q-EXTENSION OF BERNSTEIN POLYNOMIALS WITH WEIGHT $(\alpha,\ \beta)$

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Abstract In this paper, we consider a new type of q-Bernstein polynomials with fixed weight (α, β). We call them q-Bernstein polynomials with weight (α, β). We establish many interesting identities and relations satisfied by these new polynomials. These formulae are the arithmetical nature.

1. Introduction

In a recent paper [5] T. Kim proposed q-Bernstein positive linear operators and q-Bernstein polynomials. This paper seems to be a new approach in the area of q-extension of Bernstein polynomials. The main idea of our paper is to study an extension of q-Bernstein operators and polynomials which are introduced by Kim [5]. That is, we considered the extension of Kim's q-Bernstein polynomials and we call them the weighted q-Bernstein polynomials in the viewpoint of the generalization of Kim's q-Bernstein polynomials.

Throught this paper, we assume $0 < q < 1$. Define the q-numbers of x by $[x] = \frac{1-q^x}{1-q}$. Let $\mathcal{C}([0,1],\mathbb{R})$ be the space of continuous functions on $[0,1]$. For $\alpha, \beta \in \mathbb{R}$ and $f \in \mathcal{C}([0,1],\mathbb{R})$, we consider linear positive operators as follows:

$$\begin{aligned}\mathbb{B}_{n,q}(f,x|\alpha,\beta) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x]_{q^{\alpha}}^{k}[1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x,q|\alpha,\beta). \qquad (1)\end{aligned}$$

Here $\mathbb{B}_{n,q}(f,x|\alpha,\beta)$ is called q-Bernstein operator with weight (α,β) of order n. $B_{k,n}(x,q|\alpha,\beta)$ are called the q-Bernstein polynomials with weight (α,β) of degree n.

2010 Mathematics Subject Classification : 11B37, 11B65, 11C08.

Key words and phrases : q-Bernstein polynomials with weight (α,β), difference operator, Stirling numbers.

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This paper was supported by the research grant Kwangwoon University in 2011.

2. Weighted q-Bernstein polynomials

From (1) and [1,5], we note that the generating function for $B_{k,n}(x,q|\alpha,\beta)$ is given by

$$\begin{aligned} F^{(k)}_{q,\alpha,\beta}(t,x) &= \sum_{n=0}^{\infty} B_{k,n}(x,q|\alpha,\beta)\frac{t^n}{n!}, \\ &= \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!}, \end{aligned} \tag{2}$$

where $n,k \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}$ and $\alpha,\beta\in\mathbb{R}$.
By (2), we get

$$B_{k,n}(x,q|\alpha,\beta) = \begin{cases} \binom{n}{k}[x]^k_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}}, & \text{if } n\geq k, \\ 0, & \text{if } n<k, \end{cases} \tag{3}$$

Note that $\lim_{q\to 1} B_{k,n}(x,q|\alpha,\beta) = B_{k,n}(x)$ is the ordinary Bernstein polynomial of degree n.
In this section we prove the following interesting properties of weighted q-Bernstein polynomials:

Theorem 1 (Recurrence formula 1)**.** *For $n,k\in\mathbb{Z}_+$, $\alpha,\beta\in\mathbb{R}$ and $x\in[0,1]$, we have*

$$[1-x]_{q^{-\beta}}B_{k,n-1}(x,q|\alpha,\beta) + [x]_{q^\alpha}B_{k-1,n-1}(x,q|\alpha,\beta) = B_{k,n}(x,q|\alpha,\beta),$$

and

$$\frac{d}{dx}B_{k,n}(x,q|\alpha,\beta) = \left(\frac{\alpha\log q}{q^\alpha-1}q^{\alpha x}B_{k-1,n-1}(x,q|\alpha,\beta) - \frac{q^\beta\log q^\beta}{q^\beta-1}q^{-\beta+\beta x}B_{k,n-1}(x,q|\alpha,\beta)\right)n.$$

Proof of Theorem 1 : For $0\leq k\leq n$, we have

$$\begin{aligned} &[1-x]_{q^{-\beta}}B_{k,n-1}(x,q|\alpha,\beta) + [x]_{q^\alpha}B_{k-1,n-1}(x,q|\alpha,\beta) \\ &= [1-x]_{q^{-\beta}}\binom{n-1}{k}[x]^k_{q^\alpha}[1-x]^{n-k-1}_{q^{-\beta}} + [x]_{q^\alpha}\binom{n-1}{k-1}[x]^{k-1}_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}} \\ &= \binom{n-1}{k}[x]^k_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}} + \binom{n-1}{k-1}[x]^k_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}} \\ &= \binom{n}{k}[x]^k_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}} = B_{k,n}(x,q|\alpha,\beta) \end{aligned} \tag{4}$$

and the derivative of the q-Bernstein polynomials with weight (α,β) of degree n are also polynomials of degree $n-1$:

$$\begin{aligned} &\frac{d}{dx}B_{k,n}(x,q|\alpha,\beta) \\ &= k\binom{n}{k}[x]^{k-1}_{q^\alpha}[1-x]^{n-k}_{q^{-\beta}}\frac{\alpha\log q}{q^\alpha-1}q^{\alpha x} + (n-k)\binom{n}{k}[x]^k_{q^\alpha}[1-x]^{n-k-1}_{q^{-\beta}}\frac{q^\beta\log q^\beta}{1-q^\beta}q^{-\beta+\beta x} \\ &= \left(\frac{\alpha\log q}{q^\alpha-1}q^{\alpha x}B_{k-1,n-1}(x,q|\alpha,\beta) - \frac{q^\beta\log q^\beta}{q^\beta-1}q^{-\beta+\beta x}B_{k,n-1}(x,q|\alpha,\beta)\right)n. \end{aligned} \tag{5}$$

Therefore we obtain our desired theorem. □

Theorem 2 (Symmetry and various identities)**.** *For $n, k \in \mathbb{Z}_+$, $\alpha, \beta \in \mathbb{R}$ and $x \in [0,1]$, we have*

$$B_{n-k,n}(1-x, q^{-1}|\beta, \alpha) = B_{k,n}(x, q|\alpha, \beta)$$

Morover,

$$\mathbb{B}_{n,q}(1, x|\alpha, \beta) = \left(1 + [x]_{q^\alpha} - [x]_{q^\beta}\right)^n$$

and

$$\mathbb{B}_{n,q}(x, x|\alpha, \beta) = [x]_{q^\alpha}\left(1 + [x]_{q^\alpha} - [x]_{q^\beta}\right)^{n-1}.$$

Proof of Theorem 2 :

Let f be a continuous function on $[0,1]$. Then the q-Bernstein operator with weight (α, β) of degree n for f is defined by

$$\mathbb{B}_{n,q}(f, x|\alpha, \beta) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x, q|\alpha, \beta) \tag{6}$$

where $0 \le x < 1$, $n \in \mathbb{Z}_+$.
By (5) and (6), we get

$$\begin{aligned}
\mathbb{B}_{n,q}(1, x|\alpha, \beta) &= \sum_{k=0}^{n} B_{k,n}(x, q|\alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k} \\
&= \left([x]_{q^\alpha} + [1-x]_{q^{-\beta}}\right)^n = \left(1 + [x]_{q^\alpha} - [x]_{q^\beta}\right)^n.
\end{aligned}$$

Also, we get from (6) that for $f(x) = x$:

$$\begin{aligned}
\mathbb{B}_{n,q}(x, x|\alpha, \beta) &= \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k} \\
&= \sum_{k=1}^{n} \frac{k}{n} \frac{n(n-1)\cdots(n-k+1)}{k!} [x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} [x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} [x]_{q^\alpha}^{k+1} [1-x]_{q^{-\beta}}^{n-k-1} \\
&= [x]_{q^\alpha}\left([x]_{q^\alpha} + [1-x]_{q^{-\beta}}\right)^{n-1} = [x]_{q^\alpha}\left(1 + [x]_{q^\alpha} - [x]_{q^\beta}\right)^{n-1}.
\end{aligned}$$

Note that the q-Bernstein polynomials with weight (α, β) of degree n are symmetric polynomials as follows:

$$\begin{aligned}
B_{n-k,n}(1-x, q^{-1}|\beta, \alpha) &= \binom{n}{n-k} [1-x]_{(q^{-1})^\beta}^{n-k} [1-(1-x)]_{(q^{-1})^{-\alpha}}^k \\
&= \binom{n}{k} [1-x]_{q^{-\beta}}^{n-k} [x)]_{q^\alpha}^k \\
&= B_{k,n}(x, q|\alpha, \beta)
\end{aligned} \tag{7}$$

Thus we obtain our theorem. □

Theorem 3 (Explicit formula 1)**.**

$$\mathbb{B}_{n,q}(f, x|\alpha, \beta) = \sum_{m=0}^{n} \binom{n}{m} [x]_{q^\beta}^m \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f\left(\frac{k}{n}\right) \left(\frac{[x]_{q^\alpha}}{[x]_{q^\beta}}\right)^k,$$

where $f \in \mathcal{C}([0,1],\mathbb{R})$, $\alpha, \beta \in \mathbb{R}$ *and* $n \in \mathbb{Z}_+$.

Proof of Thereom 3 : From (6), we note that

$$\begin{aligned}\mathbb{B}_{n,q}(f,x|\alpha,\beta) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x,q|\alpha,\beta) \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x]_{q^{\alpha}}^{k}[1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x]_{q^{\alpha}}^{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{j}[x]_{q^{\beta}}^{j} \qquad (8)\end{aligned}$$

From the definition of binomial coefficients, we easily get

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{k+j}\binom{k+j}{k}.$$

Let $k+j=m$. Then we have

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{m}\binom{m}{k}. \qquad (9)$$

By (8) and (9), we get

$$\mathbb{B}_{n,q}(f,x|\alpha,\beta) = \sum_{m=0}^{n}\binom{n}{m}[x]_{q^{\beta}}^{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k} f\left(\frac{k}{n}\right)\left(\frac{[x]_{q^{\alpha}}}{[x]_{q^{\beta}}}\right)^{k}.$$

This yields our theorem. □

In the special case, $\alpha = \beta$, we have the following corollary.

Corollary 4. *For* $f \in \mathcal{C}([0,1],\mathbb{R})$, $\alpha \in \mathbb{R}$ *and* $n \in \mathbb{Z}_+$, *we have*

$$\mathbb{B}_{n,q}(f,x|\alpha,\alpha) = \sum_{m=0}^{n}\binom{n}{m}[x]_{q^{\alpha}}^{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k} f\left(\frac{k}{n}\right). \qquad (10)$$

It is well known that the second kind Stirling number are defined by

$$\frac{(e^t-1)^k}{k!} = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{lt} = \sum_{n=0}^{\infty} s(n,k)\frac{t^n}{n!}, \qquad (11)$$

where $k \in \mathbb{N}$ (see[3-5]).

Let Δ be the shift difference operator with

$$\Delta f(x) = f(x+1) - f(x).$$

By iterative process, we get

$$\Delta^n f(0) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} f(k), \quad \text{(see[3-5])}. \qquad (12)$$

By (11) and (12), we get

$$\frac{1}{k!}\Delta^k 0^n = s(n,k). \qquad (13)$$

From (10), (11), (12) and (13), we have the following theorem.

Theorem 5 (Difference operator)**.** *For $f \in \mathcal{C}([0,1],\mathbb{R})$, $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}_+$*

$$\mathbb{B}_{n,q}(f,x|\alpha,\alpha) = \sum_{k=0}^{n} \binom{n}{k} [x]_{q^\alpha}^k \Delta^k f\left(\frac{0}{n}\right). \tag{14}$$

In the special case, $f(x) = x^m$ $(m \in \mathbb{Z}_+)$, we have the following corollary

Corollary 6. *For $x \in [0,1]$, $m,n \in \mathbb{Z}_+$ and $\alpha,\beta \in \mathbb{R}$ we have*

$$n^m \mathbb{B}_{n,q}(x^m,x|\alpha,\alpha) = \sum_{k=0}^{n} \binom{n}{k} [x]_{q^\alpha}^k \Delta^k 0^m,$$

and

$$n^m \mathbb{B}_{n,q}(x^m,x|\alpha,\alpha) = \sum_{k=0}^{n} \binom{n}{k} [x]_{q^\alpha}^k k! s(m,k).$$

Theorem 7 (Recurrence formula 2)**.** *For $k,n \in \mathbb{Z}_+$, $\alpha,\beta \in \mathbb{R}$ and $x \in [0,1]$, we have*

$$\begin{aligned} &\frac{n+1-k}{n+1} B_{k,n+1}(x,q|\alpha,\beta) + \frac{k+1}{n+1} B_{k+1,n+1}(x,q|\alpha,\beta) \\ &= (1-[x]_{q^\beta}+[x]_{q^\alpha}) B_{k,n}(x,q|\alpha,\beta). \end{aligned} \tag{15}$$

Proof of Theorem 7 :
For $x,t \in \mathbb{C}$ and $n \in \mathbb{Z}_+$ with $n \geq k$, consider

$$\frac{n!}{2\pi i} \int_C \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!} \frac{dt}{t^{n+1}}, \tag{16}$$

where C is a circle around the origin and integration is in the positive direction. We see from the definitin of the weighted q-Bernstein polynomials and the basic theory of complex analysis including Laurent series that,

$$\begin{aligned} \int_C \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!} \frac{dt}{t^{n+1}} &= \sum_{m=0}^{\infty} \int_C \frac{B_{k,m}(x,q|\alpha,\beta) t^m}{m!} \frac{dt}{t^{n+1}} \\ &= 2\pi i \frac{B_{k,n}(x,q|\alpha,\beta)}{n!}. \end{aligned} \tag{17}$$

We get from (15) and (16) that

$$\frac{n!}{2\pi i} \int_C \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!} \frac{dt}{t^{n+1}} = B_{k,n}(x,q|\alpha,\beta) \tag{18}$$

and

$$\begin{aligned} \int_C \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!} \frac{dt}{t^{n+1}} &= \frac{[x]_{q^\alpha}^k}{k!} \sum_{m=0}^{\infty} \left(\frac{[1-x]_{q^{-\beta}}^m}{m!} \int_C t^{m-n-1+k} dt \right) \\ &= 2\pi i \frac{[x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k}}{k!(n-k)!}. \end{aligned} \tag{19}$$

By (15) and (18), we get

$$\frac{n!}{2\pi i} \int_C \frac{(t[x]_{q^\alpha})^k \exp(t[1-x]_{q^{-\beta}})}{k!} \frac{dt}{t^{n+1}} = \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\beta}}^{n-k}. \tag{20}$$

From (17) and (19), we get

$$B_{k,n}(x,q|\alpha,\beta)=\binom{n}{k}[x]_{q^\alpha}^k[1-x]_{q^{-\beta}}^{n-k}$$

where $x\in[0,1]$ and $n,k\in\mathbb{Z}_+$, $\alpha,\beta\in\mathbb{R}$.
By the definition of q-Bernstein polynomials, we see that

$$\begin{aligned}
&\frac{n-k}{n}B_{k,n}(x,q|\alpha,\beta)+\frac{k+1}{n}B_{k+1,n}(x,q|\alpha,\beta)\\
&= \frac{(n-1)!}{k!(n-k-1)!}[x]_{q^\alpha}^k[1-x]_{q^{-\beta}}^{n-k}+\frac{(n-1)!}{k!(n-k-1)!}[x]_{q^\alpha}^{k+1}[1-x]_{q^{-\beta}}^{n-k-1}\\
&= ([1-x]_{q^{-\beta}}+[x]_{q^\alpha})B_{k,n-1}(x,q|\alpha,\beta)\\
&= (1-[x]_{q^\beta}+[x]_{q^\alpha})B_{k,n-1}(x,q|\alpha,\beta).
\end{aligned}$$

Hence, we obtain our theorem. □

In the special case, $\alpha=\beta$, we obtain the following corollary.

Corollary 8. *For $k,n\in\mathbb{Z}_+$, $\alpha\in\mathbb{R}$ and $x\in[0,1]$, we have*

$$\frac{n+1-k}{n+1}B_{k,n+1}(x,q|\alpha,\alpha)+\frac{k+1}{n+1}B_{k+1,n+1}(x,q|\alpha,\alpha)=B_{k,n}(x,q|\alpha,\alpha). \tag{21}$$

We get from (3) that for $n,k\in\mathbb{N}$,

$$\begin{aligned}
&\frac{n-k+1}{k}\frac{[x]_{q^\alpha}}{[1-x]_{q^{-\beta}}}B_{k-1,n}(x,q|\alpha,\beta)\\
&= \frac{n-k+1}{k}\frac{[x]_{q^\alpha}}{[1-x]_{q^{-\beta}}}\binom{n}{k-1}[x]_{q^\alpha}^{k-1}[1-x]_{q^{-\beta}}^{n-k+1}\\
&= \frac{n!}{k!(n-k)!}[x]_{q^\alpha}^k[1-x]_{q^{-\beta}}^{n-k}\\
&= B_{k,n}(x,q|\alpha,\beta).
\end{aligned} \tag{22}$$

From (22), we obtain the following corollary.

Corollary 9.

$$\frac{n-k+1}{k}\frac{[x]_{q^\alpha}}{[1-x]_{q^{-\beta}}}B_{k-1,n}(x,q|\alpha,\beta)=B_{k,n}(x,q|\alpha,\beta),$$

where $k,n\in\mathbb{N}$, $\alpha,\beta\in\mathbb{R}$ and $x\in[0,1]$.

Theorem 10 (Explicit formula 2)**.**

$$B_{k,n}(x,q|\alpha,\beta)=\left(\frac{[x]_{q^\alpha}}{[x]_{q^\beta}}\right)^k\sum_{l=k}^{n}\binom{n}{l}\binom{l}{k}(-1)^{l-k}[x]_{q^\beta}^l.$$

where $\alpha,\beta\in\mathbb{R}$, $n,k\in\mathbb{Z}_+$ and $x\in[0,1]$.

Proof of Theorem 10 : From the definition of the q-Bernstein polynomials and binomial theorem, we note that

$$\begin{aligned}
B_{k,n}(x,q|\alpha,\beta) &= \binom{n}{k}[x]_{q^\alpha}^k[1-x]_{q^{-\beta}}^{n-k}\\
&= \binom{n}{k}[x]_{q^\alpha}^k\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l[x]_{q^\beta}^l.
\end{aligned} \tag{23}$$

It is easy to show that

$$\binom{n}{k}\binom{n-k}{l} = \binom{n}{k+l}\binom{k+l}{l}. \tag{24}$$

By (23) and (24), we get

$$\begin{aligned} B_{k,n}(x,q|\alpha,\beta) &= \left(\frac{[x]_{q^\alpha}}{[x]_{q^\beta}}\right)^k \sum_{l=k}^{n} \binom{n}{l}\binom{l}{l-k}(-1)^{l-k}[x]_{q^\beta}^l \\ &= \left(\frac{[x]_{q^\alpha}}{[x]_{q^\beta}}\right)^k \sum_{l=k}^{n} \binom{n}{l}\binom{l}{k}(-1)^{l-k}[x]_{q^\beta}^l. \end{aligned} \tag{25}$$

From (25), we obtain our theorem. □
In the special case, $\alpha = \beta$, we obtain the following corollary.

Corollary 11.

$$B_{k,n}(x,q|\alpha,\alpha) = \sum_{l=k}^{n} \binom{n}{l}\binom{l}{k}(-1)^{l-k}[x]_{q^\alpha}^l.$$

where $\alpha, \in \mathbb{R}$, $n, k \in \mathbb{Z}_+$ *and* $x \in [0,1]$.

It is possible to write $[x]_{q^\alpha}^k$ as a linear combination of the q-Bernstein polynomials by using the degree evaluation formula and mathematical induction. We see from the properties of the q-Bernstein polynomials with weight (α,β) that

$$\begin{aligned} \sum_{k=1}^{n} \frac{k}{n} B_{k,n}(x,q|\alpha,\beta) &= \sum_{k=1}^{n} \binom{n-1}{k-1}[x]_{q^\alpha}^k[1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k}[x]_{q^\alpha}^{k+1}[1-x]_{q^{-\beta}}^{n-k-1} \\ &= [x]_{q^\alpha}\left(1+[x]_{q^\alpha}-[x]_{q^\beta}\right)^{n-1}. \end{aligned}$$

In the special case, $\alpha = \beta$,

$$\sum_{k=1}^{n} \frac{k}{n} B_{k,n}(x,q|\alpha,\alpha) = [x]_{q^\alpha}.$$

By the same method, we get

$$\begin{aligned} \sum_{k=2}^{n} \frac{\binom{k}{2}}{\binom{n}{2}} B_{k,n}(x,q|\alpha,\beta) &= \sum_{k=2}^{n} \frac{\binom{k}{2}}{\binom{n}{2}}\binom{n}{k}[x]_{q^\alpha}^k[1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=2}^{n} \binom{n-2}{k-2}[x]_{q^\alpha}^k[1-x]_{q^{-\beta}}^{n-k} \\ &= \sum_{k=0}^{n-2} \binom{n-2}{k}[x]_{q^\alpha}^{k+2}[1-x]_{q^{-\beta}}^{n-k-2} \\ &= [x]_{q^\alpha}^2\left(1+[x]_{q^\alpha}-[x]_{q^\beta}\right)^{n-2}. \end{aligned}$$

In the special case, $\alpha = \beta$,

$$\sum_{k=2}^{n} \frac{\binom{k}{2}}{\binom{n}{2}} B_{k,n}(x,q|\alpha,\alpha) = [x]_{q^\alpha}^2 .$$

By continuing this process, we obtain the following theorem.

Theorem 12.

$$\sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x,q|\alpha,\beta) = [x]_{q^\alpha}^j \left(1 + [x]_{q^\alpha} - [x]_{q^\beta}\right)^{n-j} ,$$

where $n, k, j \in \mathbb{Z}_+$, $\alpha, \beta \in \mathbb{R}$ *and* $x \in [0,1]$.

In the special case, $\alpha = \beta$, we obtain the following corollary.

Corollary 13.

$$\sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x,q|\alpha,\alpha) = [x]_{q^\alpha}^j , \tag{26}$$

where $n, k, j \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$ *and* $x \in [0,1]$.

In [2], the q-Stirling numbers of the second kind are defined by

$$s_q(n,k) = \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [k-j]_q^n , \tag{27}$$

where $\binom{k}{j}_q = \frac{[k]_q!}{[j]_q![k-j]_q!}$ and $[k]_q! = [k]_q \cdots [2]_q [1]_q$.
For $m \in \mathbb{Z}_+$, we have

$$[x]_{q^\alpha}^m = \sum_{k=0}^{m} q^{\alpha\binom{k}{2}} \binom{x}{k}_{q^\alpha} [k]_{q^\alpha}! s_{q^\alpha}(k, m-k).$$

Thus, we have the following corollary.

Corollary 14.

$$\sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x,q|\alpha,\alpha) = \sum_{k=0}^{j} q^{\alpha\binom{k}{2}} \binom{x}{k}_{q^\alpha} [k]_{q^\alpha}! s_{q^\alpha}(k, j-k),$$

where $n, k, j \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$ *and* $x \in [0,1]$.

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