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Arakawa–Kaneko L-functions and generalized poly-Bernoulli polynomials

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ABSTRACT

We introduce and investigate generalized poly-Bernoulli numbers and polynomials. We state and prove several properties satisfied by these polynomials. The generalized poly-Bernoulli numbers are algebraic numbers. We introduce and study the Arakawa–Kaneko L-functions. The non-positive integer values of the complex variable s of these L-functions are expressed rationally in terms of generalized poly-Bernoulli numbers and polynomials. Furthermore, we prove difference and Raabe's type formulae for these L-functions.
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1. Generalized poly-Bernoulli polynomials

Throughout this paper, let χ be a Dirichlet character with conductor $f = f_\chi$.

1.1. Introduction

The *generalized Bernoulli numbers* $B_{m,\chi} \in \mathbb{Q}(\chi(1), \chi(2), \ldots)$ associated to χ $(m = 0, 1, \ldots)$ are defined by the generating function

$$\sum_{a=1}^{f} \chi(a) \frac{t}{e^{ft} - 1} e^{at} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}, \quad |t| < \frac{2\pi}{f}. \tag{1.1}$$

The main interest of these numbers is that they give the values at negative integers of Dirichlet L-series (cf. [8]): if $L(s, \chi) = \sum_{n \geqslant 0} \frac{\chi(n)}{n^s}$ $(\mathrm{Re}(s) > 1)$ is the L-series attached to χ, then we have the formula

$$L(-n, \chi) = -\frac{B_{n+1,\chi}}{n+1} \quad (n \geqslant 0). \tag{1.2}$$

The main purpose of this section is to introduce and investigate in details the generalized poly-Bernoulli polynomials and numbers. These polynomials include the generalized Bernoulli polynomials and numbers and the poly-Bernoulli polynomials and numbers.

Let us briefly review poly-Bernoulli polynomials [3]. For an integer $k \in \mathbb{Z}$, put

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},$$

which is the k-th polylogarithm if $k \geqslant 1$, and a rational function if $k \leqslant 0$. One knows that $\mathrm{Li}_1(z) = -\log(1 - z)$. The formal power series $\mathrm{Li}_k(z)$ can be used to define poly-Bernoulli polynomials. The polynomials $B_n^{(k)}(x)$ $(n = 0, 1, 2, \ldots)$ are said to be *poly-Bernoulli polynomials* if they satisfy

$$\frac{\mathrm{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}.$$

It should be noted that the definition of poly-Bernoulli polynomials by Coppo and Candelpergher [4] is different from ours. For any $n \geqslant 0$, we have

$$(-1)^n B_n^{(1)}(-x) = B_n(x),$$

where $B_n(x)$ are the classical Bernoulli polynomials given by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi.$$

1.2. Generalized poly-Bernoulli polynomials and numbers

Using a Dirichlet character, we generalize poly-Bernoulli polynomials and numbers as follows.

Let $k \in \mathbb{Z}$. For a Dirichlet character χ modulo f, we define *generalized poly-Bernoulli polynomials* $B_{n,\chi}^{(k)}(x)$ associated to χ by

$$\frac{1}{f}\sum_{a=1}^{f}\chi(a)\frac{\mathrm{Li}_k(1-e^{-ft})}{e^{ft}-1}e^{(x+a)t}=\sum_{n=0}^{\infty}B_{n,\chi}^{(k)}(x)\frac{t^n}{n!},\quad |t|<\frac{2\pi}{f}. \tag{1.3}$$

In particular, we put $B_{n,\chi}^{(k)} := B_{n,\chi}^{(k)}(0)$. We call them *generalized poly-Bernoulli numbers associated to* χ.

The following two identities hold:

$$B_{n,\chi}^{(1)}(x)=B_{n,\chi}(x). \tag{1.4}$$

If χ_0 is the trivial Dirichlet character, then

$$B_{n,\chi_0}^{(k)}(x)=B_n^{(k)}(x), \tag{1.5}$$

where $B_{n,\chi}(x)$ are the generalized Bernoulli polynomials associated to the Dirichlet character χ defined by

$$\sum_{a=1}^{f}\chi(a)\frac{t}{e^{ft}-1}e^{(x+a)t}=\sum_{n=0}^{\infty}B_{n,\chi}(x)\frac{t^n}{n!},\quad |t|<\frac{2\pi}{f}. \tag{1.6}$$

See [1,2,6,7,9] for (1.4), and [3] for (1.5).

We state intriguing identities. Let k be a given any integer.

The generalized poly-Bernoulli polynomials can be expressed in terms of poly-Bernoulli polynomials:

Theorem 1.1. *For any $n \geqslant 0$,*

$$B_{n,\chi}^{(k)}(x)=f^{n-1}\sum_{a=1}^{f}\chi(a)B_n^{(k)}\left(\frac{x+a}{f}-1\right).$$

In particular,

$$B_{n,\chi}^{(k)}=f^{n-1}\sum_{a=1}^{f}\chi(a)B_n^{(k)}\left(\frac{a}{f}-1\right).$$

An explicit formula for $B_{n,\chi}^{(k)}(x)$ is given as follows:

Theorem 1.2 *(Explicit formula). For any $n \geqslant 0$,*

$$B_{n,\chi}^{(k)}(x)=\frac{1}{f}\sum_{m=0}^{n}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\sum_{a=1}^{f}\chi(a)\big(x+a-(j+1)f\big)^n.$$

The generalized poly-Bernoulli polynomials satisfy the following difference identity:

Theorem 1.3 *(Difference formula). For any $n \geqslant 0$,*

$$B_{n,\chi}^{(k)}(x+f) - B_{n,\chi}^{(k)}(x) = \frac{1}{f}\sum_{m=0}^{n}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\sum_{a=1}^{f}\chi(a)(x+a-jf)^n.$$

In particular, for $k=1$, we have

$$B_{n,\chi}(x+f) - B_{n,\chi}(x) = n\sum_{a=1}^{f}\chi(a)(x+a)^{n-1}.$$

Theorem 1.4 *(Appell sequence). We have*

$$\frac{d}{dx}B_{0,\chi}^{(k)}(x) = 0,$$
$$\frac{d}{dx}B_{n+1,\chi}^{(k)}(x) = (n+1)B_{n,\chi}^{(k)}(x) \quad (n \geqslant 0).$$

Theorem 1.5 *(Addition formula). For any $n \geqslant 0$,*

$$B_{n,\chi}^{(k)}(x+y) = \sum_{m=0}^{n}\binom{n}{m}B_{m,\chi}^{(k)}(x)y^{n-m}.$$

In particular, we have

$$B_{n,\chi}^{(k)}(x) = \sum_{m=0}^{n}\binom{n}{m}B_{m,\chi}^{(k)}x^{n-m}.$$

We give recurrence formulae of two types.

Theorem 1.6 *(Recurrence formula 1). For any $n \geqslant 0$,*

$$B_{n,\chi}^{(k)}(x) = \sum_{m=0}^{n}B_{n-m}^{(k-1)}\binom{n}{m}\sum_{l=0}^{m}\frac{(-1)^{m+l}f^{n-l}}{n-l+1}\binom{m}{l}B_{l,\chi}(x).$$

Theorem 1.7 *(Recurrence formula 2). We have*

$$B_{0,\chi}^{(k)} = \begin{cases} 1 & (\chi = \chi_0), \\ 0 & (\chi \neq \chi_0), \end{cases}$$

$$B_{1,\chi}^{(k)} = \frac{1}{2}\left(B_{1,\chi}^{(k-1)}(x) + (x-f)B_{0,\chi}^{(k)} + \frac{1}{f}\sum_{a=1}^{f}\chi(a)a\right),$$

$$B_{n,\chi}^{(k)}(x) = \frac{1}{n+1}\left\{ B_{n,\chi}^{(k-1)}(x) - \sum_{m=1}^{n-1} f^{n-m}\binom{n}{m-1} B_{m,\chi}^{(k)}(x) \right.$$
$$+ (x-f)\sum_{m=0}^{n-1} f^{n-1-m}\binom{n}{m} B_{m,\chi}^{(k)}(x)$$
$$\left. + f^{n-2}\sum_{m=0}^{n-1}\binom{n}{m}\sum_{a=1}^{f}\chi(a) a B_m^{(k)}\left(\frac{x+a}{f}-1\right)\right\} \quad (n \geqslant 2).$$

By Theorems 1.5 and 1.7, $B_{n,\chi_0}^{(k)}(x)$ is a monic polynomial of degree n. Besides, if $\chi \neq \chi_0$, then $\deg B_{n,\chi}^{(k)}(x) < n$.

For $m, n \geqslant 0$, we define the following *symmetrized* Bernoulli polynomial of two variables:

$$C_{n,\chi}^{(-m)}(x,y) = \frac{1}{f^n}\sum_{b=1}^{f}\chi(b)\sum_{k=0}^{m}\binom{m}{k} B_{n,\chi}^{(-k)}(x)\left(\frac{y+b}{f}-1\right)^{m-k}.$$

Theorem 1.8 *(Symmetric formula). For $m, n \geqslant 0$, we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{n,\chi}^{(-m)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{m,\chi}^{(-n)}(y,x)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$= \frac{1}{f}\sum_{a=1}^{f}\sum_{b=1}^{f}\frac{\chi(a)\chi(b)e^{(x+a)t/f}e^{(y+b)u/f}}{e^t+e^u-e^{t+u}}.$$

As a corollary, we have the duality property for $C_{n,\chi}^{(-m)}(x)$:

Theorem 1.9 *(Duality). For $m, n \geqslant 0$,*

$$C_{n,\chi}^{(-m)}(x,y) = C_{m,\chi}^{(-n)}(y,x).$$

We deduce a closed formula from Theorem 1.8.

Theorem 1.10 *(Closed formula). For $m, n \geqslant 0$,*

$$C_{n,\chi}^{(-m)}(x,y) = \frac{1}{f}\sum_{j=0}^{\infty}(j!)^2\sum_{a=1}^{f}\sum_{b=1}^{f}\left(\sum_{k=0}^{n}\chi(a)\left(\frac{x+a}{f}\right)^{n-k}\binom{n}{k}\left\{ {k \atop j} \right\}\right)$$
$$\times\left(\sum_{l=0}^{m}\chi(b)\left(\frac{y+b}{f}\right)^{m-l}\binom{m}{l}\left\{ {l \atop j} \right\}\right),$$

where

$$\left\{ {n \atop m} \right\} = \frac{(-1)^m}{m!}\sum_{l=0}^{m}(-1)^l\binom{m}{l} l^n$$

is the Stirling number of the second kind.

2. The Arakawa–Kaneko L-functions

2.1. The Arakawa–Kaneko zeta functions: an overview [3]

Let $k \in \mathbb{Z}$. The *Arakawa–Kaneko zeta function* $Z_k(s,x)$ is by definition given by the Laplace–Mellin integral

$$Z_k(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-xt} t^{s-1}\, dt.$$

It is defined for $\mathrm{Re}(s) > 0$ and $x > 0$ if $k \geqslant 1$, and for $\mathrm{Re}(s) > 0$ and $x > |k| + 1$ if $k \leqslant 0$. If $x = 1$, then $Z_k(s,x)$ reduces to the original Arakawa–Kaneko zeta function introduced in [1]. The function $s \to Z_k(s,x)$ has analytic continuation to an entire function on the whole complex s-plane and has

$$Z_k(-n,x) = (-1)^n B_n^{(k)}(-x) \tag{2.1}$$

for all non-negative integers n and $x > 0$. Here $B_n^{(k)}(x)$ $(n = 0, 1, 2, \ldots)$ are the poly-Bernoulli polynomials. In addition, this zeta function can be rewritten to have an explicit formula as follows. For $k \in \mathbb{Z}$,

$$Z_k(s,x) = \sum_{m=0}^\infty \frac{1}{(m+1)^k} \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{1}{(x+j)^s}.$$

Note that for $k = 1$ we have

$$Z_1(s,x) = s\zeta(s+1,x), \tag{2.2}$$

where $\zeta(s,x)$ is the Hurwitz zeta function.

2.2. The Arakawa–Kaneko L-functions

For $k \in \mathbb{Z}$, the L-series attached to χ are given by

$$L_k(s,x,\chi) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{f} \sum_{a=1}^f \chi(a) \frac{\mathrm{Li}_k(1-e^{-ft})}{1-e^{-ft}} e^{-(x-a+f)t} t^{s-1}\, dt, \tag{2.3}$$

the Laplace–Mellin integral. It is defined for $\mathrm{Re}(s) > 0$ and $x > 0$ if $k \geqslant 1$, and for $\mathrm{Re}(s) > 0$ and $x > |k| + 1$ if $k \leqslant 0$. The L-series $L_k(s,x,\chi)$ are called the *Arakawa–Kaneko L-functions*. It is easy to see that the Arakawa–Kaneko L-functions include the Arakawa–Kaneko and Hurwitz zeta functions. To be exact, for the trivial Dirichlet character χ_0,

$$L_k(s,x,\chi_0) = Z_k(s,x).$$

If furthermore $k = 1$, then we have

$$L_1(s,x,\chi_0) = s\zeta(s+1,x).$$

The main purpose of this section is to investigate the fundamental properties of $L_k(s,x,\chi)$. Let us state our results.

Theorem 2.1. *For $k \in \mathbb{Z}$, we have*

$$L_k(s,x,\chi) = f^{-s-1}\sum_{a=1}^{f}\chi(a)Z_k\left(s,\frac{x-a}{f}+1\right). \tag{2.4}$$

Theorem 2.2 *(Interpolation formula). The function $s \to L_k(s,x,\chi)$ has analytic continuation to an entire function on the whole complex s-plane and for any positive integer n, we have*

$$L_k(-n,x,\chi) = (-1)^n B_{n,\chi}^{(k)}(-x). \tag{2.5}$$

Furthermore $L_k(s,x,\chi)$ satisfies the functional equation

$$\frac{\partial}{\partial x}L_k(s,x,\chi) = -sL_k(s+1,x,\chi). \tag{2.6}$$

As an immediate consequence of these theorems and an explicit formula for $Z_k(s,x)$, we get an explicit formula for $L_k(s,x,\chi)$ as follows. For $k \in \mathbb{Z}$,

$$L_k(s,x,\chi) = \frac{1}{f}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\sum_{a=1}^{f}\frac{\chi(a)}{(x-a+(j+1)f)^s}.$$

Theorem 2.3 *(Difference identity). The Arakawa–Kaneko L-functions satisfy the following difference identity*

$$\begin{aligned}&L_k(s,x+f,\chi) - L_k(s,x,\chi)\\ &\quad= \frac{1}{f}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^{j+1}\binom{m+1}{j}\sum_{a=1}^{f}\frac{\chi(a)}{(x-a+(j+1)f)^s}.\end{aligned}$$

In particular, for $k=1$, we have

$$L_k(s,x+f,\chi) - L_k(s,x,\chi) = -s\sum_{a=1}^{f}\chi(a)(x-a+f)^{-s-1}, \tag{2.7}$$

$$L_k(s,x+1,\chi_0) - L_k(s,x,\chi_0) = -sx^{-s-1}. \tag{2.8}$$

Theorem 2.4 *(Raabe's identity). For $s \in \mathbb{C}$ with $s \neq 1$ we have*

$$\begin{aligned}&\int_0^f L_k(s,x+w,\chi)\,dw\\ &\quad= -\frac{1}{s-1}\big(L_k(s-1,x+f,\chi) - L_k(s-1,x,\chi)\big) \qquad (2.9)\\ &\quad= \frac{1}{s-1}\frac{1}{f}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\sum_{a=1}^{f}\frac{\chi(a)}{(x-a+(j+1)f)^{s-1}}. \qquad (2.10)\end{aligned}$$

By combining Theorem 2.2 and Theorem 2.4, we deduce Raabe's type formulae for generalized poly-Bernoulli polynomials, Arakawa–Kaneko and Hurwitz zeta functions:

Corollary 2.5.

(1) *We have*

$$\int_0^f B_{n,\chi}^{(k)}(x-t)\,dt = \frac{1}{n+1}\frac{1}{f}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\sum_{a=1}^{f}\chi(a)\big(x+a-(j+1)f\big)^{n+1}.$$

(2) *If $s \neq 1$, then*

$$\int_0^1 Z_k(s,x+t)\,dt = \frac{1}{s-1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}(x+j)^{1-s}.$$

By (2.2), (2.8) and (2.9), we easily derive the classical Raabe's formula (cf. [5], Section 2):

$$\int_0^1 \zeta(s,x+t)\,dt = \frac{x^{1-s}}{s-1}, \quad s \in \mathbb{C}\setminus\{1\}.$$

3. Proofs of results

In this section we will give proofs for unproved results in the previous sections. The proofs of theorems in Section 1 are straightforward.

Proof of Theorem 1.1. We have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\chi}^{(k)}(x)\frac{t^n}{n!} &= \frac{1}{f}\sum_{a=1}^{f}\chi(a)\frac{\mathrm{Li}_k(1-e^{-ft})}{1-e^{-ft}}e^{((x+a)/f-1)ft}\\ &= \frac{1}{f}\sum_{a=1}^{f}\chi(a)\sum_{n=0}^{\infty}B_n^{(k)}\left(\frac{x+a}{f}-1\right)\frac{(ft)^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(f^{n-1}\sum_{a=1}^{f}\chi(a)B_n^{(k)}\left(\frac{x+a}{f}-1\right)\right)\frac{t^n}{n!}. \qquad \square\end{aligned}$$

Proof of Theorem 1.2. We proved in [3] that for any $n \geqslant 0$,

$$B_n^{(k)}(x) = \sum_{m=0}^{n}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(x-j)^n.$$

The theorem follows from this identity and Theorem 1.1. $\square$

Proof of Theorem 1.3. By Theorem 1.2, the left-hand side is

$$\frac{1}{f}\sum_{a=1}^{f}\chi(a)\sum_{m=0}^{n}\frac{1}{(m+1)^k}\left\{\sum_{j=0}^{m}(-1)^j\binom{m}{j}(x+a-jf)^n-\sum_{j=0}^{m}(-1)^j\binom{m}{j}\big(x+a-(j+1)f\big)^n\right\}.$$

The inside of the braces is

$$\begin{aligned}&(x+a)^n+(-1)^{m+1}\big(x+a-(m+1)f\big)^n+\sum_{j=1}^{m}(-1)^j\left(\binom{m}{j}+\binom{m}{j-1}\right)(x+a-jf)^n\\&=\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}(x+a-jf)^n,\end{aligned}$$

which completes the proof. □

Proof of Theorem 1.4. Differentiating (1.3) by x, the result is obtained. □

Proof of Theorem 1.5. By (1.3), we have

$$\sum_{n=0}^{\infty}B_{n,\chi}^{(k)}(x+y)\frac{t^n}{n!}=\left(\sum_{m=0}^{\infty}B_{m,\chi}^{(k)}(x)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}y^l\frac{t^l}{l!}\right),$$

which yields the result. □

Proof of Theorem 1.6. We recall

$$\frac{\mathrm{Li}_k(1-e^{-ft})}{1-e^{-ft}}=\frac{e^{ft}}{e^{ft}-1}\int_0^{ft}e^{-s}\frac{\mathrm{Li}_{k-1}(1-e^{-s})}{1-e^{-s}}\,ds.$$

Using this, the left-hand side of (1.3) becomes

$$\begin{aligned}&\frac{1}{f}\sum_{a=1}^{f}\chi(a)\frac{e^{(x+a)t}}{e^{ft}-1}\int_0^{ft}e^{-s}\frac{\mathrm{Li}_{k-1}(1-e^{-s})}{1-e^{-s}}\,ds\\&=\frac{1}{t}\sum_{a=1}^{f}\chi(a)\frac{te^{(x+a)t}}{e^{ft}-1}\int_0^{t}e^{-fu}\frac{\mathrm{Li}_{k-1}(1-e^{-fu})}{1-e^{-fu}}\,du\\&=\left(\sum_{n=0}^{\infty}B_{n,\chi}(x)\frac{t^{n-1}}{n!}\right)\int_0^{t}\left(\sum_{n=0}^{\infty}\frac{(-fu)^n}{n!}\right)\left(\sum_{m=0}^{\infty}B_m^{k-1}\frac{(fu)^m}{m!}\right)du\\&=\left(\sum_{n=0}^{\infty}B_{n,\chi}(x)\frac{t^{n-1}}{n!}\right)\int_0^{t}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}B_m^{(k-1)}\frac{(fu)^n}{n!}\,du\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{n=0}^{\infty} B_{n,\chi}(x)\frac{t^{n-1}}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^{n-m}f^n\binom{n}{m}B_m^{(k-1)}\frac{t^{n+1}}{(n+1)!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n} B_{n-l,\chi}(x)\sum_{m=0}^{l}(-1)^{l-m}f^l\binom{l}{m}B_m^{(k-1)}\frac{t^n}{(l+1)!(n-l)!}\\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{B_{n-l,\chi}(x)}{l+1}\binom{n}{l}\sum_{m=0}^{l}(-1)^{l-m}f^l\binom{l}{m}B_m^{(k-1)}\frac{t^n}{n!}\\
&\qquad \left(\binom{n}{l}\binom{l}{m}=\binom{n}{m}\binom{n-m}{n-l}\right)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}B_m^{(k-1)}\binom{n}{m}\sum_{l=m}^{n}\frac{(-1)^{l-m}f^l}{l+1}\binom{n-m}{n-l}B_{n-l,\chi}(x)\frac{t^n}{n!}\\
&\qquad (l'=n-l)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}B_m^{(k-1)}\binom{n}{m}\sum_{l'=0}^{n-m}\frac{(-1)^{n-l'-m}f^{n-l'}}{n-l'+1}\binom{n-m}{l'}B_{l',\chi}(x)\frac{t^n}{n!}\\
&\qquad (m'=n-m)\\
&= \sum_{n=0}^{\infty}\sum_{m'=0}^{n}B_{n-m'}^{(k-1)}\binom{n}{m'}\sum_{l'=0}^{m'}\frac{(-1)^{l'+m'}f^{n-l'}}{n-l'+1}\binom{m'}{l'}B_{l',\chi}(x)\frac{t^n}{n!}.
\end{aligned}
$$

This gives the result. □

Proof of Theorem 1.7. We proved in [3] the following recurrence formula for poly-Bernoulli polynomials:

$$
\begin{aligned}
&B_0^{(k)}(x)=1,\\
&B_1^{(k)}(x)=\frac{1}{2}\left(B_1^{(k-1)}(x)+xB_0^{(k)}\right),\\
&B_n^{(k)}(x)=\frac{1}{n+1}\left\{B_n^{(k-1)}(x)+xB_0^{(k)}(x)-\sum_{m=1}^{n-1}\left(\binom{n}{m-1}-\binom{n}{m}x\right)B_m^{(k)}(x)\right\}\quad (n\geqslant 2).
\end{aligned}
$$

By Theorem 1.1, we have

$$
\begin{aligned}
&B_{0,\chi}^{(k)}(x)=\frac{1}{f}\sum_{a=1}^{f}\chi(a),\\
&B_{1,\chi}^{(k)}(x)=\frac{1}{2}\left\{B_{1,\chi}^{(k-1)}(x)+\sum_{a=1}^{f}\chi(a)\left(\frac{x+a}{f}-1\right)B_0^{(k)}\left(\frac{x+a}{f}-1\right)\right\},
\end{aligned}
$$

which leads the case $n=0,1$. We next consider the case $n\geqslant 2$. By Theorem 1.1, we have

$$B_{n,\chi}^{(k)}(x)=\frac{1}{n+1}\left\{B_{n,\chi}^{(k-1)}(x)+f^{n-1}\sum_{a=1}^{f}\chi(a)\left(\frac{x+a}{f}-1\right)B_0^{(k)}\left(\frac{x+a}{f}-1\right)\right.$$
$$\left.-\sum_{m=1}^{n-1}\sum_{a=1}^{f}\left(\binom{n}{m-1}-\binom{n}{m}\left(\frac{x+a}{f}-1\right)\right)\chi(a)f^{n-1}B_m^{(k)}\left(\frac{x+a}{f}-1\right)\right\}.$$

Rewriting the inside of the braces, we obtain

$$B_{n,\chi}^{(k-1)}+f^{n-1}(x-f)B_{0,\chi}^{(k)}+f^{n-2}\sum_{a=1}^{f}\chi(a)aB_0^{(k)}\left(\frac{x+a}{f}-1\right)-\sum_{m=1}^{n-1}f^{n-m}\binom{n}{m-1}B_{m,\chi}^{(k)}(x)$$
$$+\sum_{m=1}^{n-1}f^{n-m-1}\binom{n}{m}(x-f)B_{m,\chi}^{(k)}(x)+\sum_{m=1}^{n-1}f^{n-2}\binom{n}{m}\sum_{a=1}^{f}\chi(a)aB_m^{(k)}\left(\frac{x+a}{f}-1\right).$$

This finishes the proof of the case $n\geqslant 2$. □

Proof of Theorem 1.8. We calculate the generating functions of $C_{n,\chi}^{(-m)}(x,y)$.

$$\begin{aligned}\text{L.H.S.}&=\sum_{b=1}^{f}\chi(b)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}B_{n,\chi}^{(-k)}(x)\left(\frac{y+b}{f}-1\right)^{m-k}\frac{(t/f)^n}{n!}\frac{u^m}{m!}\\&=\sum_{b=1}^{f}\chi(b)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}B_{n,\chi}^{(-k)}(x)\frac{\{(\frac{y+b}{f}-1)u\}^l}{l!}\frac{(t/f)^n}{n!}\frac{u^k}{k!}\\&=\sum_{b=1}^{f}e^{((y+b)/f-1)u}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B_{n,\chi}^{(-k)}(x)\frac{(t/f)^n}{n!}\frac{u^k}{k!}.\end{aligned}$$

By the definition of generalized Bernoulli polynomials, the last expression becomes

$$\begin{aligned}&\frac{1}{f}\sum_{a=1}^{f}\sum_{b=1}^{f}\chi(a)\chi(b)e^{((y+b)/f-1)u}\sum_{k=0}^{\infty}\frac{\mathrm{Li}_{-k}(1-e^{-t})}{1-e^{-t}}e^{((x+a)/f-1)t}\frac{u^k}{k!}\\&=\frac{1}{f}\sum_{a=1}^{f}\sum_{b=1}^{f}\chi(a)\chi(b)e^{((y+b)/f-1)u}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}B_n^{(-k)}\left(\frac{x+a}{f}-1\right)\frac{t^n}{n!}\frac{u^k}{k!}.\end{aligned}$$

Using

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}B_n^{(-k)}(x)\frac{t^n}{n!}\frac{u^k}{k!}=\frac{e^{xt}e^{t+u}}{e^t+e^u-e^{t+u}}$$

shown in [3], we have

$$\frac{1}{f}\sum_{a=1}^{f}\sum_{b=1}^{f}\frac{\chi(a)\chi(b)e^{(x+a)t/f}e^{(y+b)u/f}}{e^t+e^u-e^{t+u}},$$

which leads the result. □

Proof of Theorem 1.10. It suffices to compute the generating function of $C_{n,\chi}^{(-m)}(x,y)$.

$$\begin{aligned}
&\frac{e^{(x+a)t/f}e^{(y+b)u/f}}{e^t+e^u-e^{t+u}}\\
&\quad=\frac{e^{(x+a)t/f}e^{(y+b)u/f}}{1-(e^t-1)(e^u-1)}\\
&\quad=e^{(x+a)t/f}e^{(y+b)u/f}\sum_{j=0}^{\infty}\left(e^t-1\right)^j\left(e^u-1\right)^j\\
&\quad=\sum_{j=0}^{\infty}e^{(x+a)t/f}\left(e^t-1\right)^je^{(y+b)u/f}\left(e^u-1\right)^j\\
&\qquad\left(\text{by }\sum_{n=0}^{\infty}\left\{{n \atop k}\right\}\frac{u^n}{n!}=\frac{(e^u-1)^k}{k}\right)\\
&\quad=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n}{n!}\frac{u^m}{m!}\sum_{j=0}^{\infty}(j!)^2\left(\sum_{k=0}^{n}\left(\frac{x+a}{f}\right)^{n-k}\binom{n}{k}\left\{{k \atop j}\right\}\right)\left(\sum_{l=0}^{m}\left(\frac{y+b}{f}\right)^{m-l}\binom{m}{l}\left\{{l \atop j}\right\}\right).
\end{aligned}$$

This completes the proof. □

Proof of Theorem 2.1. It is easy to see that

$$\begin{aligned}
L_k(s,x,\chi)&=\frac{1}{f}\sum_{a=1}^{f}\chi(a)\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{\mathrm{Li}_k(1-e^{-ft})}{1-e^{-ft}}e^{-(x-a+f)t}t^{s-1}\,dt\\
&\qquad(\text{change } t \text{ by } z=ft)\\
&=\frac{1}{f}\sum_{a=1}^{f}\chi(a)\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{\mathrm{Li}_k(1-e^{-z})}{1-e^{-z}}e^{-(x-a+f)z/f}z^{s-1}f^{-s}\,dz\\
&=f^{-s-1}\sum_{a=1}^{f}\chi(a)Z_k\left(s,\frac{x-a}{f}+1\right).
\end{aligned}$$

This yields the proof of the theorem. □

Proof of Theorem 2.2. By using Theorem 2.1, to prove that $s\to L_k(s,x,\chi)$ has analytic continuation to an entire function on the whole complex s-plane, it is sufficient to show that the function $s\to Z_k(s,x)$ has such property. Since this fact comes from the first part of the Theorem 1.10 in [3], we omit it.

We now prove (2.5) of Theorem 2.2. The second part of Theorem 1.10 in [3] gives us

$$Z_k(-n,x)=(-1)^nB_n^{(k)}(-x).$$

Combining Theorems 1.1, 2.1 and this identity, (2.5) is gained. Eq. (2.6) follows from

$$\frac{\partial}{\partial x}L_k(s,x,\chi) = -\frac{1}{\Gamma(s)}\int_0^\infty \frac{1}{f}\sum_{a=1}^{f}\chi(a)\frac{\mathrm{Li}_k(1-e^{-ft})}{1-e^{-ft}}e^{-(x-a+f)t}t^s\,dt$$

$$= -\frac{s}{\Gamma(s+1)}\int_0^\infty \frac{1}{f}\sum_{a=1}^{f}\chi(a)\frac{\mathrm{Li}_k(1-e^{-ft})}{1-e^{-ft}}e^{-(x-a+f)t}t^s\,dt. \qquad \square$$

Proof of Theorem 2.3. We have

$$Z_k(s,x+1) - Z_k(s,x) = -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Li}_k\big(1-e^{-t}\big)e^{-xt}t^{s-1}\,dt$$

$$= -\frac{1}{\Gamma(s)}\sum_{m=1}^{\infty}\frac{1}{m^k}\int_0^\infty \big(1-e^{-t}\big)^m e^{-xt}t^{s-1}\,dt$$

$$= \frac{1}{\Gamma(s)}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\int_0^\infty \sum_{j=0}^{m+1}\binom{m+1}{j}(-1)^{j+1}e^{-(x+j)t}t^{s-1}\,dt$$

$$= \sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^{j+1}\binom{m+1}{j}(x+j)^{-s}.$$

By using the identity (2.4) we obtain

$$L_k(s,x+f,\chi) - L_k(s,x,\chi) = f^{-s-1}\sum_{a=1}^{f}\chi(a)\left(Z_k\left(s,\frac{x-a}{f}+2\right) - Z_k\left(s,\frac{x-a}{f}+1\right)\right)$$

$$= f^{-s-1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^{j+1}\binom{m+1}{j}\sum_{a=1}^{f}\frac{\chi(a)}{(\frac{x-a}{f}+1+j)^s}$$

$$= \frac{1}{f}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^{j+1}\binom{m+1}{j}\sum_{a=1}^{f}\frac{\chi(a)}{(x-a+(j+1)f)^s}.$$

When $k=1$, we have

$$Z_1(s,x+1) - Z_1(s,x) = -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Li}_1\big(1-e^{-t}\big)e^{-xt}t^{s-1}\,dt$$

$$= -\frac{1}{\Gamma(s)}\int_0^\infty e^{-xt}t^s\,dt \quad \big(\text{by } \mathrm{Li}_1(z) = -\log(1-z)\big)$$

$$= -\frac{\Gamma(s+1)}{\Gamma(s)}x^{-s-1}$$

$$= -sx^{-s-1}.$$

By using again the relation (2.4), we obtain our desired formula

$$L_1(s, x+f, \chi) - L_1(s, x, \chi) = -s \sum_{a=1}^{f} \chi(a)(x-a+f)^{-s-1}.$$

Furthermore, if $\chi = \chi_0$, then we get

$$L_1(s, x+1, \chi_0) - L_1(s, x, \chi_0) = -sx^{-s-1}.$$

This completes the proof of the theorem. □

Proof of Theorem 2.4. We have

$$\begin{aligned}
&\int_0^f L_k(s, x+w, \chi)\, dw \\
&= \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{f} \sum_{a=1}^{f} \chi(a) \frac{\mathrm{Li}_k(1-e^{-ft})}{1-e^{-ft}} e^{-(x-a+f)t} t^{s-1} \left(\int_0^f e^{-wt}\, dw \right) dt \\
&= \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{f} \sum_{a=1}^{f} \chi(a) \frac{\mathrm{Li}_k(1-e^{-ft})}{1-e^{-ft}} e^{-(x-a+f)t} t^{s-2} \left(1-e^{-ft}\right) dt \\
&= -\frac{\Gamma(s-1)}{\Gamma(s)} \left(L_k(s-1, x+f, \chi) - L_k(s-1, x, \chi) \right),
\end{aligned}$$

which implies the first result. Using Theorem 2.3, we get the second result. □

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Appendix A. The values of $B_n^{(k)}(x)$

In this appendix, we give the values of $B_n^{(k)}(x)$ for small k and n.

$$\begin{aligned}
B_1^{(-5)}(x) &= x + 32, \\
B_2^{(-5)}(x) &= x^2 + 64x + 454, \\
B_3^{(-5)}(x) &= x^3 + 96x^2 + 1362x + 4718, \\
B_4^{(-5)}(x) &= x^4 + 128x^3 + 2724x^2 + 18\,872x + 41\,506, \\
B_5^{(-5)}(x) &= x^5 + 160x^4 + 4540x^3 + 47\,180x^2 + 207\,530x + 329\,462, \\
B_1^{(-4)}(x) &= x + 16, \\
B_2^{(-4)}(x) &= x^2 + 32x + 146,
\end{aligned}$$

$$B_3^{(-4)}(x) = x^3 + 48x^2 + 438x + 1066,$$
$$B_4^{(-4)}(x) = x^4 + 64x^3 + 876x^2 + 4264x + 6902,$$
$$B_5^{(-4)}(x) = x^5 + 80x^4 + 1460x^3 + 10\,660x^2 + 34\,510x + 41\,506,$$
$$B_1^{(-3)}(x) = x + 8,$$
$$B_2^{(-3)}(x) = x^2 + 16x + 46,$$
$$B_3^{(-3)}(x) = x^3 + 24x^2 + 138x + 230,$$
$$B_4^{(-3)}(x) = x^4 + 32x^3 + 276x^2 + 920x + 1066,$$
$$B_5^{(-3)}(x) = x^5 + 40x^4 + 460x^3 + 2300x^2 + 5330x + 4718,$$
$$B_1^{(-2)}(x) = x + 4,$$
$$B_2^{(-2)}(x) = x^2 + 8x + 14,$$
$$B_3^{(-2)}(x) = x^3 + 12x^2 + 42x + 46,$$
$$B_4^{(-2)}(x) = x^4 + 16x^3 + 84x^2 + 184x + 146,$$
$$B_5^{(-2)}(x) = x^5 + 20x^4 + 140x^3 + 460x^2 + 730x + 454,$$
$$B_1^{(-1)}(x) = x + 2,$$
$$B_2^{(-1)}(x) = (x+2)^2,$$
$$B_3^{(-1)}(x) = (x+2)^3,$$
$$B_4^{(-1)}(x) = (x+2)^4,$$
$$B_5^{(-1)}(x) = (x+2)^5,$$
$$B_1^{(0)}(x) = x + 1,$$
$$B_2^{(0)}(x) = (x+1)^2,$$
$$B_3^{(0)}(x) = (x+1)^3,$$
$$B_4^{(0)}(x) = (x+1)^4,$$
$$B_5^{(0)}(x) = (x+1)^5,$$
$$B_1^{(1)}(x) = x + \frac{1}{2},$$
$$B_2^{(1)}(x) = x^2 + x + \frac{1}{6},$$
$$B_3^{(1)}(x) = x^3 + \frac{3}{2}x^2 + \frac{1}{2}x,$$
$$B_4^{(1)}(x) = x^4 + 2x^3 + x^2 - \frac{1}{30},$$
$$B_5^{(1)}(x) = x^5 + \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x,$$

$$B_1^{(2)}(x) = x + \frac{1}{4},$$
$$B_2^{(2)}(x) = x^2 + \frac{1}{2}x - \frac{1}{36},$$
$$B_3^{(2)}(x) = x^3 + \frac{3}{4}x^2 - \frac{1}{12}x - \frac{1}{24},$$
$$B_4^{(2)}(x) = x^4 + x^3 - \frac{1}{6}x^2 - \frac{1}{6}x + \frac{7}{450},$$
$$B_5^{(2)}(x) = x^5 + \frac{5}{4}x^4 - \frac{5}{18}x^3 - \frac{5}{12}x^2 + \frac{7}{90}x + \frac{1}{40},$$
$$B_1^{(3)}(x) = x + \frac{1}{8},$$
$$B_2^{(3)}(x) = x^2 + \frac{1}{4}x - \frac{11}{216},$$
$$B_3^{(3)}(x) = x^3 + \frac{3}{8}x^2 - \frac{11}{72}x - \frac{1}{288},$$
$$B_4^{(3)}(x) = x^4 + \frac{1}{2}x^3 - \frac{11}{36}x^2 - \frac{1}{72}x + \frac{1243}{54\,000},$$
$$B_5^{(3)}(x) = x^5 + \frac{5}{8}x^4 - \frac{55}{108}x^3 - \frac{5}{144}x^2 + \frac{1243}{10\,800}x - \frac{49}{7200},$$
$$B_1^{(4)}(x) = x + \frac{1}{16},$$
$$B_2^{(4)}(x) = x^2 + \frac{1}{8}x - \frac{49}{1296},$$
$$B_3^{(4)}(x) = x^3 + \frac{3}{16}x^2 - \frac{49}{432}x + \frac{41}{3456},$$
$$B_4^{(4)}(x) = x^4 + \frac{1}{4}x^3 - \frac{49}{216}x^2 + \frac{41}{864}x + \frac{26\,291}{3\,240\,000},$$
$$B_5^{(4)}(x) = x^5 + \frac{5}{16}x^4 - \frac{245}{648}x^3 + \frac{205}{1728}x^2 + \frac{26\,291}{648\,000}x - \frac{1921}{144\,000},$$
$$B_1^{(5)}(x) = x + \frac{1}{32},$$
$$B_2^{(5)}(x) = x^2 + \frac{1}{16}x - \frac{179}{7776},$$
$$B_3^{(5)}(x) = x^3 + \frac{3}{32}x^2 - \frac{179}{2592}x + \frac{515}{41\,472},$$
$$B_4^{(5)}(x) = x^4 + \frac{1}{8}x^3 - \frac{179}{1296}x^2 + \frac{515}{10\,368}x - \frac{216\,383}{194\,400\,000},$$
$$B_5^{(5)}(x) = x^5 + \frac{5}{32}x^4 - \frac{895}{3888}x^3 + \frac{2575}{20\,736}x^2 - \frac{216\,383}{38\,880\,000}x - \frac{183\,781}{25\,920\,000}.$$

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