FOURIER EXPANSIONS FOR APOSTOL-BERNOULLI, APOSTOL-EULER AND APOSTOL-GENOCCHI POLYNOMIALS

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Abstract. We find Fourier expansions of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. We give a very simple proof of them.

1. Introduction and statement of main results

Let $w \in \mathbb{C}$ and x a variable. The Apostol-Bernoulli polynomials $B_n(x;w)$, Apostol-Euler polynomials $E_n(x;w)$ and Apostol-Genocchi polynomials $G_n(x;w)$ are given by the generating functions

$$\sum_{n\geq 0} B_n(x;w)\frac{t^n}{n!} = \frac{te^{xt}}{we^t-1},\ |t+log(w)|<2\pi;\ \text{see [1, 4]}\ . \tag{1.1}$$

$$\sum_{n\geq 0} E_n(x;w)\frac{t^n}{n!} = \frac{2e^{xt}}{we^t+1},\ |t+log(w)|<\pi;\ \text{see [4]} \tag{1.2}$$

$$\sum_{n\geq 0} G_n(x;w)\frac{t^n}{n!} = \frac{2te^{xt}}{we^t+1},\ |t+log(w)|<\pi. \tag{1.3}$$

where

$$w=|w|e^{i\theta}, -\pi\leq\theta<\pi \ \text{ and } \ \log(w)=\log(|w|)+i\theta.$$

These polynomials are a natural extension of the classical Bernoulli, Euler and Genocchi polynomials : $B_n(x) = B_n(x;1), E_n(x) = E_n(x;1), G_n(x) = G_n(x;1)$; see [3] . These polynomials have many applications in mathematics. Our main results are

Theorem 1.1. *Let $w \in \mathbb{C}\backslash\{0\}$. For $0<x<1$ if $n=1$, $0\leq x\leq 1$ if $n\geq 2$. We have*

$$B_n(x;w) = \frac{-n!}{w^x(2\pi i)^n}\sum_{k\in\mathbb{Z}}^{*}\frac{e^{2\pi ikx}}{\left(k-\frac{log(w)}{2\pi i}\right)^n}, \tag{1.4}$$

where $\sum_{k\in\mathbb{Z}}^{*} = \sum_{k\in\mathbb{Z}\backslash\{0\}}$ *if* $w=1$ *and* $\sum_{k\in\mathbb{Z}}^{*} = \sum_{k\in\mathbb{Z}}$ *if* $w\neq 1$.

Theorem 1.2. *Let $w \in \mathbb{C}\backslash\{0\}$. For $0<x<1$ if $n=0$, $0\leq x\leq 1$ if $n\geq 1$. We have*

$$E_n(x;w) = \frac{2(n!)}{w^x(2\pi i)^{n+1}}\sum_{k\in\mathbb{Z}}^{**}\frac{e^{2\pi i\left(k-\frac{1}{2}\right)x}}{\left(k-\frac{1}{2}-\frac{log(w)}{2\pi i}\right)^{n+1}}, \tag{1.5}$$

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where $\sum_{k\in\mathbb{Z}}^{**} = \sum_{k\in\mathbb{Z}\backslash\{0\}}$ *if* $w=-1$ *and* $\sum_{k\in\mathbb{Z}}^{**} = \sum_{k\in\mathbb{Z}}$ *if* $w\neq -1$.

Theorem 1.3. *Let* $w\in\mathbb{C}\backslash\{0\}$. *For* $0<x<1$ *if* $n=0$, $0\leq x\leq 1$ *if* $n\geq 1$. *We have*

$$G_n(x;w) = \frac{2(n!)}{w^x(2\pi i)^n}\sum_{k\in\mathbb{Z}}^{**}\frac{e^{2\pi i\left(k-\frac{1}{2}\right)x}}{\left(k-\frac{1}{2}-\frac{log(w)}{2\pi i}\right)^n}. \tag{1.6}$$

Remark 1.4. Luo's proof [4], for Theorems 1.1 and 1.2, uses the Lipschitz summation formula [2] which is not easy to understand. In this paper we propose a very simple proof. On the other hand, Theorem 1.3 is new.

2. Proofs of main results

Proof of Theorem 1.1: We consider $\int_C f_n(t)\,dt$ with $f_n(t)=\frac{t^{-n}e^{xt}}{we^t-1}$, the contour C being a circle with radius $(2N+\epsilon)\pi$ (ϵ fixed real number such that $\epsilon\pi i\pm log(w)\neq 0 \pmod{2\pi i}$), centered at the origin. If $w\neq 1$, the poles of the integrand are $t_k=2\pi ik-log(w), k\in\mathbb{Z}$ and $t_\infty=0$. The residues of the functions $f_n(t)$ for $k\in\mathbb{Z}$, are easly found to be $w^{-x}(2\pi ik-log(w))^{-n}e^{2\pi ikx}$, and from Theorem 1.1 the residue at $z_\infty=0$ is seen to be $\frac{B_n(x;w)}{n!}$. The integral around the circle C tends to zero as $N\to\infty$ provided $0<x<1$ if $n=1$, $0\leq x\leq 1$ if $n\geq 2$, and by the theorem of residues we obtain

$$B_n(x;w) = \frac{-n!}{w^x(2\pi i)^n}\sum_{k\in\mathbb{Z}}\frac{e^{2\pi ikx}}{\left(k-\frac{log(w)}{2\pi i}\right)^n}.$$

If $w=1$, the poles of the integrand are $t_k=2\pi ik, k\in\mathbb{Z}$. The residues of the functions $f_n(t)$ for $k\in\mathbb{Z}\backslash\{0\}$ are easily found to be $(2\pi ik)^{-n}e^{2\pi ikx}$, and from Theorem 1.1 the residue at $z_0=0$ is seen to be $\frac{B_n(x;w)}{n!}$. The integral around the circle C tends to zero as $N\to\infty$ provided $0<x<1$ if $n=1$, $0\leq x\leq 1$ if $n\geq 2$, and by the theorem of residues we obtain

$$B_n(x;w=1) = \frac{-n!}{(2\pi i)^n}\sum_{k\in\mathbb{Z}}\frac{e^{2\pi ikx}}{k^n}.$$

This yields the theorem. □

Proof of Theorem 1.2: We apply the same method to the function $g_n(t)=\frac{t^{-(n+1)}e^{xt}}{we^t+1}$ the contour C' being a circle with radius $(2N+1+\epsilon)\pi$ (ϵ fixed real number such that $\epsilon\pi i\pm log(w)\neq 0 \pmod{\pi i}$), centered at the origin. We omit the details. □

Proof of Theorem 1.3 : We have $G_{n+1}(x;w)=(n+1)E_n(x;w)$. Thus we get Theorem 1.3 from Theorem 1.2. □

References

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