Stationary states of non-local interaction equations

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inspired by: Marco Di Francesco, Christian Schmeiser

Background

Non-local Fokker-Planck type equation

ρ individual/particle density, mass $\int_{\mathbb{R}} \rho = 1$ conserved

$$\partial_t \rho = \partial_x (\rho \, \partial_x [a(\rho) + W * \rho + V])$$

"smooth", even interaction potential $W(x) = W(-x)$

V external potential, a "diffusion"

- inelastic material $W \sim |x|^{1+\varepsilon}$
- (swarming, flocking) collective behaviour (attractive, repulsive/attractive) $W \sim e^{\pm|x|}$
- chemotaxis $W \sim \log|x|$ in 2D

Non-local interaction equations

Non-local interaction equation

measure solutions, mass $\int_{\mathbb{R}} \rho = 1$ conserved

$$\partial_t \rho = \partial_x(\rho\,\partial_x[W * \rho + V])$$

“smooth”, even interaction potential $W(x) = W(-x)$[a]

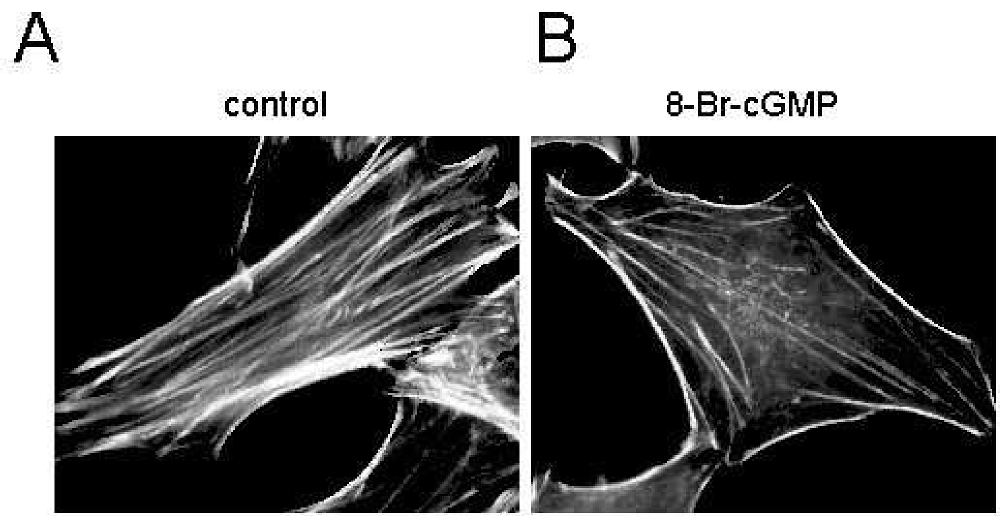
1D: consider $u(z)$ pseudo-inverse of the distribution function $u(z) = \inf\left\{x \in \mathbb{R};\ \int_{-\infty}^{x} \rho\,dx > z\right\}$, for $z \in [0,1]$

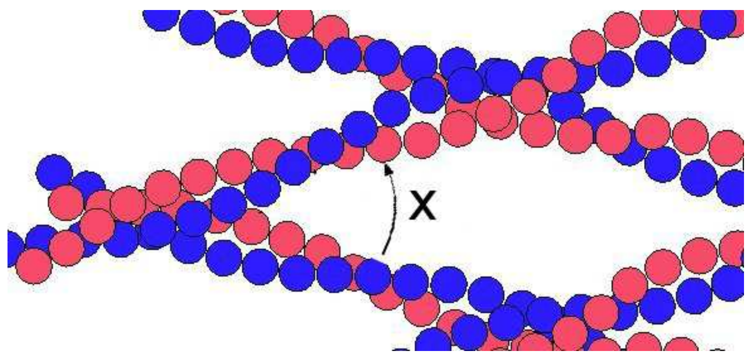
$$\partial_t u(z) = \int_0^1 W'(u(\zeta) - u(z))\,d\zeta + V'(u(z))\,,$$

[a] [Burger, Di Francesco], [Carrillo, Di Francesco, Figalli, Laurent, Slepčev]

Non-local interaction equations

Actin filaments with or without cross-linking proteins



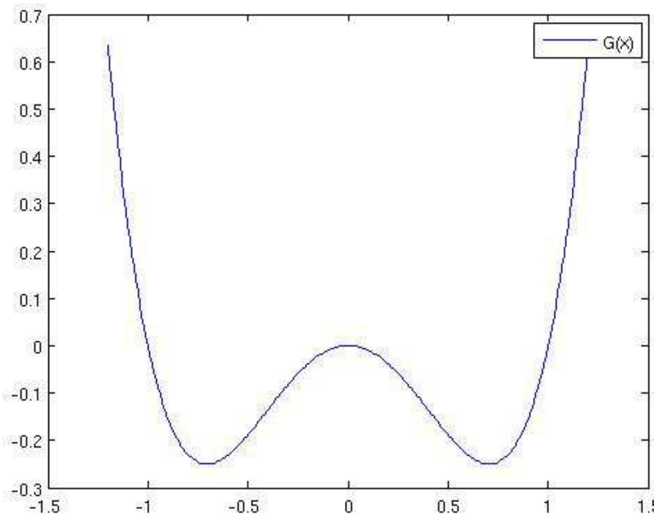


a

[a] [Kang, Perthame, Primi, Stevens, Velazquez]

Non-local repulsion-aggregation

A double-well repulsion-aggregation potential, $V = 0$



double-well: local max at $x = 0$ and local min at $x = 2x_0$

$$\beta := -\, W''(0) > 0\,, \qquad \alpha := \, W''(2x_0) > 0\,.$$

Non-local repulsion-aggregation

Conservation of (the centre of) mass, $V = 0$

$$\frac{d}{dt}\int_0^1 u(t,z)\,dz = \int_0^1\int_0^1 W'\left(u(\xi) - u(z)\right)\,d\xi\,dz = 0\,,$$

Conservation of (the centre of) mass $\int_0^1 u_{in}(z)\,dz$

$$\int_0^1 u(z,t)\,dz = \int_0^1 u_{in}(z)\,dz = 0 \qquad t \geq 0\,,$$

Non-local repulsion-aggregation

Steady states

e.g. trivial solution $\bar{u}(z) = 0 \;\Leftrightarrow\; \bar{\rho} = \delta_0$

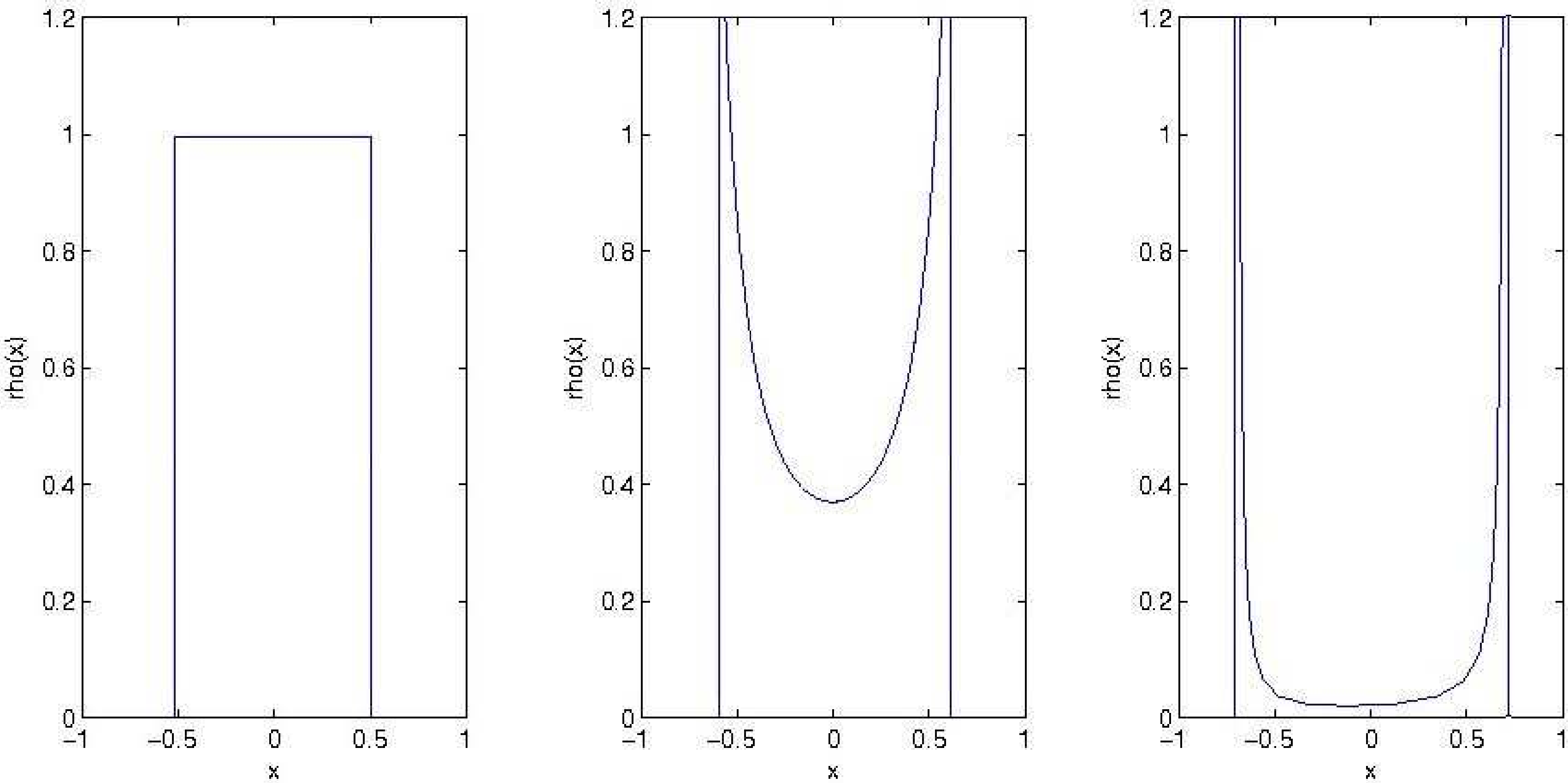
Family of monotone increasing two-valued steady states

$$
\begin{aligned}
\bar{u}(z, z_0) &= \begin{cases} -2(1 - z_0)x_0 & z < z_0\,, \\ 2z_0 x_0 & z > z_0\,. \end{cases} \\
\bar{\rho}(x, z_0) &= z_0\, \delta_{-2(1-z_0)x_0} + (1 - z_0)\, \delta_{2z_0 x_0}
\end{aligned}
$$

mass distribution parameter $z_0 \in (0, 1)$

Non-local repulsion-aggregation

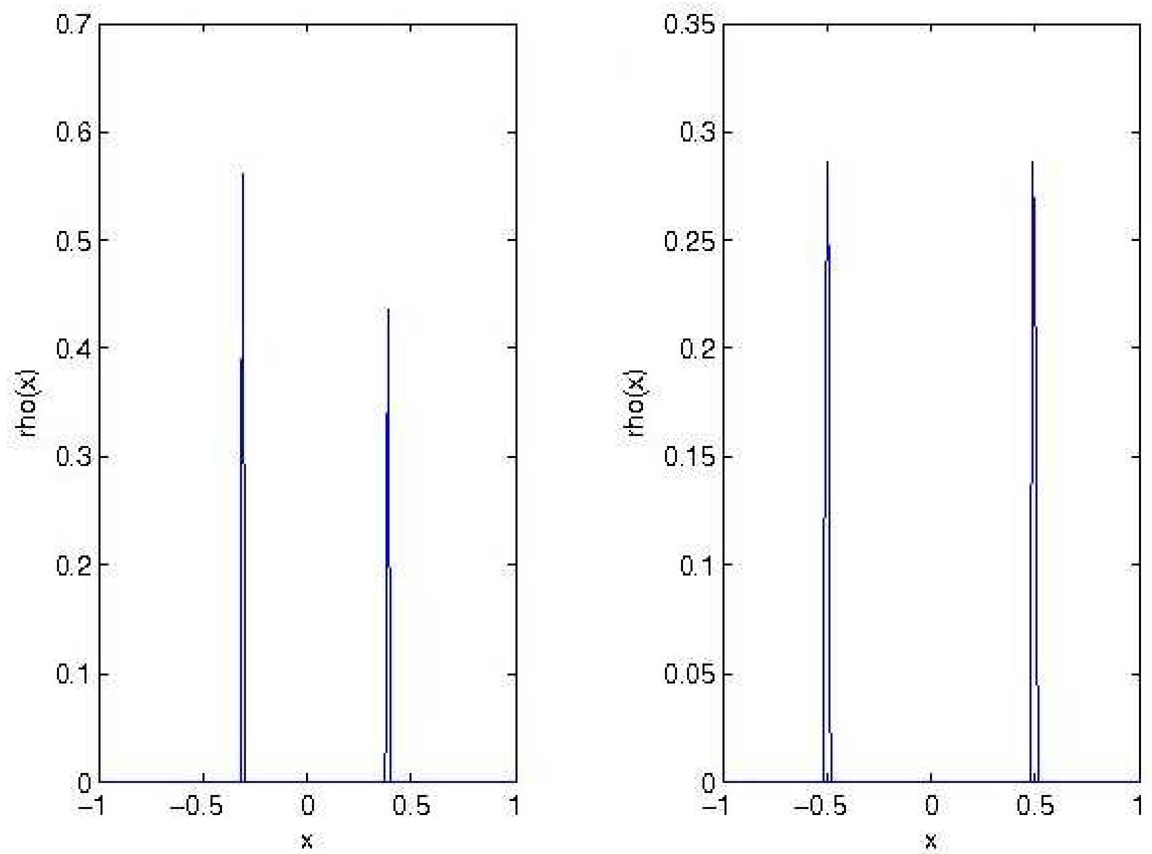
Numerics towards two-valued states



mass conserving scheme for $\partial_t u(z) = \int_0^1 W'(u(\zeta) - u(z))\, d\zeta$

Non-local repulsion-aggregation

Non-uniqueness of two-valued states



Nevertheless, both should be stable.

Non-local repulsion-aggregation

Linearised nonlocal operator

Consider mass preserving perturbations $v(z)$: $\int_0^1 v(z)\,dz = 0$.

$$F'(\bar{u}(z_0))(v(z)) = \begin{cases} \lambda_1\, v(z) - (\alpha+\beta)\int_0^{z_0} v(z)\,dz & z < z_0\,, \\ \lambda_2\, v(z) + (\alpha+\beta)\int_0^{z_0} v(z)\,dz & z > z_0\,. \end{cases}$$

where λ_1 and λ_2 denote

$$\lambda_1 := z_0\beta - (1-z_0)\alpha\,, \qquad \lambda_2 := (1-z_0)\beta - z_0\alpha\,.$$

λ_1 and λ_2 are convex combinations of β and $-\alpha$.

Non-local repulsion-aggregation

Linear stability: Ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,\varphi(z) = +(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z < z_0\,, \\ (\lambda_2 - \lambda)\,\varphi(z) = -(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z > z_0\,. \end{cases}$$

$\lambda_1 \neq \lambda \neq \lambda_2$: Then, φ is piecewise constant and

$$\lambda = -\alpha < 0\,, \qquad v(z) = \begin{cases} -\frac{1-z_0}{z_0}v_r & z < z_0\,, \\ v_r & z > z_0\,. \end{cases}$$

Stability w.r.t. “shifts” due to confinement

Non-local repulsion-aggregation

Linear stability: Ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,\varphi(z) = +(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z < z_0\,, \\ (\lambda_2 - \lambda)\,\varphi(z) = -(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z > z_0\,. \end{cases}$$

$\lambda_1 = \lambda = \lambda_2$, $z_0 = \frac{1}{2}$: Then,

$$\lambda = \frac{\beta - \alpha}{2}\,, \qquad v(z) = \begin{cases} v_l(z) : \int_0^{1/2} v_l\,dz = 0 & z < \frac{1}{2}\,, \\ v_r(z) : \int_{1/2}^{1} v_r\,dz = 0 & z > \frac{1}{2}\,. \end{cases}$$

Stability w.r.t. “reallocations” provided repulsion is controlled by aggregation.

Non-local repulsion-aggregation

Linear stability: Ansatz $v(z) = e^{\lambda t}\varphi(z)$

Eigenproblem $\lambda\varphi = F'(\bar{u}(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,\varphi(z) = +(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z < z_0\,, \\ (\lambda_2 - \lambda)\,\varphi(z) = -(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z > z_0\,. \end{cases}$$

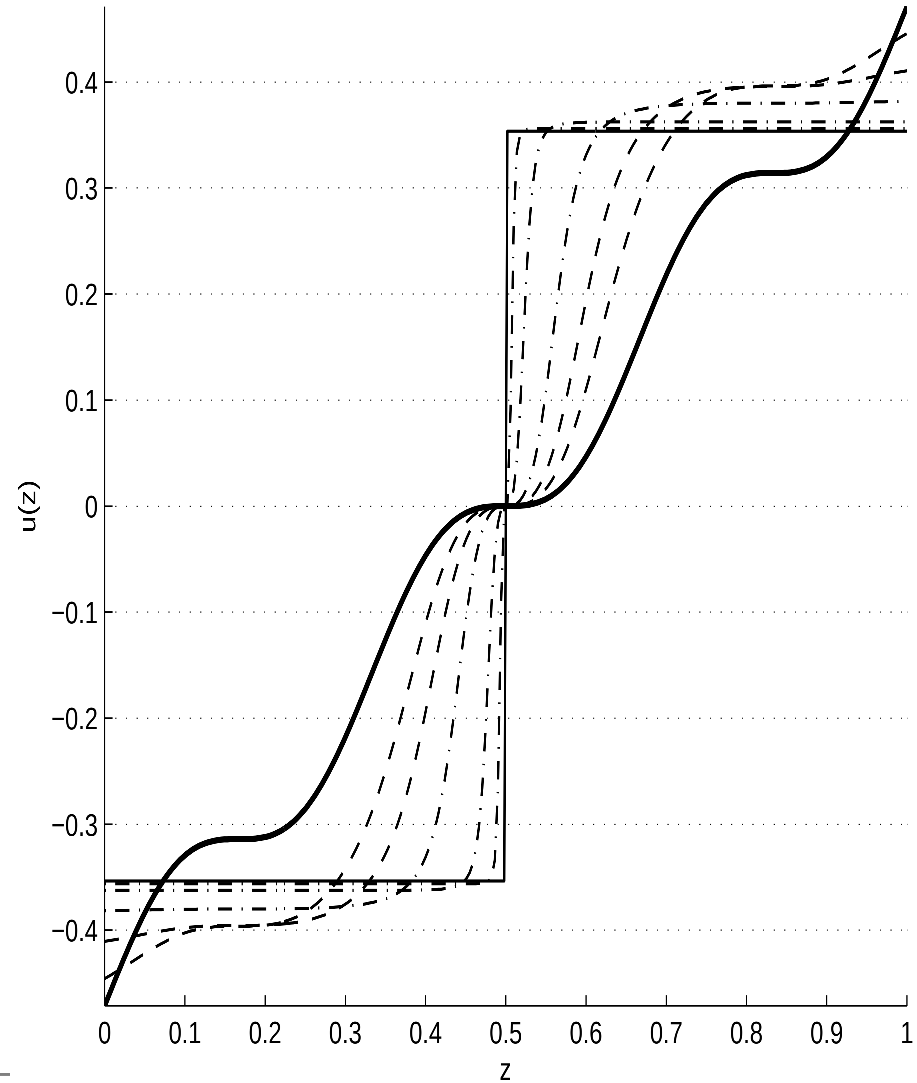
similar $\lambda_1 = \lambda$ or $\lambda_2 = \lambda$, $z_0 \neq \frac{1}{2}$ (symmetry $z_0 \to (1-z_0)$)

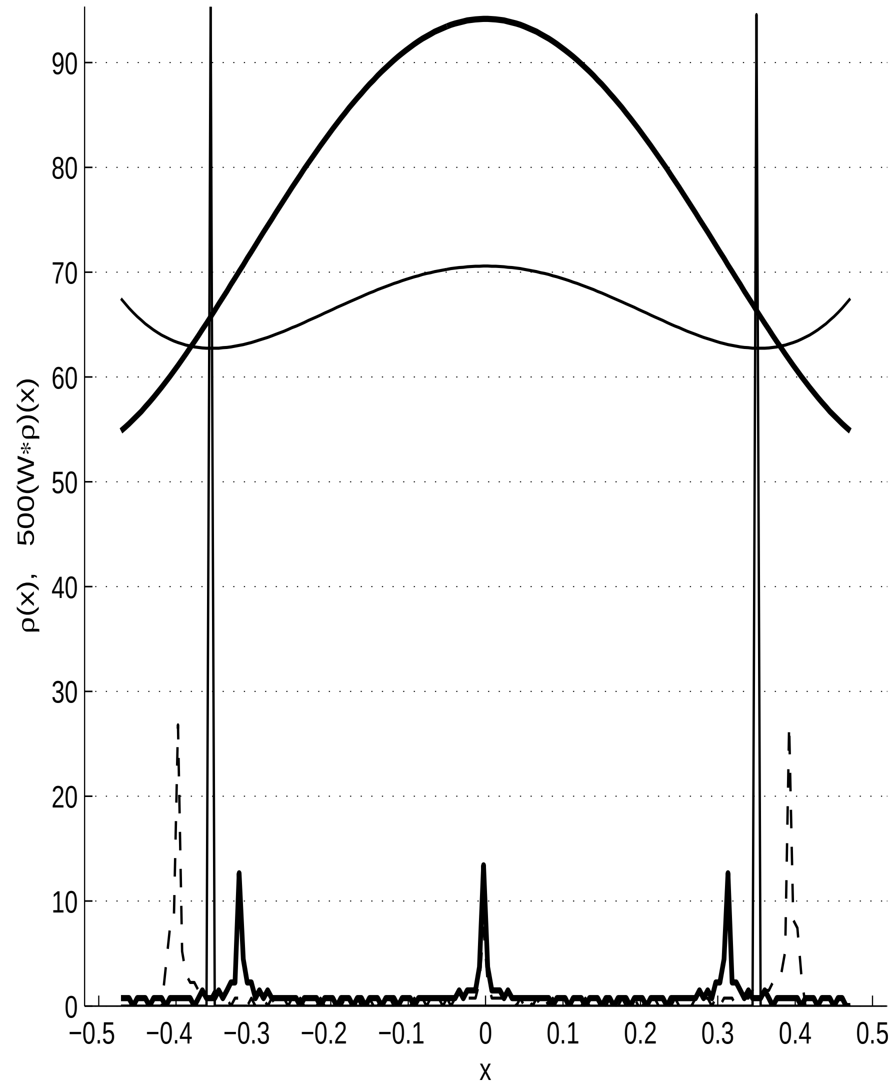
Summary: Given $\beta - \alpha < 0$ there exists an open interval of parameters z_0 with linearly stable stationary states $\bar{u}(z_0)$:

$$\max_{i=1,2}\{\lambda_i(z_0)\} < 0 \qquad \forall z_0 \in (1 - z_0^*, z_0^*)\,, \qquad z_0^* := \frac{\alpha}{\alpha+\beta} > \frac{1}{2}\,.$$

Non-local repulsion-aggregation

Stable two Dirac stationary state





Non-local repulsion-aggregation

Bifurcation for $z_0 > \frac{1}{2}$

Formal expansion if $\varepsilon := \lambda_1 \ll 1$

$$u(z) = \bar{u}(z, z_0 + \varepsilon a) + \varepsilon v(z) \, , \quad \int_0^1 v(z)\, dz = 0 \, .$$

$W'''(0) = 0$, denote $W'''(2x_0) = \gamma$, $z_0^\varepsilon := z_0 + \varepsilon a$

$$\begin{cases} \varepsilon \left[-(\alpha + \beta) \int_0^{z_0} v \, dz \right] + O(\varepsilon^2) & z < z_0^\varepsilon \\ \varepsilon \left[(\beta - \alpha) v + (\alpha + \beta) \int_0^{z_0} v \, dz \right] + O(\varepsilon^2) & z > z_0^\varepsilon \end{cases}$$

$$O(\varepsilon) : \quad \int_0^{z_0} v \, dz = \varepsilon V \qquad \text{and} \qquad v(z) = \varepsilon \tilde{v}(z) \, , \; z > z_0^\varepsilon \, ,$$

Non-local repulsion-aggregation

Bifurcation for $z_0 > \frac{1}{2}$

Integrate over $(0, z_0^\varepsilon)$ and $(z_0^\varepsilon, 1)$

$$V = -\frac{\gamma}{2}\frac{1-z_0}{2}\int_0^{z_0} v^2\,dz + O(\varepsilon)\,.$$

Reinsert

$$\begin{cases} \varepsilon^2\Big[(1+\frac{a\alpha}{z_0})v - \frac{\gamma}{2}(1-z_0)v^2 + \frac{\gamma}{2}\frac{1-z_0}{z_0}\int_0^{z_0} v^2\,dz\Big] + O(\varepsilon^3) & z < z_0^\varepsilon \\ \varepsilon^2\Big[\frac{1-2z_0}{z_0}\alpha\tilde{v} - \frac{\gamma}{2}\frac{1-2z_0}{z_0}\int_0^{z_0} v^2\,dz\Big] + O(\varepsilon^3) & z > z_0^\varepsilon \end{cases}$$

Non-local repulsion-aggregation

Bifurcation for $z_0 > \frac{1}{2}$

v can assume at most two different values for $z < z_0$.

$$v(z) = \begin{cases} v_1 := -\frac{z_0 - z_1}{z_1} v_2 & 0 < z < z_1 \\ v_2 & z_1 < z < z_0 \end{cases} .$$

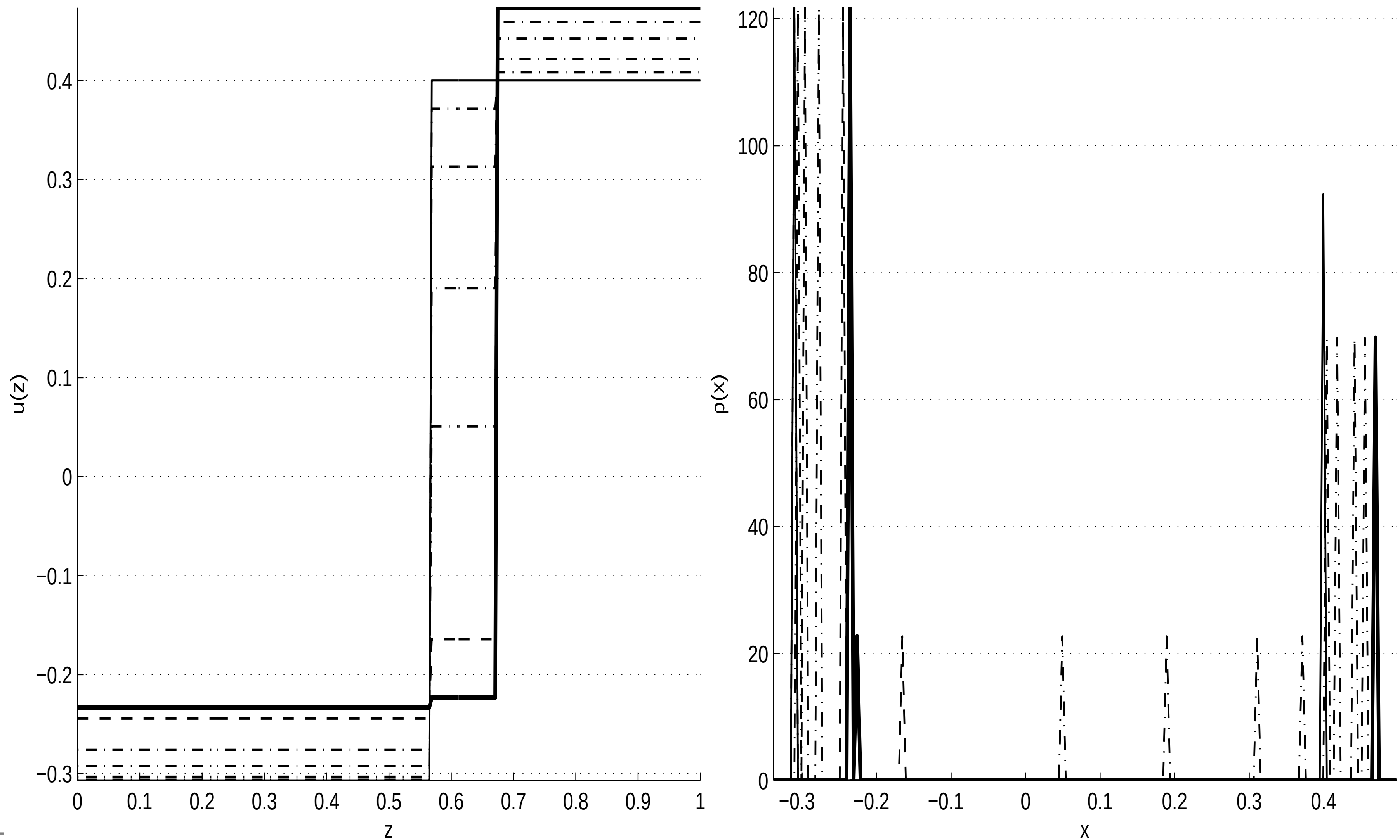
for a constant $v_2 \neq 0$ and $\int_0^{z_0} v \, dz = 0$.

$$v_1 = -\frac{2}{\gamma} \frac{z_0 + a\alpha}{(1 - z_0) z_0} \frac{z_0 - z_1}{2z_1 - z_0} , \qquad v_2 = \frac{2}{\gamma} \frac{z_0 + a\alpha}{(1 - z_0) z_0} \frac{z_1}{2z_1 - z_0} .$$

$$\tilde{v} = \frac{\gamma}{2\alpha} \frac{z_0 (z_0 - z_1)}{z_1} v_2^2 , \qquad V = \frac{\gamma}{2\alpha} \frac{(1 - z_0) z_0 (z_0 - z_1)}{z_1} v_2^2 .$$

Non-local repulsion-aggregation

Instable two Dirac stationary state



Non-local interaction equation

Structure of stationary states

- W analytic $\Rightarrow$ the stationary states are “discrete“ sums of Diracs: $\bar\rho(x) = \sum_{i=1}^n \rho_i \delta_{u_i}(x)$, $\sum_{i=1}^n \rho_i = 1$, $\rho_i > 0$, or $\bar u(z) = \sum_{i=1}^n u_i \mathbb{1}_{I_i}$, $I_i = [\sum_{j<i} \rho_j, \sum_{j\le i} \rho_j)$, $|I_i| = \rho_i$.
- $W \in C^2 \Rightarrow$ accumulating Diracs have no spectral gap.

A sum of Diracs $\bar u = \sum_1^n u_i \mathbb{1}_{I_i}$ with $|I_i| = \rho_i$ is stationary state iff

$$\sum_{j=1}^n W'(u_j - u_i)\,\rho_j = V'(u_i), \qquad i = 1, \dots, n\,.$$

Proof: $\partial_t \bar u = \int_0^1 W'(\bar u(\xi) - u_i)\,d\xi - V'(u_i)$ on $z \in I_i$

Non-local interaction equation

Three Dirac steady states for double-well, $V = 0$

A positive, normalised vector of masses

$$(\rho_1, \rho_2, \rho_3) = \frac{(-W'(u_3 - u_2), W'(u_3 - u_1), -W'(u_2 - u_1))}{-W'(u_3 - u_2) + W'(u_3 - u_1) - W'(u_2 - u_1)}$$

solves

$$\begin{pmatrix} -W'(u_3 - u_2) \\ W'(u_3 - u_1) \\ -W'(u_2 - u_1) \end{pmatrix} \times \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = 0\,,$$

for the double-well if and only if choose (u_i) such that $0 < u_2 - u_1, u_3 - u_2 < x_1$ and $u_3 - u_1 > x_1$

Local stability analysis

Linear stability steady state $\bar{\rho} = \sum_{i=1}^{n} \rho_i \delta_{u_i}$

- linear stability under small “reallocations” provided

$$0 < m_i := \sum_{j=1}^{n} W''(u_j - u_i)\rho_j + V''(u_i) \quad \forall i = 1, \ldots, n\,.$$

- linear stability under “shifts“ of the u_i, if the matrix

$$M = \operatorname{diag}(m_i) - (\rho_i W''(u_j - u_i))$$

has a positive spectrum
iff $V = 0$ then on the hyperspace $\{(w_i) : \sum_{i=1}^{n} w_i = 0\}$

Local stability analysis

Local stability without exchange of mass

$W, V \in C^{2,\alpha} \Rightarrow$ Linearly stable stationary state $\bar{\rho} = \sum_{i=1}^{n} \rho_i \delta_{u_i}$ are locally nonlinear stable w.r.t. Wasserstein W_∞, i.e.

$$\|u(0) - \bar{u}\|_\infty \leq \varepsilon \quad \Rightarrow \quad \|u(t) - \bar{u}\|_\infty \leq C\,(1 + t^{n-1})\,e^{-\eta t},$$

Proof: Consider the vector $w := \left(|v_i|, \int_{I_1} v, \cdots \int_{I_n} v\right)^T$, then

$$\frac{d}{dt}\tilde{w} = \begin{pmatrix} -\mathrm{diag}\,(m_i) & O(1) \\ 0 & -\tilde{M} \end{pmatrix} \tilde{w} + O(\|w\|^2),$$

Stability in higher dimensions via atomisation, see [CDFLS]

Singular repulsion

An explicit example

formal: local repulsion $\to$ Dirac $\Longrightarrow$ quadratic diffusion

Singular repulsive potential $W(x) = x^2 - |x|$

Steady state: $\bar{\rho} = \mathbb{I}_{[-\frac{1}{2}.\frac{1}{2}]}$

$$0 = W' * \rho = \int_{\mathbb{R}} 2(x-y)d\rho(y) - \int_{\mathbb{R}} \mathrm{sign}(x-y)d\rho(y)$$

$$= 2x - \int_{-\infty}^{x} d\rho(y) + \int_{x}^{-\infty} d\rho(y) = 2x + 1 - 2\int_{-\infty}^{x} d\rho(y)$$

Towards singular repulsion

An explicit example

evenly smoothed modulus $|x|_\varepsilon$ on the interval $(-\varepsilon, \varepsilon)$ for $\varepsilon > 0$

$$W_\varepsilon(x) = x^2 - |x|_\varepsilon, \quad W'_\varepsilon(x) = 2x - \mathrm{sign}_\varepsilon(x), \quad W''_\varepsilon(x) = 2 - 2\delta_\varepsilon(0)$$

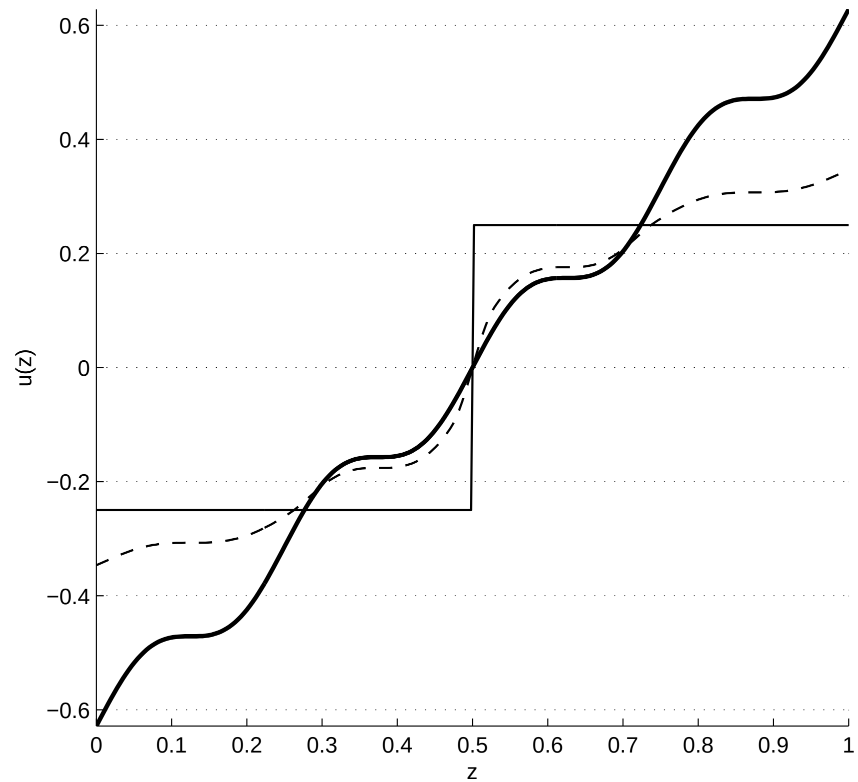
where we assume

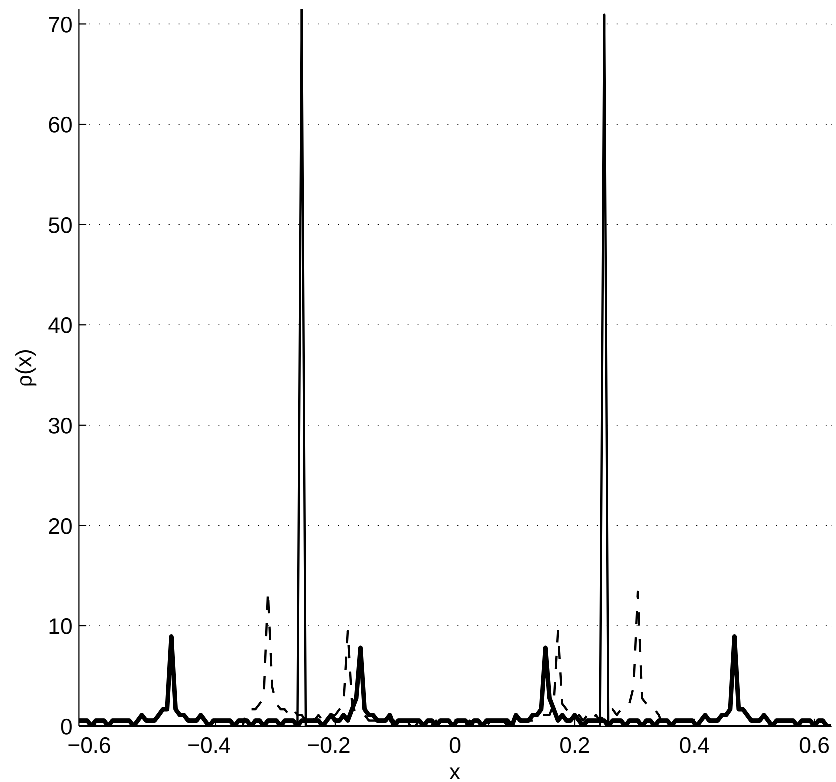
$$\mathrm{sign}_\varepsilon(0) = 0 \quad \text{and} \quad \mathrm{sign}_\varepsilon(\pm\varepsilon) = \pm 1 \qquad \delta_\varepsilon(0) \approx \frac{1}{\varepsilon}\,.$$

Towards singular repulsion

Numerics: $W_\varepsilon = x^2 - |x|_\varepsilon$ **with** $\varepsilon = 0.4$

Low repulsion: Four initial humps converge to two Diracs

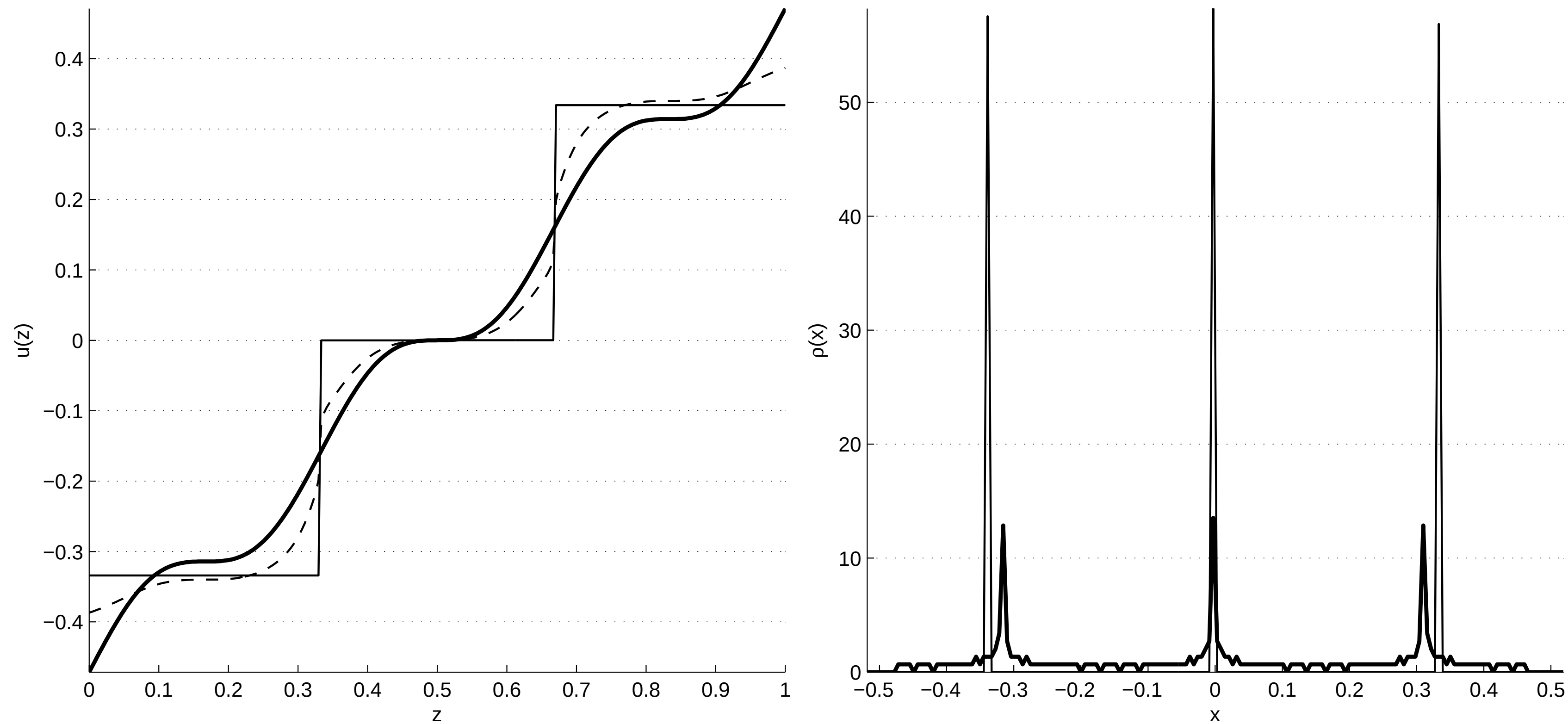




Towards singular repulsion

Numerics: $W_\varepsilon = x^2 - |x|_\varepsilon$ **with** $\varepsilon = 0.18$

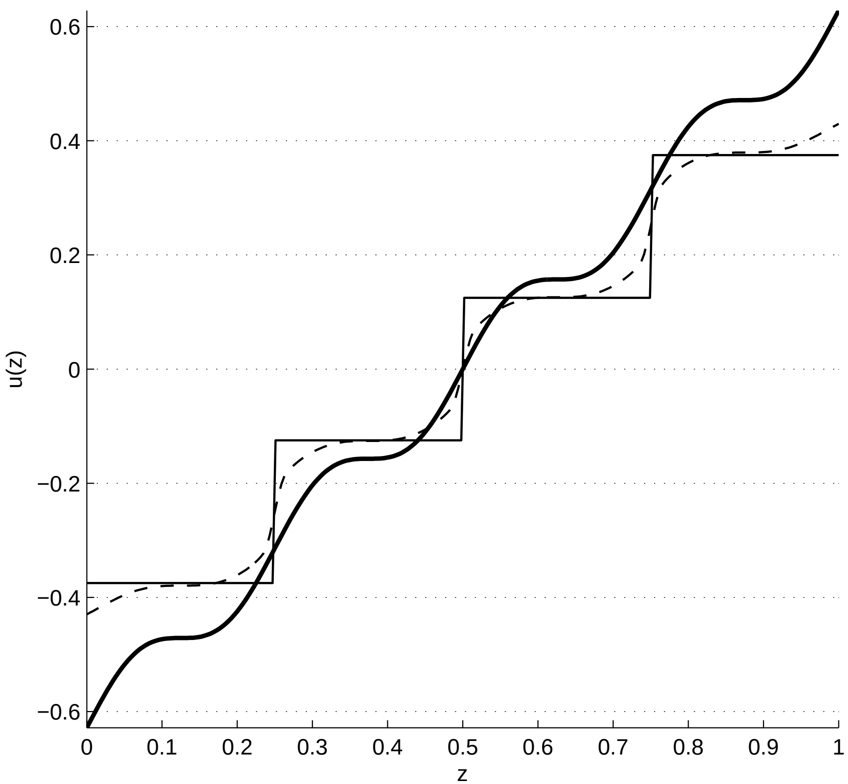
More repulsion: Three initial humps converge to three Diracs

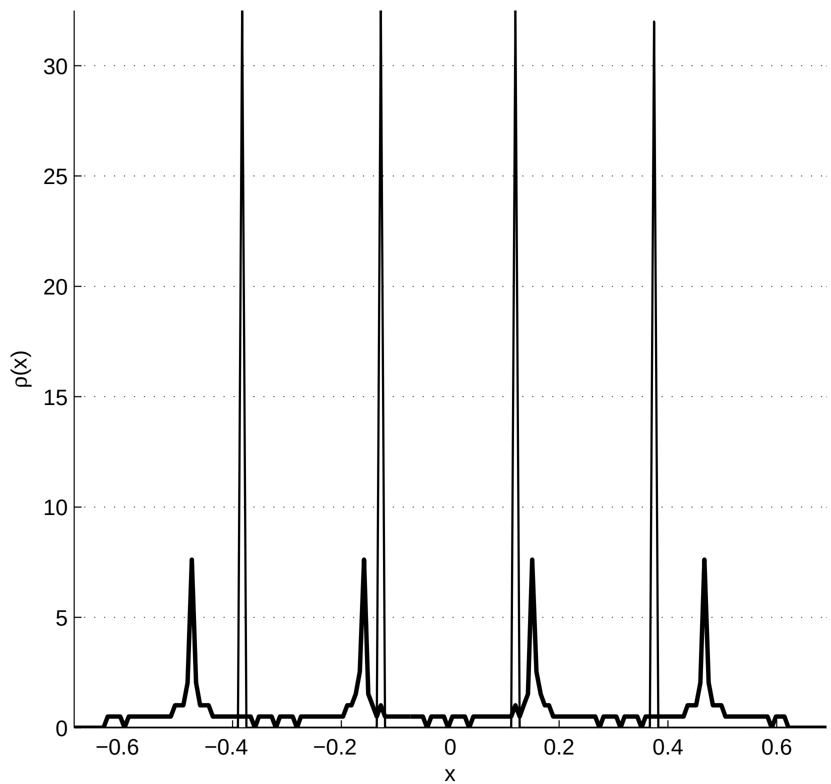


Towards singular repulsion

Numerics: $W_\varepsilon = x^2 - |x|_\varepsilon$ **with** $\varepsilon = 0.18$

More repulsion: Four initial humps converge to four Diracs

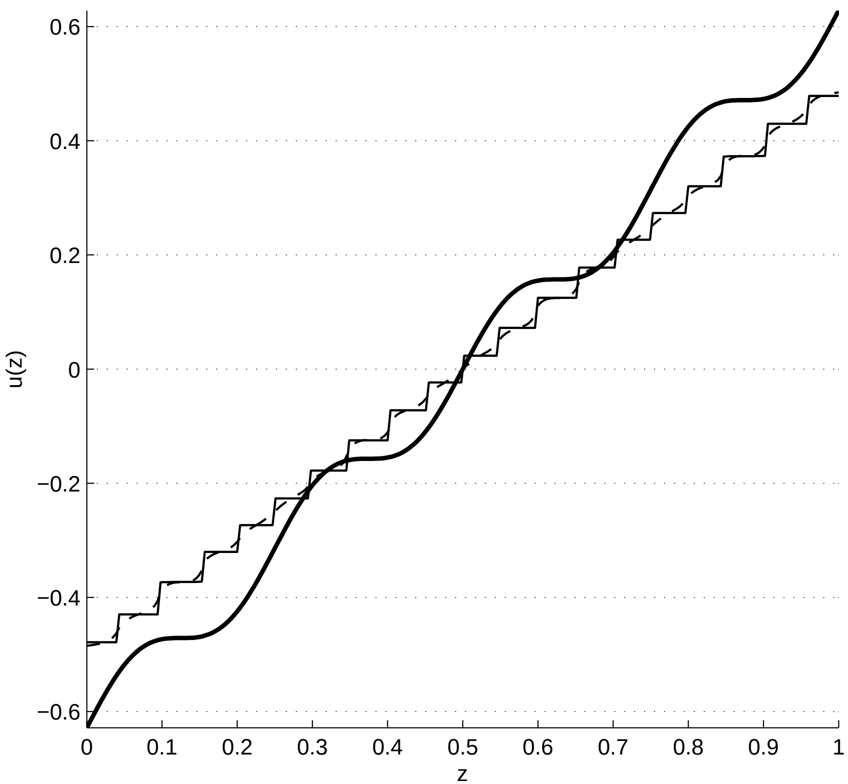


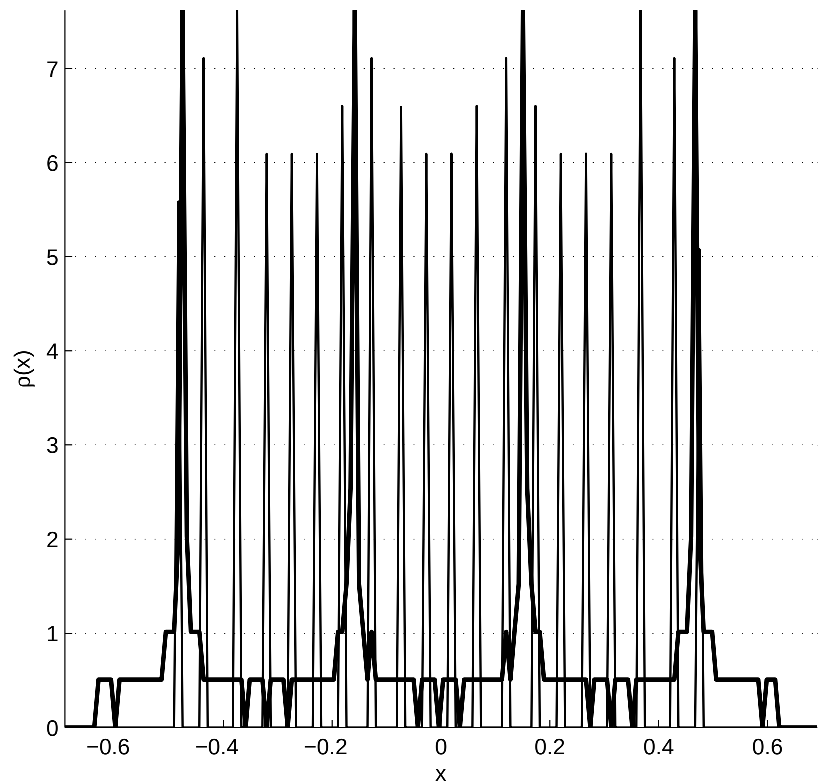


Towards singular repulsion

Numerics: $W_\varepsilon = x^2 - |x|_\varepsilon$ **with** $\varepsilon = 0.03$

High repulsion: Convergence to multiple Diracs





Singular repulsion

An explicit example

Construct stable stationary states with n Dirac masses:

$\bar{u} = \sum_{i=1}^{n} u_i \mathbb{I}_{I_i}$ with $|I_i| = \rho_i$ and $\max_i\{(u_{i+1} - u_i)\} > \varepsilon$

$$0 = \sum_{j=1}^{n} \rho_j W_\varepsilon'(u_j - u_i) = -2u_i + \sum_{j<i} \rho_j - \sum_{j>i} \rho_j$$

using $\sum_{j=1}^{n} \rho_j = 1$ and $\sum_{j=1}^{n} u_j \rho_j = 0$

Obtain multitude of stationary states

$$(u_{i+1} - u_i) = \frac{\rho_i + \rho_{i+1}}{2} > \varepsilon \quad \Rightarrow \varepsilon < \frac{1}{n}$$

stability: $m_i = \sum_{j=1}^{n} \rho_j W_\varepsilon''(u_j - u_i) = 2 - \frac{\rho_i}{\varepsilon} > 0 \Rightarrow \varepsilon > \frac{\rho_i}{2}$

Singular repulsion

Weak limit

calculate $u_1 = -\frac{1}{2} + \frac{\rho_1}{2} \longrightarrow -\frac{1}{2}$ and $u_n = \frac{1}{2} - \frac{\rho_n}{2} \longrightarrow \frac{1}{2}$ as $\rho_i < \frac{2}{n}$ for $n \to \infty$.

$$\int_{\mathbb{R}} \varphi(x)\, d\bar{\rho}(x) = \sum_{i=1}^{n} \varphi(u_i)\, \rho_i = \sum_{i=1}^{n} \int_{u_i - \frac{\rho_i}{2}}^{u_i + \frac{\rho_i}{2}} \varphi(u_i)\, dx$$

$$= \int_{u_1 - \frac{\rho_1}{2}}^{u_n + \frac{\rho_n}{2}} \sum_{i=1}^{n} \varphi(u_i) \mathbb{I}_{[u_i - \frac{\rho_i}{2}, u_i + \frac{\rho_i}{2}]}\, dx$$

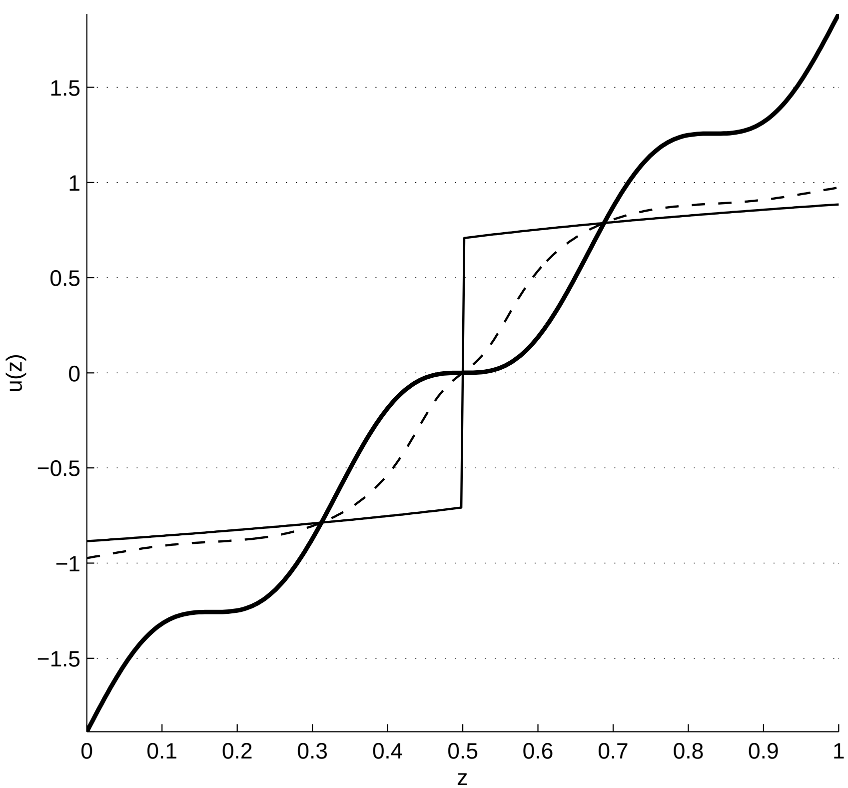
$$\to \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(x)\, dx = \int_{\mathbb{R}} \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]} \varphi(x)\ dx\,,$$

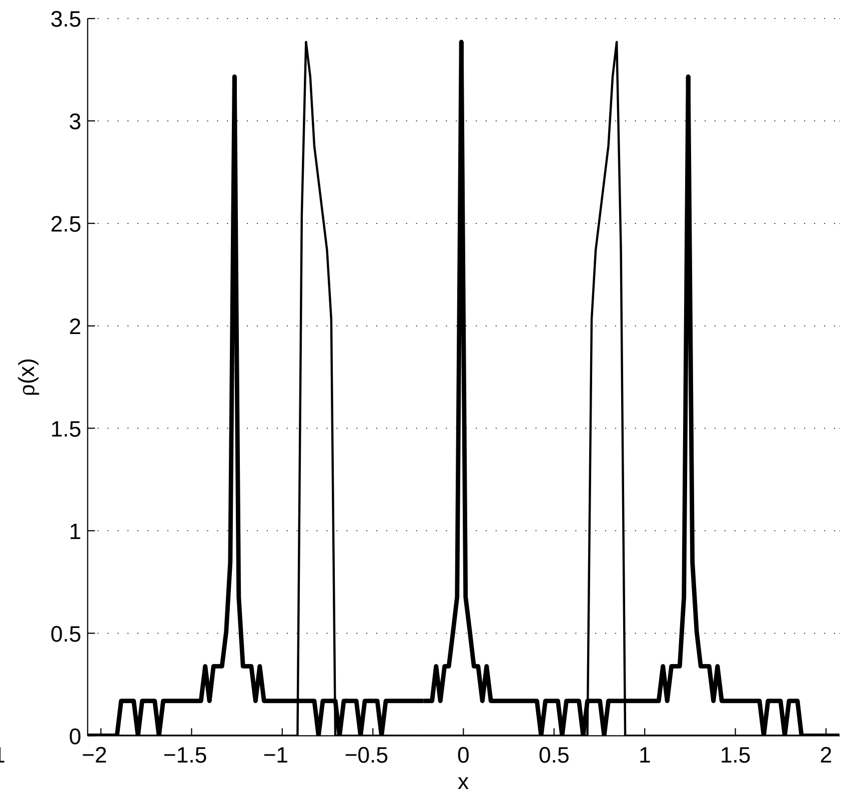
Theorem: $W_\varepsilon \to W = -|x|$, V strictly convex $\quad \Rightarrow \quad \bar{\rho}_\varepsilon \rightharpoonup \bar{\rho}$

Singular repulsion

Numerics: $W(x) = -|x|$ **and** $V(x) = x^4 - x^2$

Douplewell confinement: Separate continuous parts

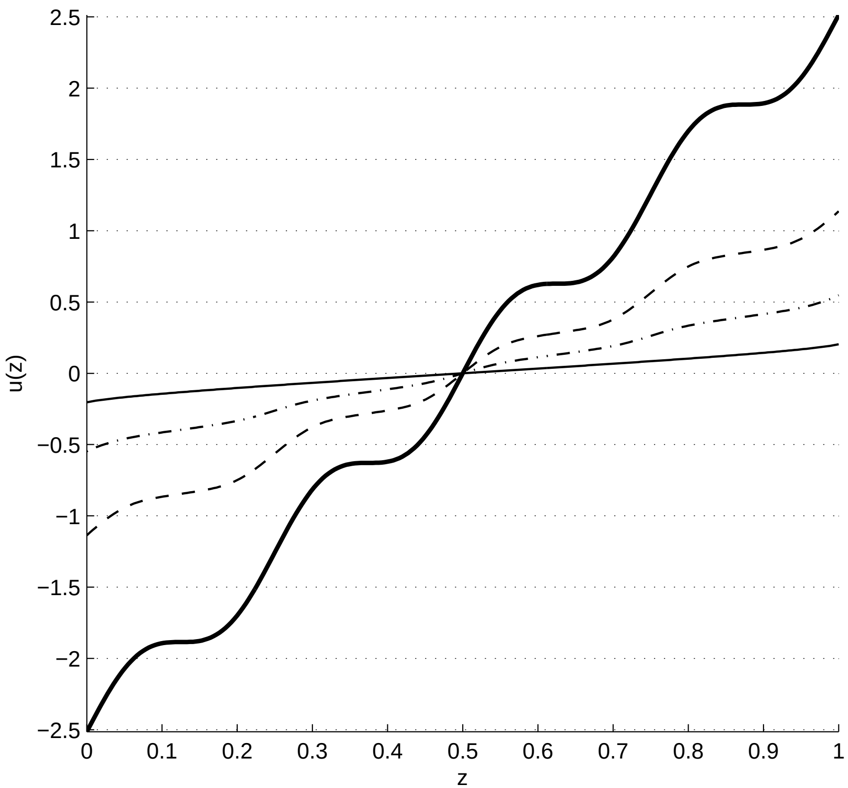


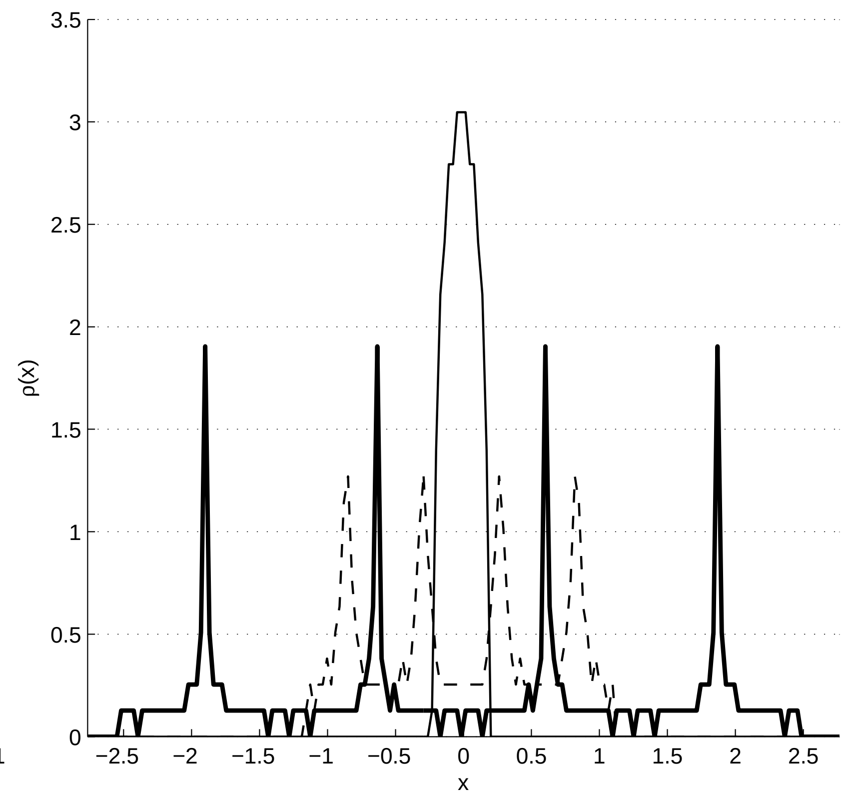


Singular repulsion

Numerics: $W(x) = x^2 - |x|^\alpha$ **with** $\alpha = 0.05$

Strong repulsion: Four initial humps converge to a lump

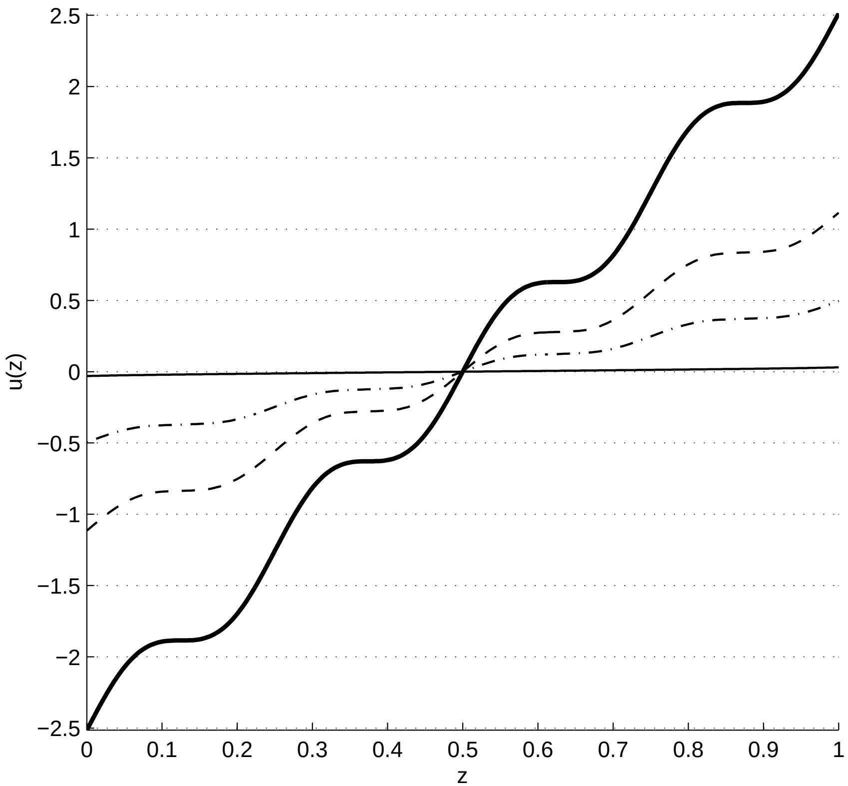


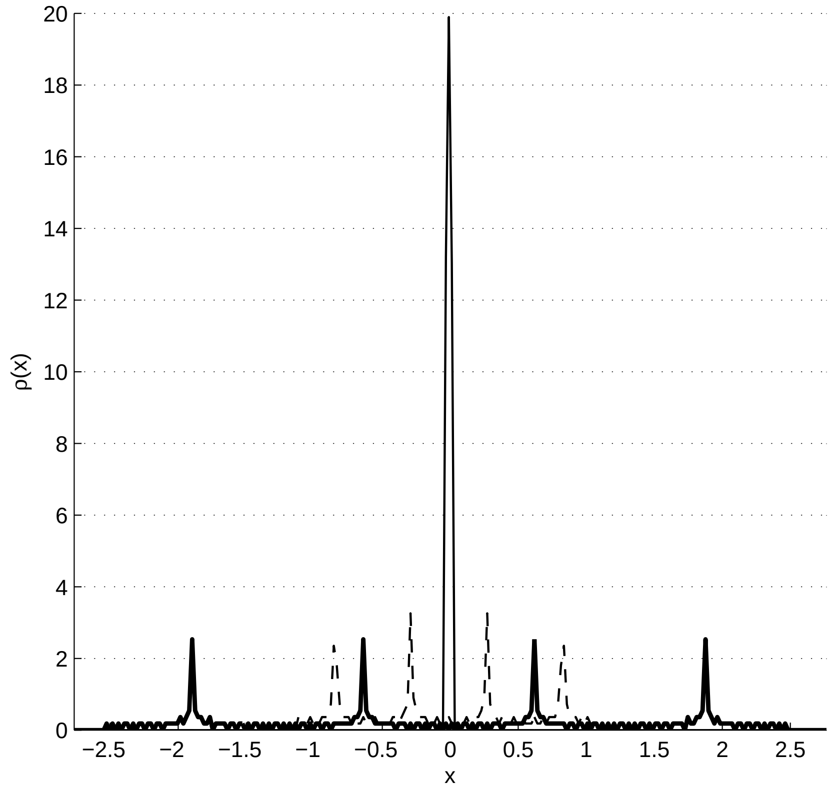


Singular repulsion

Numerics: $W(x) = x^2 - |x|^\alpha$ **with** $\alpha = 0.001$

Very strong repulsion: Four initial humps converge to a lump





Stability for singular attraction

Local non-linear stability for singular attractive potentials

$V \in C^{2,\alpha}(\mathbb{R})$, $\tilde{W}(x) := W(x) - W'(0^+)|x| \in C^{2,\alpha}(\mathbb{R})$
singular attractive: $W'(0^+) > 0$,
stationary state $\bar{u}(z) = \sum_{i=1}^{n} u_i \, \mathbb{1}_{I_i}$, linear stable w.r.t. shifts:

$$M_{ij} := \begin{cases} \rho_i W''(u_j - u_i), & \text{if } i \neq j, \\ -\sum_{k \neq i} \rho_k W''(u_k - u_i) - V''(u_i), & \text{if } i = j. \end{cases}$$

has a strictly positive spectrum
iff $V = 0$ the on the hyperspace $\{(w_i)_{i=1,\dots,n} : \sum_{i=1}^{n} w_i = 0\}$.

Then, $\|u_{in} - \bar{u}\|_\infty \le \varepsilon \Rightarrow \|u(t,\cdot) - \bar{u}\|_\infty \le C(1 + t^{n-1})\, e^{-\nu t}$.

Stability for singular repulsion

Local non-linear stability for singular repulsive potentials

$V \in C^{2,\alpha}(\mathbb{R})$, $W(x) - W'(0^+)|x| \in C^{2,\alpha}(\mathbb{R})$, $\rho_{in} \in W^{2,\infty}(\mathbb{R})$
singular repulsive: $W'(0^+) < 0$,
stationary state $\bar{u}(z) = \sum_{i=1}^{n} u_i \, \mathbb{1}_{I_i}$,

- $V'' > C > 0$ and $W|_{(0,\infty)}$ is convex,
- or $V = 0$ and $W''|_{(0,\infty)} > C > 0$.

Then, there exists a unique (up to a shift in x if $V = 0$) stationary state $\bar{\rho} \in L^1 \cap L^\infty(\mathbb{R})$

$$W_\infty(\rho(t,\cdot), \bar{\rho}) = \|u(t,\cdot) - \bar{u}\|_\infty \to 0.$$

Non-local interaction equations

Conclusions

- double-well does NOT imply that only two Dirac steady state are stable
- singular repuslsion of interaction potential at 0 acts like diffusion

THANK YOU!

K.F., G. Raoul, Stable stationary states of non-local interaction equation, to appear in M3AS.

-, Stability of stationary states of non-local equations with singular interaction potentials.