SPECIAL VALUES OF LERCH ZETA FUNCTION AND THEIR FOURIER EXPANSIONS

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Abstract. We prove a Fourier expansion for Lerch zeta transcendent Φ. The special values of this zeta function at $s = -n$ negative integers give us the so-called Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Others applications are given.

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1. Introduction and motivation

Recently, many papers are written, on Apostol-Bernoulli and Apostol-Euler numbers and their q-analogues, by T. Kim in [4, 5, 6, 7]. T. Kim in these papers used the bosonic p-adic q-integral and the fermionic p-adic q-integral. Moreover, T. Kim in [8] he gives a Fourier series of Euler polynomials. The purpose of this paper is to present a systematic study of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi numbers and also polynomials. By using the study of the Fourier expansions of the analytic continuation of Lerch zeta transcendant we investigate their special values at negative integers. From our investigation we obtain the Fourier expansions of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials.

2. Fourier expansions for Lerch zeta function Φ

The *Lerch zeta transcendent* Φ is given in [1, 2] by the series

$$\Phi(s,x,w) = \sum_{n\geq 0} \frac{w^n}{(x+n)^s}, \tag{2.1}$$

which converges for any real number $x \neq -n, n \in \mathbb{N}$, if w and s are any complex numbers with either $|w| < 1$, or $|w| = 1$ and $\Re(s) > 1$. An integral representation is

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given by

$$\Phi(s,x,w)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-xt}}{1-we^{-t}}\,dt \tag{2.2}$$

for

$$(\Re(x)>0,\Re(s)>0,\ |w|<1)\ \text{ or }\ (\Re(x)>0,\Re(s)>1,|w|=1)\,.$$

Let us recall the Lipschitz summation formula [3, 11] it states that if $x\in\mathbb{Z}$ and $\mathrm{Re}(s)>1$ or $x\in\mathbb{R}\backslash\mathbb{Z}$ and $\mathrm{Re}(s)>0$, then

$$\sum_{k\in\mathbb{Z}}\frac{e^{-2\pi ikx}}{(z+k)^s}=\frac{(-2\pi i)^s}{\Gamma(s)}\sum_{n+x>0}\frac{e^{2\pi i(n+x)z}}{(n+x)^{1-s}}, \tag{2.3}$$

for $\mathrm{Im}(z)>0$ and Γ denotes the Gamma function.
By applying this formula, it's easy to obtain the interesting result

Theorem 2.1. *Let $w\in\mathbb{C}\backslash\{0\}$ with $|w|\le 1$. For $0<x<1$ if $n=1$, $0\le x\le 1$ if $n\ge 2$. We have*

$$w^x\Phi(s,x,w)=\Gamma(1-s)(2\pi i)^{s-1}\sum_{k\in\mathbb{Z}}^{*}e^{2\pi ikx}\left(k-\frac{log(w)}{2\pi i}\right)^{s-1},\ \Re(s)<1. \tag{2.4}$$

Where $\sum_{k\in\mathbb{Z}}^{*}=\sum_{k\in\mathbb{Z}\backslash\{0\}}$ *if* $w=1$, $\sum_{k\in\mathbb{Z}}^{*}=\sum_{k\in\mathbb{Z}}$ *if* $w\neq 1$.

3. Special values of Lerch zeta function and applications

Let $w\in\mathbb{C}$ and x a variable. The Apostol-Bernoulli polynomials $B_n(x;w)$, Apostol-Euler polynomials $E_n(x;w)$ and Apostol-Genocchi polynomials $G_n(x;w)$ are given by the generating functions

$$\sum_{n\ge 0}B_n(x;w)\frac{t^n}{n!}=\frac{te^{xt}}{we^t-1},\ |t+\log(w)|<2\pi,\ \text{see [1, 10]}\ . \tag{3.1}$$

$$\sum_{n\ge 0}E_n(x;w)\frac{t^n}{n!}=\frac{2e^{xt}}{we^t+1},\ |t+\log(w)|<\pi,\ \text{see [10]} \tag{3.2}$$

$$\sum_{n\ge 0}G_n(x;w)\frac{t^n}{n!}=\frac{2te^{xt}}{we^t+1},\ |t+\log(w)|<\pi. \tag{3.3}$$

where $w=|w|e^{i\theta},-\pi\le\theta<\pi$.
These polynomials are a natural extension of the classical Bernoulli, Euler and Genocchi polynomials (corresponding to $w=1$):

$$B_n(x)=B_n(x;1),E_n(x)=E_n(x;1),G_n(x)=G_n(x;1).\ \text{See [9]}\ .$$

These polynomials have many applications in mathematics. We prove the following identities.

Theorem 3.1. *Let $x > 0$. The function $s \to \Phi(s, x, w)$ has analytic continuation to an entire function on the whole complex s-plane and for any $n \in \mathbb{N}$, we have the following formulae*

$$B_{n+1}(x; w) = -(n+1)\Phi(-n, x, w), \tag{3.4}$$

$$E_n(x; w) = 2\Phi(-n, x, -w), \tag{3.5}$$

and

$$G_n(x; w) = 2(n+1)\Phi(-n, x, -w). \tag{3.6}$$

Proof. We split $\Phi(s, x, w)$ into the sum of two integrals:

$$\Phi(s, x, w) = \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}e^{-xt}}{1 - we^{-t}}\, dt + \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}e^{-xt}}{1 - we^{-t}}\, dt.$$

The integral

$$\int_1^\infty \frac{t^{s-1}e^{-xt}}{1 - we^{-t}}\, dt$$

converges absolutely for all $s \in \mathbb{C}$ and $x > 0$. Then

$$\frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}e^{-xt}}{1 - we^{-t}}\, dt$$

vanishes at negative integers because the negative integers $s = -n$ are poles for $\Gamma(s)$, and $1/\Gamma(m) = 0$ for $m = 0, 1, 2, \ldots$. For $\mathrm{Re}(s) > 0$, the first integral may be written:

$$\begin{aligned}
\frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}e^{-xt}}{1 - we^{-t}}\, dt &= \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-2}(-t)e^{x(-t)}}{we^{-t} - 1}\, dt \\
&= \frac{1}{\Gamma(s)} \sum_{m=0}^\infty \int_0^1 \frac{(-1)^m B_m(x; w)}{m!} t^{m+s-2} dt \\
&= \frac{1}{\Gamma(s)} \sum_{m=0}^\infty (-1)^m \frac{B_m(x; w)}{m!} \cdot \frac{1}{m + s - 1},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\lim_{s \to -n} \Phi(s, x, w) &= \left(\lim_{s \to -n} \frac{(-1)^{n+1}}{\Gamma(s)(n+1)!(n+s)} \right) B_n(x; w) \\
&= -\frac{B_{n+1}(x; w)}{n+1}.
\end{aligned}$$

This proves the analytic continuation to an entire function on the complex s-plane and (3.4). The proof of the identities (3.5) and (3.6) is similar to the proof of the formula (3.4), we omit it. □

From our Theorems 2.1 and 3.1 we deduce the interesting results

Corollary 3.2. *Let $w \in \mathbb{C}\backslash\{0\}$. For $0 < x < 1$ if $n = 1$, $0 \leq x \leq 1$ if $n \geq 2$. We have*

$$B_n(x;w) = \frac{-n!}{w^x(2\pi i)^n} \sum_{k\in\mathbb{Z}}^{*} \frac{e^{2\pi ikx}}{\left(k - \frac{log(w)}{2\pi i}\right)^n}. \tag{3.7}$$

where $\sum_{k\in\mathbb{Z}}^{*} = \sum_{k\in\mathbb{Z}\backslash\{0\}}$ *if* $w = 1$ *and* $\sum_{k\in\mathbb{Z}}^{*} = \sum_{k\in\mathbb{Z}}$ *if* $w \neq 1$.

Corollary 3.3. *Let $w \in \mathbb{C}\backslash\{0\}$. For $0 < x < 1$ if $n = 0$, $0 \leq x \leq 1$ if $n \geq 1$. We have*

$$E_n(x;w) = \frac{2(n!)}{w^x(2\pi i)^{n+1}} \sum_{k\in\mathbb{Z}}^{**} \frac{e^{2\pi i\left(k-\frac{1}{2}\right)x}}{\left(k - \frac{1}{2} - \frac{log(w)}{2\pi i}\right)^{n+1}}, \tag{3.8}$$

and

$$G_n(x;w) = \frac{2(n!)}{w^x(2\pi i)^n} \sum_{k\in\mathbb{Z}}^{**} \frac{e^{2\pi i\left(k-\frac{1}{2}\right)x}}{\left(k - \frac{1}{2} - \frac{log(w)}{2\pi i}\right)^{n}}. \tag{3.9}$$

Where $\sum_{k\in\mathbb{Z}}^{**} = \sum_{k\in\mathbb{Z}\backslash\{0\}}$ *if* $w = -1$ *and* $\sum_{k\in\mathbb{Z}}^{**} = \sum_{k\in\mathbb{Z}}$ *if* $w \neq -1$.

The identities 3.2 and 3.8 were essentially discovered for the first time by Luo [10]. Luo's proof [10] is slightly different to our.

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