

IDENTITIES INVOLVING VALUES OF BERNSTEIN, q -BERNOULLI AND q -EULER POLYNOMIALS

A. BAYAD AND T. KIM

Abstract In this paper we give some relations involving values of q -Bernoulli, q -Euler and Bernstein polynomials. From these relations, we obtain some interesting identities on the q -Bernoulli, q -Euler and Bernstein polynomials.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, let p be a fixed odd prime number. The symbols, \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p denote the ring of p -adic integers, the field of p -adic numbers and the field of p -adic completion of the algebraic closure of \mathbb{Q}_p , respectively. When one talks of q -extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p -adic $q \in \mathbb{C}_p$. Let \mathbb{N} be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $x = p^r \frac{s}{t}$ where $r \in \mathbb{Q}$ and $(s, t) = (p, s) = (p, t) = 1$. Then the p -adic absolute value is defined by $|x|_p = p^{-r}$. If $q \in \mathbb{C}$, we assume $|q| < 1$, and if $q \in \mathbb{C}_p$, we always assume $|1 - q|_p < 1$.

Let $C_{p^n} = \{\xi \mid \xi^{p^n} = 1\}$ be the cyclic group of order p^n . Then, the p -adic locally constant space is defined as $T_p = \lim_{n \rightarrow \infty} C_{p^n}$ (see [10]). Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on \mathbb{Z}_p . For $f \in UD(\mathbb{Z}_p)$, the bosonic p -adic invariant integral on \mathbb{Z}_p is defined by

$$\begin{aligned} I_1(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) &= \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \mu(x + p^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (\text{see [17, 10, 11, 12]}). \end{aligned} \tag{1}$$

From (1), we note that

$$I_1(f_1) = \int_{\mathbb{Z}_p} f(x+1) d\mu(x) = \int_{\mathbb{Z}_p} f(x) d\mu(x) + f'(0), \tag{2}$$

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where $f_1(x) = f(x+1)$, (see [14, 10]).

Let $f \in UD(\mathbb{Z}_p)$. Then the fermionic integral on \mathbb{Z}_p is given by

$$\begin{aligned} I_{-1}(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \mu_{-1}(x + p^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x, \quad (\text{see [18, 17, 9]}). \end{aligned} \quad (3)$$

As known results, by (3), we get

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \quad (\text{see [15]}). \quad (4)$$

By using (2) and (4), we investigate some properties for the q -Bernoulli polynomials and the q -Euler polynomials.

Let $C[0, 1]$ be denote by the space of continuous functions on $[0, 1]$. For $f \in C[0, 1]$, Bernstein introduced the following well-known linear positive operator in the field of real numbers \mathbb{R} :

$$\mathbb{B}_n(f|x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x),$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$ (see [1, 3, 18, 17, 14, 12]). Here, $\mathbb{B}_n(f|x)$ is called the Bernstein operator of order n for f . For $k, n \in \mathbb{Z}_+$, the Bernstein polynomials of degree n are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } x \in [0, 1]. \quad (5)$$

For $x \in \mathbb{Z}_p$, the p -adic extension of Bernstein polynomials are given by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{where } k, n \in \mathbb{Z}_+, \quad (\text{see [18, 22, 5]}). \quad (6)$$

The purpose of this paper is to give some relations for the q -Bernoulli, q -Euler and Bernstein polynomials.

From these relations, we obtain some interesting identities on the q -Bernoulli, q -Euler and Bernstein polynomials.

2. SOME IDENTITIES ON THE q -BERNOULLI, q -EULER AND BERNSTEIN POLYNOMIALS

In this section we assume that $q \in T_p$. Let $f(x) = q^x e^{xt}$. From (1) and (2), we have

$$\int_{\mathbb{Z}_p} f(x) d\mu(x) = \frac{t}{qe^t - 1}, \quad (\text{see [10]}). \quad (7)$$

The q -Bernoulli numbers are defined by

$$\frac{t}{qe^t - 1} = e^{B(q)t} = \sum_{n=0}^{\infty} B_n(q) \frac{t^n}{n!}, \quad (\text{see [10]}), \quad (8)$$

with usual convention about replacing $B_n(q)$ by $B_n(q)$.

By (7) and (8), we get Witt's formula for the q -Bernoulli numbers as follows:

$$\int_{\mathbb{Z}_p} q^x x^n d\mu(x) = B_n(q), \quad n \in \mathbb{Z}_+, \quad (\text{ see [10] }). \quad (9)$$

From (8), we can derive the following recurrence sequence:

$$B_0(q) = 0, \text{ and } q(B(q) + 1)^n - B_n(q) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad (10)$$

with usual convention about replacing $B^n(q)$ by $B_n(q)$.

Now, we define the q -Bernoulli polynomials as follows:

$$\frac{t}{qe^t - 1} e^{xt} = e^{B(x|q)t} = \sum_{n=0}^{\infty} B_n(x|q) \frac{t^n}{n!}, \quad (11)$$

with usual convention about replacing $B^n(x|q)$ by $B_n(x|q)$.

By (2) and (11), we easily get

$$\int_{\mathbb{Z}_p} q^y e^{(x+y)t} d\mu(y) = \frac{t}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x|q) \frac{t^n}{n!}. \quad (12)$$

Thus, we also obtain Witt's formula for the q -Bernoulli polynomials as follows:

$$\int_{\mathbb{Z}_p} q^y (x+y)^n d\mu(y) = B_n(x|q), \quad \text{for } n \in \mathbb{Z}_+. \quad (13)$$

By (9) and (13), we easily get

$$\begin{aligned} B_n(x|q) &= \sum_{l=0}^n \binom{n}{l} x^{n-l} B_l(q) \\ &= (x + B(q))^n, \quad \text{for } n \in \mathbb{Z}_+. \end{aligned}$$

It is easy to show that

$$\begin{aligned} \frac{t}{qe^t - 1} &= \left(\frac{t}{q(1 - q^{-1})} \right) \left(\frac{1 - q^{-1}}{e^t - q^{-1}} \right) \\ &= \frac{1}{q(1 - q^{-1})} \sum_{n=0}^{\infty} H_n(q^{-1})(n+1) \frac{t^{n+1}}{(n+1)!}, \end{aligned} \quad (14)$$

where $H_n(q^{-1})$ are the n -th Frobenius-Euler numbers (see [17]).

By (8), (10) and (15), we get

$$\frac{B_{n+1}(q)}{n+1} = \frac{H_n(q^{-1})}{q(1 - q^{-1})} = \frac{H_n(q^{-1})}{q-1}. \quad (15)$$

Therefore, by (15), we obtain the following proposition.

Proposition 1. *For any $n \in \mathbb{Z}_+$, we have*

$$\frac{B_{n+1}(q)}{n+1} = \frac{H_n(q^{-1})}{q(1 - q^{-1})} = \frac{H_n(q^{-1})}{q-1},$$

where $H_n(q^{-1})$ are the n -th Frobenius-Euler numbers.

From (11) and (12), we can derive the following equation:

$$\frac{qt}{qe^t - 1} e^{(1-x)t} = \frac{-t}{q^{-1}e^{-t} - 1} e^{-xt} = \sum_{n=0}^{\infty} B_n(x|q^{-1}) (-1)^n \frac{t^n}{n!}, \quad (16)$$

and

$$\frac{qt}{qe^t - 1} e^{(1-x)t} = qe^{B(1-x|q)t} = q \sum_{n=0}^{\infty} B_n(1-x|q) \frac{t^n}{n!}. \quad (17)$$

By comparing the coefficients on the both sides of (16) and (17), we obtain the following theorem.

Theorem 2. *Let $n \in \mathbb{Z}_+$. Then we have*

$$qB_n(1-x|q) = (-1)^n B_n(x|q^{-1}).$$

From (10) and (13), we have

$$B_n(2|q) = (B(q) + 1 + 1)^n = \sum_{l=0}^n \binom{n}{l} B_l(1|q) \quad (18)$$

$$= B_0(q) + nB_1(1|q) + \frac{1}{q} \sum_{l=2}^n \binom{n}{l} qB_l(1|q) \quad (19)$$

$$\begin{aligned} &= \frac{1}{q} \sum_{l=2}^n \binom{n}{l} B_l(q) + \frac{n}{q} (1 + B_1(q)) \\ &= \frac{n}{q} + \frac{1}{q^2} qB_n(1|q) = \frac{n}{q} + \frac{1}{q^2} B_n(q), \text{ while } n > 1. \end{aligned}$$

Therefore, by (18), we obtain the following theorem.

Theorem 3. *For $n \in \mathbb{Z}_+$ with $n > 1$, we have*

$$q^2 B_n(2|q) = B_n(q) + qn.$$

By (9) and (13), we easily see that

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{-x} (1-x)^n d\mu(x) &= (-1)^n \int_{\mathbb{Z}_p} q^{-x} (x-1)^n d\mu(x) \\ &= q \int_{\mathbb{Z}_p} (x+2)^n q^x d\mu(x) \\ &= \frac{1}{q} \int_{\mathbb{Z}_p} x^n q^x d\mu(x) + n. \end{aligned} \quad (20)$$

Therefore, by (19), we obtain the following theorem.

Theorem 4. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\int_{\mathbb{Z}_p} q^{-x} (1-x)^n d\mu(x) = \frac{1}{q} \int_{\mathbb{Z}_p} x^n q^x d\mu(x) + n.$$

Let $f(x) = q^x e^{xt}$. Then, from (3) and (4), we note that

$$\int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x) = \frac{2}{qe^t + 1}, \quad (\text{ see [18, 19, 4, 21]}). \quad (21)$$

In [18], the q -Euler numbers are defined by

$$\frac{2}{qe^t + 1} = e^{E(q)t} = \sum_{n=0}^{\infty} E_n(q) \frac{t^n}{n!}, \quad (22)$$

with usual convention about replacing $E^n(q)$ by $E_n(q)$.

Let us define the q -Euler polynomials as follows:

$$\frac{2}{qe^t + 1} e^{xt} = e^{E(x|q)t} = \sum_{n=0}^{\infty} E_n(x|q) \frac{t^n}{n!}. \quad (23)$$

From (3) and (4), we note that

$$\int_{\mathbb{Z}_p} q^y e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x|q) \frac{t^n}{n!}. \quad (24)$$

Thus, by (22) and (23), we get

$$\int_{\mathbb{Z}_p} q^y (x+y)^n d\mu_{-1}(y) = E_n(x|q), \quad \text{for } n \in \mathbb{Z}_+. \quad (25)$$

By (20), (21) and (24), we get

$$E_n(x|q) = \sum_{l=0}^n \binom{n}{l} x^{n-l} E_{l,q} = (x + E(q))^n, \quad (26)$$

with usual convention $E(q)^n$ by $E_n(q)$.

In [2], it is known that

$$qE_n(2|q) = 2 + \frac{1}{q} E_n(q), \quad (27)$$

and

$$\int_{\mathbb{Z}_p} q^{-x} (1-x)^n d\mu_{-1}(x) = 2 + \frac{1}{q} \int_{\mathbb{Z}_p} x^n q^x d\mu_{-1}(x), \quad (\text{ see [13] }).$$

For $x \in \mathbb{Z}_p$, by (6), we have

$$\begin{aligned} B_n(1-x|q) &= \sum_{l=0}^n \binom{n}{l} (1-x)^{n-l} B_l(q) \\ &= \sum_{l=0}^n \binom{n}{l} (1-x)^{n-l} x^l B_l(q) x^{-l} \\ &= \sum_{l=0}^n B_{l,n}(x) B_l(q) x^{-l}, \end{aligned} \quad (28)$$

where $B_{l,n}(x)$ are Bernstein polynomials of degree n .

Therefore, by (27), we obtain the following proposition.

Proposition 5. *For $n \in \mathbb{Z}_+$, we have*

$$B_n(1-x|q) = \sum_{l=0}^n B_{l,n}(x) B_l(q) x^{-l},$$

where $B_{l,n}(x)$ are Bernstein polynomials of degree n .
From (7) and (20), we can derive the following equation:

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} q^{x+y} e^{(x+y)t} d\mu_{-1}(x) d\mu(y) = \frac{2t}{q^2 e^{2t} - 1}. \quad (29)$$

By (8), we get

$$\frac{2t}{q^2 e^{2t} - 1} = e^{B(q^2)2t} = \sum_{n=0}^{\infty} 2^n B_n(q^2) \frac{t^n}{n!}. \quad (30)$$

By comparing the coefficients on the both sides of (28) and (29), we obtain the following theorem.

Theorem 6. *For $n \in \mathbb{Z}_+$, we have*

$$\frac{1}{2^n} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} q^{x+y} (x+y)^{n+1} d\mu_{-1}(x) d\mu(y) = B_n(q^2).$$

By using (9) and (24), we obtain the following corollary.

Corollary 7. *For $n \in \mathbb{Z}_+$, we have*

$$\frac{1}{2^n} \sum_{l=0}^n \binom{n}{l} E_l(q) B_{n-l}(q) = B_n(q^2).$$

From the definition of Bernstein polynomials, we note that

$$B_{k,n}(x) = B_{n-k,n}(1-x), \quad \text{for } n, k \in \mathbb{Z}_+.$$

Thus, we have

$$B_{k,n}\left(\frac{1}{2}\right) = B_{n-k,n}\left(\frac{1}{2}\right) = \binom{n}{k} \left(\frac{1}{2}\right)^{n-k} \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^n \binom{n}{k}.$$

Therefore, we obtain the following lemma.

Lemma 8. *Let $n, k \in \mathbb{Z}_+$. Then we have*

$$B_{k,n}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^n \binom{n}{k}.$$

By Lemma 8 and Corollary 7, we obtain the following corollary.

Corollary 9. *For $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned} B_n(q^2) &= \frac{1}{2^n} \sum_{l=0}^n \binom{n}{l} E_l(q) B_{n-l}(q) \\ &= \sum_{l=0}^n B_{l,n}\left(\frac{1}{2}\right) E_l(q) B_{n-l}(q). \end{aligned}$$

For the right side of (28), we have

$$\begin{aligned} \frac{2t}{q^2 e^{2t} - 1} &= -2t \sum_{m=0}^{\infty} q^{2m} e^{2mt} = -2t \sum_{n=0}^{\infty} \left(2^n \sum_{m=0}^{\infty} q^{2m} m^n \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(-2^{n+1} (n+1) \sum_{m=0}^{\infty} q^{2m} m^n \right) \frac{t^{n+1}}{(n+1)!}. \end{aligned} \quad (31)$$

In (29), we see that $B_0(q^2) = 0$. By comparing coefficients on the both sides of (29) and (30), we obtain the following theorem.

Theorem 10. *For $n \in \mathbb{Z}_+$, we have*

$$-\frac{1}{2^{(n+1)}(n+1)} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} q^{x+y} (x+y)^n d\mu_{-1}(x) d\mu(y) = \sum_{m=0}^{\infty} q^{2m} m^n.$$

By (9), (24) and binomial theorem, we obtain the following corollary

Corollary 11. *For $n \in \mathbb{Z}_+$, we have*

$$\sum_{m=0}^{\infty} q^{2m} m^n = -\frac{1}{2^{(n+1)}(n+1)} \sum_{l=0}^{n+1} \binom{n+1}{l} E_l(q) B_{n+1-l}(q).$$

By Lemma 7, we obtain the following corollary.

Corollary 12. *For $n \in \mathbb{Z}_+$, we have*

$$\sum_{m=0}^{\infty} q^{2m} m^n = -\frac{1}{n+1} \sum_{l=0}^{n+1} B_{n+1,l} \left(\frac{1}{2}\right) E_l(q) B_{n+1-l}(q).$$

It seems to be important to study double p -adic integral representation of bosonic and fermionic on the Bernstein polynomials associated with q -Bernoulli and q -Euler polynomials. Theorem 10 is useful to study those integral representation on \mathbb{Z}_p related to Bernstein polynomials. Now, we take bosonic p -adic invariant integral on \mathbb{Z}_p for one Bernstein polynomials in (6) as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x) q^x d\mu(x) &= \int_{\mathbb{Z}_p} \binom{n}{k} (1-x)^{n-k} x^k q^x d\mu(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} x^{k+l} q^x d\mu(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l B_{k+l}(q). \end{aligned} \tag{32}$$

From the reflection symmetry of Bernstein polynomials, we have

$$B_{k,n}(x) = B_{n-k,n}(1-x), \quad \text{where } n, k \in \mathbb{Z}_+. \tag{33}$$

Let $n, k(>1) \in \mathbb{N}$ with $n > k+1$. Then, by (32), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x) q^x d\mu(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x) q^x d\mu(x) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} (1-x)^{n-l} q^x d\mu(x) \\ &= q \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left(q \int_{\mathbb{Z}_p} q^{-x} x^{n-l} d\mu(x) + n-l \right) \\ &= q \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} B_{n-l}(q^{-1}). \end{aligned} \tag{34}$$

Therefore, we obtain the following theorem.

Theorem 13. *Let $n, k(> 1) \in \mathbb{Z}_+$ with $n > k + 1$. Then we have*

$$\int_{\mathbb{Z}_p} B_{k,n}(x) q^{1-x} d\mu(x) = 2^k \binom{n}{k} \sum_{l=0}^k B_{l,k}(\frac{1}{2}) (-1)^{k-l} B_{n-l}(q).$$

By (31), we obtain the following corollary

Corollary 14. *For $n, k(> 1) \in \mathbb{Z}_+$ with $n > k + 1$, we have*

$$2^{2k} \binom{n}{k} B_{l,k}(\frac{1}{2}) (-1)^{k-l} B_{n-l}(q) = 2^n q \sum_{l=0}^{n-k} B_{l,n-k}(\frac{1}{2}) (-1)^l B_{k+l}(q).$$

Let $m, n, k(> 1) \in \mathbb{Z}_+$, with $m + n > k + 1$. Then we have

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{-x} d\mu(x) &= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \int_{\mathbb{Z}_p} q^{-x} (1-x)^{n+m-l} d\mu(x) \\ &= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} q \int_{\mathbb{Z}_p} q^x (x+2)^{n+m-l} d\mu(x) \\ &= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left(\frac{1}{q} B_{n+m-l}(q) + n + m - l \right). \end{aligned} \tag{35}$$

Therefore, by (34), we obtain the following theorem.

Theorem 15. *Let $m, n, k(> 1) \in \mathbb{Z}_+$ with $m + n > k + 1$. Then we have*

$$\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{1-x} d\mu(x) = \binom{n}{k} \binom{m}{k} 2^{2k} \sum_{l=0}^{2k} B_{l,2k}(\frac{1}{2}) (-1)^{l+2k} B_{n+m-l}(q).$$

By binomial theorem, we easily get

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{1-x} d\mu(x) &= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \int_{\mathbb{Z}_p} x^{l+2k} q^{1-x} d\mu(x) \\ &= q \binom{n}{k} \binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l B_{l+2k}(\frac{1}{q}). \end{aligned} \tag{36}$$

By (35), we obtain the following corollary.

Corollary 16. *Let $m, n, k(> 1) \in \mathbb{Z}_+$ with $n + m > 2k + 1$. Then we have*

$$2^{4k} \sum_{l=0}^{2k} (-1)^{l+2k} B_{l,2k}(\frac{1}{2}) B_{n+m-l}(q) = 2^{n+m} q \sum_{l=0}^{n+m-2k} (-1)^l B_{l,n+m-2k}(\frac{1}{2}) B_{l+2k}(\frac{1}{q}).$$

For $s \in \mathbb{N}$, let $n_1, n_2, \dots, n_s, k(> 1) \in \mathbb{Z}_+$ with $n_1 + n_2 + \dots + n_s > sk + 1$.

$$\begin{aligned}
& \int_{\mathbb{Z}_p} B_{k,n_1}(x) \cdots B_{k,n_s}(x) q^{-x} d\mu(x) = \binom{n_1}{k} \cdots \binom{n_s}{k} \int_{\mathbb{Z}_p} x^{sk} (1-x)^{n_1+\dots+n_s} q^{-x} d\mu(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} (-1)^{sk+l} \int_{\mathbb{Z}_p} (1-x)^{n_1+\dots+n_s-l} q^{-x} d\mu(x) \\
&= q \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \int_{\mathbb{Z}_p} (x+2)^{n_1+\dots+n_s-l} q^x d\mu(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left(\frac{1}{q} B_{n_1+\dots+n_s-l}(q) + n_1 + \dots + n_s - l \right). \tag{37}
\end{aligned}$$

Therefore, by (36), we obtain the following theorem.

Theorem 17. For $s \in \mathbb{N}$, let $n_1, n_2, \dots, n_s, k(> 1) \in \mathbb{Z}_+$ with $n_1 + n_2 + \dots + n_s > sk + 1$. Then we have

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x) \cdots B_{k,n_s}(x) q^{1-x} d\mu(x) = \binom{n_1}{k} \cdots \binom{n_s}{k} 2^{sk} \sum_{l=0}^{sk} (-1)^{sk+l} B_{l,sk} \left(\frac{1}{2} \right) B_{n_1+\dots+n_s-l}(q).$$

From binomial theorem, we can easily derive the following equation:

$$\begin{aligned}
& \int_{\mathbb{Z}_p} B_{k,n_1}(x) \cdots B_{k,n_s}(x) q^{-x} d\mu(x) \tag{38} \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{n_1+\dots+n_s-sk}{l} (-1)^l \int_{\mathbb{Z}_p} x^{l+sk} q^{-x} d\mu(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{n_1+\dots+n_s-sk}{l} (-1)^l B_{l+sk}(q^{-1}).
\end{aligned}$$

By (37), we obtain the following corollary.

Corollary 18. Let $s \in \mathbb{N}$ and $n_1, n_2, \dots, n_s, k(> 1) \in \mathbb{Z}_+$ with $n_1 + n_2 + \dots + n_s > sk + 1$. Then we have

$$\begin{aligned}
& 2^{n_1+\dots+n_s-2sk} q \sum_{l=0}^{n_1+\dots+n_s-sk} (-1)^l B_{l,n_1+\dots+n_s-sk} \left(\frac{1}{2} \right) B_{l+sk}(q^{-1}) \\
&= \sum_{l=0}^{sk} B_{l,sk} \left(\frac{1}{2} \right) (-1)^{l+sk} B_{n_1+\dots+n_s-l}(q).
\end{aligned}$$

3. FURTHER REMARKS

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. The q -Euler polynomials and q -Bernoulli polynomials are defined by the generating functions as follows:

$$F_q^E(t, x) = \frac{2}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(q|x) \frac{t^n}{n!}, \quad |t + \log q| < \pi,$$

and

$$F_q^B(t, x) = \frac{t}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(q|x) \frac{t^n}{n!}, \quad |t + \log q| < 2\pi. \tag{39}$$

In the special case $x = 0$, $E(0|q) = E_n(q)$ are called the n -th q -Euler numbers and $B_n(0|q) = B_n(q)$ are also called the n -th q -Bernoulli numbers.

As usual convention, let us define $F_q^B(t, 0) = F_q^B(t)$ and $F_q^E(t, 0) = F_q^E(t)$.

For $s \in \mathbb{C}$, we have

$$\begin{aligned} \frac{q}{\Gamma(s)} \int_0^\infty F_q^B(-t, 1) t^{s-2} dt &= \frac{q}{\Gamma(s)} \int_0^\infty \frac{e^{-t}}{1 - qe^{-t}} t^{s-1} dt \\ &= \sum_{m=0}^\infty q^{m+1} \frac{1}{\Gamma(s)} \int_0^\infty e^{-(m+1)t} t^{s-1} dt \\ &= \sum_{m=0}^\infty \frac{q^{m+1}}{(m+1)^s} = \sum_{m=1}^\infty \frac{q^m}{m^s}. \end{aligned} \quad (40)$$

From (39), for any $|q| < 1$ we define q -zeta function as follows:

$$\zeta_q(s) = \sum_{m=1}^\infty \frac{q^m}{m^s}.$$

Note that $\zeta_q(s)$ is analytic in whole complex s -plane. By (38), (39) and elementary complex integral, we get

$$\zeta_q(1-n) = (-1)^n \frac{qB_n(1|q)}{n} = \begin{cases} -(1+B_1(q)) & \text{if } n=1, \\ (-1)^n \frac{B_n(q)}{n} & \text{if } n>1. \end{cases} \quad (41)$$

From (38), we note that

$$\left(\frac{2}{qe^t + 1} \right) \left(\frac{t}{qe^t - 1} \right) = \frac{2t}{q^2 e^{2t} - 1} = \sum_{n=0}^\infty 2^n B_n(q^2) \frac{t^n}{n!},$$

and

$$\left(\frac{2}{qe^t + 1} \right) \left(\frac{t}{qe^t - 1} \right) = e^{(B(q)+E(q))t} = \sum_{n=0}^\infty (B(q) + E(q))^n \frac{t^n}{n!}.$$

Thus, we have

$$2^n B_n(q^2) = (B(q) + E(q))^n = \sum_{l=0}^n \binom{n}{l} B_l(q) E_{n-l}(q), \quad (42)$$

with usual convention about replacing $B^n(q)$ by $B_n(q)$.

By (42), we get

$$B_n(q^2) = \frac{1}{2^n} \sum_{l=0}^n \binom{n}{l} B_l(q) E_{n-l}(q), \quad (\text{cf. [16]}). \quad (43)$$

From (40) and (41), we can derive the following equation:

$$\begin{aligned} \zeta_{q^2}(-n) &= (-1)^{n+1} \frac{q^2 B_{n+1}(q^2)}{n+1} = -(-1)^n \frac{B_{n+1}(q^2)}{n+1} \\ &= -(-1)^n \frac{1}{2^{n+1}(n+1)} \sum_{l=0}^{n+1} \binom{n+1}{l} B_l(q^2) E_{n+1-l}(q^2). \end{aligned}$$

REFERENCES

- [1] M. Acikgoz, S. Araci, *A study on the integral of the product of several type Bernstein polynomials*, IST Transaction of Applied Mathematics-Modelling and Simulation, 2010, Vol. 1, No.1(2), ISBN 1913-8342, 10–14.
- [2] A. Bayad, T. Kim, B. Lee, S.-H. Rim, *Some identities on the Bernstein polynomials associated with q -Euler polynomials* (Communicated).
- [3] S. Bernstein, *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités*, Commun. Soc. Math. Kharkow **13** (1912), 1–2.
- [4] I. N. Cangul, V. Kurt, H. Ozden, Y. Simsek, *On the higher-order w - q -Genocchi numbers*, Adv. Stud. Contemp. Math. **19** (2009), 39–57.
- [5] H. W. Gould, *A theorem concerning the Bernstein polynomials*, Mathematics Magazine, Vol. 31, No. 5 (1958), pp. 259–264.
- [6] N. K. Govil, V. Gupta, *Convergence of q -Meyer-König-Zeller-Durrmeyer operators*, Adv. Stud. Contemp. Math. **19** (2009), 97–108.
- [7] L.C. Jang, W.-J. Kim, Y. Simsek, *A study on the p -adic integral representation on \mathbb{Z}_p associated with Bernstein and Bernoulli polynomials*, Advances in Difference Equations, Vol 2010 (2010), Article ID 163219, 6pp.
- [8] K. I Joy, *Bernstein polynomials*, On-line Geometric Modelling Notes, http://en.wikipedia.org/wiki/Bernstein_polynomials.
- [9] T. Kim, *Barnes-type multiple q -zeta functions and q -Euler polynomials*, J. Physics A : Math. Theor., **43** (2010), 255201, 11pp.
- [10] T. Kim, *An analogue of Bernoulli numbers and their congruences*, Rep. Fac. Sci. Engrg. Saga Univ. Math.22(1994), N.2, 21–26.
- [11] T. Kim, *On the symmetries of the q -Bernoulli polynomials*, Abstract and Applied Analysis, Vol.2008(2008), Article ID 914367, 7pp.
- [12] T. Kim, *A new approach to p -adic q - L -functions*, Advanced Studies in Contemporary Mathematics **12** (2006), N.1, 61–72.
- [13] T. Kim, *q -Bernoulli numbers and polynomials associated with Gaussian binomial coefficients*, Russian Journal of Mathematical physics **15** (2008), 51–59.
- [14] T. Kim, J. Choi, Y. H. Kim, *q -Bernstein polynomials associated with q -Stirling numbers and Carlit's q -Bernoulli numbers*, Abstract and Applied Analysis Volume 2010, Article ID 150975, 11 pages doi:10.1155/2010/150975 .
- [15] T. Kim, *Some identities on the q -Euler polynomials of higher order and q -Stirling numbers by fermionic p -adic integral on \mathbb{Z}_p* , Russian Journal of Mathematical physics **16** (2009), 484–491.
- [16] T. Kim, *Note on the Euler q -zeta functions*, Journal of Number Theory **129** (2009), 1798–1804.
- [17] T. Kim, B. Lee, *Some identities of the Frobenius-Euler polynomials*, Abstract and Applied Analysis 2009 (2009), Article ID 639439, 7 pp.
- [18] T. Kim, J. Choi, Y.H. Kim, C. S. Ryoo, *On the fermionic p -adic integral representation of Bernstein polynomials associated with Euler numbers and polynomials*, Journal of inequalities and Applications, Vol 2010 (2010), Article ID 864249, 12 pp.
- [19] B. A. Kupershmidt, *Reflection symmetries of q -Bernoulli polynomials*, J. Non-linear Math. Phys. **12**(2005), 412–422.
- [20] G.G. Lorentz, *Bernstein polynomials*, (2nd edition) printed in USA (1986).

- [21] S-H. Rim, J-H. Jin, E-J. Moon, S-J. Lee, *On multiple interpolation functions of the q -Genocchi polynomials*, Journal of Inequalities and Applications, Vol 2010 (2010), Article ID 351419, 13 pp..
- [22] S. Zorlu, H. Aktuglu, and M. A. Ozarslan, *An estimation to the solution of an initial value problem via q -Bernstein polynomials*, Journal of Computational Analysis and Applications, Vol. 12, N. 3, (2010), 637645.

ABDELMEJID BAYAD. DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ D'EVRY VAL D'ESSONNE, BD. F. MITTERRAND, 91025 EVRY CEDEX, FRANCE,
E-mail address: `abayad@maths.univ-evry.fr`

TAEKYUN KIM. DIVISION OF GENERAL EDUCATION-MATHEMATICS, KWANG-
WOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA,
E-mail address: `tkkim@kw.ac.kr`