# A finite speed of propagation approximation for the incompressible Navier-Stokes equations

Imène HACHICHA

**Abstract.** In this paper, we introduce a finite propagation speed perturbation of the incompressible Navier-Stokes equations $(NS)$. The model we consider is inspired by a hyperbolic perturbation of the heat equation due to Cattaneo in [5] and by an equation that Višik and Fursikov investigated in [8] in order to find statistical solutions to $(NS)$. We prove that the solutions to the perturbed Navier-Stokes equation approximate those to $(NS)$.
We use refined energy methods involving fractional Sobolev spaces and precise estimates on the nonlinear term due to the dyadic Littlewood-Paley decomposition.

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## 1. Introduction

The purpose of this paper is to approximate the solutions to the incompressible Navier-Stokes equations with quasi-critical regularity initial datum by solutions to a nonlinear wave equation with a finite speed of propagation which is obtained by penalizing the incompressibility constraint. First, let us recall the Navier-Stokes equations which govern the motion of an incompressible, viscous and homogeneous Newtonian fluid whose velocity and pressure are denoted by $v$ and $p$ respectively.

$$(NS)\ \partial_t v(t,x) - \nu\Delta v(t,x) + (v.\nabla)v(t,x) = -\nabla p(t,x)\ ,\ \operatorname{div} v(t,x) = 0,$$

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where $t > 0$ and $x \in \mathbb{R}^n$ for $n = 2, 3$. The velocity is a $\mathbb{R}^n$-valued vector field and the pressure is scalar. The coefficient of the Laplacian is the viscosity and, without loss of generality, is assumed to be 1 in the following.
Applying the Leray projector $\mathbb{P}$ which maps $L^2$ into $L^2_\sigma := \left\{u \in L^2 : \operatorname{div} u = 0\right\}$ to $(NS)$, we obtain the equations

$$(NS)\ \partial_t v - \Delta v + \mathbb{P}(v.\nabla)v = 0\ ,\ \operatorname{div} v = 0$$

from which we can recover the pressure $p$.
A first hyperbolic perturbation of $(NS)$ has been obtained after relaxation of the Euler equations and rescaling variables (see [3] and references therein):

$$(HNS^\varepsilon)\ \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon + \mathbb{P}(u^\varepsilon.\nabla)u^\varepsilon = 0\ ,\ \operatorname{div} u^\varepsilon = 0.$$

In [5], Cattaneo obtained this equation (without the nonlinear term) as a perturbation of the linear heat equation by introducing a delayed version for the heat flux, see for example [6, 13]. In [3] and [15], the authors approximate the solutions to $(NS)$ by solutions to $(HNS^\varepsilon)$ under some assumptions on the size and the regularity of the initial data. In [9], we improve the results of [3] and [15]. Under weaker assumptions on the initial data size, we prove the convergence of solutions to $(HNS^\varepsilon)$ towards solutions to $(NS)$ in the critical Sobolev space norm, that is $\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)$, as $\varepsilon$ goes to 0. More precisely, the theorem is:

**Theorem 1.1.** *Let $n = 2$ or $3$ and $0 < s, \delta < 1$. Let $v_0 \in H^{\frac{n}{2}-1+s}(\mathbb{R}^n)^n$ be a divergence-free vector field and $(u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac{n}{2}+\delta}(\mathbb{R}^n)^n \times H^{\frac{n}{2}-1+\delta}(\mathbb{R}^n)^n$ be a sequence of initial data for problem $(HNS^\varepsilon)$. Assume*

$$\begin{cases} \|u_0^\varepsilon - v_0\|_{\dot{H}^{\frac{n}{2}-1}} + \varepsilon\|u_1^\varepsilon\|_{\dot{H}^{\frac{n}{2}-1}} + \varepsilon^{\frac{1}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{n}{2}}} + \varepsilon^{\frac{1+\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{n}{2}+\delta}} + \varepsilon^{\frac{\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{n}{2}-1+\delta}} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{1+\frac{\delta}{2}}\|u_1^\varepsilon\|_{\dot{H}^{\frac{n}{2}-1+\delta}} = o(1). \end{cases}$$

*Moreover, if $n = 3$, assume that $\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3} < \frac{1}{16}$.*
*Then, for $\varepsilon$ small enough, there exists a global solution $u^\varepsilon$ to system $(HNS^\varepsilon)$ that converges, when $\varepsilon$ goes to $0$, in the $L^\infty_{loc}(\mathbb{R}^+; \dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)^n)$ norm, towards the unique solution $v$ to $(NS)$, with $v_0$ as initial datum. Moreover, for all positive $T$, there exists a constant $C_T$, depending only on $T$ and $v$, such that*

$$\sup_{t\in[0,T]} \|u^\varepsilon - v\|^2_{\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)^n}\, dx \le C_T \varepsilon^{\left(\frac{s}{2}\right)^-}.$$

*Remark* 1.2. As a consequence of the assumptions $\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < \frac{1}{16}$ and $\|u_0^\varepsilon - v_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right)$, we obtain the smallness of $\|v_0\|_{\dot{H}^{\frac{1}{2}}}$, which is a necessary condition to the existence of a global strong solution to the Navier-Stokes equations in $\mathbb{R}^3$.

Many other approximations of the incompressible Navier-Stokes equations have been investigated, see for example [14, 4, 10, 16, 17, 7, 19, 18]. One of them is the quasilinear delayed hyperbolic Navier-Stokes model:

$$\begin{aligned} \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon + \big((\varepsilon\partial_t u^\varepsilon + u^\varepsilon).\nabla\big)u^\varepsilon + \varepsilon(u^\varepsilon.\nabla)\partial_t u^\varepsilon + \nabla(p^\varepsilon + \varepsilon\partial_t p^\varepsilon) &= 0 \\ \operatorname{div} u^\varepsilon &= 0 \end{aligned}$$

which is close to $(HNS^\varepsilon)$ as it arises from Cattaneo's version of the constitutive law of the deformation tensor in $(NS)$. This model has been investigated in the papers by Racke and Saal [16, 17], by Fan and Ozawa [7] and by Schöwe [19, 18]. The authors prove local and global existence results for these equations as well as convergence estimates. Let us notice that this model has a finite speed of propagation in vorticity.

In this paper, we shall consider another hyperbolic perturbation of $(NS)$ which is inspired by Cattaneo's perturbation $(HNS^\varepsilon)$ and by Višik and Fursikov's weakly compressible equations in [8]. The main interest of this model is to get around the difficulties which come from the Leray projector $\mathbb{P}$ (or, equivalently, from the pressure) on the one hand and, on the other hand, from the infinite propagation speed due to the heat kernel. The equation we introduce in this paper is:

$$(HNS^{\varepsilon,\alpha})\ \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} = -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} + \frac{1}{\alpha}\nabla(\operatorname{div} u^{\varepsilon,\alpha}),$$

where $\varepsilon$ and $\alpha$ are positive numbers. Under the same assumptions as in Theorem 1.1, we prove that $(HNS^{\varepsilon,\alpha})$ is globally well-posed and that its solutions approximate those to $(NS)$. Notice that we do not need any more restrictions involving $\alpha$ on the initial data.

**Theorem 1.3.** *Let $n = 2$ or $3$ and $0 < s, \delta < 1$. Let $v_0 \in H^{\frac{n}{2}-1+s}(\mathbb{R}^n)^n$ be a divergence-free vector field and $(u_0^{\varepsilon,\alpha}, u_1^{\varepsilon,\alpha}) = (u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac{n}{2}+\delta}(\mathbb{R}^n)^n \times H^{\frac{n}{2}-1+\delta}(\mathbb{R}^n)^n$ be a sequence of divergence-free initial data for problem $(HNS^{\varepsilon,\alpha})$, independent of $\alpha$. Assume*

$$\begin{cases} \|u_0^{\varepsilon,\alpha} - v_0\|_{\dot{H}^{\frac{n}{2}-1}} + \varepsilon\|u_1^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{n}{2}-1}} + \varepsilon^{\frac{1}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{n}{2}}} &= o(1) \\ \varepsilon^{\frac{1+\delta}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{n}{2}+\delta}} + \varepsilon^{\frac{\delta}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{n}{2}-1+\delta}} + \varepsilon^{1+\frac{\delta}{2}}\|u_1^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{n}{2}-1+\delta}} &= o(1). \end{cases} \tag{1.1}$$

*Moreover, if $n = 3$, assume that $\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3} < \frac{1}{36K_2^3}$, where $K_2$ is the best constant satisfying, for all $f \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$,*

$$\|f\|_{L^3(\mathbb{R}^3)} \leq K_2\|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}.$$

*Then, for $\varepsilon, \alpha$ small enough and for all positive $T$, there exists a global solution $u^{\varepsilon,\alpha}$ to system $(HNS^{\varepsilon,\alpha})$ that converges, when $\varepsilon, \alpha$ go to $0$, in the $L^\infty_{loc}(\mathbb{R}^+; \dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)^n)$ norm, towards the unique solution $v$ to $(NS)$, with $v_0$ as initial datum.*

*Remark* 1.4. A possible choice of initial data is $\widehat{u_0^{\varepsilon,\alpha}}(\xi) = \widehat{v_0}(\xi)\mathbf{1}_{\{\sqrt{\varepsilon}|\xi|<1\}}$ and $u_1^{\varepsilon,\alpha} = 0$.

In this paper, we shall prove that the solutions to $(HNS^{\varepsilon,\alpha})$ with initial data $(u_0^\varepsilon, u_1^\varepsilon)$ converge, as $\alpha$ goes to 0, towards the solutions to $(HNS^\varepsilon)$ with the same initial data. Then we conclude according to Theorem 1.1.

The next section is dedicated to the introduction of the model. In section 3, we shall prove that $(HNS^{\varepsilon,\alpha})$ has a finite speed of propagation using the results of section 4 where we prove local existence. Then, we focus on the 2D case in section 5: first, we recall in subsection 5.1 some important estimates on the solutions $u^\varepsilon$ to $(HNS^\varepsilon)$ then we prove global existence for $(HNS^{\varepsilon,\alpha})$ in part 5.2 and we show that its solutions approximate those to $(HNS^\varepsilon)$ in subsection 5.3. Finally, section 6 is devoted to the 3D case and follows the same plan as section 5: in subsection 6.1 we recall the useful regularity results on $u^\varepsilon$ then we prove that the local solutions to $(HNS^{\varepsilon,\alpha})$ are global in subsection 6.2 and that the global solutions approximate those to $(HNS^\varepsilon)$ in subsection 6.3. Finally, Theorem 1.1 allows to conclude the proof of Theorem 1.3. Some important estimates coming from the framework of Littlewood-Paley theory are recalled in appendix A.

**Notation.** In the following, the spaces $L^p(\mathbb{R}^+;\mathcal{E})$ and $L^p([0,T];\mathcal{E})$ will be denoted by $L^p\mathcal{E}$ and $L^p_T\mathcal{E}$ respectively.

## 2. Introducing the model

In this section we shall introduce the finite speed of propagation equation that we will study in the next sections. We will work in the setting of $\mathbb{R}^2$. The 3D case is similar.

First, let us perturb the Navier-Stokes system $(NS)$ into the damped non-linear wave equation $(HNS^\varepsilon)$ which we recall:

$$(HNS^\varepsilon)\ \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon = -(u^\varepsilon.\nabla)u^\varepsilon - \nabla p^\varepsilon\ ,\ \operatorname{div} u^\varepsilon = 0.$$

Then, applying the $\operatorname{div} = \sum_{i=1}^n \partial_i(.)_i$ operator to this equation , we obtain

$$0 = \operatorname{div} f(u^\varepsilon) - \Delta p^\varepsilon, \tag{2.1}$$

where $f(u) = -(u.\nabla)u$. Let us now consider the stationary problem

$$-\Delta w^\varepsilon = f - \nabla p^\varepsilon\ ,\ \operatorname{div} w^\varepsilon = 0 \tag{2.2}$$

which is related to (2.1). Indeed, applying div to (2.2), we obtain (2.1). Given $f \in \dot{H}^{-1}$, we look for a solution $w^\varepsilon \in \dot{H}^1$ to (2.2), *i.e.* such that

$$J^\varepsilon(w^\varepsilon) = \min\left\{J^\varepsilon(v)\ :\ v \in \dot{H}^1, \operatorname{div} v = 0\right\},$$

where $J^\varepsilon(v) = \int \left(\frac{1}{2}|\nabla v|^2 - f.v\right)\,dx$. Now, using a penalization method, we change the problem into minimizing

$$J^{\varepsilon,\alpha}(v) = J^\varepsilon(v) + \frac{1}{2\alpha}\int |\operatorname{div} v|^2\,dx$$

in the space $\dot{H}^1$ so that the constraint $\operatorname{div} w^\varepsilon = 0$ is integrated to the functional $J^{\varepsilon,\alpha}$ to minimize. Let us call the minimizer $w^{\varepsilon,\alpha}$. Letting $\alpha$ go to zero in $w^{\varepsilon,\alpha}$, we obtain the desired solution $w^{\varepsilon}$. Since $w^{\varepsilon,\alpha}$ minimizes $J^{\varepsilon,\alpha}$, we know that it solves

$$-\Delta w^{\varepsilon,\alpha} = f + \frac{1}{\alpha}\nabla(\mathrm{div}\, w^{\varepsilon,\alpha}) = 0. \tag{2.3}$$

Now, recall that $f \in \dot{H}^{-1}$ and write

$$\begin{aligned}\int_{\mathbb{R}^2}\left(|\nabla w^{\varepsilon,\alpha}|^2 + \frac{1}{\alpha}|\mathrm{div}\, w^{\varepsilon,\alpha}|^2\right) dx &= \int_{\mathbb{R}^2} f.w^{\varepsilon,\alpha}\, dx \\ &\leq \|f\|_{\dot{H}^{-1}}\|w^{\varepsilon,\alpha}\|_{\dot{H}^1} \\ &\leq \frac{1}{2}\|f\|^2_{\dot{H}^{-1}} + \frac{1}{2}\|w^{\varepsilon,\alpha}\|^2_{\dot{H}^1}.\end{aligned}$$

So we immediately deduce that

$$\|\mathrm{div}\, w^{\varepsilon,\alpha}\|^2_{L^2} \leq C\alpha\|f\|^2_{\dot{H}^{-1}} = \mathcal{O}(\alpha). \tag{2.4}$$

On this basis, we shall consider the equation

$$(HNS^{\varepsilon,\alpha})\ \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} = -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} + \frac{1}{\alpha}\nabla(\mathrm{div}\, u^{\varepsilon,\alpha}).$$

Due to (2.4), we say that $(HNS^{\varepsilon,\alpha})$ is weakly compressible. Let us point out here that this model reminds of the one studied by Višik and Fursikov in [8] and, later, by Basson [2] and Lelièvre [11] in order to study statistical solutions to the Navier-Stokes equations.

In the next section, we shall prove that $(HNS^{\varepsilon,\alpha})$ has a finite propagation speed through Picard's fixed point theorem and energy inequalities.

## 3. Finite speed of propagation

In order to prove that the equation

$$(HNS^{\varepsilon,\alpha})\ \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} = -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} + \frac{1}{\alpha}\nabla(\mathrm{div}\, u^{\varepsilon,\alpha})$$

has a finite propagation speed, let us consider the Helmholtz-Hodge decomposition of $u^{\varepsilon,\alpha}$:

$$u^{\varepsilon,\alpha} = w^{\varepsilon,\alpha} + z^{\varepsilon,\alpha},$$

where $w^{\varepsilon,\alpha} = \mathbb{Q}u^{\varepsilon,\alpha} := \frac{1}{\Delta}\nabla\mathrm{div}\, u^{\varepsilon,\alpha}$ is irrotational and $z^{\varepsilon,\alpha} = \mathbb{P}u^{\varepsilon,\alpha} := u^{\varepsilon,\alpha} - w^{\varepsilon,\alpha}$ is divergence-free. We obtain the system

$$\left\{\begin{aligned}\varepsilon\partial_{tt}z^{\varepsilon,\alpha} + \partial_t z^{\varepsilon,\alpha} - \Delta z^{\varepsilon,\alpha} &= -\mathbb{P}\left(\left((w^{\varepsilon,\alpha} + z^{\varepsilon,\alpha}).\nabla\right)(w^{\varepsilon,\alpha} + z^{\varepsilon,\alpha})\right) \\ \varepsilon\partial_{tt}w^{\varepsilon,\alpha} + \partial_t w^{\varepsilon,\alpha} - \tfrac{\alpha+1}{\alpha}\Delta w^{\varepsilon,\alpha} &= -\mathbb{Q}\left(\left((w^{\varepsilon,\alpha} + z^{\varepsilon,\alpha}).\nabla\right)(w^{\varepsilon,\alpha} + z^{\varepsilon,\alpha})\right)\end{aligned}\right.$$

from which we can deduce a Duhamel formula for the Cauchy problem

$$(P)\ \left\{\begin{array}{l}(HNS^{\varepsilon,\alpha}) \\ u|_{t=0} = u_0\ ,\ \partial_t u|_{t=0} = u_1\end{array}\right. .$$

In section 4, we show local existence through Picard's contraction theorem in a small ball of the complete metric space

$$X_T = \left\{(u, \partial_t u) \in L^\infty_T\left(\dot{H}^{\frac{n}{2}+\delta} \cap \dot{H}^{\frac{n}{2}+\delta-1}(\mathbb{R}^n)^n\right) \times L^\infty_T \dot{H}^{\frac{n}{2}+\delta-1}(\mathbb{R}^n)^n\right\}.$$

The contractive map argument is detailed in section 4. In the following, we shall denote $u^{\varepsilon,\alpha}$ by $u$ to alleviate the notations. Picard's fixed point theorem gives a sequence

$$(u^j, \partial_t u^j) \xrightarrow{X_T} (u, \partial_t u)$$

defined by

$$\varepsilon \partial_{tt} u^{j+1} - \Delta u^{j+1} = -\partial_t u^j - (u^j.\nabla) u^j + \frac{1}{\alpha} \nabla \mathrm{div}\, u^j. \tag{3.1}$$

Now, set $\tilde{u} = \mathrm{div}\, u$ and apply the div operator to $(HNS^{\varepsilon,\alpha})$. Doing so, we obtain the following system:

$$(S) \begin{cases} \varepsilon \partial_{tt} \tilde{u} - \frac{\alpha+1}{\alpha} \Delta \tilde{u} & = & -\partial_t \tilde{u} - \mathrm{div}\, (u.\nabla) u \\ \varepsilon \partial_{tt} u - \ \Delta u & = & -\partial_t u - (u.\nabla) u + \frac{1}{\alpha} \nabla \tilde{u} \end{cases}.$$

Then the Cauchy problem $(P)$ is equivalent to

$$(P') \begin{cases} (S) \\ u|_{t=0} = u_0 \ , \ \partial_t u|_{t=0} = u_1 \ , \ \tilde{u}|_{t=0} = \mathrm{div}\, u_0 \ , \ \partial_t \tilde{u}|_{t=0} = \mathrm{div}\, u_1 \end{cases}.$$

*Remark* 3.1. As we shall see in the following, this way of writing the equation $(HNS^{\varepsilon,\alpha})$ is convenient for the proof of finite speed of propagation but, unless we assume that $u$ and $\tilde{u}$ are smooth, we cannot prove directly that the Duhamel formula related to system $(S)$ is locally contractive (due to the term $\nabla \tilde{u}$). Besides, we cannot prove that $(HNS^{\varepsilon,\alpha})$ has a finite speed of propagation through the Helmholtz-Hodge decomposition since the operators $\mathbb{P}$ and $\mathbb{Q}$ are non-local.

From now on, we assume $u_0$ and $u_1$ to be supported in a ball $B(0, R)$ and we set

$$c_1 = \sqrt{\frac{\alpha+1}{\alpha\varepsilon}} \ , \ c_2 = \frac{1}{\sqrt{\varepsilon}}.$$

Let us recall the standard energies associated to the wave equations in $(S)$:

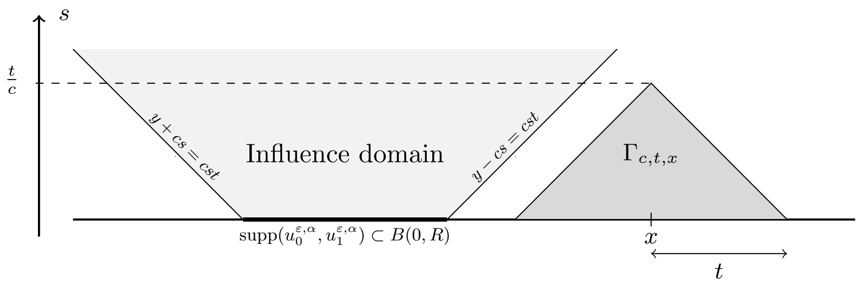
$$E_{c_1}(\tilde{u})(t) = \int_{\Gamma_{c_1,t,x}} e_{c_1}(\tilde{u})(t,y)\,\mathrm{d}y \ , \ E_{c_2}(u)(t) = \int_{\Gamma_{c_2,t,x}} e_{c_2}(u)(t,y)\,\mathrm{d}y,$$

where the densities are

$$\begin{aligned} e_{c_1}(\tilde{u})(t,y) &= \frac{\varepsilon}{2} |\partial_t \tilde{u}(t,y)|^2 + \frac{\alpha+1}{2\alpha} |\nabla \tilde{u}(t,y)|^2, \\ e_{c_2}(u)(t,y) &= \frac{\varepsilon}{2} |\partial_t u(t,y)|^2 + \frac{1}{2} |\nabla u(t,y)|^2 \end{aligned}$$

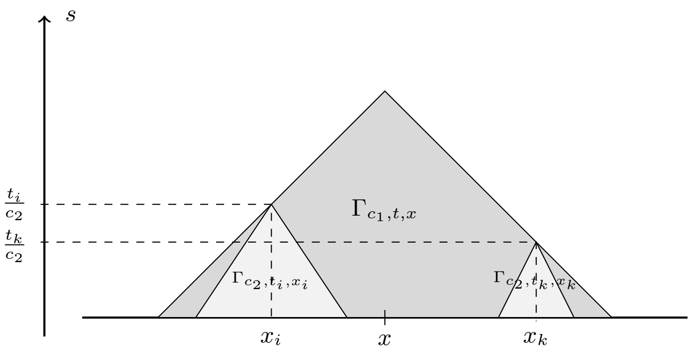
and the cones $\Gamma_{c_i,t,x}$ are defined by

$$\Gamma_{c_i,t,x} = \left\{ (s,y) \ : \ 0 \le s \le \frac{t}{c_i}, \ |y-x| \le t - c_i s \right\}.$$




Notice that, since $c_1 > c_2$, we have $\Gamma_{c_1,t,x} \subset \Gamma_{c_2,t,x}$. Applying a standard energy argument to the first equation in $(S)$, we prove that, for $j \in \mathbb{N}$, $\tilde{u}^j, \partial_t \tilde{u}^j = 0$ on the cone $\Gamma_{c_1,x,t}$.

Now, in order to handle the second equation in $(S)$, we cover the cone $\Gamma_{c_1,x,t}$ with cones of the type $\Gamma_{c_2,t_i,x_i}$ and integrate the second equation in $(S^j)$ mutiplied by $\partial_t u^{j+1}$ on these cones.




Doing so, we obtain that $u^{j+1}, \partial_t u^{j+1} = 0$ on $\Gamma_{c_1,t,x}$ . We have proven that $(HNS^{\varepsilon,\alpha})$ has a finite speed of propagation $c(\varepsilon,\alpha) \geq c_1 \to +\infty$ as $\alpha$ goes to 0 .

## 4. Local existence for equation $(HNS^{\varepsilon,\alpha})$

### 4.1. Introduction

Let us consider the Cauchy problem

$$\begin{cases} (HNS^{\varepsilon,\alpha})\ \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} - \frac{1}{\alpha}\nabla(\operatorname{div} u^{\varepsilon,\alpha}) = f(u^{\varepsilon,\alpha}) \\ u^{\varepsilon,\alpha}|_{t=0} = u_0^{\varepsilon,\alpha} \in H^{\frac{n}{2}+\delta}(\mathbb{R}^n)\ ,\ \partial_t u^{\varepsilon,\alpha}|_{t=0} = u_1^{\varepsilon,\alpha} \in H^{\frac{n}{2}+\delta-1}(\mathbb{R}^n) \end{cases}, \quad (4.1)$$

where $n = 2,3$ and $f(u) = -(u.\nabla)u$. First, let us assume that $f = 0$ and split the solution $u^{\varepsilon,\alpha}$ to equation $(HNS^{\varepsilon,\alpha})$ into its irrotational part $w^{\varepsilon,\alpha}$ and its divergence-free part $z^{\varepsilon,\alpha}$:

$$u^{\varepsilon,\alpha}(t,x) = w^{\varepsilon,\alpha}(t,x) + z^{\varepsilon,\alpha}(t,x).$$

More precisely, $w^{\varepsilon,\alpha}$ and $z^{\varepsilon,\alpha}$ are defined as follows:

$$\begin{aligned} w^{\varepsilon,\alpha} &= \mathbb{Q}u^{\varepsilon,\alpha} := \frac{1}{\Delta}\nabla(\operatorname{div} u^{\varepsilon,\alpha}) \\ z^{\varepsilon,\alpha} &= \mathbb{P}u^{\varepsilon,\alpha} := (\mathbf{1}-\mathbb{Q})u^{\varepsilon,\alpha} = u^{\varepsilon,\alpha} - w^{\varepsilon,\alpha}. \end{aligned}$$

Then $w^{\varepsilon,\alpha}$ and $z^{\varepsilon,\alpha}$ solve the following equations:

$$\begin{cases} \varepsilon\partial_{tt}w^{\varepsilon,\alpha} + \partial_t w^{\varepsilon,\alpha} - \left(1+\frac{1}{\alpha}\right)\Delta w^{\varepsilon,\alpha} = 0 \\ w^{\varepsilon,\alpha}|_{t=0} = w_0^{\varepsilon,\alpha} := \mathbb{Q}u_0^{\varepsilon,\alpha}\ ,\ \partial_t w^{\varepsilon,\alpha}|_{t=0} = w_1^{\varepsilon,\alpha} := \mathbb{Q}u_1^{\varepsilon,\alpha}, \end{cases} \tag{4.2}$$

$$\begin{cases} \varepsilon\partial_{tt}z^{\varepsilon,\alpha} + \partial_t z^{\varepsilon,\alpha} - \Delta z^{\varepsilon,\alpha} = 0 \\ z^{\varepsilon,\alpha}|_{t=0} = z_0^{\varepsilon,\alpha} := \mathbb{P}u_0^{\varepsilon,\alpha}\ ,\ \partial_t z^{\varepsilon,\alpha}|_{t=0} = z_1^{\varepsilon,\alpha} := \mathbb{P}u_1^{\varepsilon,\alpha}. \end{cases} \tag{4.3}$$

Since (4.2) is a wave equation, we know that

$$w^{\varepsilon,\alpha}(t,x) = A_{\mathbb{Q}}(t)w_0^{\varepsilon,\alpha}(x) + B_{\mathbb{Q}}(t)w_1^{\varepsilon,\alpha}(x) - \int_0^t B_{\mathbb{Q}}(t-s)\partial_t w^{\varepsilon,\alpha}(s)\,ds,$$

where $A_{\mathbb{Q}}$ and $B_{\mathbb{Q}}$ are defined as follows.

$$A_{\mathbb{Q}}(t) = \cos\left(\sqrt{\frac{\alpha+1}{\varepsilon\alpha}}t\Lambda\right)\ ,\ B_{\mathbb{Q}}(t) = \frac{\sin\left(\sqrt{\frac{\alpha+1}{\varepsilon\alpha}}t\Lambda\right)}{\sqrt{\frac{\alpha+1}{\varepsilon\alpha}}\Lambda}.$$

Similarly, (4.3) is a wave equation and we have

$$z^{\varepsilon,\alpha}(t,x) = A_{\mathbb{P}}(t)z_0^{\varepsilon,\alpha}(x) + B_{\mathbb{P}}(t)z_1^{\varepsilon,\alpha}(x) - \int_0^t B_{\mathbb{P}}(t-s)\partial_t z^{\varepsilon,\alpha}(s)\,ds,$$

where $A_{\mathbb{P}}$ and $B_{\mathbb{P}}$ are

$$A_{\mathbb{P}}(t) = \cos\left(\frac{t\Lambda}{\sqrt{\varepsilon}}\right)\ ,\ B_{\mathbb{P}}(t) = \sqrt{\varepsilon}\frac{\sin\left(\frac{t\Lambda}{\sqrt{\varepsilon}}\right)}{\Lambda}.$$

Now, setting $A(t) = A_{\mathbb{Q}}(t)\mathbb{Q} + A_{\mathbb{P}}(t)\mathbb{P}$ and $B(t) = B_{\mathbb{Q}}(t)\mathbb{Q} + B_{\mathbb{P}}(t)\mathbb{P}$, we can write Duhamel's formula for the Cauchy problem (4.1) with $f=0$:

$$\phi(u^{\varepsilon,\alpha})(t) = A(t)u_0^{\varepsilon,\alpha} + B(t)u_1^{\varepsilon,\alpha} - \int_0^t B(t-s)\partial_t u^{\varepsilon,\alpha}(s)\,ds.$$

Adding the source term $f(u^{\varepsilon,\alpha}) = -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}$, the formula becomes

$$\phi(u^{\varepsilon,\alpha})(t) = A(t)u_0^{\varepsilon,\alpha} + B(t)u_1^{\varepsilon,\alpha} + \int_0^t B(t-s)\left(f(u^{\varepsilon,\alpha}) - \partial_t u^{\varepsilon,\alpha}\right)(s)\,ds.$$

### 4.2. Contraction argument

We shall show local existence for (4.1) in the complete metric space

$$\begin{aligned} X_T(a) = \Big\{ &(u,\partial_t u) \in L_T^\infty\left(\dot H^{\frac{n}{2}+\delta} \cap \dot H^{\frac{n}{2}+\delta-1}(\mathbb{R}^n)^n\right) \times L_T^\infty \dot H^{\frac{n}{2}+\delta-1}(\mathbb{R}^n)^n \ : \\ &\|u\|_{X_T} := \|u\|_{L_T^\infty \dot H^{\frac{n}{2}+\delta}} + \|u\|_{L_T^\infty \dot H^{\frac{n}{2}+\delta-1}} + \|\partial_t u\|_{L_T^\infty \dot H^{\frac{n}{2}+\delta-1}} \le a \Big\}, \end{aligned}$$

with $a > 0$ and $0 < T < 1$ to be chosen later.

In order to estimate $\|\phi(u)\|_{X_T}$, we need to prove the following lemma.

**Lemma 4.1.** *Let $u \in \dot{H}^{\frac{n}{2}+\delta} \cap \dot{H}^{\frac{n}{2}+\delta-1}(\mathbb{R}^n)$ and $div u \neq 0$. Then we have*

$$\|div u\; u\|_{\dot{H}^{\frac{n}{2}+\delta-1}} \leq 2C\|u\|_{\dot{H}^{\frac{n}{2}+\delta}}\|u\|_{L^\infty}.$$

*Proof.* Applying the paraproduct decomposition (see Lemma A.2 in the appendix), we can write

$$\begin{aligned}
\|\partial_i u\; u\|_{\dot{H}^{\frac{n}{2}+\delta-1}} &\leq \sum_p 2^{p\left(\frac{n}{2}+\delta-1\right)}\|\Delta_p \partial_i u S_{p+1}u\|_{L^2} + \sum_q 2^{q\left(\frac{n}{2}+\delta-1\right)}\|\Delta_q u S_{p+1}\partial_i u\|_{L^2} \\
&\leq \sum_p 2^{p\left(\frac{n}{2}+\delta-1\right)}\|\Delta_p \partial_i u\|_{L^2}\|S_{p+1}u\|_{L^\infty} + \\
&\qquad + \sum_q 2^{q\left(\frac{n}{2}+\delta-1\right)}\|\Delta_q u\|_{L^2}\|S_{p+1}\partial_i u\|_{L^\infty} \\
&\leq \sum_p 2^{p\left(\frac{n}{2}+\delta\right)}\|\Delta_p u\|_{L^2}\|u\|_{L^\infty} + \sum_q 2^{q\left(\frac{n}{2}+\delta\right)}\|\Delta_q u\|_{L^2}\|u\|_{L^\infty} \\
&\leq 2C\|u\|_{\dot{H}^{\frac{n}{2}+\delta}}\|u\|_{L^\infty}.
\end{aligned}$$

The Lemma is thereby proved. □

Then, using standard estimates, we obtain

$$\begin{aligned}
\|\phi(u)\|_{X_T} \leq\; & \left(2 + \frac{1}{\sqrt{\varepsilon}} + \sqrt{\frac{\alpha+1}{\alpha\varepsilon}}\right)\|u_0\|_{\dot{H}^{\frac{n}{2}+\delta}} + 2\|u_0\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + \\
& + \left(2 + \sqrt{\varepsilon} + \sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right)\|u_1\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + 2T\|u_1\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + \\
& + \left(2 + 2T + \sqrt{\varepsilon} + \sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right) aT(1+4Ca).
\end{aligned}$$

Now, let us set

$$\frac{a}{2} = \left(2 + \frac{1}{\sqrt{\varepsilon}} + \sqrt{\frac{\alpha+1}{\alpha\varepsilon}}\right)\|u_0\|_{\dot{H}^{\frac{n}{2}+\delta}} + 2\|u_0\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + \left(2 + \sqrt{\varepsilon} + \sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right)\|u_1\|_{\dot{H}^{\frac{n}{2}+\delta-1}}.$$

So we have

$$\|\phi(u)\|_{X_T} \leq \frac{a}{2} + \frac{aT}{2 + \sqrt{\varepsilon} + \sqrt{\frac{\alpha\varepsilon}{\alpha+1}}} + \left(2 + 2T + \sqrt{\varepsilon} + \sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right) aT(1+4Ca).$$

Finally, we have the following bound on the local existence time:

$$T \leq \frac{C}{1 + \left[\left(C_\varepsilon + \sqrt{\frac{\alpha+1}{\alpha\varepsilon}}\right)\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{n}{2}+\delta}} + 2\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + \left(\tilde{C}_\varepsilon + \sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right)\|u_1^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{n}{2}+\delta-1}}\right]}.$$

In order to prove that the solutions obtained in this section are global, we shall prove that the denominator

$$\left(C_\varepsilon + \sqrt{\frac{\alpha+1}{\alpha\varepsilon}}\right)\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{n}{2}+\delta}} + 2\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + \left(\tilde{C}_\varepsilon + \sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right)\|u_1^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{n}{2}+\delta-1}}$$

remains bounded for all fixed $\alpha$ and $\varepsilon$.

In subsections 5.2 and 6.2,we will globalize the solutions obtained in this section and then, in subsections 5.3 and 6.3, we will prove that they converge towards the solutions to $(HNS^\varepsilon)$ as $\alpha$ goes to 0.

# 5. The 2D case

## 5.1. Preliminary estimates

In this part, we shall recall the regularity results we have on the solution $u^\varepsilon$ to $(HNS^\varepsilon)$.

**Lemma 5.1.** *Let $T>0$ and $u^\varepsilon \in L^\infty_T(\dot H^{1+\delta}\cap \dot H^\delta)(\mathbb{R}^2)^2$ be the global solution to $(HNS^\varepsilon)$ with initial data $(u_0^\varepsilon, u_1^\varepsilon)\in H^{1+\delta}\times H^\delta(\mathbb{R}^2)^2$. Then we have*

$$u^\varepsilon \in L^\infty_T \dot H^{1+\delta}\cap L^2_T\dot H^{1+\delta}\cap L^2_T L^2\cap L^\infty_T L^2\cap L^2_T\dot H^1$$

*and*

$$\partial_t u^\varepsilon \in L^2_T L^2.$$

We sketch here the proof of this lemma since all the details can be found in [9]. In the following, we shall denote by $C$ all the constants, even those depending on $T$.

*Proof.* First, let us introduce the energy

$$E^\delta_\varepsilon(t) := \frac{1}{2}\|u^\varepsilon+\varepsilon\partial_t u^\varepsilon\|^2_{\dot H^\delta}+\frac{\varepsilon^2}{2}\|\partial_t u^\varepsilon\|^2_{\dot H^\delta}+\varepsilon\|\nabla u^\varepsilon\|^2_{\dot H^\delta},$$

for non-negative $\delta$. Then, according to [9], we know that

$$\exists C>0\ :\ \forall t\geq 0\ ,\ E^\delta_\varepsilon(t)\leq C\varepsilon^{-\delta}.$$

From this inequality, we immediately deduce that

$$\varepsilon\|u^\varepsilon\|_{L^\infty_T\dot H^{1+\delta}}+\varepsilon\|u^\varepsilon\|_{L^2_T\dot H^{1+\delta}}\leq C\varepsilon^{-\delta} \text{ and } \|u^\varepsilon\|_{L^\infty_T L^\infty}=o\left(\frac{1}{\sqrt{\varepsilon}}\right). \tag{5.1}$$

Now, let us compute the time derivative of $E^0_\varepsilon$. We have

$$\frac{d}{dt}E^0_\varepsilon(t)+\varepsilon\int_{\mathbb{R}^2}|\partial_t u^\varepsilon+\nabla:(u^\varepsilon\otimes u^\varepsilon)|^2\,dx+\int_{\mathbb{R}^2}\left(|\nabla u^\varepsilon|^2-\varepsilon|\nabla(u^\varepsilon\otimes u^\varepsilon)|^2\right)\,dx=0.$$

Due to the control of the norm $\|u^\varepsilon\|_{L^\infty_T L^\infty}$ by $\dfrac{1}{C\sqrt{\varepsilon}}$ for any $C$ provided that $\varepsilon$ is small enough, the last term in the left hand side is lower bounded by $\int_{\mathbb{R}^2}\frac{1}{2}|\nabla u^\varepsilon|^2\,\mathrm{d}x$. Now, integrating in time, we have

$$E^0_\varepsilon(T)+\int_0^T\int_{\mathbb{R}^2}\varepsilon|\partial_t u^\varepsilon+\nabla:(u^\varepsilon\otimes u^\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\mathbb{R}^2}\frac{1}{2}|\nabla u^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\leq E^0_\varepsilon(0)\leq C.$$

So, we have that $\varepsilon\|\partial_t u^\varepsilon+\nabla:(u^\varepsilon\otimes u^\varepsilon)\|^2_{L^2_T L^2}\leq C$ and

$$\frac{1}{2}\|u^\varepsilon\|^2_{L^2_T\dot H^1}\leq C. \tag{5.2}$$

The later yields

$$\varepsilon\|\nabla:(u^\varepsilon\otimes u^\varepsilon)\|^2_{L^2_TL^2}\le C\varepsilon\|u^\varepsilon\|^2_{L^\infty_TL^\infty}\|u^\varepsilon\|^2_{L^2_T\dot H^1}\le C.$$

Consequently, we have

$$\|\sqrt{\varepsilon}\partial_tu^\varepsilon\|^2_{L^2_TL^2}\le 2\varepsilon\|\partial_tu^\varepsilon+\nabla:(u^\varepsilon\otimes u^\varepsilon)\|^2_{L^2_TL^2}+2\varepsilon\|\nabla:(u^\varepsilon\otimes u^\varepsilon)\|^2_{L^2_TL^2}\le C. \tag{5.3}$$

We have thereby proven that

$$\sqrt{\varepsilon}\partial_tu^\varepsilon\in L^2_TL^2 \text{ uniformly in } \varepsilon.$$

Finally, notice that $\|u^\varepsilon\|^2_{L^2}\le 2E^0_\varepsilon(t)\le 2C_0$, so that

$$u^\varepsilon\in L^2_TL^2. \tag{5.4}$$

### 5.2. Globalization

As in [9], let us define the energy

$$E^\delta_{\varepsilon,\alpha}(t)=\int_{\mathbb{R}^2}\frac{1}{2}|\Lambda^\delta(u^{\varepsilon,\alpha}+\varepsilon\partial_tu^{\varepsilon,\alpha})|^2+\frac{\varepsilon^2}{2}|\Lambda^\delta\partial_tu^{\varepsilon,\alpha}|^2+\varepsilon|\Lambda^\delta\nabla u^{\varepsilon,\alpha}|^2+\frac{\varepsilon}{\alpha}|\Lambda^\delta\mathrm{div}\,u^{\varepsilon,\alpha}|^2.$$

In view of the dependence of the local time existence on the initial data, we know that proving that $E^\delta_{\varepsilon,\alpha}$ is bounded yields the global existence for $(HNS^{\varepsilon,\alpha})$. We shall first show that it is true on a time interval $[0,T)$ then prove that $T=+\infty$.

First, let us point out that $\dot H^{1+\delta}\cap L^\infty(\mathbb{R}^2)$ is an algebra and that the product estimate

$$\|fg\|_{\dot H^{1+\delta}(\mathbb{R}^2)}\le C_1\left(\|f\|_{\dot H^{1+\delta}}\|g\|_\infty+\|g\|_{\dot H^{1+\delta}}\|f\|_\infty\right). \tag{5.5}$$

holds (see Proposition A.3 in the appendix or [1]) for all functions $f,g\in\dot H^{1+\delta}\cap L^\infty(\mathbb{R}^2)$. Moreover, we know that the homogeneous Besov[a] space $\dot B^1_{2,1}(\mathbb{R}^2)$ embeds into $L^\infty(\mathbb{R}^2)$ and, interpolating, we obtain

$$\|f\|_\infty\ \le\ \tilde C\|f\|_{\dot B^1_{2,1}}\ \le\ C_2\,\|f\|^\delta_{\dot H^\delta}\,.\,\|f\|^{1-\delta}_{\dot H^{1+\delta}}\,. \tag{5.6}$$

Finally, notice that $\mathrm{div}\,u^{\varepsilon,\alpha}\neq 0$, so that we have

$$(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}=\sum_{i=1}^2u_i^{\varepsilon,\alpha}\partial_iu^{\varepsilon,\alpha}=\sum_{i=1}^2\partial_i(u_i^{\varepsilon,\alpha}.u^{\varepsilon,\alpha})-\mathrm{div}\,u^{\varepsilon,\alpha}\times u^{\varepsilon,\alpha}. \tag{5.7}$$

but we still can prove the estimate

$$\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{\dot H^\delta}\le C_3\|u^{\varepsilon,\alpha}\|_{L^\infty}\|u^{\varepsilon,\alpha}\|_{\dot H^{1+\delta}} \tag{5.8}$$

using that $\dot H^{1+\delta}\cap L^\infty(\mathbb{R}^2)$ is an algebra for the first term in (5.7) and due to Lemma 4.1 for the second term.

Now, let us define $T^{\max}$ the maximal existence time of $(HNS^{\varepsilon,\alpha})$ and prove the following lemma.

[a] For definitions and properties of the Besov spaces, see the book by P.-G. Lemarié-Rieusset [12]

**Lemma 5.2.** *Assume the following, when $\varepsilon$ goes to zero :*

$$(H)\left\{\begin{array}{ll} i) & \varepsilon^{\frac{1+\delta}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot H^{1+\delta}}+\varepsilon^{\frac{\delta}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot H^{\delta}}=o(1)\\ ii) & \varepsilon^{\frac12}\|u_0^{\varepsilon,\alpha}\|_{\dot H^{1}}+\varepsilon\|u_1^{\varepsilon,\alpha}\|_{L^2}=o(1).\end{array}\right.$$

*Let us define $0\le T\le T^{max}$ by*

$$T=\sup\left\{0\le\tau\le T^{max}\ :\ \forall\, t\in[0,\tau),\ \|u^{\varepsilon,\alpha}(t)\|_{L^\infty}<\frac{1}{2C_3\sqrt{\varepsilon}}\right\}. \tag{5.9}$$

*Then, for $\varepsilon$ small enough, there exists a constant $A=A(\delta)$ such that, for all $0\le t<T$,*

$$E^\delta_{\varepsilon,\alpha}(t)\le A\,E^\delta_{\varepsilon,\alpha}(0). \tag{5.10}$$

*Proof.* Let us compute the time derivative of $E^\delta_{\varepsilon,\alpha}$:

$$\begin{aligned}\frac{\mathrm d}{\mathrm dt}E^\delta_{\varepsilon,\alpha}(t) &= \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot H^\delta}-\|\nabla u^{\varepsilon,\alpha}\|^2_{\dot H^\delta}-\int_{\mathbb R^2}\Lambda^\delta u^{\varepsilon,\alpha}.\Lambda^\delta(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm dx-\\ &\quad -\varepsilon\|\partial_t u^{\varepsilon,\alpha}+(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot H^\delta}-\frac1\alpha\|\mathrm{div}\,u^{\varepsilon,\alpha}\|^2_{\dot H^\delta}\\ &\le \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot H^\delta}-\|\nabla u^{\varepsilon,\alpha}\|^2_{\dot H^\delta}-\int_{\mathbb R^2}\Lambda^\delta u^{\varepsilon,\alpha}.\Lambda^\delta(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm dx.\end{aligned}$$

Using (5.8), the derivative of the energy estimates as follows:

$$\frac{\mathrm d}{\mathrm dt}E^\delta_{\varepsilon,\alpha}(t)\le\left(C_3^2\varepsilon\|u^{\varepsilon,\alpha}\|^2_\infty-1\right)\|u^{\varepsilon,\alpha}\|^2_{\dot H^{1+\delta}}-\int_{\mathbb R^2}\Lambda^\delta u^{\varepsilon,\alpha}.\Lambda^\delta(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm dx.$$

Since we assume $i)$ and using inequality (5.6), we can write, for $\varepsilon$ small enough,

$$\|u_0^{\varepsilon,\alpha}\|_{L^\infty}\le\frac{1}{2C_3\sqrt\varepsilon}. \tag{5.11}$$

Now, by continuity of the (local) solution $u^{\varepsilon,\alpha}$ with respect to $t$, we deduce that $T>0$ and that the inequality

$$\frac{\mathrm d}{\mathrm dt}E^\delta_{\varepsilon,\alpha}(t)\le-\frac14\|u^{\varepsilon,\alpha}\|^2_{\dot H^{1+\delta}}+C\|u^{\varepsilon,\alpha}\|_{\dot H^\delta}\|u^{\varepsilon,\alpha}\|_\infty\|u^{\varepsilon,\alpha}\|_{\dot H^{1+\delta}}$$

holds on $[0,T)$. Then, using the interpolation inequalities

$$\|u^{\varepsilon,\alpha}\|_{\dot H^\delta}\le C_4\|u^{\varepsilon,\alpha}\|^{1-\delta}_{L^2}\|u^{\varepsilon,\alpha}\|^\delta_{\dot H^1} \tag{5.12}$$

and (5.6), we obtain the estimate

$$\frac{d}{dt}E^\delta_{\varepsilon,\alpha}(t)\le-\frac14\|u^{\varepsilon,\alpha}\|^2_{\dot H^{1+\delta}}+C\|u^{\varepsilon,\alpha}\|_2^{1-\delta}\left(\|u^{\varepsilon,\alpha}\|_{\dot H^1}\|u^{\varepsilon,\alpha}\|_{\dot H^\delta}\right)^\delta\|u^{\varepsilon,\alpha}\|^{2-\delta}_{\dot H^{1+\delta}}.$$

Finally, a Young inequality yields

$$\frac{\mathrm d}{\mathrm dt}E^\delta_{\varepsilon,\alpha}(t)\le C\|u^{\varepsilon,\alpha}\|_2^{2\frac{1-\delta}{\delta}}\|u^{\varepsilon,\alpha}\|^2_{\dot H^1}E^\delta_{\varepsilon,\alpha}(t). \tag{5.13}$$

In order to show that $E^\delta_{\varepsilon,\alpha}$ is bounded, we shall use the decay of $E^0_{\varepsilon,\alpha}$. So, let us estimate its time derivative:

$$\frac{\mathrm d}{\mathrm dt}E^0_{\varepsilon,\alpha}(t)\le\varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{L^2}-\|u^{\varepsilon,\alpha}\|^2_{\dot H^1}-\int_{\mathbb R^2}u^{\varepsilon,\alpha}.(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm dx-\frac1\alpha\|\mathrm{div}\,u^{\varepsilon,\alpha}\|^2_{L^2}.$$

First, by Lemma 4.1, we immediately have

$$\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{L^2} \leq C_3\|u^{\varepsilon,\alpha}\|_{L^\infty}\|u^{\varepsilon,\alpha}\|_{\dot{H}^1}.$$

Besides, by integrations by parts, we obtain

$$\int_{\mathbb{R}^2} u^{\varepsilon,\alpha}.(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^2} u^{\varepsilon,\alpha}.\operatorname{div} u^{\varepsilon,\alpha}\; u^{\varepsilon,\alpha} = -\frac{1}{2}\int_{\mathbb{R}^2} \operatorname{div} u^{\varepsilon,\alpha}\; |u^{\varepsilon,\alpha}|^2.$$

Recall that $\operatorname{div} u^{\varepsilon,\alpha} \in L^2(\mathbb{R}^2)$ and $u^{\varepsilon,\alpha} \in L^2 \cap \dot{H}^1 \subset \dot{H}^{\frac{1}{2}} \subset L^4(\mathbb{R}^2)$. Thus the integral estimates

$$\begin{aligned}
|\int_{\mathbb{R}^2} u^{\varepsilon,\alpha}.(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm{d}x| &= |\frac{1}{2}\int_{\mathbb{R}^2} \operatorname{div} u^{\varepsilon,\alpha}\; |u^{\varepsilon,\alpha}|^2| \\
&\leq \frac{1}{2}\|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2}\||u^{\varepsilon,\alpha}|^2\|_{L^2} \\
&\leq \frac{1}{2}\|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2}\|u^{\varepsilon,\alpha}\|_{L^4}^2 \\
&\leq \frac{K}{2}\|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2}\|u^{\varepsilon,\alpha}\|_{L^2}\|u^{\varepsilon,\alpha}\|_{\dot{H}^1} \\
&\leq \frac{K^2}{8}\|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2}^2\|u^{\varepsilon,\alpha}\|_{L^2}^2 + \frac{1}{2}\|u^{\varepsilon,\alpha}\|_{\dot{H}^1}^2,
\end{aligned}$$

where we have used the Gagliardo-Nirenberg inequality

$$\|f\|_{L^4}^2 \leq K\|f\|_{L^2}\|f\|_{\dot{H}^1}$$

for $f \in H^1(\mathbb{R}^2)$. We therefore obtain the following estimate on $E^0_{\varepsilon,\alpha}$ on $[0,T)$ for $\varepsilon$ small enough:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E^0_{\varepsilon,\alpha}(t) &\leq \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{L^2}^2 - \|u^{\varepsilon,\alpha}\|_{\dot{H}^1}^2 - \int_{\mathbb{R}^2} u^{\varepsilon,\alpha}.(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm{d}x - \frac{1}{\alpha}\|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2}^2 \\
&\leq \left(C_3^2\varepsilon\|u^{\varepsilon,\alpha}\|_{L^\infty}^2 - \frac{1}{2}\right)\|u^{\varepsilon,\alpha}\|_{\dot{H}^1}^2 + \left(\frac{K^2}{8}\|u^{\varepsilon,\alpha}\|_{L^2}^2 - \frac{1}{\alpha}\right)\|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2}^2 \\
&\leq -\frac{1}{4}\|u^{\varepsilon,\alpha}\|_{\dot{H}^1}^2 + \left(\frac{K^2}{8}\|u^{\varepsilon,\alpha}\|_{L^2}^2 - \frac{1}{\alpha}\right)\|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2}^2.
\end{aligned}$$

Now, recall that we have $\|u_0^{\varepsilon,\alpha}\|_{L^2}^2 = \|v_0\|_{L^2}^2 + o(1)$ as $\varepsilon$ goes to 0, where $v_0$ is the initial datum for $(NS)$. Moreover, let us assume that $\alpha \leq \dfrac{2}{K^2(1+\|v_0\|_{L^2}^2)}$, so that we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E^0_{\varepsilon,\alpha}(0) \leq -\frac{1}{4}\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^1}^2.$$

Thus $E^0_{\varepsilon,\alpha}$ decreases on some time interval $[0,\tau]$, where $\tau < T$. Then the smallness assumptions *ii)* on the initial data yield

$$\|u^{\varepsilon,\alpha}(\tau)\|_{L^2}^2 \;\leq\; 2E^0_{\varepsilon,\alpha}(\tau) \;\leq\; 2E^0_{\varepsilon,\alpha}(0) \;\overset{ii)}{\leq}\; 2(1+\|v_0\|_{L^2}^2)$$

and $E^0_{\varepsilon,\alpha}$ is therefore decreasing on $[0,T)$ since, for all $0 \leq t < T$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E^0_{\varepsilon,\alpha}(t) \leq -\frac{1}{4}\|u^{\varepsilon,\alpha}(t)\|_{\dot{H}^1}^2. \tag{5.14}$$

Let us notice that we easily deduce from (5.14) that

$$\|u^{\varepsilon,\alpha}\|^2_{L^2_T\dot{H}^1} \leq E^0_{\varepsilon,\alpha}(0) \leq 1 + \|v_0\|^2_{L^2}.$$

Now, since we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E^\delta_{\varepsilon,\alpha}(t) \leq C\|u^{\varepsilon,\alpha}(t)\|_2^{2\frac{1-\delta}{\delta}}\|u^{\varepsilon,\alpha}(t)\|^2_{\dot{H}^1}E^\delta_{\varepsilon,\alpha}(t),$$

where $u^{\varepsilon,\alpha} \in L^\infty_T L^2 \cap L^2_T\dot{H}^1$ uniformly in $\varepsilon, \alpha$, we can apply Gronwall's Lemma and we obtain the following estimate.

$$E^\delta_{\varepsilon,\alpha}(t) \leq E^\delta_{\varepsilon,\alpha}(0)\exp\left(C_\delta\int_0^t \|u^{\varepsilon,\alpha}(\tau)\|^2_{\dot{H}^1}\,\mathrm{d}\tau\right)$$

and $E^\delta_{\varepsilon,\alpha}$ therefore satisfies the estimate (5.10) for all $t \in [0,T)$. □

The aim of the following lemma is to ensure the control of $\|u^{\varepsilon,\alpha}(t)\|_{L^\infty}$ throughout the time, so that we can reiterate the reasoning.

**Lemma 5.3.** *Assume that*

$$\varepsilon^{1+\frac{\delta}{2}}\|u_1^{\varepsilon,\alpha}\|_{\dot{H}^\delta} \longrightarrow 0\ ,\ \varepsilon \to 0 \tag{5.15}$$

*in addition to the assumptions* $(H)$ *in Lemma* 5.2*. Then, if* $\varepsilon$ *is small enough, there exists a constant* $C = C(\delta)$ *such that* $E^\delta_{\varepsilon,\alpha}(0) \leq C(\delta)$*. Moreover, for all* $t \in [0,T)$*,*

$$\|u^{\varepsilon,\alpha}\|_{L^\infty} \leq \frac{1}{4C_3\sqrt{\varepsilon}}. \tag{5.16}$$

We skip the proof of this lemma since it can be found in [9].

*Remark* 5.4. Notice that, under the assumptions (1.1) in Theorem 1.3, the conditions $(H)$ in Lemma 5.2 and (5.15) in Lemma 5.3 are fulfilled.

As a consequence of Lemmas 5.2 and 5.3, we obtain that $E^\delta_\varepsilon$ remains bounded on the whole existence interval $[0,T^{\max})$. Therefore $(HNS^{\varepsilon,\alpha})$ has a global solution.

### 5.3. Convergence

Let $u^{\varepsilon,\alpha}$ and $u^\varepsilon$ be the solutions to

$$\begin{aligned}(HNS^{\varepsilon,\alpha})\ \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} &= -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} + \frac{1}{\alpha}\nabla\mathrm{div}\,u^{\varepsilon,\alpha}\\ (HNS^\varepsilon)\ \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon &= -(u^\varepsilon.\nabla)u^\varepsilon + \nabla p^\varepsilon\ ,\ \mathrm{div}\,u^\varepsilon = 0\end{aligned}$$

with the same initial data $u_0^{\varepsilon,\alpha} = u_0^\varepsilon \in H^{1+\delta}(\mathbb{R}^2)^2$ and $u_1^{\varepsilon,\alpha} = u_1^\varepsilon \in H^\delta(\mathbb{R}^2)^2$. Let us now define the following modulated energy

$$E_{\varepsilon,\alpha,u^\varepsilon} = \frac{1}{2}\|u^{\varepsilon,\alpha} - u^\varepsilon + \varepsilon\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)\|^2_{L^2} + \frac{\varepsilon^2}{2}\|\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)\|^2_{L^2} + \varepsilon\|u^{\varepsilon,\alpha} - u^\varepsilon\|^2_{\dot{H}^1} + \frac{\varepsilon}{\alpha}\|\mathrm{div}\,u^{\varepsilon,\alpha}\|^2_{L^2}$$

which is inspired by the Dafermos modulated energy (see [3, 9]). Notice that

$$E_{\varepsilon,\alpha,u^\varepsilon}(u^{\varepsilon,\alpha}) = E_{\varepsilon,\alpha}(u^{\varepsilon,\alpha} - u^\varepsilon).$$

Through a Gronwall estimate on this energy, we shall prove that $u^{\varepsilon,\alpha}$ converges to $u^\varepsilon$ in the $L^\infty_T L^2(\mathbb{R}^2)$ norm, as $\alpha$ goes to 0. To this end, let us estimate the time derivative of the modulated energy $E_{\varepsilon,\alpha,u^\varepsilon}$ using the equations $(HNS^\varepsilon)$ and $(HNS^{\varepsilon,\alpha})$. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_{\varepsilon,\alpha,u^\varepsilon} &= \int_{\mathbb{R}^2}\Big((\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon)+\varepsilon\partial_{tt}(u^{\varepsilon,\alpha}-u^\varepsilon)).(u^{\varepsilon,\alpha}-u^\varepsilon+\varepsilon\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon))\\
&+\varepsilon^2\partial_{tt}(u^{\varepsilon,\alpha}-u^\varepsilon).\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon)+2\varepsilon\partial_t\nabla(u^{\varepsilon,\alpha}-u^\varepsilon).\nabla(u^{\varepsilon,\alpha}-u^\varepsilon)\\
&-\frac{2\varepsilon}{\alpha}\partial_t u^{\varepsilon,\alpha}.\nabla(\operatorname{div}u^{\varepsilon,\alpha})\Big)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^2}\Big((\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon)+\varepsilon\partial_{tt}(u^{\varepsilon,\alpha}-u^\varepsilon)).(u^{\varepsilon,\alpha}-u^\varepsilon+2\varepsilon\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon))\\
&-\varepsilon|\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon)|^2-2\varepsilon\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon).\Delta(u^{\varepsilon,\alpha}-u^\varepsilon)-\frac{2\varepsilon}{\alpha}\partial_t u^{\varepsilon,\alpha}.\nabla(\operatorname{div}u^{\varepsilon,\alpha})\Big)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^2}\Big(-|\nabla(u^{\varepsilon,\alpha}-u^\varepsilon)|^2+((u^\varepsilon.\nabla)u^\varepsilon-(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}).(u^{\varepsilon,\alpha}-u^\varepsilon)-\varepsilon|\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon)|^2\\
&+2\varepsilon\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon).\big((u^\varepsilon.\nabla)u^\varepsilon-(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\big)+\frac{1}{\alpha}\nabla(\operatorname{div}u^{\varepsilon,\alpha}).(u^{\varepsilon,\alpha}-u^\varepsilon)+\\
&+\frac{2\varepsilon}{\alpha}\nabla(\operatorname{div}u^{\varepsilon,\alpha}).\big(\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon)-\partial_t u^{\varepsilon,\alpha}\big)-\nabla p^\varepsilon.(u^{\varepsilon,\alpha}-u^\varepsilon)-2\varepsilon\nabla p^\varepsilon.\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon)\Big)\,\mathrm{d}x.
\end{aligned}$$

Now, using that $\operatorname{div}u^\varepsilon=0$ and the inequality

$$-\varepsilon|\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon)|^2+2\varepsilon\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon).\big((u^\varepsilon.\nabla)u^\varepsilon-(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\big)\le\varepsilon|(u^\varepsilon.\nabla)u^\varepsilon-(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}|^2$$

and integrating in time, we finally obtain the estimate

$$\begin{aligned}
E_{\varepsilon,\alpha,u^\varepsilon}(T) \le\ & E_{\varepsilon,\alpha,u^\varepsilon}(0)-\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{L^2_T\dot H^1}+\int_0^T\!\!\int_{\mathbb{R}^2}(u^{\varepsilon,\alpha}-u^\varepsilon).\big((u^\varepsilon.\nabla)u^\varepsilon-(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\big)\,\mathrm{d}x\,\mathrm{d}t+\\
&+\varepsilon\|(u^\varepsilon.\nabla)u^\varepsilon-(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{L^2_TL^2}-\int_0^T\!\!\int_{\mathbb{R}^2}\nabla p^\varepsilon.(u^{\varepsilon,\alpha}+2\varepsilon\partial_t u^{\varepsilon,\alpha})\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

In the following subsections, we shall estimate the terms in the right hand side.

**Notation:** We shall write $f=\mathcal{O}(1)$ if there exists a constant $C=C(\varepsilon,u^\varepsilon)$ such that $f\le C$. Similarly, $f=\mathcal{O}(\sqrt{\alpha})$ means that $f\le C(\varepsilon,u^\varepsilon)\sqrt{\alpha}$.

**5.3.1. Estimate on $\int_0^T\int_{\mathbb{R}^2}(u^{\varepsilon,\alpha}-u^\varepsilon).((u^\varepsilon.\nabla)u^\varepsilon-(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha})\,\mathbf{d}x\mathbf{d}t$.** From the estimates on the energies $E^0_{\varepsilon,\alpha}$ and $E^0_\varepsilon$, we know that

$$u^{\varepsilon,\alpha},u^\varepsilon\in L^\infty_TL^2(\mathbb{R}^2)\cap L^2_T\dot H^1(\mathbb{R}^2)$$

and the boundedness of $E^0_{\varepsilon,\alpha}$ yields

$$\|\operatorname{div}u^{\varepsilon,\alpha}\|_{L^2_TL^2}=\mathcal{O}(\sqrt{\alpha}).$$

Besides, let us recall Ladyzhenskaya's inequality:

$$\|f\|^2_{L^4}\le c\|f\|_{L^2}\|f\|_{\dot H^1}\tag{5.17}$$

which holds for all $f\in H^1(\mathbb{R}^2)$. Hence we can estimate the integral

$$I=\int_{\mathbb{R}^2}(u^{\varepsilon,\alpha}-u^\varepsilon).((u^\varepsilon.\nabla)u^\varepsilon-(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha})\,\mathrm{d}x$$

as follows:

$$
\begin{aligned}
I &= \int_{\mathbb{R}^2} \left[ (u^{\varepsilon,\alpha} - u^{\varepsilon}).\left( (u^{\varepsilon} - u^{\varepsilon,\alpha}).\nabla \right) u^{\varepsilon} + (u^{\varepsilon,\alpha} - u^{\varepsilon}).(u^{\varepsilon,\alpha}.\nabla)(u^{\varepsilon} - u^{\varepsilon,\alpha}) \right] \mathrm{d}x \\
&= \int_{\mathbb{R}^2} (u^{\varepsilon,\alpha} - u^{\varepsilon}).\left( (u^{\varepsilon} - u^{\varepsilon,\alpha}).\nabla \right) u^{\varepsilon} \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^2} \operatorname{div} u^{\varepsilon,\alpha} \; |u^{\varepsilon,\alpha} - u^{\varepsilon}|^2 \, \mathrm{d}x \\
&\leq \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{L^4}^2 \|\nabla u^{\varepsilon}\|_{L^2} + \frac{1}{2} \|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2} \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{L^4}^2 \\
&\overset{(5.17)}{\leq} C \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{L^2} \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{\dot{H}^1} \left( \|u^{\varepsilon}\|_{\dot{H}^1} + \|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2} \right) \\
&\leq \frac{C^2}{\eta} \left( \|u^{\varepsilon}\|_{\dot{H}^1} + \|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2} \right)^2 \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{L^2}^2 + \eta \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{\dot{H}^1}^2 \\
&\leq \frac{2C^2}{\eta} \left( \|u^{\varepsilon}\|_{\dot{H}^1}^2 + \|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2}^2 \right) \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{L^2}^2 + \eta \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{\dot{H}^1}^2,
\end{aligned}
$$

where $\eta > 0$ is a small number to be chosen in the conclusion.

**5.3.2. Estimate on** $\varepsilon \|(u^{\varepsilon}.\nabla)u^{\varepsilon} - (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{L_T^2 L^2}^2 - \frac{1}{2}\|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{L_T^2 \dot{H}^1}^2$**.** Let us set

$$
A^{\varepsilon,\alpha}(t) = \varepsilon \|(u^{\varepsilon}.\nabla)u^{\varepsilon} - (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{L^2}^2 - \frac{1}{2}\|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{\dot{H}^1}^2
$$

and write

$$
(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} - (u^{\varepsilon}.\nabla)u^{\varepsilon} = \big(u^{\varepsilon,\alpha}.\nabla\big)(u^{\varepsilon,\alpha} - u^{\varepsilon}) + \big((u^{\varepsilon,\alpha} - u^{\varepsilon}).\nabla\big)u^{\varepsilon}
$$

then, using Lemma 4.1, estimate

$$
\|(u^{\varepsilon}.\nabla)u^{\varepsilon} - (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{L^2} \leq \|u^{\varepsilon,\alpha}\|_{L^\infty} \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{\dot{H}^1} + \|\big((u^{\varepsilon,\alpha} - u^{\varepsilon}).\nabla\big)u^{\varepsilon}\|_{L^2}.
$$

The second term on the right hand side estimates using the Sobolev embeddings

$$
\dot{H}^s \subset L^{\frac{2}{1-s}}(\mathbb{R}^2)
$$

as follows. Notice that $\nabla u^{\varepsilon} \in L_T^\infty \dot{H}^\delta \subset L_T^\infty L^{\frac{2}{1-\delta}}$ and

$$
u^{\varepsilon,\alpha} - u^{\varepsilon} \in L_T^2 L^2 \cap L_T^2 \dot{H}^1 \subset L_T^2 \dot{H}^{1-\delta} \subset L_T^2 L^{\frac{2}{\delta}}.
$$

So we have

$$
\begin{aligned}
\|\big((u^{\varepsilon,\alpha} - u^{\varepsilon}).\nabla\big)u^{\varepsilon}\|_{L^2} &\leq \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{L^{\frac{2}{\delta}}} \|\nabla u^{\varepsilon}\|_{L^{\frac{2}{1-\delta}}} \\
&\leq C \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{L^2}^{\delta} \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{\dot{H}^1}^{1-\delta} \|\nabla u^{\varepsilon}\|_{\dot{H}^\delta}.
\end{aligned}
$$

Now, integrating in time the squared norm and performing a Young inequality, we obtain

$$
\|\big(u^{\varepsilon,\alpha} - u^{\varepsilon}).\nabla\big)u^{\varepsilon}\|_{L_T^2 L^2}^2 \leq \frac{C_\delta}{\eta} \int_0^T \|u^{\varepsilon}\|_{\dot{H}^{1+\delta}}^2 E_{\varepsilon,\alpha,u^{\varepsilon}}(t) \, \mathrm{d}t + \tilde{C}_\delta \eta \|u^{\varepsilon}\|_{L_T^\infty \dot{H}^{1+\delta}}^2 \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{L_T^2 \dot{H}^1}^2.
$$

Finally, recalling that $\varepsilon\left(\|u^\varepsilon\|^2_{L^\infty_T\dot H^{1+\delta}}+\|u^\varepsilon\|^2_{L^2_T\dot H^{1+\delta}}\right)=\mathcal{O}(1)$, we obtain the inequality

$$\begin{aligned}\int_0^T A^{\varepsilon,\alpha}\mathrm{d}t &\le \left(C\varepsilon\|u^{\varepsilon,\alpha}\|^2_{L^\infty_T L^\infty}+\tilde C_\delta\varepsilon\eta\|u^\varepsilon\|^2_{L^\infty_T\dot H^{1+\delta}}-\frac{1}{2}\right)\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{L^2_T\dot H^1}+\\ &\quad+\frac{C_\delta}{\eta}\int_0^T\varepsilon\|u^\varepsilon\|^2_{\dot H^{1+\delta}}E_{\varepsilon,\alpha,u^\varepsilon}(t)\mathrm{d}t\\ &\le \frac{C_\delta}{\eta}\int_0^T\varepsilon\|u^\varepsilon\|^2_{\dot H^{1+\delta}}E_{\varepsilon,\alpha,u^\varepsilon}(t)\mathrm{d}t\end{aligned}$$

if $\varepsilon$ and $\eta$ are small enough.

**5.3.3. Estimate on $\int_0^T\int_{\mathbb{R}^2}\nabla p^\varepsilon.u^{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}t$.** First, notice that, applying the div operator to equation $(HNS^\varepsilon)$, we obtain the identity

$$\Delta p^\varepsilon=\mathrm{div}\,(u^\varepsilon.\nabla)u^\varepsilon=\mathrm{div}\,\nabla:u^\varepsilon\otimes u^\varepsilon \tag{5.18}$$

from which we deduce that $p^\varepsilon$ has the same regularity as $u^\varepsilon\otimes u^\varepsilon$ and

$$\|p^\varepsilon\|_{L^2_TL^2}\le C\|u^\varepsilon\otimes u^\varepsilon\|_{L^2_TL^2}.$$

Since $u^\varepsilon\in L^2_T\dot H^1\cap L^\infty_TL^2\subset L^4_TL^4(\mathbb{R}^2)^2$, we immediately conclude that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^2}\nabla p^\varepsilon.u^{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}t &= -\int_0^T\int_{\mathbb{R}^2}p^\varepsilon.\mathrm{div}\,u^{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}t \le \|p^\varepsilon\|_{L^2_TL^2}\|\mathrm{div}\,u^{\varepsilon,\alpha}\|_{L^2_TL^2}\\ &\le C\sqrt{\alpha}\|p^\varepsilon\|_{L^2_TL^2}\le C\sqrt{\alpha}\|u^\varepsilon\otimes u^\varepsilon\|_{L^2_TL^2}\\ &\le C\sqrt{\alpha}\|u^\varepsilon\|^2_{L^4_TL^4}=\mathcal{O}(\sqrt{\alpha}).\end{aligned}$$

**5.3.4. Estimate on $2\varepsilon\int_0^T\int_{\mathbb{R}^2}\nabla p^\varepsilon.\partial_t u^{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}t$.** From identity (5.18), we know that

$$\partial_t p^\varepsilon=\sum_{i,j=1}^2\partial_t\frac{\partial_i\partial_j}{\Delta}(u_i^\varepsilon u_j^\varepsilon).$$

Besides, due to the control of the energies $E^0_\varepsilon$ and $E^\delta_\varepsilon$, we have

$$\begin{aligned}\partial_t u^\varepsilon &\in L^2_TL^2\\ u^\varepsilon &\in L^\infty_T H^{1+\delta}\subset L^\infty_T L^\infty.\end{aligned}$$

After two integrations by parts (one in each variable), we obtain an integral which easily estimates:

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^2}\nabla p^\varepsilon.\partial_t u^{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^2}\partial_t p^\varepsilon.\mathrm{div}\,u^{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}t\\ &\le \|\partial_t p^\varepsilon\|_{L^2_TL^2}\|\mathrm{div}\,u^{\varepsilon,\alpha}\|_{L^2_TL^2}\\ &\le C\sqrt{\alpha}\|\partial_t u^\varepsilon\|_{L^2_TL^2}\|u^\varepsilon\|_{L^\infty_TL^\infty}=\mathcal{O}(\sqrt{\alpha}).\end{aligned}$$

**5.3.5. Conclusion.** First, notice that since we take the same initial data for $(HNS^{\varepsilon,\alpha})$ and $(HNS^{\varepsilon})$, we have in particular $\operatorname{div} u_0^{\varepsilon,\alpha} = 0$ so

$$E_{\varepsilon,\alpha,u^\varepsilon}(0) = 0.$$

Now, gathering the estimates in the previous subsections, we obtain that

$$\begin{aligned} E_{\varepsilon,\alpha,u^\varepsilon}(T) \quad &\leq \quad \mathcal{O}(\sqrt{\alpha}) + C_{\delta,\eta} \int_0^T \left(\|u^\varepsilon\|_{\dot H^1}^2 + \varepsilon \|u^\varepsilon\|_{\dot H^{1+\delta}}^2 + \mathcal{O}(\alpha)\right) E_{\varepsilon,\alpha,u^\varepsilon}(t)\,\mathrm{d}t \\ &\quad + \left(\eta + (1-\delta)\varepsilon\eta \|u^\varepsilon\|_{L^\infty_T \dot H^{1+\delta}}^2 - \frac{1}{2}\right) \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{L^2_T \dot H^1}^2 . \end{aligned}$$

So, choosing $\eta$ small enough, Gronwall's lemma yields, for all positive $T$,

$$E_{\varepsilon,\alpha,u^\varepsilon}(T) = \mathcal{O}(\sqrt{\alpha}).$$

Now, recall that Theorem 1.1 tells that $u^\varepsilon$ converges towards the solution $v$ to $(NS)$. Theorem 1.3 is now proved.

## 6. The 3D case

### 6.1. Preliminary estimates

As in subsection 5.1, we start by recalling the regularity results we have on $u^\varepsilon$. All the proofs are in [9].

**Lemma 6.1.** *Let $u^\varepsilon \in L^\infty_T(\dot H^{\frac32+\delta} \cap \dot H^{\frac12+\delta})(\mathbb{R}^3)^3$ be the global solution to $(HNS^\varepsilon)$ with initial data $(u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac32+\delta} \times H^{\frac12+\delta}(\mathbb{R}^3)^3$. Then we have*

$$u^\varepsilon \in L^\infty_T \dot H^{\frac32+\delta} \cap L^2_T \dot H^{\frac32+\delta} \cap L^2_T \dot H^{\frac32} \cap L^\infty_T \dot H^{\frac12} \cap L^\infty_T \dot H^{\frac32}$$

*and*

$$\partial_t u^\varepsilon \in L^2_T \dot H^{\frac12} \cap L^\infty_T \dot H^{\frac12+\delta}.$$

*Remark* 6.2. In this paper, we are not interested in the dependence of the norms on $\varepsilon$.

*Proof.* Here is a sketch of the proof. First, let us define the 3D energy

$$E_\varepsilon^{\frac12+\delta}(t) = \frac{1}{2}\|u^\varepsilon + \varepsilon\partial_t u^\varepsilon\|_{\dot H^{\frac12+\delta}}^2 + \frac{\varepsilon^2}{2}\|\partial_t u^\varepsilon\|_{\dot H^{\frac12+\delta}}^2 + \varepsilon\|\nabla u^\varepsilon\|_{\dot H^{\frac12+\delta}}^2$$

for non-negative $\delta$. In order to obtain that the solutions $u^\varepsilon$ to $(HNS^\varepsilon)$ are global, we proved in [9] that

$$\exists C > 0 \ : \ \forall t \geq 0 \ , \ E_\varepsilon^{\frac12+\delta}(t) \leq C\varepsilon^{-\delta}. \tag{6.1}$$

Directly from the expression of $E_\varepsilon^\delta$ and from (6.1), we deduce that

$$\partial_t u^\varepsilon \in L^\infty_T \dot H^{\frac12+\delta} \ , \ u^\varepsilon \in L^\infty_T \dot H^{\frac32+\delta} \cap L^2_T \dot H^{\frac32+\delta}.$$

Besides, let us consider the time derivative of $E_\varepsilon^{\frac12}$. According to [9], we have

$$E_\varepsilon^{\frac12}(T) + \varepsilon\|\partial_t u^\varepsilon + (u^\varepsilon.\nabla)u^\varepsilon\|_{L^2_T \dot H^{\frac12}}^2 + \|u^\varepsilon\|_{L^2_T \dot H^{\frac32}}^2 - \varepsilon\|(u^\varepsilon.\nabla)u^\varepsilon\|_{L^2_T \dot H^{\frac12}}^2 = E_\varepsilon^{\frac12}(0) + I(\varepsilon, T),$$

where

$$I(\varepsilon,T) \quad := \quad \int_0^T\int_{\mathbb{R}^3} \Lambda^{\frac12}\left((u^\varepsilon.\nabla)u^\varepsilon\right).\Lambda^{\frac12}u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \quad \|u^\varepsilon\|_{L^\infty_T\dot H^{\frac12}}\|u^\varepsilon\|^2_{L^2_T\dot H^{\frac32}}.$$

The $\Lambda$ above is the Fourier multiplier defined by $\widehat{\Lambda f}(\xi)=|\xi|\hat f(\xi)$.
So, since $E_\varepsilon^{\frac12}(0)\leq C$ and $\|u^\varepsilon\|_{L^\infty_T\dot H^{\frac12}}\leq 2\|u_0^\varepsilon\|_{\dot H^{\frac12}}\leq\frac18$ by the decay of $E_\varepsilon^{\frac12}$, we obtain the inequality

$$E_\varepsilon^{\frac12}(T)+\varepsilon\|\partial_t u^\varepsilon+(u^\varepsilon.\nabla)u^\varepsilon\|^2_{L^2_T\dot H^{\frac12}}+\frac12\|u^\varepsilon\|^2_{L^2_T\dot H^{\frac32}}\leq C$$

if $\varepsilon$ is small enough. Now, we complete the reasoning as in the 2D case and deduce that

$$\partial_t u^\varepsilon\in L^2_T\dot H^{\frac12}\ ,\ u^\varepsilon\in L^2_T\dot H^{\frac32}.$$

Finally, since $E_\varepsilon^{\frac12}$ is bounded, we immediately have

$$u^\varepsilon\in L^\infty_T\dot H^{\frac12}\cap L^\infty_T\dot H^{\frac32}$$

**6.2. Globalization**

As in the 2D case, we shall prove that the energy

$$E^{\frac12+\delta}_{\varepsilon,\alpha}(t)=\frac12\|u^{\varepsilon,\alpha}+\varepsilon\partial_t u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12+\delta}}+\frac{\varepsilon^2}{2}\|\partial_t u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12+\delta}}+\varepsilon\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32+\delta}}+\frac{\varepsilon}{\alpha}\|\mathrm{div}\,u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12+\delta}}$$

is bounded in order to show that the local solution obtained by the contraction argument is global. First, let us define $0\leq T\leq T^{\max}$ by

$$T=\sup\left\{0\leq\tau\leq T^{\max}\ :\ \forall\ t\in[0,\tau),\ \|u^{\varepsilon,\alpha}(t)\|_{L^\infty}<\frac{1}{2K_1\sqrt{\varepsilon}}\right\},\qquad(6.2)$$

where $K_1=\max(K,\tilde K)$ such that

$$\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{\dot H^{\frac12}}\quad\leq\quad K\|u^{\varepsilon,\alpha}\|_{L^\infty}\|u^{\varepsilon,\alpha}\|_{\dot H^{\frac32}},\qquad(6.3)$$
$$\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{\dot H^{\frac12+\delta}}\quad\leq\quad \tilde K\|u^{\varepsilon,\alpha}\|_{L^\infty}\|u^{\varepsilon,\alpha}\|_{\dot H^{\frac32+\delta}}.\qquad(6.4)$$

We shall start by proving that the energy $E^{\frac12}_{\varepsilon,\alpha}$ decreases on $[0,T)$. To this end, we estimate its time derivative:

$$\frac{\mathrm{d}E^{\frac12}_{\varepsilon,\alpha}}{\mathrm{d}t}\quad\leq\quad\varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12}}-\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32}}-\int_{\mathbb{R}^3}\Lambda^{\frac12}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac12}u^{\varepsilon,\alpha}\,\mathrm{d}x$$
$$\leq\quad\left(K_1^2\varepsilon\|u^{\varepsilon,\alpha}\|^2_{L^\infty}-1\right)\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32}}-\int_{\mathbb{R}^3}\Lambda^{\frac12}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac12}u^{\varepsilon,\alpha}\,\mathrm{d}x.$$

So, for $\varepsilon$ small enough, we have

$$\frac{\mathrm{d}E^{\frac12}_{\varepsilon,\alpha}}{\mathrm{d}t}\leq-\frac34\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32}}+\int_{\mathbb{R}^3}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda u^{\varepsilon,\alpha}\,\mathrm{d}x$$

on $[0,T)$. Then, recalling that $\operatorname{div} u^{\varepsilon,\alpha} \neq 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda u^{\varepsilon,\alpha}\,\mathrm{d}x &= \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i^{\varepsilon,\alpha}\,\partial_i u_j^{\varepsilon,\alpha}\,\partial_j u_i^{\varepsilon,\alpha}\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3}\left(\sum_{i,j=1}^{3}\partial_j u_i^{\varepsilon,\alpha}\,\partial_i u_j^{\varepsilon,\alpha}\,u_i^{\varepsilon,\alpha} + \sum_{i=1}^{3}(u_i^{\varepsilon,\alpha})^2\,\partial_i \operatorname{div} u^{\varepsilon,\alpha}\right)\mathrm{d}x \\
&= \sum_{i=1}^{3}\int_{\mathbb{R}^3} u_i^{\varepsilon,\alpha}\,\partial_i u_i^{\varepsilon,\alpha}\,\operatorname{div} u^{\varepsilon,\alpha}\mathrm{d}x \\
&\leq \sum_{i=1}^{3}\|u_i^{\varepsilon,\alpha}\|_{L^3}\,\|\partial_i u_i^{\varepsilon,\alpha}\|_{L^3}\,\|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^3} \\
&\leq 3K_2^3\|u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}}\,\|u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{3}{2}}}\,\|\operatorname{div} u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}} \\
&\leq 9K_2^3\|u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}}\,\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}}},
\end{aligned}$$

where $K_2$ is the constant such that

$$\|f\|_{L^3(\mathbb{R}^3)} \leq K_2\|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}.$$

Now, assume that

$$\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{1}{36K_2^3}. \tag{6.5}$$

Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{\frac{1}{2}}_{\varepsilon,\alpha}(0) \leq \left(-\frac{3}{4} + 9K_2^3\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}}\right)\|u_0^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}}} < -\frac{1}{4}\|u_0^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}}}$$

and $E^{\frac{1}{2}}_{\varepsilon,\alpha}$ therefore decreases on an interval $[0,\tau]$. Set

$$\tau = \sup\left\{t < T \;:\; E^{\frac{1}{2}}_{\varepsilon,\alpha} \text{ decreases on } [0,t]\right\}.$$

Since $E^{\frac{1}{2}}_{\varepsilon,\alpha}$ decreases on $[0,\tau]$, we have

$$\|u^{\varepsilon,\alpha}(\tau)\|^2_{\dot{H}^{\frac{1}{2}}} \leq 2E^{\frac{1}{2}}_{\varepsilon,\alpha}(\tau) \leq E^{\frac{1}{2}}_{\varepsilon,\alpha}(0) \leq 4\|u_0^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{1}{2}}} \leq \left(\frac{1}{18K_2^3}\right)^2 \tag{6.6}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{\frac{1}{2}}_{\varepsilon,\alpha}(\tau) \leq -\frac{1}{4}\|u^{\varepsilon,\alpha}(\tau)\|^2_{\dot{H}^{\frac{3}{2}}}.$$

Thus $E^{\frac{1}{2}}_{\varepsilon,\alpha}$ decreases on $[0,\tau']$, where $\tau' > \tau$. We have proved by contradiction that the energy decays on the whole interval $[0,T)$.

Now, we can prove the following lemma:

**Lemma 6.3.** *Assume that* $\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}} < \dfrac{1}{36K_2^3}$ *and*

$$(H')\ i)\ \varepsilon^{\frac{1+\delta}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{3}{2}+\delta}} = o(1)\ ,\ ii)\ \varepsilon^{\frac{\delta}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}+\delta}} = o(1). \tag{6.7}$$

*when $\varepsilon$ goes to zero.*
*Now, recall that $0 \leq T \leq T^{max}$ is defined by*

$$T = \sup\left\{0 \leq \tau \leq T^{max} \ : \ \forall\ t \in [0,\tau),\ \|u^{\varepsilon,\alpha}(t)\|_{L^\infty} < \frac{1}{2K_1\sqrt{\varepsilon}}\right\}.$$

*Then $T > 0$ and there exists a constant $\tilde{A} = \tilde{A}(\delta)$ such that*

$$E_\varepsilon^{\frac{1}{2}+\delta}(t) \leq \tilde{A}\, E_\varepsilon^{\frac{1}{2}+\delta}(0) \tag{6.8}$$

*for all $t \in [0,T)$ and $\varepsilon$ small enough.*

*Proof.* First, let us compute the time derivative of this energy:

$$\begin{aligned}
\frac{d}{dt}E_{\varepsilon,\alpha}^{\frac{1}{2}+\delta}(t) &= \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{1}{2}+\delta}} - \|u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}+\delta}} - \int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}+\delta}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac{1}{2}+\delta}u^{\varepsilon,\alpha}\,\mathrm{d}x - \\
&\quad -\varepsilon\|\partial_t u^{\varepsilon,\alpha} + (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{1}{2}+\delta}} - \frac{1}{\alpha}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{1}{2}+\delta}} \\
&\leq \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{1}{2}+\delta}} - \|u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}+\delta}} - \int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}+\delta}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac{1}{2}+\delta}u^{\varepsilon,\alpha}\,\mathrm{d}x \\
&\leq \left(K_1^2\varepsilon\|u^{\varepsilon,\alpha}\|^2_{L^\infty} - 1\right)\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}+\delta}} - \int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}+\delta}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac{1}{2}+\delta}u^{\varepsilon,\alpha}\,\mathrm{d}x.
\end{aligned}$$

On the time interval $[0,T)$, we have $K_1^2\varepsilon\|u^{\varepsilon,\alpha}\|^2_{L^\infty} - 1 \leq -\frac{1}{2}$ and we can estimate the integral on the right hand side by

$$\begin{aligned}
\int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}+\delta}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac{1}{2}+\delta}u^{\varepsilon,\alpha}\,\mathrm{d}x &\leq \|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}+\delta}}\|u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}+\delta}} \\
&\leq K_1\|u^{\varepsilon,\alpha}\|_{L^\infty}\|u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{3}{2}+\delta}}\|u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}+\delta}}
\end{aligned}$$

due to Lemma 4.1. Using that followed by (5.6), we obtain

$$\begin{aligned}
\frac{d}{dt}E_{\varepsilon,\alpha}^{\frac{1}{2}+\delta}(t) &\leq -\frac{1}{2}\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}+\delta}} + K_1\|u^{\varepsilon,\alpha}\|_{L^\infty}\|u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{3}{2}+\delta}}\|u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}+\delta}} \\
&\leq -\frac{1}{2}\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}+\delta}} + C\|u^{\varepsilon,\alpha}\|^{1+\delta}_{\dot{H}^{\frac{1}{2}+\delta}}\|u^{\varepsilon,\alpha}\|^{2-\delta}_{\dot{H}^{\frac{3}{2}+\delta}} \\
&\leq -\frac{1}{2}\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}+\delta}} + C\|u^{\varepsilon,\alpha}\|^{1-\delta}_{\dot{H}^{\frac{1}{2}}}\|u^{\varepsilon,\alpha}\|^{\delta}_{\dot{H}^{\frac{3}{2}}}\|u^{\varepsilon,\alpha}\|^{\delta}_{\dot{H}^{\frac{1}{2}+\delta}}\|u^{\varepsilon,\alpha}\|^{2-\delta}_{\dot{H}^{\frac{3}{2}+\delta}} \\
&\leq C_\delta\|u^{\varepsilon,\alpha}\|^{2\frac{1-\delta}{\delta}}_{\dot{H}^{\frac{1}{2}}}\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}}}E_{\varepsilon,\alpha}^{\frac{1}{2}+\delta}(t),
\end{aligned}$$

where we have used standard interpolations and a Young inequality. Now, since we do not know if the energy $E_{\varepsilon,\alpha}^{\frac{1}{2}+\delta}$ decays, we will use that $E_{\varepsilon,\alpha}^{\frac{1}{2}}$ satisfies the inequality

$$\frac{d}{dt}E_{\varepsilon,\alpha}^{\frac{1}{2}}(t) \leq -\frac{1}{4}\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}}}$$

on the interval $[0,T)$. So we have $u^{\varepsilon,\alpha} \in L^2_T\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)^3$ uniformly in $\varepsilon,\alpha$. Besides, we already know that $u^{\varepsilon,\alpha} \in L^\infty_T\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$ uniformly in $\varepsilon,\alpha$ by (6.6). Therefore Gronwall's Lemma applies and we obtain the following estimate:

$$E_{\varepsilon,\alpha}^{\frac{1}{2}+\delta}(t) \leq \exp(C_\delta T)\, E_{\varepsilon,\alpha}^{\frac{1}{2}+\delta}(0),$$

which finishes the proof of the lemma. □

As in the 2D case, the smallness assumptions on the initial data yield the boundedness of $E^{\frac12+\delta}_{\varepsilon,\alpha}$ for $\alpha$ and $\varepsilon$ small enough. From this, we deduce that the solutions to $(HNS^{\varepsilon,\alpha})$ are global.

### 6.3. Convergence

Let $u^{\varepsilon,\alpha}$ and $u^\varepsilon$ be the solutions to $(HNS^{\varepsilon,\alpha})$ and $(HNS^\varepsilon)$ respectively with the same initial data

$$(u_0^{\varepsilon,\alpha},u_1^{\varepsilon,\alpha})=(u_0^\varepsilon,u_1^\varepsilon)\in H^{\frac32+\delta}(\mathbb{R}^3)^3\times H^{\frac12+\delta}(\mathbb{R}^3)^3.$$

In order to prove that $u^{\varepsilon,\alpha}$ converges to $u^\varepsilon$ in the $L^\infty_T\dot H^{\frac12}$ norm, as $\alpha$ goes to 0, we shall use, as in the 2D case, a variant of the Dafermos modulated energy:

$$\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(t):=\frac12\|u^{\varepsilon,\alpha}-u^\varepsilon+\varepsilon\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon)\|^2_{\dot H^{\frac12}}+\frac{\varepsilon^2}{2}\|\partial_t(u^{\varepsilon,\alpha}-u^\varepsilon)\|^2_{\dot H^{\frac12}}+\varepsilon\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{\dot H^{\frac32}}+\frac{\varepsilon}{\alpha}\|\operatorname{div}u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12}}.$$

As done in section 5.3, we shall compute the time derivative of $\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}$ and use equations $(HNS^{\varepsilon,\alpha})$ and $(HNS^\varepsilon)$. Doing so, we obtain the following estimate:

$$\begin{aligned}\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(T)\ \le\ &\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(0)+\varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}-(u^\varepsilon.\nabla)u^\varepsilon\|^2_{L^2_T\dot H^{\frac12}}-\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{L^2_T\dot H^{\frac32}}-\\&-\int_0^T\int_{\mathbb{R}^3}\Lambda^{\frac12}p^\varepsilon.\Lambda^{\frac12}\operatorname{div}u^{\varepsilon,\alpha}\,dxdt+2\varepsilon\int_0^T\int_{\mathbb{R}^3}\Lambda^{\frac12}\partial_tp^\varepsilon.\Lambda^{\frac12}\operatorname{div}u^{\varepsilon,\alpha}\,dxdt+\\&+\int_0^T\int_{\mathbb{R}^3}\left[(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}-(u^\varepsilon.\nabla)u^\varepsilon\right].\Lambda(u^{\varepsilon,\alpha}-u^\varepsilon)\,dx\,dt-\frac1\alpha\|\operatorname{div}u^{\varepsilon,\alpha}\|^2_{L^2_T\dot H^{\frac12}}.\end{aligned}$$

#### 6.3.1. Estimate on $\varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}-(u^\varepsilon.\nabla)u^\varepsilon\|^2_{L^2_T\dot H^{\frac12}}$.

First, $A:=(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}-(u^\varepsilon.\nabla)u^\varepsilon$ writes

$$(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}-(u^\varepsilon.\nabla)u^\varepsilon=\big((u^{\varepsilon,\alpha}-u^\varepsilon).\nabla\big)u^\varepsilon+\big(u^{\varepsilon,\alpha}.\nabla\big)(u^{\varepsilon,\alpha}-u^\varepsilon).$$

Let us estimate the RHS using the dyadic Littlewood-Paley decomposition. First, by Lemma A.2 in the appendix, we have

$$\begin{aligned}\|(u^{\varepsilon,\alpha}-u^\varepsilon).\nabla u^\varepsilon\|^2_{\dot H^{\frac12}}\ &\le\ \sum_{i,j=1}^3\sum_{p\in\mathbb{Z}}2^p\|\Delta_p(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\,.\,S_{p+1}(\partial_iu_j^\varepsilon)\|^2_{L^2}\\&+\ \sum_{i,j=1}^3\sum_{q\in\mathbb{Z}}2^q\|\Delta_q(\partial_iu_j^\varepsilon)\,.\,S_q(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\|^2_{L^2}.\end{aligned}$$

The first term estimates easily:

$$\begin{aligned}\sum_p2^p\|\Delta_p(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\,.\,S_{p+1}(\partial_iu_j^\varepsilon)\|^2_{L^2}\ &\le\ \sum_p2^p\|\Delta_p(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\|^2_{L^2}\,\|S_{p+1}(\partial_iu_j^\varepsilon)\|^2_{L^\infty}\\&\le\ \|u^\varepsilon\|^2_{L^\infty}\sum_p2^{3p}\,\|\Delta_p(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\|^2_{L^2}\\&\le\ \|u^\varepsilon\|^2_{L^\infty}\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{\dot H^{\frac32}}\end{aligned}$$

due to property (A.1) in the appendix. Now, notice that Sobolev embeddings imply that $u^\varepsilon \in \dot{H}^{\frac{3}{2}+\delta} \subset W^{\frac{3}{2},\alpha}(\mathbb{R}^3)$, where $\alpha = \frac{6}{3-2\delta}$. Then, if $\bar{\alpha} = \frac{3}{\delta}$, we have

$$\begin{aligned}
\sum_q 2^q \|\Delta_q(\partial_i u_j^\varepsilon)\,.\,S_q(u_i^{\varepsilon,\alpha} - u_i^\varepsilon)\|_{L^2}^2 &\leq \sum_q 2^q \|\Delta_q(\partial_i u_j^\varepsilon)\|_{L^\alpha}^2 \, \|S_q(u_i^{\varepsilon,\alpha} - u_i^\varepsilon)\|_{L^{\bar{\alpha}}}^2 \\
&\leq \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{L^{\bar{\alpha}}}^2 \sum_q 2^{q+3q\left(1-\frac{2}{\alpha}\right)} \|\Delta_q(\partial_i u_j^\varepsilon)\|_{L^2}^2 \\
&\leq C\|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot{H}^{\frac{3}{2}-\delta}}^2 \sum_q 2^{3q+3q\left(1-\frac{2}{\alpha}\right)} \|\Delta_q u_j^\varepsilon\|_{L^2}^2 \\
&= C\|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot{H}^{\frac{3}{2}-\delta}}^2 \sum_q 2^{6q-\frac{6q}{\alpha}} \|\Delta_q u_j^\varepsilon\|_{L^2}^2 \\
&= C\|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot{H}^{\frac{3}{2}-\delta}}^2 \sum_q 2^{2q\left(\frac{3}{2}+\delta\right)} \|\Delta_q u_j^\varepsilon\|_{L^2}^2 \\
&\leq C\|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^\delta \, \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot{H}^{\frac{3}{2}}}^{1-\delta} \|u^\varepsilon\|_{\dot{H}^{\frac{3}{2}+\delta}} \\
&\leq C_\delta \left( \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} + \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot{H}^{\frac{3}{2}}} \right) \|u^\varepsilon\|_{\dot{H}^{\frac{3}{2}+\delta}}.
\end{aligned}$$

Now, let us estimate

$$\begin{aligned}
\|u^{\varepsilon,\alpha}.\nabla(u^{\varepsilon,\alpha} - u^\varepsilon)\|_{\dot{H}^{\frac{1}{2}}} \leq& \sum_{i,j=1}^{3} \sum_{p\in\mathbb{Z}} 2^p \|\Delta_p u_i^{\varepsilon,\alpha}\,.\,S_{p+1}\partial_i(u_j^{\varepsilon,\alpha} - u_j^\varepsilon)\|_{L^2}^2 \\
&+ \sum_{i,j=1}^{3} \sum_{q\in\mathbb{Z}} 2^q \|\Delta_q \partial_i(u_j^{\varepsilon,\alpha} - u_j^\varepsilon)\,.\,S_q u_i^{\varepsilon,\alpha}\|_{L^2}^2.
\end{aligned}$$

We know that $u^{\varepsilon,\alpha} \in \dot{H}^{\frac{3}{2}+\delta} \subset \dot{W}^{\frac{3}{2},\alpha}$, where $\alpha = \frac{6}{3-2\delta}$. Let $\bar{\alpha} = \frac{3}{\delta}$. Then

$$\begin{aligned}
\sum_p 2^p \|\Delta_p u_i^{\varepsilon,\alpha}\,.\,S_{p+1}\partial_i(u_j^{\varepsilon,\alpha} - u_j^\varepsilon)\|_{L^2}^2 &\leq \sum_p 2^p \|\Delta_p u_i^{\varepsilon,\alpha}\|_{L^\alpha}^2 \|S_{p+1}\partial_i(u_j^{\varepsilon,\alpha} - u_j^\varepsilon)\|_{L^{\bar{\alpha}}}^2 \\
&\leq \sum_p 2^{3p} \|\Delta_p u_i^{\varepsilon,\alpha}\|_{L^\alpha}^2 \|S_{p+1}(u_j^{\varepsilon,\alpha} - u_j^\varepsilon)\|_{L^{\bar{\alpha}}}^2 \\
&\leq \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{L^{\bar{\alpha}}}^2 \sum_p 2^{3p} \|\Delta_p u_i^{\varepsilon,\alpha}\|_{L^\alpha}^2 \\
&\leq \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{L^{\bar{\alpha}}}^2 \sum_p 2^{3p+6p\left(\frac{1}{2}-\frac{1}{\alpha}\right)} \|\Delta_p u_i^{\varepsilon,\alpha}\|_{L^2}^2 \\
&= \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{L^{\bar{\alpha}}}^2 \sum_p 2^{2p\left(\frac{3}{2}+\delta\right)} \|\Delta_p u_i^{\varepsilon,\alpha}\|_{L^2}^2 \\
&\leq C\|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot{H}^{\frac{3}{2}-\delta}}^2 \|u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{3}{2}+\delta}}^2
\end{aligned}$$

since $\dot{H}^{\frac{3}{2}-\delta} \subset L^{\bar{\alpha}}(\mathbb{R}^3)$. So, by interpolation and a Young inequality, we get

$$\sum_p 2^p \|\Delta_p u_i^{\varepsilon,\alpha}.S_{p+1}\partial_i(u_j^{\varepsilon,\alpha} - u_j^\varepsilon)\|_{L^2}^2 \leq C_\delta \|u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{3}{2}+\delta}} \left( \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} + \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot{H}^{\frac{3}{2}}} \right).$$

Immediately, the remaining term estimates

$$\sum_q 2^q \|\Delta_q \partial_i (u_j^{\varepsilon,\alpha} - u_j^\varepsilon)\,.\,S_q u_i^{\varepsilon,\alpha}\|_{L^2}^2 \leq C \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot H^{\frac32}}^2 \|u^{\varepsilon,\alpha}\|_{L^\infty}^2. \tag{6.9}$$

Finally, we obtain

$$\begin{aligned}
\varepsilon \|A\|^2_{L^2_T \dot H^{\frac12}} &\leq C_{\delta,\eta}\varepsilon \int_0^T \Big( \|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32+\delta}} + \|u^\varepsilon\|^2_{\dot H^{\frac32+\delta}} \Big) \mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(t)\,\mathrm{d}t + \\
&+ C\varepsilon \left( \|u^{\varepsilon,\alpha}\|^2_{L^\infty} + \|u^\varepsilon\|^2_{L^\infty} \right) \|u^{\varepsilon,\alpha} - u^\varepsilon\|^2_{L^2_T \dot H^{\frac32}} + \\
&+ C_\delta \eta \varepsilon \left( \|u^{\varepsilon,\alpha}\|^2_{L^\infty_T \dot H^{\frac32+\delta}} + \|u^\varepsilon\|^2_{L^\infty_T \dot H^{\frac32+\delta}} \right) \|u^{\varepsilon,\alpha} - u^\varepsilon\|^2_{L^2_T \dot H^{\frac32}},
\end{aligned}$$

where we use that $\varepsilon \|u^{\varepsilon,\alpha}\|^2_{L^\infty_T \dot H^{\frac32+\delta}} = \mathcal{O}(1)$ (see subsection 6.1).

**6.3.2. Estimate on $\int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac12} \nabla p^\varepsilon . \Lambda^{\frac12} u^{\varepsilon,\alpha}\, \mathbf{d}x\, \mathbf{d}t$.** Applying the div operator to $(HNS^\varepsilon)$, we obtain the identity

$$p^\varepsilon = -\frac{1}{\Delta} \mathrm{div}\, (u^\varepsilon . \nabla) u^\varepsilon.$$

Then, recall that $\mathrm{div}\, u^{\varepsilon,\alpha} \in L^2_T \dot H^{\frac12}$ and $\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2_T \dot H^{\frac12}} = \mathcal{O}(\sqrt{\alpha})$. Now, we easily estimate the integral as follows:

$$\begin{aligned}
\left| \int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac12} \nabla p^\varepsilon . \Lambda^{\frac12} u^{\varepsilon,\alpha}\, \mathrm{d}x\, \mathrm{d}t \right| &= \left| \int_0^T \int_{\mathbb{R}^3} \Lambda p^\varepsilon . \mathrm{div}\, u^{\varepsilon,\alpha}\, \mathrm{d}x\, \mathrm{d}t \right| \\
&\leq \|\Lambda p^\varepsilon\|_{L^2_T L^{\frac32}} \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2_T L^3} \\
&\leq C \sum_{i,j,k=1}^3 \|\big(\partial_k u_i^\varepsilon\big) u_j^\varepsilon\|_{L^2_T L^{\frac32}} \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2_T L^3} \\
&\leq C \sum_{i,j,k=1}^3 \|\partial_k u_i^\varepsilon\|_{L^2_T L^3} \|u_j^\varepsilon\|_{L^\infty_T L^3} \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2_T L^3} \\
&\leq C \|u^\varepsilon\|_{L^2_T \dot H^{\frac32}} \|u^\varepsilon\|_{L^\infty_T \dot H^{\frac12}} \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2_T \dot H^{\frac12}} \\
&= \mathcal{O}(\sqrt{\alpha}).
\end{aligned}$$

**6.3.3. Estimate on $2\varepsilon \int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac12} \nabla p^\varepsilon . \Lambda^{\frac12} \partial_t u^{\varepsilon,\alpha}\, \mathbf{d}x\, \mathbf{d}t$.** First, two integrations by parts (one in space and another in time) give

$$I := 2\varepsilon \int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac12} \nabla p^\varepsilon . \Lambda^{\frac12} \partial_t u^{\varepsilon,\alpha}\, \mathrm{d}x\, \mathrm{d}t = 2\varepsilon \int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac12} \partial_t p^\varepsilon . \Lambda^{\frac12} \mathrm{div}\, u^{\varepsilon,\alpha}\, \mathrm{d}x\, \mathrm{d}t$$

which is easier to estimate.
Notice that $\Lambda^{\frac12} \mathrm{div}\, u^{\varepsilon,\alpha} \in L^2_T L^2(\mathbb{R}^3)$. Now, recall that

$$\Lambda^{\frac12} \partial_t p^\varepsilon = \Lambda^{\frac12} \partial_t \frac{1}{\Delta} \mathrm{div}\, (u^\varepsilon . \nabla) u^\varepsilon.$$

So we have

$$\|\Lambda^{\frac12}\partial_t p^\varepsilon\|_{L^2_TL^2} \le C\|\partial_t(u^\varepsilon.\nabla)u^\varepsilon\|_{L^2_T\dot H^{-\frac12}} = C\|\partial_t\big(u^\varepsilon\otimes u^\varepsilon\big)\|_{L^2_T\dot H^{\frac12}} \le C\|\partial_t u^\varepsilon\otimes u^\varepsilon\|_{L^2_T\dot H^{\frac12}}.$$

We shall estimate this term by a dyadic Littlewood-Paley decomposition. Using Lemma A.2 in the appendix, we have

$$\|u^\varepsilon\otimes\partial_t u^\varepsilon\|^2_{\dot H^{\frac12}} \le \sum_{i,j}\sum_{p\in\mathbb{Z}} 2^p\|\Delta_p u_i^\varepsilon\, S_{p+1}\partial_t u_j^\varepsilon\|^2_{L^2} + \sum_{i,j}\sum_{q\in\mathbb{Z}} 2^q\|\Delta_q\partial_t u_j^\varepsilon\, S_q u_i^\varepsilon\|^2_{L^2}.$$

Then we estimate each term separately, using Bernstein inequalities. First, we have

$$\begin{aligned}
\sum_p 2^p\|\Delta_p u_i^\varepsilon\, S_{p+1}\partial_t u_j^\varepsilon\|^2_{L^2} &\le \sum_p 2^p\|\Delta_p u_i^\varepsilon\|^2_{L^2}\|S_{p+1}\partial_t u_j^\varepsilon\|^2_{L^\infty}\\
&\le C\|u^\varepsilon\|^2_{\dot H^{\frac32}}\sum_p 2^{-2p}\Bigg\|\sum_{-1\le k\le p}\Delta_k\partial_t u_j^\varepsilon\Bigg\|^2_{L^\infty}\\
&\le C\|u^\varepsilon\|^2_{\dot H^{\frac32}}\sum_p 2^{-2p}\sum_{-1\le k\le p} 2^{3k}\left\|\Delta_k\partial_t u_j^\varepsilon\right\|^2_{L^2}\\
&\le C\|u^\varepsilon\|^2_{\dot H^{\frac32}}\,\|\partial_t u^\varepsilon\|^2_{\dot H^{\frac12}}\,.
\end{aligned}$$

Similarly, we obtain that

$$\sum_q 2^q\|\Delta_q\partial_t u_j^\varepsilon\, S_q u_i^\varepsilon\|^2_{L^2} \le \sum_q 2^q\|\Delta_q\partial_t u_j^\varepsilon\|^2_{L^2}\|S_q u_i^\varepsilon\|^2_{L^\infty} \le C\|\partial_t u^\varepsilon\|^2_{\dot H^{\frac12+\delta}}\|u^\varepsilon\|^2_{\dot H^{\frac32+\delta}}.$$

So finally, we have

$$\begin{aligned}
I &\le C\varepsilon\left(\|u^\varepsilon\|_{L^\infty_T\dot H^{\frac32}}\left\|\partial_t u_j^\varepsilon\right\|_{L^2_T\dot H^{\frac12}} + \|\partial_t u^\varepsilon\|_{L^\infty_T\dot H^{\frac12+\delta}}\|u^\varepsilon\|_{L^2_T\dot H^{\frac32+\delta}}\right)\|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2_T\dot H^{\frac12}}\\
&= \mathcal{O}\left(\sqrt{\alpha}\right).
\end{aligned}$$

**6.3.4. Estimate on $\int_0^T\int_{\mathbb{R}^3}\Lambda(u^{\varepsilon,\alpha}-u^\varepsilon).\,(u^\varepsilon.\nabla u^\varepsilon - u^{\varepsilon,\alpha}.\nabla u^{\varepsilon,\alpha})\,\mathbf{d}x\,\mathbf{d}t$.** Let us write $u^\varepsilon.\nabla u^\varepsilon - u^{\varepsilon,\alpha}.\nabla u^{\varepsilon,\alpha} = (u^\varepsilon - u^{\varepsilon,\alpha}).\nabla u^\varepsilon + u^{\varepsilon,\alpha}.\nabla(u^\varepsilon - u^{\varepsilon,\alpha})$. First, using the Sobolev embeddings $\dot H^1\subset L^6(\mathbb{R}^3)$ and $\dot H^{\frac12}\subset L^3(\mathbb{R}^3)$, we estimate the integral

$$\tilde A := \int_0^T\!\!\int_{\mathbb{R}^3}\Lambda(u^{\varepsilon,\alpha}-u^\varepsilon).(u^\varepsilon - u^{\varepsilon,\alpha}).\nabla u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$

as follows:

$$\begin{aligned}
\tilde A &\le \int_0^T\|\Lambda(u^{\varepsilon,\alpha}-u^\varepsilon)\|_{L^2}\,\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{L^6}\,\|\nabla u^\varepsilon\|_{L^3}\,\mathrm{d}t\\
&\le C\int_0^T\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{\dot H^1}\,\|u^\varepsilon\|_{\dot H^{\frac32}}\,\mathrm{d}t\\
&\le C\int_0^T\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^{\frac12}}\,\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^{\frac32}}\,\|u^\varepsilon\|_{\dot H^{\frac32}}\,\mathrm{d}t\\
&\le \frac{C}{\eta}\int_0^T\|u^\varepsilon\|^2_{\dot H^{\frac32}}\,\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(t)\,\mathrm{d}t + \eta\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{L^2_T\dot H^{\frac32}}
\end{aligned}$$

Now, we are left with the term:

$$\Lambda(u^{\varepsilon,\alpha}-u^\varepsilon).u^{\varepsilon,\alpha}.\nabla(u^\varepsilon-u^{\varepsilon,\alpha}) = \sum_{i,j=1}^{3}\partial_j(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\,(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\,\partial_i(u_j^{\varepsilon,\alpha}-u_j^\varepsilon)+ + \sum_{i,j=1}^{3}\partial_j(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\,u_i^\varepsilon\,\partial_i(u_j^{\varepsilon,\alpha}-u_j^\varepsilon) = I_1+I_2.$$

One can easily check that the first part estimates as follows:

$$\begin{aligned}
\int_{\mathbb{R}^3} I_1\,\mathrm{d}x &= \frac{1}{2}\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)^2\mathrm{div}\,u^{\varepsilon,\alpha}\,\mathrm{d}x\\
&\le C\sum_{i=1}^{3}\|u_i^{\varepsilon,\alpha}-u_i^\varepsilon\|_{L^3}\|\partial_i(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\|_{L^3}\|\mathrm{div}\,u^{\varepsilon,\alpha}\|_{L^3}\\
&\le C\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^{\frac12}}\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^{\frac32}}\|\mathrm{div}\,u^{\varepsilon,\alpha}\|_{\dot H^{\frac12}}\\
&\le \frac{C\alpha}{\eta}\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}+\eta\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{\dot H^{\frac32}},
\end{aligned}$$

where $\eta$ is small. Besides, by integrations by parts, we obtain for the second term

$$\begin{aligned}
\int_{\mathbb{R}^3} I_2\,\mathrm{d}x &= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\partial_j u_i^\varepsilon\partial_i(u_j^{\varepsilon,\alpha}-u_j^\varepsilon)\,\mathrm{d}x-\sum_{i=1}^{3}\int_{\mathbb{R}^3}(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)u_i\partial_i\mathrm{div}\,u^{\varepsilon,\alpha}\,\mathrm{d}x\\
&= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\partial_j u_i^\varepsilon\partial_i(u_j^{\varepsilon,\alpha}-u_j^\varepsilon)\,\mathrm{d}x\\
&\quad+\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\,u_i^\varepsilon\,\mathrm{div}\,u^{\varepsilon,\alpha}\,\mathrm{d}x\\
&\quad+\sum_{i=1}^{3}\int_{\mathbb{R}^3}(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\,\partial_i u_i^\varepsilon\,\mathrm{div}\,u^{\varepsilon,\alpha}\,\mathrm{d}x\\
&= I+II+III
\end{aligned}$$

We estimate these three terms as follows:

$$\begin{aligned}
I &\le \sum_{i,j=1}^{3}\|u_i^{\varepsilon,\alpha}-u_i^\varepsilon\|_{L^3}\,\|\partial_j u_i^\varepsilon\|_{L^3}\,\|\partial_i(u_j^{\varepsilon,\alpha}-u_j^\varepsilon)\|_{L^3}\\
&\le C\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^{\frac12}}\,\|u^\varepsilon\|_{\dot H^{\frac32}}\,\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^{\frac32}}\\
&\le \frac{C}{\eta}\|u^\varepsilon\|^2_{\dot H^{\frac32}}\,\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}+\eta\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{\dot H^{\frac32}}.
\end{aligned}$$

$$
\begin{aligned}
II &\leq \sum_{i=1}^{3} \|\partial_i(u_i^{\varepsilon,\alpha} - u_i^\varepsilon)\|_{L^3}\, \|u_i^\varepsilon\|_{L^3}\, \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^3} \\
&\leq C\|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot H^{\frac32}}\, \|u^\varepsilon\|_{\dot H^{\frac12}}\, \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{\dot H^{\frac12}} \\
&\leq C\|u_0^\varepsilon\|_{\dot H^{\frac12}}\, \|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot H^{\frac32}}\, \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{\dot H^{\frac12}} \\
&\leq \eta\|u^{\varepsilon,\alpha} - u^\varepsilon\|^2_{\dot H^{\frac32}} + \frac{C}{\eta}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12}}.
\end{aligned}
$$

$$
\begin{aligned}
III &\leq \sum_{i=1}^{3} \|u_i^{\varepsilon,\alpha} - u_i^\varepsilon\|_{L^3}\, \|\partial_i u_i^\varepsilon\|_{L^3}\, \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^3} \\
&\leq C\|u^{\varepsilon,\alpha} - u^\varepsilon\|^2_{\dot H^{\frac12}}\, \|u^\varepsilon\|^2_{\dot H^{\frac32}} + \frac12\|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12}} \\
&\leq C\|u^\varepsilon\|^2_{\dot H^{\frac32}}\, \mathcal{E}_{\varepsilon,\alpha,u^\varepsilon} + \frac12\|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12}}.
\end{aligned}
$$

Summarizing, we have obtained that

$$
\begin{aligned}
\int_0^T\!\!\int_{\mathbb{R}^3} \Lambda(u^{\varepsilon,\alpha} - u^\varepsilon).\,(u^\varepsilon.\nabla u^\varepsilon - u^{\varepsilon,\alpha}.\nabla u^{\varepsilon,\alpha})\,\mathrm{d}x\,\mathrm{d}t \;\leq\; &\int_0^T \Big(C_\eta\|u^\varepsilon\|^2_{\dot H^{\frac32}} + C_\eta\alpha\Big)\,\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(t)\mathrm{d}t + \\
&+ C\eta\|u^{\varepsilon,\alpha} - u^\varepsilon\|^2_{L^2_T\dot H^{\frac32}} + \mathcal{O}(\alpha).
\end{aligned}
$$

**6.3.5. Conclusion.** Since $\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(0) = 0$, if we choose $\eta$ and $\varepsilon$ small enough, we obtain the estimate

$$
\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(T) \leq \mathcal{O}(\sqrt{\alpha}) + \int_0^T \Big[C_{\delta,\eta}\varepsilon\left(\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32+\delta}} + \|u^\varepsilon\|^2_{\dot H^{\frac32+\delta}}\right) + C_\eta\|u^\varepsilon\|^2_{\dot H^{\frac32}} + C_\eta\alpha\Big]\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(t)\mathrm{d}t.
$$

Now, notice that $u^\varepsilon \in L^2_T\dot H^{\frac32} \cap L^2_T\dot H^{\frac32+\delta}$ and that $\varepsilon\|u^{\varepsilon,\alpha}\|^2_{L^2_T\dot H^{\frac32+\delta}} = \mathcal{O}(1)$ then apply the Gronwall's lemma and obtain that, for all positive $T$,

$$
\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(T) = \mathcal{O}(\sqrt{\alpha}).
$$

As in the 2D case, Theorem 1.1 concludes the proof of Theorem 1.3.

## Appendix A. Littlewood-Paley theory

One of the main tools we use in this paper is the dyadic Littlewood-Paley decomposition. In this subsection, we briefly recall some important results. Our main references for the subject are the books by Alinhac and Gérard [1] and by Lemarié-Rieusset [12].

First, recall that the homogeneous Sobolev norm $\dot H^s$ writes

$$
\|u\|^2_{\dot H^s} = \int_{\mathbb{R}^d} |\xi|^{2s}|\hat u(\xi)|^2\,\mathrm{d}\xi = \sum_{p\in\mathbb{Z}} \int_{2^p\leq|\xi|\leq 2^{p+1}} |\xi|^{2s}|\hat u(\xi)|^2\,\mathrm{d}\xi.
$$

In particular, we can estimate this norm in terms of the $L^2$ norm of $u|_{\{2^p\leq|\xi|\leq 2^{p+1}\}}$ as follows:

$$\sum_{p\in\mathbb{Z}} 2^{2ps}\int_{2^p\leq|\xi|\leq 2^{p+1}} |\hat{u}(\xi)|^2\,\mathrm{d}\xi \leq \|u\|_{\dot{H}^s}^2 \leq 2^{2s}\sum_{p\in\mathbb{Z}} 2^{2ps}\int_{2^p\leq|\xi|\leq 2^{p+1}} |\hat{u}(\xi)|^2\,\mathrm{d}\xi.$$

Now, we shall approximate the functions $\mathbf{1}_{\{2^p\leq|\xi|\leq 2^{p+1}\}}$ by smooth ones. So let us consider a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi(\xi) = 1$ if $|\xi| \leq 1$ and $\varphi(\xi) = 0$ if $|\xi| \geq 1+\varepsilon$. Then define the function

$$\psi : \xi \mapsto \varphi\left(\frac{\xi}{2}\right) - \varphi(\xi)$$

which is supported in the annulus $\{1 \leq |\xi| \leq 2(1+\varepsilon)\}$.
For all tempered distribution $u$, that is $u \in \mathcal{S}'(\mathbb{R}^d)$, we define $\Delta_p u$ the $p^{\text{th}}$ dyadic block of $u$ by

$$\widehat{\Delta_p u}(\xi) = \psi(2^{-p}\xi)\hat{u}(\xi)$$

and we denote the partial sums by

$$S_p u = \sum_{q<p} \Delta_q u$$

for all $p \in \mathbb{Z}$.
Now, notice that $\forall\xi \neq 0\ ,\ \sum_{p\in\mathbb{Z}} \psi(2^{-p}\xi) = 1$. So, in particular for all the functions $u$ considered in this work, we have

$$\forall\xi \neq 0\ ,\ \hat{u}(\xi) = \sum_{p\in\mathbb{Z}} \widehat{\Delta_p u}(\xi).$$

*Remark* A.1. Every dyadic block is in $H^{+\infty}$ once $u \in H^s$.

We recall here some important properties of this dyadic decomposition. The results are stated for homogeneous Sobolev spaces but hold also for classical Sobolev spaces.

- (Almost-orthogonality) For all $u \in L^2$,

$$\sum_{p\in\mathbb{Z}} \|\Delta_p u\|_{L^2}^2 \leq \|u\|_{L^2}^2 \leq 2\sum_{p\in\mathbb{Z}} \|\Delta_p u\|_{L^2}^2.$$

- There exists a constant $C$ such that for all $s \in \mathbb{R}$ and $p \in \mathbb{Z}$,

$$\frac{1}{C}\sum_{p\in\mathbb{Z}} 2^{2ps}\|\Delta_p u\|_{L^2}^2 \leq \|u\|_{\dot{H}^s}^2 \leq C\sum_{p\in\mathbb{Z}} 2^{2ps}\|\Delta_p u\|_{L^2}^2. \tag{A.1}$$

  We will use this property several times in this work.
- There exists a constant $C$ such that, for all $\alpha \in \mathbb{N}^d$ and $p \in \mathbb{Z}$,

$$\begin{aligned} \|\partial^\alpha \Delta_p u\|_{L^2} &\leq C2^{p|\alpha|}\|u\|_{L^2} \quad , \quad \|\partial^\alpha S_p u\|_{L^2} \leq C2^{p|\alpha|}\|u\|_{L^2}, \\ \|\partial^\alpha \Delta_p u\|_{L^\infty} &\leq C2^{p|\alpha|}\|u\|_{L^\infty} \quad , \quad \|\partial^\alpha S_p u\|_{L^\infty} \leq C2^{p|\alpha|}\|u\|_{L^\infty}. \end{aligned}$$

Using this theory, we can prove some important product estimates. The following lemma tells how to write a product of tempered distributions:

**Lemma A.2 (Paraproduct).** *Given two tempered distributions $u, v \in \mathcal{S}'(\mathbb{R}^d)$, we can write their product, if it exists, as follows:*

$$uv = \sum_{p\in\mathbb{Z}} \Delta_p u \, S_{p+1} v + \sum_{q\in\mathbb{Z}} \Delta_q v \, S_q u.$$

Then, the following proposition is very useful to estimate some products.

**Proposition A.3.** *Let $s > 0$ and $u, v \in L^\infty \cap \dot{H}^s$. Then the product $uv$ is also in $L^\infty \cap \dot{H}^s$ and*

$$\|uv\|_{L^\infty \cap \dot{H}^s} \leq C \left( \|u\|_{L^\infty} \|v\|_{\dot{H}^s} + \|u\|_{\dot{H}^s} \|v\|_{L^\infty} \right).$$

For a detailed proof of this proposition, see the book by Alinhac and Gérard.
In this paper, we also use Bernstein inequalities:

**Proposition A.4.** *There exists a positive constant $C_0$ such that, for all $u \in \mathcal{S}'(\mathbb{R}^d)$, we have*

$$\begin{cases} \forall k \geq 0, \ \forall p \geq 1, \ 2^{qk} C_0^{-k} \|\Delta_q u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^k \Delta_q u\|_{L^p} \leq 2^{qk} C_0^k \|\Delta_q u\|_{L^p} \\ \forall p' \geq p \geq 1, \ \|\Delta_q u\|_{L^{p'}} \leq C_0 2^{dq\left(\frac{1}{p} - \frac{1}{p'}\right)} \|\Delta_q u\|_{L^p} \end{cases}$$

*and, for the partial sums:*

$$\begin{cases} \forall k \geq 0, \ \forall p \geq 1, \ \sup_{|\alpha|=k} \|\partial^k S_q u\|_{L^p} \leq 2^{qk} C_0^k \|S_q u\|_{L^p} \\ \forall p' \geq p \geq 1, \ \|S_q u\|_{L^{p'}} \leq C_0 2^{dq\left(\frac{1}{p} - \frac{1}{p'}\right)} \|S_q u\|_{L^p} \end{cases}$$

Using the dyadic blocks, we can easily define the homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^d)$, where $s \in \mathbb{R}$, $p, r \geq 1$ and $d \geq 1$,

$$\dot{B}^s_{p,r}(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) \ : \ \|u\|_{\dot{B}^s_{p,r}} := \left( \sum_{j\geq -1} 2^{jsr} \|\Delta_j u\|^r_{L^p} \right)^{\frac{1}{r}} \right\}.$$

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Imène HACHICHA
Laboratoire Analyse et Probabilités
Université d'Evry-Val d'Essonne
23 bd de France, 91037 Evry Cedex
France
e-mail: `hachicha@math.univ-paris13.fr`