &= \overline{E}_m(X)\end{aligned}$$

2.3 Jacobi forms

Our elliptic Bernoulli and Euler functions are connected to Jacobi forms in three variables. Precisely, we have

Theorem 2.3.1

i) The generating function of $\overline{B}_m(\varphi, L)$ is the Jacobi form given by

$$D_L(z;\varphi) = e\Big(z\varphi_1\Big)\frac{\theta'_\tau(0)\theta_\tau\left(z+\varphi\right)}{\theta_\tau\left(z\right)\theta_\tau\left(\varphi\right)}$$

ii) The generating function of $\overline{E}_m(\varphi, L)$ is the following Jacobi form given by

$$F_L(z;\varphi) = -\frac{1}{\pi i}e\Big((z-\frac{1}{2})\varphi_1\Big)\frac{\theta'_\tau(0)\theta_\tau\left(z+\varphi-\frac{1}{2}\right)}{\theta_\tau\left(z-\frac{1}{2}\right)\theta_\tau\left(\varphi\right)}$$

where

$$\theta_\tau(z) = \sum_{n\in\mathbb{Z}} e\left(\frac{1}{2}(n+\frac{1}{2})^2\tau + (n+\frac{1}{2})(z+\frac{1}{2})\right)$$

Proof :

i) The generating function of $\overline{B}_m(\varphi, L)$ is

$$\sum_{m\geqslant 0}\overline{B}_m(\varphi, L)\frac{(2\pi i)^m}{m!}z^{m-1}$$

Now, we are going to prove a). First, we prove the equlaity

$$D_L(z;\varphi) = \sum_{\omega\in L}^{(e)}\frac{e(-E_L(\omega,\varphi))}{z+\omega}. \tag{2.3.15}$$

Where $\sum_{\omega\in L}^{(e)}$ is the Eisenstein summation equal to

$$\sum_{\omega\in L}^{(e)} = \lim_{M,N\to\infty}\sum_{m=-M}^{m=M}\sum_{m=-N}^{m=N}, \text{ Where } \omega = m\tau + n.$$

The functions $z \to D_L(z;\varphi)$ and $F_\varphi : z \to \sum_{\omega\in L}^{(e)}\frac{e(-E_L(\omega,\varphi))}{z+\omega}$ are meromorphic functions with only simple poles in $\omega \in L$. Moreover,

$$\begin{aligned} D_L(z+\rho;\varphi) &= e(E_L(\rho,\varphi))D_L(z;\varphi) \\ F_\varphi(z+\rho) &= e(E_L(\rho,\varphi))F_\varphi(z)\forall\rho\in L. \end{aligned}$$

and finally, they have the same residue at $z = \omega, \omega \in L$:

$$\mathrm{Res}\Big(F_\varphi(z)dz, z = \omega\Big) = \mathrm{Res}\Big(D_L(z;\varphi)dz, z = \omega\Big) = e\Big(E_L(\omega,\varphi)\Big)$$

We introduce the quotient function $z \to \frac{F_\varphi(z)}{D_L(z;\varphi)}$. It is meromorphic, periodic with periods the lattice L and modulo L it has only one pole (a simple one) at $z = -\varphi$. Then, from Liouville's Theorem, the functions $z \to D_L(z;\varphi)$ and $F_\varphi(z)$ are proportional. But, we know that they have the same residue at $z = \omega, \omega \in L$. Thus we get our desired equality 2.3.15.

We use the Laurent expansion of $z \to \frac{1}{z+\omega}$. For $\omega \in L\backslash\{0\}$ we have

$$\frac{1}{z+\omega} = \sum_{k\geqslant 0}(-1)^k\left(\frac{z}{\omega}\right)^k \tag{2.3.16}$$

For our convenance we use $-\omega$ instead ω. Then,

$$D_{L_\tau}(z;\varphi) = \frac{1}{z} + \sum_{m\geqslant 1}\left(\sum_{\substack{\omega\in L\\ \omega\neq 0}}^{(e)}\frac{e(E_L(\omega,\varphi))}{\omega^m}\right)z^{m-1}$$

Now, we know that

$$\overline{B}_m(\varphi, L) = -\frac{m!}{(2\pi i)^m}\sum_{\omega\in L\backslash\{0\}}\frac{e\left(E_L(\omega,\varphi)\right)}{\omega^m},$$

Then, we conclude that the Jacobi form $D_{L_\tau}(z;\varphi)$ is the generating function of $\overline{B}_m(\varphi, L)$.

ii) Similarly, we can prove the property b).
In fact, the generating function of $\overline{E}_m(\varphi, L)$ is defined by

$$\sum_{m\geqslant 0}\overline{E}_m(\varphi, L)\frac{(2\pi i z)^m}{m!}$$

We remark that

$$F_L(z;\varphi) = -\frac{1}{\pi i}D_L(z-\frac{1}{2};\varphi) \tag{2.3.17}$$

Then,

$$F_L(z;\varphi) = -\frac{1}{\pi i}\sum_{\omega\in L}^{(e)}\frac{e(-E_L(\omega,\varphi))}{z+\omega-\frac{1}{2}}$$

which is equal to, thanks to (2.3.16),

$$F_L(z;\varphi) = -\frac{1}{\pi i}\sum_{m\geqslant 0}\left(\sum_{\omega\in L}^{(e)}\frac{e(E_L(\omega,\varphi))}{\left(\omega+\frac{1}{2}\right)^{m+1}}\right)z^m$$

Now,

$$\overline{E}_m(X,L) = 2\frac{m!}{(2\pi i)^{m+1}}\sum_{\omega\in L}\frac{e\left(E_L(\omega,X)\right)}{\left(\omega+\frac{1}{2}\right)^{m+1}}$$

Hence

$$F_L(z;\varphi) = \sum_{m\geqslant 0}\overline{E}_m(\varphi,L)\frac{(2\pi i z)^m}{m!}$$

Thus, it's give us our desired result.

□

For more details concerning the Jacobi form $D_L(z;\varphi)$, see [7, 9, 10, 11, 12, 14, 15].

3 Elliptic Raabe formulas

The aim of this section, is to study the distribution formulas (elliptic Raabe formulas) satisfied by elliptic Bernoulli and elliptic Euler functions.

3.1 Elliptic Raabe formulae for elliptic Bernoulli functions

Theorem 3.1.1 *(***Elliptic Bernoulli functions***)*

i)

$$\overline{B}_m(\lambda\varphi,\lambda L) = \lambda^{-m}\overline{B}_m(\varphi,L), \forall\lambda\in\mathbb{C}\backslash\{0\}.$$

ii)

$$\overline{B}_m\left(\frac{\varphi}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^m\overline{B}_m(\varphi;\tau),\ \forall\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$$

iii)

$$\overline{B}_m(\varphi+\rho;\tau)=\overline{B}_m(\varphi;\tau),\forall\rho\in\mathbb{Z}\tau+\mathbb{Z}$$

iv)

$$\overline{B}_m(-\varphi;\tau)=(-1)^{m-1}\overline{B}_m(\varphi;\tau)$$

v) For L, Λ lattices such that : $L\subset\Lambda$. Then,

$$\sum_{\bar{t}\in\Lambda/L}\overline{B}_m(\varphi+t;L)=[\Lambda:L]^{1-m}\overline{B}_m(\varphi;\Lambda),m\geqslant 1,$$

We start by recalling some basic properties de Jacobi form D_L. We quote from [7, 10, 11, 12, 14] the following theorem

Theorem 3.1.2 *(***Jacobi forms***)*

i) D_L is meromorphic in the first variable z, and only real analytic on the second variable φ.

ii) D_L is homogenous of degre -1

$$D_{\lambda L}(\lambda z;\lambda\varphi)=\lambda^{-1}D_L(z;\varphi),\forall\lambda\in\mathbb{C}\backslash\{0\}.$$

In particular, we have the following symmetry

$$D_L(-z;-\varphi)=-D_L(z;\varphi). \tag{3.1.18}$$

*iii) (***Periodicity** *of $D_L(z;\varphi)$):*

$$\begin{cases} D_L(z;\varphi+\rho)=D_L(z;\varphi)\\ D_L(z+\rho;\varphi)=e(E_L(\rho,\varphi))D_L(z;\varphi)\end{cases},\forall\rho\in L \tag{3.1.19}$$

*iv) (***Functional Equation***):* $D_L(z;\varphi)e(-E_L(z,\varphi))=D_L(\varphi;z)$.

*v) (***Modularity***): D_L is a Jacobi modular form for $SL_2(\mathbb{Z})$, with index 0 and weight 1 i.e*

$$D_{\frac{a\tau+b}{c\tau+d}}\left(\frac{z}{c\tau+d};\frac{\varphi}{c\tau+d}\right)=(c\tau+d)D_\tau(z;\varphi),\forall\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z}) \tag{3.1.20}$$

where

$$D_\tau(z;\varphi):=D_{L_\tau}(z;\varphi),L_\tau=\mathbb{Z}\tau+\mathbb{Z},\tau\in\mathcal{H}.$$

vi) For L, Λ complex lattices such that : $L \subset \Lambda$, $[\Lambda : L] = l$. We have:

$$\sum_{\bar{t}\in\Lambda/L} D_L(lz;\varphi+t) = D_\Lambda(z;\varphi) \tag{3.1.21}$$

Proof of theorem 3.1.1:

We give the proof of property (e):
We start from the relation 3.1.21. From this relation, we use the Laurent expansion of the functions $z \to D_L(lz;\varphi+t)$ and $z \to D_\Lambda(z;\varphi)$. We compare their coeficients, then we obtain our distribution formulae

$$\sum_{\bar{t}\in\Lambda/L} \overline{B}_m(\varphi+t;L) = [\Lambda : L]^{1-m}\overline{B}_m(\varphi;\Lambda).$$

We prove the modularity property (b):
We know that, the Jacobi form D_L is a modular form for $\mathrm{SL}_2(\mathbb{Z})$, with index 0 and weight 1 3.1.20.
Again from the relation 3.1.20, we use the Laurent expansion of the functions $z \to D_{\frac{a\tau+b}{c\tau+d}}\left(\frac{z}{c\tau+d};\frac{\varphi}{c\tau+d}\right)$ and $z \to D_\Lambda(z;\varphi)$. We compare their coeficients, then we obtain our distribution modularity formulae

$$\overline{B}_m\left(\frac{\varphi}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^m\overline{B}_m(\varphi;\tau),\ \forall\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Also the periodicity 3.1.18, symmetry 3.1.18 and homogeneity 3.1.18 of $\overline{B}_m(\varphi;L)$ coming from , in the same way, from the periodicity 3.1.19 , symmetry 3.1.18 and homogeneity 3.1.18 of the Jacobi form D_L. □

3.2 Elliptic Raabe formulae for elliptic Euler functions

In this section we state the the analogues of Raabe formulas for the elliptic Euler functions. Here the situation is different from the elliptic Bernoulli functions, we must to distinguish two cases: odd or even index. More precisely, we have

Theorem 3.2.1

i) For L, Λ lattices such that : $L \subset \Lambda$ with $[\Lambda : L]$ **odd.** *Then, for $m \geqslant 1$*

$$\overline{E}_m(\varphi;\Lambda) = [\Lambda : L]^m \sum_{\bar{t}\in\Lambda/L} e\left(\frac{1}{2}E_\Lambda(1,\varphi+t) - \frac{1}{2}E_L(1,\varphi+t)\right)\overline{E}_m(\varphi+t;L).$$

ii) For L, Λ lattices such that : $L \subset \Lambda$ with $[\Lambda : L]$ **even.** *Then, for $m \geqslant 1$*

$$\overline{E}_m(\varphi;\Lambda) = -2\frac{[\Lambda : L]^{-m}}{m+1} \sum_{\bar{t}\in\Lambda/L} e\left(\frac{1}{2}E_\Lambda(1,\varphi+t)\right)\overline{B}_m(\varphi+t;L).$$

Proof : The proof of this theorem is similar to the proof of theorem 3.1.1 . In fact, in place of the Jacobi form D_L we consider the new one $F_L(z;\varphi)$. From 2.3.17, we have

$$F_L(z;\varphi) = -\frac{1}{\pi i}D_L(z-\frac{1}{2};\varphi)$$

Now we change z by $z-\frac{1}{2}$ and by using the distribution formulae 3.1.21 satisfied by the Jacobi form D_L we obtain

$$\sum_{\bar{t}\in\Lambda/L} D_L(lz-\frac{l}{2};\varphi+t) = D_\Lambda(z-\frac{1}{2};\varphi) \tag{3.2.22}$$

Now, we are going to use the periodicity property of the Jacobi form D_L 3.1.19. We distinguish two cases:

i) We assume that $[\Lambda : L] = l$ is **odd.** Then

$$\sum_{\bar{t}\in\Lambda/L} D_L(lz-\frac{1}{2}+\frac{l-1}{2};\varphi+t) = D_\Lambda(z-\frac{1}{2};\varphi)$$

give us

$$\sum_{\bar{t}\in\Lambda/L} D_L(lz-\frac{1}{2};\varphi+t)e\left(\frac{1}{2}E_\Lambda(1,\varphi+t)-\frac{1}{2}E_L(1,\varphi+t)\right) = D_\Lambda(z-\frac{1}{2};\varphi)$$

By writing the Laurent expansions of the functions $z \to D_L(lz-\frac{1}{2};\varphi+t)e\left(\frac{1}{2}E_\Lambda(1,\varphi+t)-\frac{1}{2}E_L(1,\varphi+t)\right)$ and $z \to D_\Lambda(z-\frac{1}{2};\varphi)$, at $z=0$, we obtain from their coefficients our desired equality

$$\overline{E}_m(\varphi;\Lambda) = [\Lambda : L]^m \sum_{\bar{t}\in\Lambda/L} e\left(\frac{1}{2}E_\Lambda(1,\varphi+t)-\frac{1}{2}E_L(1,\varphi+t)\right)\overline{E}_m(\varphi+t;L).$$

ii) We assume that $[\Lambda : L] = l$ is **even.** Then

$$\sum_{\bar{t}\in\Lambda/L} D_L(lz-\frac{l}{2};\varphi+t) = D_\Lambda(z-\frac{1}{2};\varphi)$$

give us

$$\sum_{\bar{t}\in\Lambda/L} D_L(lz;\varphi+t)e\left(\frac{1}{2}E_L(l,\varphi+t)\right) = D_\Lambda(z-\frac{1}{2};\varphi)$$

Thanks to $E_\Lambda = [\Lambda : L]E_L = lE_L$ and by camparing Laurent expansions of the functions $z \to D_L(lz;\varphi+t)e\left(\frac{1}{2}E_L(l,\varphi+t)\right)$, $D_\Lambda(z-\frac{1}{2};\varphi)$ we obtain the second elliptic Raabe relation

$$\overline{E}_m(\varphi;\Lambda) = -2\frac{[\Lambda : L]^{-m}}{m+1}\sum_{\bar{t}\in\Lambda/L} e\left(\frac{1}{2}E_\Lambda(1,\varphi+t)\right)\overline{B}_m(\varphi+t;L).$$

□

4 Elliptic Dedekind Reciprocity Laws

The aim of this section is to state and prove the elliptic Dedekind reciprocity laws attached to these functions. We start by recalling the classical motivations and results.

4.1 Classical Dedekind Reciprocity law: An overview.

Dedekind-Rademacher sums have many applications. In particular, they can be used

i) to prove the Gauss quadratic reciprocity law [34]

ii) to prove quadratic reciprocity law in the quadratic imaginary fields by Bayad [8]

iii) to compute the number of lattice points in a pyramid of dimension $n+1$ [13], [41].

iv) to obtain the class number formulas [35], Hardy-Ramanujan partition formula [37], index formula of Hirzebruch (signature default) and Riemann-Roch Theorem [4, 5, 6, 24, 28, 30].

v) to study the Eisenstein cohomology of $\mathrm{SL}_2(\mathbb{Z})$ [34] and $\mathrm{SL}_2(O_K)$ [39, 31, 32, 43], where O_K is the ring of integers of imaginary qudratic field.

vi) to express special values of partial zeta functions associated to totally real number fields in terms of these sums via the Eisenstein cocycle [26, 43].

If a, b are coprime integers the classical Dedekind sum $S(a,b)$ is definde by

$$S(a,b) = \sum_{t=0}^{a-1} \overline{B}_1\left(\frac{bt}{a}\right)\overline{B}_1\left(\frac{t}{a}\right).$$

Dedekind introduced this sum in connection with transformation formula for the Dedekind η-function and deduced from this his reciprocity formula

$$S(a,b) + S(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) . \tag{4.1.23}$$

The Dedekind sum has been generalized, notably by Rademacher, who introduced the homogeneous sums $S(a,b,c)$ in which the fraction $\frac{t}{a}$ in the left-hand Bernoulli function above becomes $\frac{ct}{a}$. Precisely, we set

$$S(a,b,c) := \sum_{t=0}^{a-1} \overline{B}_1\left(\frac{bt}{a}\right)\overline{B}_1\left(\frac{ct}{a}\right)$$

Rademacher's reciprocity formula, which is not implied by Dedekind's, is for $a, b, c \in \mathbb{N}^*$ are pairwise coprime

$$S(a,b,c) + S(b;c,a) + S(c;a,b) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{c}{ab} + \frac{b}{ac}\right) . \tag{4.1.24}$$

4.2 Elliptic Dedekind Reciprocity laws

In this section we state and prove elliptic analogues to Dedekind and Rademacher theorems 4.1.23 and 4.1.24.

Definition 4.2.1 *Let a, b, c be three positive integers with no common and x, y, z three complex numbers.*

i) The Dedekind-Rademacher elliptic sums attached to elliptic Bernoulli functions are defined by

$$S_{m,k}(a,b,c|x,y,z) = \frac{1}{c}\sum_{\overline{t}\in L/cL} \overline{B}_m\left(a\frac{z+t}{c} - x; L\right)\overline{B}_k\left(b\frac{z+t}{c} - y; L\right).$$

ii) The Dedekind-Euler-Rademacher elliptic sums attached to elliptic Euler functions are defined by

$$E_{m,k}(a,b,c|x,y,z) = \frac{1}{c}\sum_{\overline{t}\in L/cL+\frac{1}{2}} \overline{E}_m\left(a\frac{z+t}{c}-x;L\right)\overline{E}_k\left(b\frac{z+t}{c}-y;L\right).$$

We want to prove that these sums, which are elliptic analogues to the classical Dedekind Rademacher sums (writing in terms of values of Bernoulli and Euler functions), satisfy certain reciprocity relations under simultaneous cyclic permuations of (a,b,c) and (x,y,z). However, these reciprocity relations mix pairs of indices (m,n), so they are most conveniently stated in terms of their generating functions. Therefore, we obtain

4.3 Dedekind Reciprocity law for elliptic Dedekind-Rademacher sums

Theorem 4.3.1 *Let a,b,c be three positive integers with no common factor and x,y,z three complex numbers.*

Then, we have

i) The generating function

$$\mathfrak{S}\left(a,b,c|x,y,z|X,Y,Z\right) := \sum_{m,n\geqslant 0}\frac{S_{m,n}(a,b,c|x,y,z)}{m!n!}\left(\frac{2\pi iX}{\overline{a}}\right)^{m-1}\left(\frac{2\pi iY}{\overline{b}}\right)^{n-1}$$

is equal to

$$\mathfrak{S}\left(a,b,c|x,y,z|X,Y,Z\right) = \frac{1}{c}\sum_{\overline{t}\in L/cL} D_L\left(\frac{X}{\overline{a}},a\frac{z+t}{c}-x\right)D_L\left(\frac{Y}{\overline{b}},b\frac{z+t}{c}-y\right)$$

ii) Let X,Y,Z three variables such that

$$X+Y+Z=0, az-cx\notin <a,c>_L, ay-bx\notin <a,b>_L, bz-cy\notin <b,c>_L .$$

Then, we have the elliptic Dedekind reciprocity Law

$$\mathfrak{S}_3\left(a,b,c|x,y,z|X,Y,Z\right)+\mathfrak{S}_3\left(b,c,a|y,z,x|Y,Z,X\right)+\mathfrak{S}_3\left(c,a,b|z,x,y|Z,X,Y\right)=0$$

Proof : We compute the generating function of $S_{m,n}(a,b,c|x,y,z)$.

$$\begin{aligned}
\mathfrak{S}\left(a,b,c|x,y,z|X,Y,Z\right) &= \sum_{m,n\geqslant 0}\frac{S_{m,n}(a,b,c|x,y,z)}{m!n!}\left(\frac{2\pi iX}{\overline{a}}\right)^{m-1}\left(\frac{2\pi iY}{\overline{b}}\right)^{n-1}\\
&= \frac{1}{c}\sum_{\overline{t}\in L/cL}\sum_{m\geqslant 0}\frac{\overline{B}_m\left(a\frac{z+t}{c}-x;L\right)}{m!}\left(\frac{2\pi iX}{\overline{a}}\right)^{m-1}\times\\
&\quad\sum_{n\geqslant 0}\frac{\overline{B}_n\left(b\frac{z+t}{c}-y;L\right)}{m!}\left(\frac{2\pi iY}{\overline{a}}\right)^{n-1}\\
&= \frac{1}{c}\sum_{\overline{t}\in L/cL}D_L\left(\frac{X}{\overline{a}},a\frac{z+t}{c}-x\right)D_L\left(\frac{Y}{\overline{b}},b\frac{z+t}{c}-y\right)
\end{aligned}$$

Thus, we obtain (a).
We prove now (b). For this, we consider the function

$$F(z)=\prod_{1\leqslant j\leqslant 3}D_{L_j}\left(a_jz;\frac{\varphi_j}{\overline{a_j}}\right),\quad \text{where } a_1=a,a_2=b,a_3=c$$

This function is meromorphic, and periodic with periods the Lattice L, because $\varphi_1+\varphi_2+\varphi_3=0$. Thanks to the hypothesis $az-cx\notin< a,c >_L$ $,ay-bx\notin< a,b >_L, bz-cy\notin< b,c >_L$, the poles of this functions are (modulo L) $z=\frac{t}{a_j},\overline{t}\in L/a_jL$ all its poles have a simple multiplicity. Then, by residue Theorem, we obtain

$$\sum_{k=1}^{3}\frac{1}{a_k}\sum_{\overline{t_k}\in L_k/a_kL\setminus\{\overline{0}\}}\prod_{1\leqslant j\neq k\leqslant 3}D_L\left(\frac{\varphi_j}{\overline{a_j}};a_j\frac{\tilde{t}_k}{a_k}\right)=0$$

Thus, we deduce

$$\mathfrak{S}_3\left(a,b,c|x,y,z|X,Y,Z\right)+\mathfrak{S}_3\left(b,c,a|y,z,x|Y,Z,X\right)+\mathfrak{S}_3\left(c,a,b|z,x,y|Z,X,Y\right)=0$$

This is our desired result in (b). □

4.4 Dedekind Reciprocity law for elliptic Dedekind-Euler-Rademacher sums

Theorem 4.4.1 *Let a,b,c be three positive integers with no common factor and x,y,z three complex numbers.*

i) The generating function of $E_{m,n}(a,b,c|x,y,z)$ *is*

$$\mathfrak{P}\left(a,b,c|x,y,z|X,Y,Z\right) := \sum_{m,n\geqslant 0} \frac{E_{m,n}(a,b,c|x,y,z)}{m!n!}\left(\frac{2\pi i X}{\overline{a}}\right)^m \left(\frac{2\pi i Y}{\overline{b}}\right)^n$$

$$\mathfrak{P}\left(a,b,c|x,y,z|X,Y,Z\right) = \frac{1}{c}\sum_{\overline{t}\in L/cL+\frac{1}{2}} F_L\left(\frac{X}{\overline{a}}, a\frac{z+t}{c}-x\right) F_L\left(\frac{Y}{\overline{b}}, b\frac{z+t}{c}-y\right)$$

where

$$F_L(z;\varphi) = -\frac{2}{2\pi i} e\Big((z-1/2)\varphi_1\Big)\frac{\theta'_\tau(0)\theta_\tau\left(z+\varphi-\frac{1}{2}\right)}{\theta_\tau\left(z-\frac{1}{2}\right)\theta_\tau\left(\varphi\right)}$$

ii) Let X,Y,Z *three variables such that*

$$X+Y+Z = \frac{\overline{a}+\overline{b}+\overline{c}}{2}, az-cx \notin < a,c >_L, ay-bx \notin < a,b >_L, bz-cy \notin < b,c >_L .$$

Then, we obtain the elliptic Dedekind reciprocity Law given by

$$\mathfrak{P}\left(a,b,c|x,y,z|X,Y,Z\right) + \mathfrak{P}\left(b,c,a|y,z,x|Y,Z,X\right) + \mathfrak{P}\left(c,a,b|z,x,y|Z,X,Y\right) = 0$$

Proof :

i) The proof of i) is similar to the proof of the property (a) in theorem 4.3.1.

ii) The proofs of this theorem are similars to the proofs of theorem 3.1.1 and theorem 4.3.1. In fact, in place of the Jacobi form D_L we consider the new one $F_L(z;\varphi)$ given by 2.3.17.

□

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