Arakawa-Kaneko L-functions and generalized poly-Bernoulli polynomials

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1 Introduction

The aim of this exposition is to generalize poly-Bernoulli numbers and Arakawa-Kaneko zeta functions that are introduced and studied in [1], [2], [5]. Our work is a continuation of the previous work [3], which introduces and discusses poly-Bernoulli polynomials and generalized Arakawa-Kaneko zeta functions. It should be noted that [4] and [6] treat the topics relevant to our results.

2 Generalized poly-Bernoulli polynomials

Throughout this report, let χ be a Dirichlet character with conductor $f = f_\chi$. The *generalized Bernoulli numbers* $B_{m,\chi} \in \mathbb{Q}\big(\chi(1), \chi(2), ...\big)$ associated to χ ($m = 0, 1, \ldots$) are defined by the generating function

$$\sum_{a=1}^{f} \chi(a) \frac{t}{e^{ft}-1} e^{at} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}, \quad \mid t \mid < \frac{2\pi}{f}.$$

The main interest of these numbers is that they give the values at negative integers of Dirichlet L-series: if $L(s,\chi) = \displaystyle\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ ($\mathrm{Re}(s) > 1$) is the L-series attached to χ, then we have the formula

$$L(-n,\chi) = -\frac{B_{n+1,\chi}}{n+1} \qquad (n > 0).$$

The main purpose of this section is to introduce and investigate the generalized poly-Bernoulli polynomials and numbers. These polynomials include the generalized Bernoulli polynomials and numbers and the poly-Bernoulli polynomials and numbers. Let us briefly review poly-Bernoulli polynomials ([3]). For an integer $k \in \mathbb{Z}$, put

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},$$

which is the k-th polylogarithm if $k \geq 1$, and a rational function if $k \leq 0$. One knows that $\mathrm{Li}_1(z) = -\log(1-z)$. The formal power series $\mathrm{Li}_k(z)$ can be used to define poly-Bernoulli polynomials. The polynomials $B_n^{(k)}(x)$ ($n = 0, 1, 2, \ldots$) are said to be *poly-Bernoulli polynomials* if they satisfy

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}.$$

It should be noted that the definition of poly-Bernoulli polynomials by Coppo-Candelpergher [4] is different from ours. For any $n \geq 0$, we have

$$(-1)^n B_n^{(1)}(-x) = B_n(x),$$

where $B_n(x)$ are the classical Bernoulli polynomials given by

$$\frac{t}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \mid t \mid < 2\pi.$$

Using a Dirichlet character, we generalize poly-Bernoulli polynomials and numbers as follows. Let $k \in \mathbb{Z}$. For a Dirichlet character χ modulo f, we define *generalized poly-Bernoulli polynomials* $B_{n,\chi}^{(k)}(x)$ associated to χ by

$$\frac{1}{f} \sum_{a=1}^{f} \chi(a) \frac{\mathrm{Li}_k(1-e^{-ft})}{e^{ft}-1} e^{(x+a)t} = \sum_{n=0}^{\infty} B_{n,\chi}^{(k)}(x) \frac{t^n}{n!}, \quad \mid t \mid < \frac{2\pi}{f}.$$

In particular, we put $B_{n,\chi}^{(k)} := B_{n,\chi}^{(k)}(0)$. We call them *generalized poly-Bernoulli numbers associated to* χ.

The following two identities hold:

$$B_{n,\chi}^{(1)}(x) = B_{n,\chi}(x).$$

If χ_0 is the trivial Dirichlet character, then

$$B_{n,\chi_0}^{(k)}(x) = B_n^{(k)}(x),$$

where $B_{n,\chi}(x)$ are the generalized Bernoulli polynomials associated to the Dirichlet character χ defined by

$$\sum_{a=1}^{f} \chi(a) \frac{t}{e^{ft}-1} e^{(x+a)t} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}, \quad |\, t \,| < \frac{2\pi}{f}.$$

We state some identities. Let k be a given any integer. The generalized poly-Bernoulli polynomials can be expressed in terms of poly-Bernoulli polynomials:

Theorem 2.1 *For any* $n \geq 0$,

$$B_{n,\chi}^{(k)}(x) = f^{n-1} \sum_{a=1}^{f} \chi(a) B_n^{(k)} \left(\frac{x+a}{f} - 1 \right).$$

In particular,

$$B_{n,\chi}^{(k)} = f^{n-1} \sum_{a=1}^{f} \chi(a) B_n^{(k)} \left(\frac{a}{f} - 1 \right).$$

An explicit formula for $B_{n,\chi}^{(k)}(x)$ is given as follows:

Theorem 2.2 (Explicit formula) *For any* $n \geq 0$,

$$B_{n,\chi}^{(k)}(x) = \frac{1}{f} \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{a=1}^{f} \chi(a) \left(x + a - (j+1)f \right)^n.$$

The generalized poly-Bernoulli polynomials satisfy the following difference identity:

Theorem 2.3 (Difference formula) *For any* $n \geq 0$,

$$B_{n,\chi}^{(k)}(x+f) - B_{n,\chi}^{(k)}(x) = \frac{1}{f} \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \sum_{a=1}^{f} \chi(a)(x+a-jf)^n.$$

We provide recurrence formulae of two types.

Theorem 2.4 (Recurrence formula 1) *For any* $n \geq 0$,

$$B_{n,\chi}^{(k)}(x) = \sum_{m=0}^{n} B_{n-m}^{(k-1)} \binom{n}{m} \sum_{l=0}^{m} \frac{(-1)^{m+l} f^{n-l}}{n-l+1} \binom{m}{l} B_{l,\chi}(x).$$

Theorem 2.5 (Recurrence formula 2) *We have*

$$\begin{aligned}
B_{0,\chi}^{(k)} &= \begin{cases} 1 & (\chi = \chi_0) \\ 0 & (\chi \neq \chi_0) \end{cases}, \\
B_{1,\chi}^{(k)} &= \frac{1}{2} \left(B_{1,\chi}^{(k-1)}(x) + (x-f) B_{0,\chi}^{(k)} + \frac{1}{f} \sum_{a=1}^{f} \chi(a) a \right), \\
B_{n,\chi}^{(k)}(x) &= \frac{1}{n+1} \Bigg\{ B_{n,\chi}^{(k-1)}(x) - \sum_{m=1}^{n-1} f^{n-m} \binom{n}{m-1} B_{m,\chi}^{(k)}(x) \\
&\qquad + (x-f) \sum_{m=0}^{n-1} f^{n-1-m} \binom{n}{m} B_{m,\chi}^{(k)}(x) \\
&\qquad + f^{n-2} \sum_{m=0}^{n-1} \binom{n}{m} \sum_{a=1}^{f} \chi(a) a B_m^{(k)} \left(\frac{x+a}{f} - 1 \right) \Bigg\}. \\
&\qquad\qquad (n \geq 2)
\end{aligned}$$

For $m, n \geq 0$, we define the following *symmetrized* Bernoulli polynomial of two variables:

$$C_{n,\chi}^{(-m)}(x,y) = \frac{1}{f^n}\sum_{b=1}^{f}\chi(b)\sum_{k=0}^{m}\binom{m}{k}B_{n,\chi}^{(-k)}(x)\left(\frac{y+b}{f}-1\right)^{m-k}.$$

Theorem 2.6 (Symmetric formula) *For $m, n \geq 0$, we have*

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_{n,\chi}^{(-m)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_{m,\chi}^{(-n)}(y,x)\frac{t^n}{n!}\frac{u^m}{m!}\\ &= \frac{1}{f}\sum_{a=1}^{f}\sum_{b=1}^{f}\frac{\chi(a)\chi(b)e^{(x+a)t/f}e^{(y+b)u/f}}{e^t+e^u-e^{t+u}}.\end{aligned}$$

As a corollary, we have the duality property for $C_{n,\chi}^{(-m)}(x)$:

Theorem 2.7 (Duality) *For $m, n \geq 0$,*

$$C_{n,\chi}^{(-m)}(x,y) = C_{m,\chi}^{(-n)}(y,x).$$

We deduce a closed formula from Theorem 1.6.

Theorem 2.8 (Closed formula) *For $m, n \geq 0$,*

$$\begin{aligned}&C_{n,\chi}^{(-m)}(x,y)\\ &\quad= \frac{1}{f}\sum_{j=0}^{\min(n,m)}(j!)^2\sum_{a=1}^{f}\sum_{b=1}^{f}\left(\sum_{k=0}^{n}\chi(a)\left(\frac{x+a}{f}\right)^{n-k}\binom{n}{k}\left\{{k \atop j}\right\}\right)\left(\sum_{l=0}^{m}\chi(b)\left(\frac{y+b}{f}\right)^{m-l}\binom{m}{l}\left\{{l \atop j}\right\}\right),\end{aligned}$$

where

$$\left\{{n \atop m}\right\} = \frac{(-1)^m}{m!}\sum_{l=0}^{m}(-1)^l\binom{m}{l}l^n$$

is the Stirling number of the second kind.

3 Arakawa-Kaneko L-functions

Firstly, we overview Arakawa-Kaneko zeta functions (cf. [3], [4].) Let $k \in \mathbb{Z}$. The *Arakawa-Kaneko zeta function* $Z_k(s,x)$ is by definition the Laplace-Mellin integral

$$Z_k(s,x) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{-xt}t^{s-1}dt.$$

It is defined for $\mathrm{Re}(s) > 0$ and $x > 0$ if $k \geq 1$, and for $\mathrm{Re}(s) > 0$ and $x > |k|+1$ if $k \leq 0$. If $x = 1$, then $Z_k(s,x)$ reduces to the original Arakawa-Kaneko zeta function introduced in [1]. The function $s \mapsto Z_k(s,x)$ has analytic continuation to an entire function on the whole complex s-plane and has

$$Z_k(-n,x) = (-1)^n B_n^{(k)}(-x),$$

for all non-negative integers n and $x > 0$. Here $B_n^{(k)}(x)$ $(n = 0, 1, 2, \ldots)$ are the poly-Bernoulli polynomials. In addition, this zeta function can be rewritten to have an explicit formula as follows. For $k \in \mathbb{Z}$,

$$Z_k(s,x) = \sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{1}{(x+j)^s}.$$

Note that for $k = 1$ we have

$$Z_1(s,x) = s\zeta(s+1,x),$$

where $\zeta(s,x)$ is the Hurwitz zeta function.

We next introduce Arakawa-Kaneko L-functions. For $k \in \mathbb{Z}$, the L-series attached to χ are given by

$$L_k(s,x,\chi) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{f} \sum_{a=1}^{f} \chi(a) \frac{\mathrm{Li}_k(1-e^{-ft})}{1-e^{-ft}} e^{-(x-a+f)t} t^{s-1} dt,$$

the Laplace-Mellin integral. It is defined for $\mathrm{Re}(s) > 0$ and $x > 0$ if $k \geq 1$, and for $\mathrm{Re}(s) > 0$ and $x > |k| + 1$ if $k \leq 0$. The L-series $L_k(s,x,\chi)$ are called the *Arakawa-Kaneko L-functions*. It is easy to see that the Arakawa-Kaneko L-functions include the Arakawa-Kaneko and Hurwitz zeta functions. To be exact, for the trivial Dirichlet character χ_0,

$$L_k(s,x,\chi_0) = Z_k(s,x).$$

If furthermore $k = 1$, then we have

$$L_1(s,x,\chi_0) = s\zeta(s+1,x).$$

We state the fundamental properties of $L_k(s,x,\chi)$.

Theorem 3.1 *For $k \in \mathbb{Z}$, we have*

$$L_k(s,x,\chi) = f^{-s-1} \sum_{a=1}^{f} \chi(a) Z_k\left(s, \frac{x-a}{f} + 1\right).$$

Theorem 3.2 (Interpolation formula) *The function $s \mapsto L_k(s,x,\chi)$ has analytic continuation to an entire function on the whole complex s-plane and for any positive integer n, we have*

$$L_k(-n,x,\chi) = (-1)^n B_{n,\chi}^{(k)}(-x).$$

Furthermore $L_k(s,x,\chi)$ satisfies the functional equation

$$\frac{\partial}{\partial x} L_k(s,x,\chi) = -sL_k(s+1,x,\chi).$$

As an immediate consequence of these theorems and an explicit formula for $Z_k(s,x)$, we get an explicit formula for $L_k(s,x,\chi)$ as follows. For $k \in \mathbb{Z}$,

$$L_k(s,x,\chi) = \frac{1}{f} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{a=1}^{f} \frac{\chi(a)}{(x-a+(j+1)f)^s}.$$

Theorem 3.3 (Difference identity) *The Arakawa-Kaneko L-functions satisfy the following difference identity*

$$L_k(s,x+f,\chi) - L_k(s,x,\chi) = \frac{1}{f} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^{j+1} \binom{m+1}{j} \sum_{a=1}^{f} \frac{\chi(a)}{(x-a+(j+1)f)^s}.$$

Theorem 3.4 (Raabe's identity) *For $s \in \mathbb{C}$ with $s \neq 1$ we have*

$$\begin{aligned}
&\int_0^f L_k(s,x+w,\chi)\, dw \\
&\quad = -\frac{1}{s-1} \left(L_k(s-1,x+f,\chi) - L_k(s-1,x,\chi) \right) \\
&\quad = \frac{1}{s-1} \frac{1}{f} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \sum_{a=1}^{f} \frac{\chi(a)}{(x-a+(j+1)f)^{s-1}}.
\end{aligned}$$

We remark that combining Theorem 3.2 and Theorem 3.4, we deduce Raabe's type formulae for generalized poly-Bernoulli polynomials, Arakawa-Kaneko and Hurwitz zeta functions.

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