Selection-mutation Models

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Basic Modeling ideas

- Each individual is characterized by a phenotypic trait.
- But for mutations, an individual inherits the trait of his parent $\longrightarrow$ asexual reproduction.
- Time scale for mutations $>>$ Time scale for reproduction.
- Many mutations of small amplitude.

The first model

Denote $u_\varepsilon(t,x)$ the density of individuals with trait $x \in \mathbb{R}$. Following D. J. M. P.,

$$\partial_t u_\varepsilon = \frac{1}{\varepsilon} \left(\sum_{i=1}^{k} I_i^\varepsilon(t)\, \eta_i(x) - 1 \right) u_\varepsilon + M_\varepsilon(u_\varepsilon), \tag{1}$$

where M_ε is the mutation kernel

$$M_\varepsilon(f) = \frac{1}{\varepsilon} \int_{\mathbb{R}} K(z)\, (f(x+\varepsilon z) - f(x))\, dz, \tag{2}$$

The I_i^ε are ressources,the environnement, that are used by the population. The simplest equation for them is

$$I_i^\varepsilon(t) = \frac{1}{1 + \int_{\mathbb{R}} \eta_i(x)\, u_\varepsilon(x)\, dx}. \tag{3}$$

The limit $\varepsilon \longrightarrow 0$

Following again D.J.M.P., put

$$u_\varepsilon(t,x) = \exp(\phi_\varepsilon(t,x)/\varepsilon), \quad \text{or} \quad \phi_\varepsilon = \varepsilon \ \log u_\varepsilon .$$

Then ϕ_ε is a solution to

$$\partial_t \phi_\varepsilon = \sum_{i=1}^{k} I_i^\varepsilon(t)\, \eta_i(x) - 1 + H_\varepsilon(\phi_\varepsilon),$$

with

$$H_\varepsilon(\phi_\varepsilon) = \int K(z) \exp\left(\frac{\phi_\varepsilon(t, x + \varepsilon z) - \phi_\varepsilon(t,x)}{\varepsilon} \right) \, dz.$$

It is moreover quite straightforward to obtain uniform bounds on ϕ_ε. For example $\phi_\varepsilon \in W^{1,\infty}([0, \ T] \times \mathbb{R})$.

It is hence easy to pass to the limit, in the appropriate sense to get $\phi_\varepsilon \longrightarrow \phi$

$$\partial_t \phi = \sum_{i=1}^{k} I_i(t)\, \eta_i(x) - 1 + H(\partial_x \phi),$$

with

$$H(\xi) = \int K(z)\, \exp\left(\xi\, z\right)\, dz.$$

However we do not have anymore the equation on the I_i. One can try to replace it by the constraint

$$\max \phi(t, .) = 0.$$

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Indeed there are k unknowns, the I_i : As a Lagrange multiplier, the constraint $\max \phi(t, .) = 0$ should be of dimension k.

Dimension of $\max \phi = 0$?
Formally equal to the cardinal of $\{x,\ \phi(x) = 0\}$.

Therefore this is formally well posed only if this number is larger than k.
$\Longrightarrow$ Uniqueness only if $k = 1$, see Barles-Perthame.

Long time behaviour for selection only

The aim is to study the long time behaviour of

$$\partial_t u = \left(\sum_{i=1}^{k} I_i(t)\, \eta_i(x) - 1 \right) u,$$

with still the relations

$$I_i(t) = \frac{1}{1 + \int_{\mathbb{R}} \eta_i(x)\, u(x)\, dx}.$$

Under certain conditions on the η_i, one can show that $u \longrightarrow \mu$. μ is the unique non negative measure satisfying

$$\operatorname{supp} \mu \subset \operatorname{supp} u^0, \quad \bar{I}_i = \frac{1}{1 + \int_{\mathbb{R}} \eta_i(x)\, d\mu(x)},$$

$$\sum_i I_i\, \eta_i - 1 \leq 0 \quad \text{on } \operatorname{supp} u^0, \qquad = 0 \quad \text{on } \operatorname{supp} \mu.$$

Idea of the proof

For uniqueness for example, assume μ_1 and μ_2 satisfy the definition with the

$$I_i^\alpha = \frac{1}{1 + \int_{\mathbb{R}} \eta_i(x)\, d\mu^\alpha(x)}.$$

Then write

$$\begin{aligned} 0 &\geq \int (\sum_i I_i^1, \eta_i - 1)\, d\mu^2 + \int (\sum_i I_i^2, \eta_i - 1)\, d\mu^1 \\ &= \int (\sum_i (I_i^1 - I_i^2)\, \eta_i)\, d\mu^2 + \int (\sum_i (I_i^2 - I_i^1)\, \eta_i)\, d\mu^1 \\ &= \sum_i \frac{(I_i^1 - I_i^2)^2}{I_i^1\, I_i^2}. \end{aligned}$$

The result follows if one has the correct assumptions on the η_i.

For the long time behaviour, one uses the following convex entropy

$$L(u) = \sum_{i=1}^{k} \log I_i + \int u(t,x)\, dx$$

$$= -\sum_{i=1}^{k} \log\left(1 + \int \eta_i\, u(t,x)\, dx\right) + \int u(t,x)\, dx.$$

Then it satisfies

$$\frac{d}{dt} L(u(t,.)) = -\int \left(\sum_i I_i(t) - 1\right)^2 u(t,x)\, dx.$$

The missing equation

The equilibrium measure that was just introduced, depends only on $\{u^0 > 0\}$. Thus define the application

$$\omega \subset \mathbb{R} \longrightarrow \mu(\omega).$$

One then obtains the full system

$$\partial_t \phi = \sum_{i=1}^{k} I_i(t)\, \eta_i(x) - 1 + H(\partial_x \phi),$$

with

$$I_i(t) = \frac{1}{1 + \int_{\mathbb{R}} \eta_i(x)\, d\mu(\{\phi(t,.) = 0\})}.$$

Perspectives and open problems

The ratio between the maximal value of u_ε and its minimal value is of order

$$\exp(C/\varepsilon).$$

If ε is really small (too small with respect to the total number of individuals), this means that most subpopulations are represented by less than one individual.

It would therefore be required to take the finite size of the population into account in the model...