Kimura's model : an integro-differential model to study evolution

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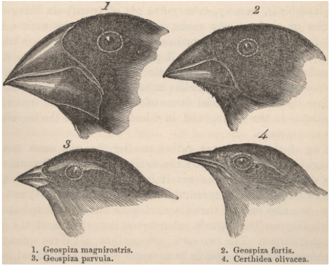
A population structured by a phenotypic trait

- We consider a phenotypic trait $x \in \mathbb{R}$, that is a physical characteristic, which can be measured on each individual.



- At time $t \geq 0$, A population is a measure [1] $f(t, \cdot)$ on the set of phenotypic traits $\mathbb{R}$.

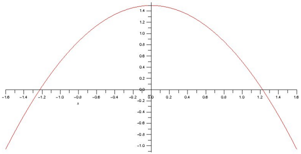
[1] Bürger R, The Mathematical theory of selection, recombination and mutation. *Wiley*, New-York (2000).

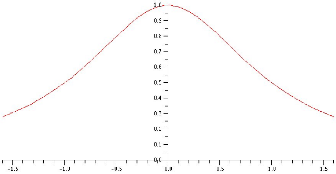
The fitness

Definition

We denote by $s[\mu](x)$ the fitness of an individual of trait x living in an environment where the population is $\mu \in M_+^1(X)$. We consider logistic fitnesses of the type :

$$s[\mu](x) = a(x) - \int_{\mathbb{R}} b(x,y)\mu(y)\,dy.$$

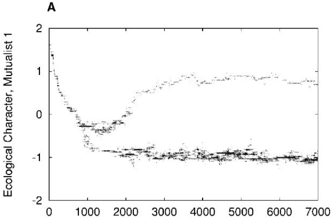




typical shapes of $x \mapsto a(x)$ and $(x-y) \mapsto b(x,y) = B(x-y)$.

A first class of models : stochastic models

At each time step δt, an individual of trait x dies, and gives birth to a random number of offsprings, with an expectancy of $1 + \delta t(s[f(t,\cdot)](x) - 1)$. The offsprings have a trait x, except if a mutation occurs.



Kimura's model[2]

We consider an initial population $f^0(\cdot) \in M^1_+(\mathbb{R})$, $f^0 > 0$. The population $f(t,\cdot)\,dx$ is then a solution of :

$$\begin{cases} f(0,x) = f^0(x) > 0, \\ \partial_t f(t,x) = s[f(t,\cdot)](x) f(t,x) \quad (+\text{mutations}), \\ \qquad\quad = \left(a(x) - \int_{\mathbb{R}} b(x,y) f(t,y)\,dy \right) f(t,x) \quad (+\text{mutations}). \end{cases}$$

Mutations can be either neglected, or :

$$\text{mutations} = \varepsilon \Delta_x f(t,x), \quad \text{mutations} = \varepsilon \int m(x,y) f(t,y)\,dy.$$

[2] Kimura M, A stochastic model concerning the maintenance of genetic variability in quantitative characters. *Proc. Natl. Acad. Sci. USA* **54**, 731–736 (1965).

The case of one resource[3]

If only one ressource is available, the fitness writes :

$$s[\mu](x) = a(x) - b_1(x) \int_{\mathbb{R}} b_2(y)\mu(y)\,dy.$$

Then, using the change of variable $f_\varepsilon(t,x) = e^{\frac{1}{\varepsilon}u_\varepsilon(t,x)}$, and letting $\varepsilon \longrightarrow 0$, one obtains the following H-J equation :

$$\partial_t u(t,x) - |D_x u(t,x)|^2 = a(x) - b_1(x) I_1(t),$$

where the limit $I_1(t)$ of $\int_{\mathbb{R}} b_2(y) f_\varepsilon(t,y)\,dy$ is determined by the constraint :

$$\forall t \geq 0, \quad \max_{x\in\mathbb{R}} u(t,x) = 0.$$

[3] Barles, Diekmann, Jabin, Mirrahimi, Mischler, Perthame

The case of several ressources[4]

If several ressources are available, the fitness writes :

$$s[\mu](x) = a(x) - \sum_{i=1}^{n} b_i(x) \int_{\mathbb{R}} b_i(y)\mu(y)\,dy.$$

Then,

- There exist only one stable steady-state $\bar{f}$,
- There exist an entropy :

$$F(t) = \int \left[\bar{f}(x) \log\left(\frac{1}{f(t,x)}\right) + f(t,x) \right] dx,$$

and then $\bar{f}$ is globally stable (with a speed $O\left(\frac{\log t}{t}\right)$) and structurally stable.

Generalisation : if $\forall g \in M^1 \setminus \{0\}, \quad \int b(x,y)g(x)g(y)\,dy\,dx > 0.$

[4] Jabin, R

The general case

In the general case, where :

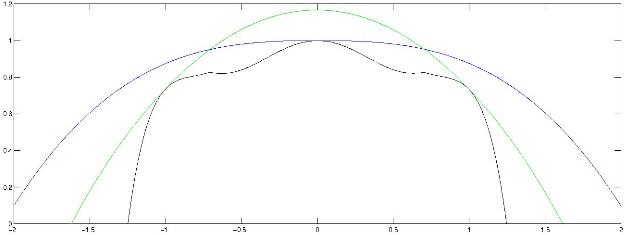
$$s[\mu](x) = a(x) - \int b(x,y)\mu(y)\,dy,$$

there is :

- no uniqueness of steady-states,
- no entropy (a priori).

Non uniqueness of steady-states

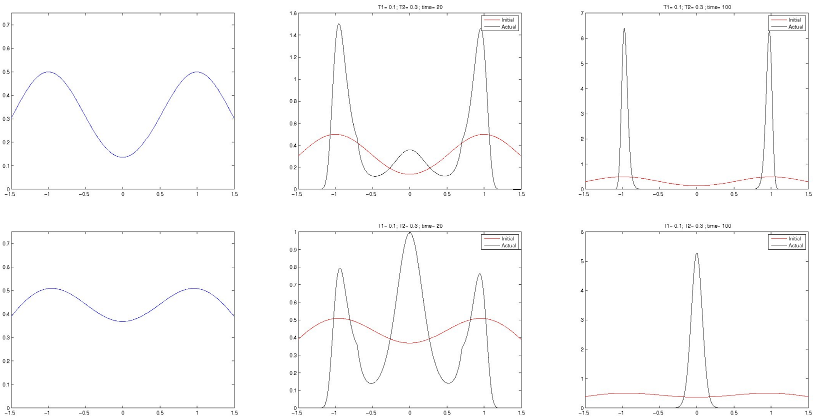
$$s[f(t,\cdot)](x) = a(x) - \int_{\mathbb{R}} b(x-y) f(t,y)\, dy$$



a in black, b in blue, and $\frac{2}{3}\left(b(\cdot - 1) + b(\cdot + 1)\right)$ in green

Then, both $\bar{f}_1 = \delta_0$ and $\bar{f}_2 = \frac{2}{3}(\delta_{-1} + \delta_1)$ are stable steady-states.

Non uniqueness of steady-states



ESD

Definition

$\bar{f} \in M^1$ is called an Evolutionary stable Distribution (ESD) if :

$$\begin{cases} \forall x \in \mathrm{supp}\ \bar{f}, \quad s[\bar{f}](x) = 0, \\ \forall x, \quad s[\bar{f}](x) \leq 0. \end{cases}$$

That is $\bar{f}$ is a steady-state, and no mutant can invade.

Rescaling

$$\partial_t f(t,x) = s[f(t,\cdot)](x) f(t,x) + \varepsilon m *_x f_\varepsilon(t,x),$$

where $m \geq 0$, $m(0) > 0$.

We scaled time as follows : $t' = \frac{t}{\varepsilon}$. Then,

$$\begin{cases} f_\varepsilon(0,x) = f^0(x) \\ \partial_t f_\varepsilon(t,x) = \frac{1}{\varepsilon} s[f_\varepsilon(t,\cdot)](x) f_\varepsilon(t,x) + m *_x f_\varepsilon(t,x), \end{cases}$$

Notice that the time is more accelerated than usual[5] (which would be $t' = \frac{t}{\sqrt{\varepsilon}}$).

[5] Perthame, S. Genieys, Concentration in the nonlocal Fisher equation : the Hamilton-Jacobi limit. *Math. Model. Nat. Phenom.* **2**, no. 4, 135–151, (2007).

An integral form of the equation

$$\partial_t f_\varepsilon(t,x) = \frac{1}{\varepsilon} s[f_\varepsilon(t,\cdot)](x) f_\varepsilon(t,x) + m *_x f_\varepsilon(t,x).$$

f writes :

$$\begin{aligned} f_\varepsilon(t,x) &= f^0(x) e^{\frac{1}{\varepsilon}\int_0^t s[f_\varepsilon(\tau,\cdot)](x)\,d\tau} \\ &\quad + \int_0^t (m *_x f_\varepsilon)(\sigma,x) e^{\frac{1}{\varepsilon}\int_\sigma^t s[f_\varepsilon(\tau,\cdot)](x)\,d\tau}\,d\sigma, \end{aligned}$$

where $\int_\sigma^t s[f_\varepsilon(\tau,\cdot)](x)\,d\tau = \int_\sigma^t \left[a(x) - \int b(x,y) f_\varepsilon(\tau,y)\,dy\right]\,d\tau$.

An asymptotic limit

Theorem (Desvillettes, Jabin, Mischler, R.)

If $0 \le f^0 \in L^1(\mathbb{R})$, $f^0 \neq 0$, then there exist a unique solution f_ε to Kimura's model with mutations, a subsequence $f_{\varepsilon'}$ of f_ε, and $f \in L^\infty([0,\infty), M^1_+)$, such that :

$$\begin{cases} f_\varepsilon \rightharpoonup f \quad L^\infty(w*]0,T[;\sigma(M^1,Cb)(\mathbb{R})) \\ \left(\int_s^t s[f_\varepsilon(\sigma,\cdot)](x)\,d\sigma\right) \to \left(\int_s^t s[f(\sigma,\cdot)](x)\,d\sigma\right) \\ \qquad \textit{uniformly on } \mathbb{R}_+ \times \mathbb{R}_+ \times X. \end{cases}$$

Convergence to an ESD

Proposition (R.)

If $0 \leq f^0 \in L^1(\mathbb{R})$, $f^0 \neq 0$, f_ε *the solution of Kimura's model with mutations, and* f *a limit (up to an extraction) of* (f_ε), *then, for* $t \in \mathbb{R}_+$ *a.e.,* $f(t,\cdot)$ *is an ESD, that is :*

$$\forall x \in X, \quad s[f(t,\cdot)](x) \leq 0,$$

$$supp\ f(t,\cdot) \subset \{x \in X; s[f(t,\cdot)](x) = 0\}.$$

Proof

$$
\begin{aligned}
f_\varepsilon(t,x) &= f^0(x)e^{\frac{1}{\varepsilon}\int_0^t s[f_\varepsilon(\tau,\cdot)](x)\,d\tau} \\
&\quad + \int_0^t (m *_x f_\varepsilon)(\sigma,x)e^{\frac{1}{\varepsilon}\int_\sigma^t s[f_\varepsilon(\tau,\cdot)](x)\,d\tau}\,d\sigma.
\end{aligned}
$$

We show that $f_\varepsilon \geq C\varepsilon^\eta$, and then :

- for a.e. $t_0 \geq 0$ and $x_0 \in \mathbb{R}$, $s[f(t_0,\cdot)](x_0) \leq 0$, otherwise, for s, t close to t_0 and x close to x_0,

$$
e^{\frac{1}{\varepsilon}\int_\sigma^t s[f_\varepsilon(\tau,\cdot)](x)\,d\tau} \to_{\varepsilon\to\infty} +\infty,
$$

- If $s[f(t_0,\cdot)](x_0) < 0$, then $f(t_0,x_0) = 0$.

Corollaries

Corollary (R.)

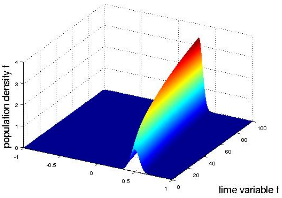
For a large class of coefficients a, b, an ESD exists.

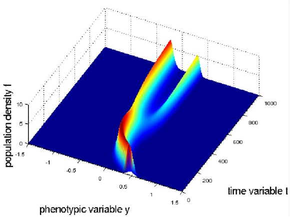
Corollary (R.)

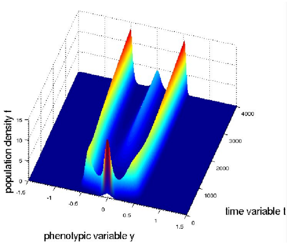
If there is at most one $\bar{f} \in M^1(X)$, then the sequence (f_ε) of solutions of Kimura's model converges to $\bar{f}$ (for any mutation kernel m) :

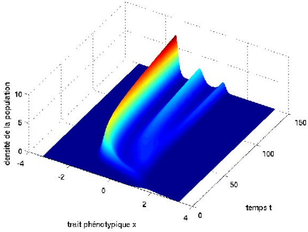
$$f^\varepsilon \longrightarrow_{\varepsilon \to 0} \bar{f} \quad L^\infty\left(\omega * [0,T], \sigma(M^1, C_b)(X)\right)$$

Numerical simulations









An example[6]

Example

parasite insects laying their eggs in larva, structured by a 2-dimensional trait :

- *strength of insects,*
- *time spend on the nest to protect eggs.*

[6] Goubault M, Mack AFS, Hardy ICW, Encountering competitors reduces clutch size and increases offspring size in a parasitoid with female-female fighting. *Proc. Roy. Soc.*

An example[7]

Example

Two closely related species exists :

- *strong insects that protect carefully their eggs,*
- *weak insect that lay eggs in as many larva as possible.*

[7] Goubault M, Mack AFS, Hardy ICW, Encountering competitors reduces clutch size and increases offspring size in a parasitoid with female-female fighting. *Proc. Roy. Soc. B* **274**, 2571–2577 (2007).

Adaptive Dynamics Theory

Adaptive Dynamics[8] is a theory developed by biologists in the 1980's to study evolution. In this theory, we consider only populations consisting in a finite number of Dirac masses :

$$\bar{f} = \sum_{i=1}^{n} \bar{\rho}_i \delta_{\bar{x}_i}.$$

Under which conditions is this population stable ? They consider two special perturbations :

- introduction of a mutant : $\left(\sum_{i=1}^n \bar{\rho}_i \delta_{\bar{x}_i}\right) \to \left(\sum_{i=1}^n \bar{\rho}_i \delta_{\bar{x}_i}\right) + \varepsilon \delta_x$,
- small shifts of the Dirac masses : $(\bar{x}_i) \to (\tilde{x}_i)$

[8] Diekmann O, Beginner's guide to adaptive dynamics. *Banach Center Publ.* **63**, Polish Acad. Sci., Warsaw, 47–86 (2004).

Evolutionary Attractor

A steady population $\bar{f} = \sum_{i=1}^n \bar{\rho}_i \delta_{\bar{x}_i}$ is an Evolutionary Attractor if :

- $\forall j,\ \partial_x s[\bar{f}](x_j) = 0$,
- $\forall j,\ \partial^2_{xx} s[\bar{f}](x_j) < 0$,
- there exists $\nu > 0$ such that $\forall u \in \mathbb{R}^n$,

$$
{}^t u \, diag\left(\left(\frac{1}{-\partial^2_{xx} s[\sum_{j=1}^n \bar{\rho}_j \delta_{\bar{x}_j}](\bar{x}_i)}\right)_{i=1,\dots,n}\right) DG((\bar{x}_i)_{i=1,\dots,n})\, u
< -\nu \|u\|^2,
$$

where $G(x_i) = (\partial_x s[\sum_{i=1}^n \rho_i \delta_{x_i}](x_i))_{i=1,\dots,n}$.

An asymptotic limit

$$\partial_t f_\varepsilon(t,x) = \frac{1}{\varepsilon} s[f_\varepsilon(t,\cdot)](x) f_\varepsilon(t,x),$$

$$f_\varepsilon(t,x) = f^0(x) e^{\frac{1}{\varepsilon}\int_0^t s[f_\varepsilon(\tau,\cdot)](x)\, d\tau}$$

Theorem (Desvillettes, Jabin, Mischler, R.)

Let $a \in C^1(\mathbb{R})$, $0 < \delta < b \in C^0(\mathbb{R})$, we assume that there exists x_0 such that $a(x_0) > 0$. If $0 < f^0 \in L^1(\mathbb{R})$, then there exist a unique solution f_ε of Kimura's model, and a subsequence f_ε of f_ε, such that :

$$\begin{cases} f_\varepsilon \rightharpoonup f \quad L^\infty(w*]0,T[;\sigma(M^1,Cb)(\mathbb{R})) \\ \int_0^t s[f_\varepsilon(\sigma,\cdot)](x)\, d\sigma \to \int_0^t s[f(\sigma,\cdot)](x)\, d\sigma \ \textit{uniformly on } \mathbb{R}_+ \times X, \end{cases}$$

Local stability of Evolutionary Attractors

Assumption : $f^0 \in M^1(\mathbb{R})$ is close to $\bar{f} = \sum_{i=1}^N \bar{\rho}_i \delta_{\bar{x}_i}$, in the sense that :

$$supp\left(f^0\right) = (\cup_i \{x_i\}) + B(0,\lambda) =: \cup_i I_i,$$
$$\forall i = 1, \dots, N, \left|\bar{\rho}_i - \int_{\bar{x}_i + B(0,\varepsilon)} f^0\right| \leq \lambda.$$

Proposition (R.)

If $\bar{f} = \sum_{i=1}^N \bar{\rho}_i \delta_{\bar{x}_i}$ *is an Evolutionary Attractor and if* f^0 *satisfies the Assumption for* $\lambda > 0$ *small enough, then the asymptotic limit* f *of* (f_ε) *is :*

$$f = \bar{f} = \sum_{i=1}^N \bar{\rho}_i \delta_{\bar{x}_i}.$$

Proof

- The neighbourhoods of $\bar{f}$ are stable sets for the equation,
- $f(t,\cdot) = \sum_{i=1}^{n} \rho_i(t)\delta_{x_i(t)}$, and for any $i = 1, \ldots, n$, $t \geq 0$,

$$\int_0^t s[f(\sigma,\cdot)](x_i(t))\,d\sigma = 0, \quad \frac{d^2}{dx^2}\int_0^t s[f(\sigma,\cdot)](x_i(t))\,d\sigma \leq c < 0,$$

- the x_i are regular and even satisfy an ODE,
- finally, at all times, $x_i(t) = \bar{x}_i$, $\rho_i(t) = \bar{\rho}_i$.

Adaptive Dynamics and its limitations

Adaptive Dynamics is a popular theory, easy to use for biologists. The root of this theory is that the population is a finite sum of Dirac masses. Thus :

- Adaptive Dynamics cannot be used to explain why "continuums of species" are not found in nature,
- Adaptive Dynamics does not describe the phenotypic variability existing in species.

Assumptions on a, b

$$\partial_t f_\varepsilon(t,x) = \frac{1}{\varepsilon} s[f_\varepsilon(t,\cdot)](x) f_\varepsilon(t,x) + m *_x f_\varepsilon(t,x).$$

Assumption : a reaches its maximum at $x = 0$, for $x \in X$, $\partial_1 b(x,x) = 0$, $m > 0$, and for some $1 > \bar{\varepsilon} > 0$,

$$\max_X a'' + \frac{\max_X a + \bar{\varepsilon} \max_{y \in X} \|m(\cdot,y)\|_{L^1(X)}}{\inf_{X \times X} b} \|\partial_{11}^2 b\|_\infty \leq -\delta < 0. \qquad (1)$$

Then, for any initial population $f^0 > 0$, the solution of Kimura's model converges to $\bar{f} = \frac{a(0)}{b(0,0)} \delta_0$.

An estimate on the diversity in the species

Theorem (Calsina, Cuadrado, Desvillettes, R.)

For $\varepsilon \in (0, \bar{\varepsilon})$, let $f^\varepsilon \geq 0$ be a steady-state of Kimura's model with mutations. Then, there exist $\bar{x}^\varepsilon = O(\varepsilon^{-1/3})$, such that :

$$\varepsilon f^\varepsilon(\varepsilon(\bar{x}^\varepsilon + x)) = \frac{m(0,0)\frac{a(0)}{b(0,0)} + O(\sqrt{\varepsilon}) + O(\varepsilon x)}{\frac{\frac{1}{2}m(0,0)\pi^2}{a''(0) - \frac{a(0)}{b(0,0)}\partial^2_{11}b(0,0)} + C\,x^2},$$

where $C = \frac{1}{2}\left(a''(0) - \frac{a(0)}{b(0,0)}\partial^2_{11}b(0,0) + O(\sqrt{\varepsilon}) + O(\varepsilon x)\right)$.

Proof if $b \equiv 1$, $m \equiv 1$

Let $f_\varepsilon(x) = \frac{1}{\varepsilon} u^\varepsilon(\frac{x}{\varepsilon})$ be a steady-state of Kimura's model with mutations :

$$0 = \left(a(\varepsilon x) - \int u^\varepsilon(y)\,dy \right) u^\varepsilon(x) + \varepsilon \int u^\varepsilon(y)\,dy.$$

Then,

$$u^\varepsilon(x) \sim \frac{\int u^\varepsilon(y)\,dy}{Q_\varepsilon + |a''(0)|x^2},$$

where $Q_\varepsilon = \frac{-1}{\varepsilon^2}\left(a(0) - \int u^\varepsilon(y)\,dy\right)$ can be estimated thanks to an integration of the above formula.

Conclusion

- Matematical models play a significant role in the biology of evolution,
- Kimura's model can be used to improve Adaptive Dynamics Theory,
- many questions remain open : evolving populations, evolutionary branchings.