IDENTITIES FOR THE BERNOULLI, THE EULER AND THE GENOCCHI NUMBERS AND POLYNOMIALS

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Abstract. The purpose of this paper is to give some new identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials.

1. Introduction

The second Author in [2] he consider the equations of the fermionic p-adic invariant integral on $\mathbb{Z}_p$ for **even** $d \in \mathbb{Z}_+$. From those equations, he derive the following interesting and valuable identities for the Euler, the Genocchi and the Bernoulli numbers and polynomials. For $d \in \mathbb{Z}_+$, with $d \equiv 0 \ (mod\ 2)$, he obtain the identities

$$\frac{E_n}{2} = \frac{d^n}{n+1} \sum_{l=0}^{d-1} (-1)^{l-1} B_{n+1}(\frac{l}{d}). \tag{1}$$

$$\sum_{l=0}^{d-1} (-1)^{l-1} \left(\frac{l}{d}\right)^n = \frac{1}{n+1} \left(\sum_{l=0}^{d-1} (-1)^{l-1} \left(B_{n+1}(\frac{l}{d}+1) - B_{n+1}(\frac{l}{d}) \right) \right). \tag{2}$$

$$\frac{G_n(x)}{2} = d^{n-1} \sum_{l=0}^{d-1} (-1)^{l-1} B_n(\frac{l+x}{d}), \text{ for } n \in \mathbb{Z}_+. \tag{3}$$

Indeed, for any χ Dirichlet character with even conductor $d \in \mathbb{Z}_+^*$ we have

$$\frac{G_{n,\chi}(x)}{2} = d^{n-1} \sum_{l=0}^{d-1} (-1)^{l-1} \chi(l) B_n(\frac{l+x}{d}). \tag{4}$$

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where $G_{n,\chi}(x)$ and $E_{n,\chi}(x)$ are the Genocchi and Euler polynomials associated to Dirichlet character χ with conductor d given by

$$\frac{2t\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1}e^{xt}=\sum_{n=0}^{\infty}G_{n,\chi}\frac{t^n}{n!},\ \text{for } |t|<\frac{\pi}{d}$$
$$\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1}e^{xt}=\sum_{n=0}^{\infty}E_{n,\chi}(x)\frac{t^n}{n!},\ \text{for } |t|<\frac{\pi}{d}.$$

and for any $a,b\in\mathbb{Z}_+^*$ we have

$$\begin{aligned}&\sum_{i=0}^{l}\binom{l}{i}\frac{d^i}{i+1}\sum_{l=0}^{d-1}(-1)^{l-1}\chi(l)B_{i+1}(\frac{l+bx}{d})T_{l-i,\chi}(da-1)a^ib^{l-i}\\ &=\sum_{i=0}^{l}\binom{l}{i}\frac{d^i}{i+1}\sum_{l=0}^{d-1}(-1)^{l-1}\chi(l)B_{i+1}(\frac{l+ax}{d})T_{l-i,\chi}(db-1)b^ia^{l-i}.\end{aligned}\tag{5}$$

where $T_{k,\chi}(n)=\sum_{l=0}^{n}(-1)^{n-1}\chi(l)l^k$.

For more details on the definitions of various polynomials cited here see [1–9].

The main purpose of this paper is to extend all these results (1), (2), (3), (4) and (5) to **arbitrary** d (**even or odd**). Our approach here is based on the theory of generating functions.

2. Identities of the Bernoulli, the Euler and the Genocchi numbers and polynomials

We have the following theorems.

Theorem 1. *For $n\in\mathbb{Z}_+$, we have*

$$\sum_{l=0}^{d-1}(-1)^{l-1}B_{n+1}\left(\frac{l}{d}\right)=\frac{1}{2d^{n+1}}\sum_{j=0}^{n+1}\binom{n+1}{j}d^j\Big((-1)^dB_j(1)-B_j(0)\Big)E_{n+1-j}(0).\tag{1}$$

More explicitely, we know that

$$B_1(1)=\frac{1}{2},B_1(0)=-\frac{1}{2},B_j(1)=B_j(0),\forall j\in\mathbb{Z}_+\backslash\{1\}.$$

Then, we have to consider the following two cases

1) (Even case): If $d\equiv 0\ (mod\ 2)$

$$\sum_{l=0}^{d-1}(-1)^{l-1}B_{n+1}\left(\frac{l}{d}\right)=\frac{(n+1)}{2d^n}E_n(0)$$

2) (Odd case): If $d\equiv 1\ (mod\ 2)$

$$\sum_{l=0}^{d-1}(-1)^{l-1}B_{n+1}\left(\frac{l}{d}\right)=-\frac{1}{d^{n+1}}\sum_{j=0}^{n+1}\binom{n+1}{j}d^j\overline{B}_j(0)E_{n+1-j}(0)$$

where $\overline{B}_j(0) = B_j(0)$ if $j \neq 1$ and $\overline{B}_1(0) = 0$.

Proof of theorem 1: We consider the exponential generating function of the left hand side of (1)

$$\begin{aligned}\sum_{n\geq 0}\left(\sum_{l=0}^{d-1}(-1)^{l-1}B_{n+1}\left(\frac{l}{d}\right)\right)\frac{t^n}{n!} &= \sum_{l=0}^{d-1}(-1)^{l-1}\left(\sum_{n\geq 0}B_{n+1}\left(\frac{l}{d}\right)\frac{t^n}{n!}\right)\\ &= \frac{t}{e^t-1}\sum_{l=0}^{d-1}(-1)^{l-1}e^{\frac{lt}{d}}\\ &= \frac{t}{e^{\frac{t}{d}}+1}.\frac{(-1)^d e^t-1}{e^t-1}\\ &= \frac{1}{e^{\frac{t}{d}}+1}\left((-1)^d\frac{te^t}{e^t-1}-\frac{t}{e^t-1}\right)\end{aligned} \tag{2}$$

It is easy to see that

$$\frac{1}{e^{\frac{t}{d}}+1} = \frac{1}{2}\sum_{n\geq 0}\frac{E_n(0)}{d^n}\frac{t^n}{n!} \tag{3}$$

and

$$\frac{(-1)^d te^t - t}{e^t-1} = \sum_{n\geq 0}\Big((-1)^d B_n(1) - B_n(0)\Big)\frac{t^n}{n!} \tag{4}$$

Now, from the equalities (2), (3) and (4) we obtain

$$\sum_{l=0}^{d-1}(-1)^{l-1}B_n\left(\frac{l}{d}\right) = \frac{1}{2d^n}\sum_{j=0}^{n}\binom{n}{j}d^j\Big((-1)^d B_j(1) - B_j(0)\Big)E_{n-j}(0).$$

Thus, we obtain our theorem 1.

Theorem 2. *For $d, n \in \mathbb{Z}_+$, we have*

$$\sum_{l=0}^{d-1}(-1)^{l-1}\left(\frac{l}{d}\right)^n = \frac{1}{n+1}\left(\sum_{l=0}^{d-1}(-1)^{l-1}\left(B_{n+1}\left(\frac{l}{d}+1\right) - B_{n+1}\left(\frac{l}{d}\right)\right)\right),$$

and precisely

1) If $d \equiv 0 \ (mod\ 2)$,

$$\sum_{l=0}^{d-1}(-1)^{l-1}l^n = \frac{1}{2}\left(E_n(d) - E_n(0)\right).$$

2) If $d \equiv 1 \ (mod\ 2)$,

$$\sum_{l=0}^{d-1}(-1)^{l-1}l^n = -\frac{1}{n+1}\sum_{j=0}^{n+1}\binom{n+1}{j}d^{j-1}\overline{B}_j(0)\Big(E_{n+1-j}(d) - E_{n+1-j}(0)\Big)$$

Proof of theorem 2: We use the difference formula for Bernoulli polynomials

$$B_{n+1}(X+1) - B_{n+1}(X) = (n+1)X^n, \ \forall n \geq 1$$

Then

$$\left(\frac{l}{d}\right)^n = \frac{1}{n+1}\left(B_{n+1}\left(\frac{l}{d}+1\right) - B_{n+1}\left(\frac{l}{d}\right)\right)$$

Hence,

$$\sum_{l=0}^{d-1}(-1)^{l-1}\left(\frac{l}{d}\right)^n = \frac{1}{n+1}\left(\sum_{l=0}^{d-1}(-1)^{l-1}\left(B_{n+1}\left(\frac{l}{d}+1\right) - B_{n+1}\left(\frac{l}{d}\right)\right)\right).$$

Now, for the rest of the theorem 2, we use the result of theorem 1.

Theorem 3.
Let d and $n \in \mathbb{Z}_+$. We have

$$\sum_{l=0}^{d-1}(-1)^{l-1}B_n\left(\frac{l+x}{d}\right) = \frac{1}{2d^n}\sum_{j=0}^{n}\frac{1}{n+1}\binom{n+1}{j}d^j\Big((-1)^d B_j(1) - B_j(0)\Big)G_{n+1-j}(x). \tag{5}$$

In particular, we have the following simple formulas

1) If $d \equiv 0 \ (mod\ 2)$, we have

$$\sum_{l=0}^{d-1}(-1)^{l-1}B_n\left(\frac{l+x}{d}\right) = \frac{1}{d^{n-1}}.\frac{G_n(x)}{2},$$

In terms of Euler's polynomials, we have

$$\sum_{l=0}^{d-1}(-1)^{l-1}B_{n+1}\left(\frac{l+x}{d}\right) = \frac{n}{d^n}.\frac{E_n(x)}{2},$$

2) If $d \equiv 1 \ (mod\ 2)$, we have

$$\sum_{l=0}^{d-1}(-1)^{l-1}B_n\left(\frac{l+x}{d}\right) = \frac{1}{d^n}\sum_{j=0}^{n}\frac{1}{n+1}\binom{n+1}{j}d^j\overline{B}_j(0)G_{n+1-j}(x)$$

In terms of Euler's polynomials, we have

$$\sum_{l=0}^{d-1}(-1)^{l-1}B_n\left(\frac{l+x}{d}\right) = \frac{1}{d^n}\sum_{j=0}^{n}\binom{n}{j}d^j\overline{B}_j(0)E_{n-j}(x)$$

Proof of theorem 3: We consider the exponential generating function of the left hand side of (5)

$$\begin{aligned}\sum_{n\geq 0}\left(\sum_{l=0}^{d-1}(-1)^{l-1}B_{n+1}\left(\frac{x+l}{d}\right)\right)\frac{t^n}{n!} &= \sum_{l=0}^{d-1}(-1)^{l-1}\left(\sum_{n\geq 0}B_{n+1}\left(\frac{x+l}{d}\right)\frac{t^n}{n!}\right)\\ &= \frac{t}{e^t-1}\sum_{l=0}^{d-1}(-1)^{l-1}e^{\frac{(x+l)t}{d}}\\ &= \frac{te^{\frac{xt}{d}}}{e^{\frac{t}{d}}+1}.\frac{(-1)^d e^t-1}{e^t-1}\\ &= \frac{e^{\frac{xt}{d}}}{e^{\frac{t}{d}}+1}\left((-1)^d\frac{te^t}{e^t-1}-\frac{t}{e^t-1}\right)\end{aligned} \tag{6}$$

It is easy to see that

$$\frac{te^{\frac{xt}{d}}}{e^{\frac{t}{d}}+1} = \frac{1}{2}\sum_{n\geq 0}\frac{G_n(x)}{d^n}\frac{t^n}{n!} \tag{7}$$

and

$$\frac{(-1)^d e^t-1}{e^t-1} = \sum_{n\geq 0}\Big((-1)^d B_n(1)-B_n(0)\Big)\frac{t^{n-1}}{n!} \tag{8}$$

Now, from the equalities (6), (7) and (8) we obtain

$$\sum_{l=0}^{d-1}(-1)^{l-1}B_n\left(\frac{l+x}{d}\right) = \frac{1}{2d^n}\sum_{j=0}^{n}\frac{1}{n+1}\binom{n+1}{j}d^j\Big((-1)^d B_j(1)-B_j(0)\Big)G_{n+1-j}(x).$$

Thus, we obtain our theorem 3.

Theorem 4. *Let χ a Dirichlet character with conductor $d\in\mathbb{Z}_+$, we have*

$$\frac{G_{n+1,\chi}(x)}{n+1} = d^n\sum_{l=0}^{d-1}(-1)^l\chi(l)E_n\left(\frac{l+x}{d}\right) = E_{n,\chi}(x).$$

From the definiton of $E_{n,\chi}(x)$ and $G_{n+1,\chi}(x)$ it is easy to see that

$$E_{n,\chi}(x) = \frac{G_{n+1,\chi}(x)}{n+1}.$$

We write

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,\chi}(x)\frac{t^n}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1}e^{xt}\\ &= \sum_{l=0}^{d-1}(-1)^l\chi(l)\frac{2e^{\left(\frac{x+l}{d}\right)dt}}{e^{dt}+1}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{d-1}(-1)^l\chi(l)E_n\left(\frac{x+l}{d}\right)d^n\right)\frac{t^n}{n!}\end{aligned} \tag{9}$$

Now, by camparing the coefficients of the left and right hand side of this equality (9) we obtain the theorem 4.

For the next result we define the alternating sums

$$\tilde{T}_{k,\chi}(n) = \sum_{i=0}^{d-1}\sum_{j=0}^{n-1}(-1)^{i+j}\chi(j)(i+jd)^k.$$

we can also easily see that

$$(-1)^{n-1}E_{k,\chi}(nd) + E_{k,\chi} = 2\tilde{T}_{k,\chi}(n).$$

Note that if χ_0 is the trivial Dirichlet's character then

$$\tilde{T}_k(n) := \tilde{T}_{k,\chi_0}(n) = \sum_{j=1}^{n}(-1)^{i-1}i^k = \frac{1}{2}\Big((-1)^{n-1}E_k(n) + E_k\Big)$$

Theorem 5.

Let χ be the Dirichlet's character with an arbitrary conductor $d \in \mathbb{Z}_+^*$. For $a, b \in \mathbb{Z}_+$ with same parity we have

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n+1}{j}b^j a^{n-j}G_{n+1-j,\chi}(bX)\sum_{k=0}^{j}\binom{j}{k}\tilde{T}_{k,\chi}(a)\\ &= \sum_{j=0}^{n}\binom{n+1}{j}a^j b^{n-j}G_{n+1-j,\chi}(aX)\sum_{k=0}^{j}\binom{j}{k}\tilde{T}_{k,\chi}(b).\end{aligned} \tag{10}$$

Proof of theorem 5:

For $a, b \in \mathbb{Z}_+$ with $a \equiv b \pmod 2$, let us consider the following functional equation:

$$\begin{aligned}F_d(a,b;X) &= \frac{1}{2}\left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jat}}{e^{dat}+1}\right)e^{abXt}\left(1-(-1)^a e^{dabt}\right)\\ &\times\left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jbt}}{e^{dbt}+1}\right)e^{abYt}\\ &= \left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jat}}{e^{dat}+1}\right)e^{abXt}\left(\frac{1-(-1)^a e^{dabt}}{e^{bdt}+1}\right)\\ &\left(\sum_{j=0}^{d-1}\chi(j)(-1)^j e^{jbt}\right)e^{abYt}.\end{aligned} \tag{11}$$

It is easy to see that

$$\begin{aligned}&\left(\frac{1-(-1)^a e^{abdt}}{e^{bdt}+1}\right)\left(\sum_{i=0}^{d-1}\chi(i)e^{ibt}(-1)^i\right) = \left(\sum_{l=0}^{a-1}e^{ldbt}(-1)^l\right)\left(\sum_{i=0}^{d-1}\chi(i)(-1)^i e^{ibt}\right)\\ &= \sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}e^{(id+j)bt} = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}(id+j)^k\right)\frac{b^k t^k}{k!}\\ &= \sum_{k=0}^{\infty}\tilde{T}_{k,\chi}(a)\frac{b^k t^k}{k!}.\end{aligned} \tag{12}$$

By (11) and (12), we have

$$
\begin{aligned}
F_d(a,b;X) &= \left(\sum_{i=0}^{\infty} E_{i,\chi}(bX)\frac{a^i t^i}{i!}\right)\left(\sum_{l=0}^{\infty} \tilde{T}_{l,\chi}(a)\frac{b^l t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} b^j a^{n-j} E_{n-j,\chi}(bX)\sum_{k=0}^{j}\tilde{T}_{k,\chi}(a)\binom{j}{k}\right)\frac{t^n}{n!}.
\end{aligned} \tag{13}
$$

By the symmetry of $F_d(a,b;X)$ in a and b (because $a \equiv b(\mod 2)$) , we also see that

$$
F_d(a,b;X) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} a^j b^{n-j} E_{n-j,\chi}(aX)\sum_{k=0}^{j}\binom{j}{k}\tilde{T}_{k,\chi}(b)\right)\frac{t^n}{n!}. \tag{14}
$$

By comparing the coefficients on the both sides of (13) and (14), we obtain the theorem 5.

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