## INTRODUCTION TO MATHEMATICS OF CREDIT RISK MODELING

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# Introduction

The goal of this survey is to provide an introduction to the area of mathematical modeling of credit risk. It is largely based on the following works by Bielecki et al. [2, 3, 4, 5] and some sections from the monograph by Bielecki and Rutkowski [7].

Credit risk embedded in a financial transaction is the risk that at least one of the parties involved in the transaction will suffer a financial loss due to default or decline in the creditworthiness of the counter-party to the transaction, or perhaps of some third party. For example:

- A holder of a corporate bond bears a risk that the (market) value of the bond will decline due to decline in credit rating of the issuer.
- A bank may suffer a loss if a bank's debtor defaults on payment of the interest due and (or) the principal amount of the loan.
- A party involved in a trade of a credit derivative, such as a credit default swap (CDS), may suffer a loss if a reference credit event occurs.
- The market value of individual tranches constituting a collateralized debt obligation (CDO) may decline as a result of changes in the correlation between the default times of the underlying defaultable securities (i.e., of the collateral).

The most extensively studied form of credit risk is the *default risk* – that is, the risk that a counterparty in a financial contract will not fulfil a contractual commitment to meet her/his obligations stated in the contract. For this reason, the main tool in the area of credit risk modeling is a judicious specification of the random time of default. A large part of the present text is devoted to this issue.

Our main goal is to present the most important mathematical tools that are used for the arbitrage valuation of defaultable claims, which are also known under the name of credit derivatives. We also examine briefly the important issue of hedging these claims.

These notes are organized as follows:

- In Chapter 1, we provide a concise summary of the main developments within the so-called *structural approach* to modeling and valuation of credit risk. We also study very briefly the case of a random barrier.
- Chapter 2 is devoted to the study of a simple model of credit risk within the *hazard function* framework. We also deal here with the issue of replication of single- and multi-name credit derivatives in the stylized CDS market.
- Chapter 3 deals with the so-called *reduced-form approach* in which the main tool is the *hazard rate* process. This approach is of a purely probabilistic nature and, technically speaking, it has a lot in common with the reliability theory.
- Chapter 4 studies hedging strategies for defaultable claims under assumption that some primary defaultable assets are traded. We discuss some preliminary results in a semimartingale set-up and we develop the PDE approach in a Markovian set-up.

Let us mention that the proofs of most results can be found in Bielecki et al. [2, 3, 4, 5], Bielecki and Rutkowski [7] and Jeanblanc and Rutkowski [42]. We quote some of the seminal papers; the reader can also refer to books by Bruyère [17], Bluhm et al. [10], Bielecki and Rutkowski [7], Cossin and Pirotte [22], Duffie and Singleton [29], Frey, McNeil and Embrechts [35], Lando [45], or Schönbucher [59] for more information.

Finally, it should be acknowledged that several results (especially within the reduced-form approach) were obtained independently by various authors, who worked under different set of assumptions and/or within different set-ups. For this reason, we decided to omit detailed credentials in most cases. We hope that our colleagues will accept our apologies for this deficiency and we stress that this by no means signifies that any result given in what follows that is not explicitly attributed is ours.

'Begin at the beginning, and go on till you come to the end: then stop.'

Lewis Carroll, Alice's Adventures in Wonderland

## Chapter 1

# Structural Approach

In this chapter, we present the so-called *structural approach* to modeling credit risk, which is also known as the *value-of-the-firm approach*. This methodology refers directly to economic fundamentals, such as the capital structure of a company, in order to model credit events (a default event, in particular). As we shall see in what follows, the two major driving concepts in the structural modeling are: the total value of the firm's assets and the default triggering barrier. It is worth noting that this was historically the first approach used in this area – it goes back to the fundamental papers by Black and Scholes [9] and Merton [53].

## **1.1** Basic Assumptions

We fix a finite horizon date  $T^* > 0$ , and we suppose that the underlying probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , endowed with some (reference) filtration  $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}$ , is sufficiently rich to support the following objects:

- The *short-term interest rate* process r, and thus also a default-free term structure model.
- The *firm's value process V*, which is interpreted as a model for the total value of the firm's assets.
- The barrier process v, which will be used in the specification of the default time  $\tau$ .
- The promised contingent claim X representing the firm's liabilities to be redeemed at maturity date  $T \leq T^*$ .
- The process A, which models the *promised dividends*, i.e., the liabilities stream that is redeemed continuously or discretely over time to the holder of a defaultable claim.
- The recovery claim  $\widetilde{X}$  representing the recovery payoff received at time T, if default occurs prior to or at the claim's maturity date T.
- The recovery process Z, which specifies the recovery payoff at time of default, if it occurs prior to or at the maturity date T.

## 1.1.1 Defaultable Claims

**Technical assumptions.** We postulate that the processes V, Z, A and v are progressively measurable with respect to the filtration  $\mathbb{F}$ , and that the random variables X and  $\tilde{X}$  are  $\mathcal{F}_T$ -measurable. In addition, A is assumed to be a process of finite variation, with  $A_0 = 0$ . We assume without mentioning that all random objects introduced above satisfy suitable integrability conditions. **Probabilities**  $\mathbb{P}$  and  $\mathbb{Q}$ . The probability  $\mathbb{P}$  is assumed to represent the *real-world* (or *statistical*) probability, as opposed to a *martingale measure* (also known as a *risk-neutral probability*). Any *martingale measure* will be denoted by  $\mathbb{Q}$  in what follows.

**Default time.** In the structural approach, the default time  $\tau$  will be typically defined in terms of the firm's value process V and the barrier process v. We set

$$\tau = \inf \{ t > 0 : t \in \mathcal{T} \text{ and } V_t \leq v_t \}$$

with the usual convention that the infimum over the empty set equals  $+\infty$ . In main cases, the set  $\mathcal{T}$  is an interval [0,T] (or  $[0,\infty)$  in the case of perpetual claims). In first passage structural models, the default time  $\tau$  is usually given by the formula:

$$\tau = \inf \{ t > 0 : t \in [0, T] \text{ and } V_t \leq \bar{v}(t) \},\$$

where  $\bar{v}: [0,T] \to \mathbb{R}_+$  is some deterministic function, termed the *barrier*.

**Predictability of default time.** Since the underlying filtration  $\mathbb{F}$  in most structural models is generated by a standard Brownian motion,  $\tau$  will be an  $\mathbb{F}$ -predictable stopping time (as any stopping time with respect to a Brownian filtration): there exists a sequence of increasing stopping times announcing the default time.

**Recovery rules.** If default does not occur before or at time T, the promised claim X is paid in full at time T. Otherwise, depending on the market convention, either (1) the amount  $\tilde{X}$  is paid at the maturity date T, or (2) the amount  $Z_{\tau}$  is paid at time  $\tau$ . In the case when default occurs at maturity, i.e., on the event  $\{\tau = T\}$ , we postulate that only the recovery payment  $\tilde{X}$  is paid. In a general setting, we consider simultaneously both kinds of recovery payoff, and thus a generic defaultable claim is formally defined as a quintuple  $(X, A, \tilde{X}, Z, \tau)$ .

#### 1.1.2 Risk-Neutral Valuation Formula

Suppose that our financial market model is arbitrage-free, in the sense that there exists a martingale measure (risk-neutral probability)  $\mathbb{Q}$ , meaning that price process of any tradeable security, which pays no coupons or dividends, becomes an  $\mathbb{F}$ -martingale under  $\mathbb{Q}$ , when discounted by the savings account B, given as

$$B_t = \exp\Big(\int_0^t r_u \, du\Big).$$

We introduce the jump process  $H_t = \mathbb{1}_{\{\tau \leq t\}}$ , and we denote by D the process that models all cash flows received by the owner of a defaultable claim. Let us denote

$$X^{d}(T) = X \mathbb{1}_{\{\tau > T\}} + X \mathbb{1}_{\{\tau \le T\}}.$$

**Definition 1.1.1** The dividend process D of a defaultable contingent claim  $(X, A, \tilde{X}, Z, \tau)$ , which settles at time T, equals

$$D_t = X^d(T) \mathbb{1}_{\{t \ge T\}} + \int_{]0,t]} (1 - H_u) \, dA_u + \int_{]0,t]} Z_u \, dH_u.$$

It is apparent that D is a process of finite variation, and

$$\int_{]0,t]} (1 - H_u) \, dA_u = \int_{]0,t]} \mathbb{1}_{\{\tau > u\}} \, dA_u = A_{\tau -} \mathbb{1}_{\{\tau \le t\}} + A_t \mathbb{1}_{\{\tau > t\}}.$$

Note that if default occurs at some date t, the promised dividend  $A_t - A_{t-}$ , which is due to be paid at this date, is not received by the holder of a defaultable claim. Furthermore, if we set  $\tau \wedge t = \min \{\tau, t\}$  then

$$\int_{]0,t]} Z_u \, dH_u = Z_{\tau \wedge t} \mathbb{1}_{\{\tau \le t\}} = Z_\tau \mathbb{1}_{\{\tau \le t\}}.$$

#### 1.1. BASIC ASSUMPTIONS

**Remark 1.1.1** In principle, the promised payoff X could be incorporated into the promised dividends process A. However, this would be inconvenient, since in practice the recovery rules concerning the promised dividends A and the promised claim X are different, in general. For instance, in the case of a defaultable coupon bond, it is frequently postulated that in case of default the future coupons are lost, but a strictly positive fraction of the face value is usually received by the bondholder.

We are in the position to define the ex-dividend price  $S_t$  of a defaultable claim. At any time t, the random variable  $S_t$  represents the current value of all future cash flows associated with a given defaultable claim.

**Definition 1.1.2** For any date  $t \in [0, T[$ , the ex-dividend price of the defaultable claim  $(X, A, \tilde{X}, Z, \tau)$  is given as

$$S_t = B_t \mathbb{E}_{\mathbb{Q}} \Big( \int_{]t,T]} B_u^{-1} dD_u \, \Big| \, \mathcal{F}_t \Big).$$

$$(1.1)$$

In addition, we always set  $S_T = X^d(T)$ . The discounted ex-dividend price  $S_t^*, t \in [0,T]$ , satisfies

$$S_t^* = S_t B_t^{-1} - \int_{]0,t]} B_u^{-1} \, dD_u, \quad \forall t \in [0,T],$$

and thus it follows a supermartingale under  $\mathbb{Q}$  if and only if the dividend process D is increasing. The process  $S_t + B_t \int_{[0,t]} B_u^{-1} dD_u$  is also called the cum-dividend process.

## 1.1.3 Defaultable Zero-Coupon Bond

Assume that  $A \equiv 0$ ,  $Z \equiv 0$  and X = L for some positive constant L > 0. Then the value process S represents the arbitrage price of a *defaultable zero-coupon bond* (also known as the *corporate discount bond*) with the face value L and recovery at maturity only. In general, the price D(t,T) of such a bond equals

$$D(t,T) = B_t \mathbb{E}_{\mathbb{Q}} \left( B_T^{-1}(L\mathbb{1}_{\{\tau > T\}} + \widetilde{X}\mathbb{1}_{\{\tau \le T\}}) \, \middle| \, \mathcal{F}_t \right).$$

It is convenient to rewrite the last formula as follows:

$$D(t,T) = LB_t \mathbb{E}_{\mathbb{Q}} \left( B_T^{-1}(\mathbb{1}_{\{\tau > T\}} + \delta(T)\mathbb{1}_{\{\tau \le T\}}) \, \middle| \, \mathcal{F}_t \right),$$

where the random variable  $\delta(T) = \widetilde{X}/L$  represents the so-called *recovery rate upon default*. It is natural to assume that  $0 \leq \widetilde{X} \leq L$  so that  $\delta(T)$  satisfies  $0 \leq \delta(T) \leq 1$ . Alternatively, we may re-express the bond price as follows:

$$D(t,T) = L\Big(B(t,T) - B_t \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1}w(T)\mathbb{1}_{\{\tau \le T\}} \,\big|\, \mathcal{F}_t\Big)\Big).$$

where

$$B(t,T) = B_t \mathbb{E}_{\mathbb{Q}}(B_T^{-1} \mid \mathcal{F}_t)$$

is the price of a unit default-free zero-coupon bond, and  $w(T) = 1 - \delta(T)$  is the writedown rate upon default. Generally speaking, the time-t value of a corporate bond depends on the joint probability distribution under  $\mathbb{Q}$  of the three-dimensional random variable  $(B_T, \delta(T), \tau)$  or, equivalently,  $(B_T, w(T), \tau)$ .

**Example 1.1.1** Merton [53] postulates that the recovery payoff upon default (that is, when  $V_T < L$ , equals  $\tilde{X} = V_T$ , where the random variable  $V_T$  is the firm's value at maturity date T of a corporate bond. Consequently, the random recovery rate upon default equals  $\delta(T) = V_T/L$ , and the writedown rate upon default equals  $w(T) = 1 - V_T/L$ .

**Expected writedowns.** For simplicity, we assume that the savings account B is non-random – that is, the short-term rate r is deterministic. Then the price of a default-free zero-coupon bond equals  $B(t,T) = B_t B_T^{-1}$ , and the price of a zero-coupon corporate bond satisfies

$$D(t,T) = L_t(1 - w^*(t,T)),$$

where  $L_t = LB(t,T)$  is the present value of future liabilities, and  $w^*(t,T)$  is the *conditional expected* writedown rate under  $\mathbb{Q}$ . It is given by the following equality:

$$w^*(t,T) = \mathbb{E}_{\mathbb{Q}}\big(w(T)\mathbb{1}_{\{\tau \leq T\}} \,|\, \mathcal{F}_t\big).$$

The conditional expected writedown rate upon default equals, under  $\mathbb{Q}$ ,

$$w_t^* = \frac{\mathbb{E}_{\mathbb{Q}}\left(w(T)\mathbb{1}_{\{\tau \le T\}} \mid \mathcal{F}_t\right)}{\mathbb{Q}\{\tau \le T \mid \mathcal{F}_t\}} = \frac{w^*(t,T)}{p_t^*}$$

where  $p_t^* = \mathbb{Q}\{\tau \leq T \mid \mathcal{F}_t\}$  is the conditional risk-neutral probability of default. Finally, let  $\delta_t^* = 1 - w_t^*$  be the conditional expected recovery rate upon default under  $\mathbb{Q}$ . In terms of  $p_t^*, \delta_t^*$  and  $p_t^*$ , we obtain

$$D(t,T) = L_t(1-p_t^*) + L_t p_t^* \delta_t^* = L_t(1-p_t^* w_t^*).$$

If the random variables w(T) and  $\tau$  are conditionally independent with respect to the  $\sigma$ -field  $\mathcal{F}_t$ under  $\mathbb{Q}$ , then we have  $w_t^* = \mathbb{E}_{\mathbb{Q}}(w(T) \mid \mathcal{F}_t)$ .

**Example 1.1.2** In practice, it is common to assume that the recovery rate is non-random. Let the recovery rate  $\delta(T)$  be constant, specifically,  $\delta(T) = \delta$  for some real number  $\delta$ . In this case, the writedown rate  $w(T) = w = 1 - \delta$  is non-random as well. Then  $w^*(t,T) = wp_t^*$  and  $w_t^* = w$  for every  $0 \le t \le T$ . Furthermore, the price of a defaultable bond has the following representation

$$D(t,T) = L_t(1-p_t^*) + \delta L_t p_t^* = L_t(1-wp_t^*).$$

We shall return to various recovery schemes later in the text.

## **1.2** Classic Structural Models

Classic structural models are based on the assumption that the risk-neutral dynamics of the value process of the assets of the firm V are given by the SDE:

$$dV_t = V_t \left( (r - \kappa) \, dt + \sigma_V \, dW_t \right), \quad V_0 > 0,$$

where  $\kappa$  is the constant payout (dividend) ratio, and the process W is a standard Brownian motion under the martingale measure  $\mathbb{Q}$ .

## 1.2.1 Merton's Model

We present here the classic model due to Merton [53].

**Basic assumptions.** A firm has a single liability with promised terminal payoff L, interpreted as the zero-coupon bond with maturity T and face value L > 0. The ability of the firm to redeem its debt is determined by the total value  $V_T$  of firm's assets at time T. Default may occur at time T only, and the default event corresponds to the event  $\{V_T < L\}$ . Hence, the stopping time  $\tau$  equals

$$\tau = T \mathbb{1}_{\{V_T < L\}} + \infty \mathbb{1}_{\{V_T \ge L\}}.$$

Moreover A = 0, Z = 0, and

$$X^{d}(T) = V_{T} \mathbb{1}_{\{V_{T} < L\}} + L \mathbb{1}_{\{V_{T} \ge L\}}$$

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so that  $\tilde{X} = V_T$ . In other words, the payoff at maturity equals

$$D_T = \min(V_T, L) = L - \max(L - V_T, 0) = L - (L - V_T)^+.$$

The latter equality shows that the valuation of the corporate bond in Merton's setup is equivalent to the valuation of a European put option written on the firm's value with strike equal to the bond's face value. Let D(t,T) be the price at time t < T of the corporate bond. It is clear that the value  $D(V_t)$  of the firm's debt equals

$$D(V_t) = D(t,T) = LB(t,T) - P_t,$$

where  $P_t$  is the price of a put option with strike L and expiration date T. It is apparent that the value  $E(V_t)$  of the firm's equity at time t equals

$$E(V_t) = V_t e^{-\kappa(T-t)} - D(V_t) = V_t e^{-\kappa(T-t)} - LB(t,T) + P_t = C_t,$$

where  $C_t$  stands for the price at time t of a call option written on the firm's assets, with strike price L and exercise date T. To justify the last equality above, we may also observe that at time T we have

$$E(V_T) = V_T - D(V_T) = V_T - \min(V_T, L) = (V_T - L)^+.$$

We conclude that the firm's shareholders are in some sense the holders of a call option on the firm's assets.

Merton's formula. Using the option-like features of a corporate bond, Merton [53] derived a closed-form expression for its arbitrage price. Let N denote the standard Gaussian cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du, \quad \forall x \in \mathbb{R}.$$

**Proposition 1.2.1** For every  $0 \le t < T$  the value D(t,T) of a corporate bond equals

$$D(t,T) = V_t e^{-\kappa(T-t)} N(-d_+(V_t,T-t)) + L B(t,T) N(d_-(V_t,T-t))$$

where

$$d_{\pm}(V_t, T-t) = \frac{\ln(V_t/L) + \left(r - \kappa \pm \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V \sqrt{T-t}}$$

The unique replicating strategy for a defaultable bond involves holding, at any time  $0 \le t < T$ ,  $\phi_t^1 V_t$ units of cash invested in the firm's value and  $\phi_t^2 B(t,T)$  units of cash invested in default-free bonds, where

$$\phi_t^1 = e^{-\kappa(T-t)} N (-d_+(V_t, T-t))$$

and

$$\phi_t^2 = \frac{D(t,T) - \phi_t^1 V_t}{B(t,T)} = LN(d_-(V_t, T - t))$$

**Credit spreads.** For notational simplicity, we set  $\kappa = 0$ . Then Merton's formula becomes:

$$D(t,T) = LB(t,T) \big( \Gamma_t N(-d) + N(d - \sigma_V \sqrt{T-t}) \big),$$

where we denote  $\Gamma_t = V_t / LB(t,T)$  and

$$d = d(V_t, T - t) = \frac{\ln(V_t/L) + (r + \sigma_V^2/2)(T - t)}{\sigma_V \sqrt{T - t}}.$$

Since LB(t,T) represents the current value of the face value of the firm's debt, the quantity  $\Gamma_t$  can be seen as a proxy of the asset-to-debt ratio  $V_t/D(t,T)$ . It can be easily verified that the inequality

D(t,T) < LB(t,T) is valid. This property is equivalent to the positivity of the corresponding *credit* spread (see below).

Observe that in the present setup the continuously compounded yield r(t,T) at time t on the T-maturity Treasury zero-coupon bond is constant, and equal to the short-term rate r. Indeed, we have

$$B(t,T) = e^{-r(t,T)(T-t)} = e^{-r(T-t)}$$

Let us denote by  $r^{d}(t,T)$  the continuously compounded yield on the corporate bond at time t < T, so that

$$D(t,T) = Le^{-r^{d}(t,T)(T-t)}.$$

From the last equality, it follows that

$$r^{d}(t,T) = -\frac{\ln D(t,T) - \ln L}{T-t}.$$

For t < T the credit spread S(t,T) is defined as the excess return on a defaultable bond:

$$S(t,T) = r^{d}(t,T) - r(t,T) = \frac{1}{T-t} \ln \frac{LB(t,T)}{D(t,T)}.$$

In Merton's model, we have

$$S(t,T) = -\frac{\ln\left(N(d - \sigma_V \sqrt{T - t}) + \Gamma_t N(-d)\right)}{T - t} > 0.$$

This agrees with the well-known fact that risky bonds have an expected return in excess of the riskfree interest rate. In other words, the yields on corporate bonds are higher than yields on Treasury bonds with matching notional amounts. Notice, however, when t tends to T, the credit spread in Merton's model tends either to infinity or to 0, depending on whether  $V_T < L$  or  $V_T > L$ . Formally, if we define the *forward short spread at time* T as

$$FSS_T = \lim_{t \uparrow T} S(t, T)$$

then

$$FSS_T(\omega) = \begin{cases} 0, & \text{if } \omega \in \{V_T > L\},\\ \infty, & \text{if } \omega \in \{V_T < L\}. \end{cases}$$

## 1.2.2 Black and Cox Model

By construction, Merton's model does not allow for a premature default, in the sense that the default may only occur at the maturity of the claim. Several authors put forward structural-type models in which this restrictive and unrealistic feature is relaxed. In most of these models, the time of default is given as the *first passage time* of the value process V to either a deterministic or a random barrier. In principle, the bond's default may thus occur at any time before or on the maturity date T. The challenge is to appropriately specify the lower threshold v, the recovery process Z, and to explicitly evaluate the conditional expectation that appears on the right-hand side of the risk-neutral valuation formula

$$S_t = B_t \mathbb{E}_{\mathbb{Q}} \Big( \int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{F}_t \Big),$$

which is valid for  $t \in [0, T[$ . As one might easily guess, this is a non-trivial mathematical problem, in general. In addition, the practical problem of the lack of direct observations of the value process V largely limits the applicability of the first-passage-time models based on the value of the firm process V.

**Corporate zero-coupon bond.** Black and Cox [8] extend Merton's [53] research in several directions, by taking into account such specific features of real-life debt contracts as: safety covenants,

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debt subordination, and restrictions on the sale of assets. Following Merton [53], they assume that the firm's stockholders receive continuous dividend payments, which are proportional to the current value of firm's assets. Specifically, they postulate that

$$dV_t = V_t ((r - \kappa) dt + \sigma_V dW_t), \quad V_0 > 0,$$

where W is a Brownian motion (under the risk-neutral probability  $\mathbb{Q}$ ), the constant  $\kappa \geq 0$  represents the payout ratio, and  $\sigma_V > 0$  is the constant volatility. The short-term interest rate r is assumed to be constant.

**Safety covenants.** Safety covenants provide the firm's bondholders with the right to force the firm to bankruptcy or reorganization if the firm is doing poorly according to a set standard. The standard for a poor performance is set by Black and Cox in terms of a time-dependent deterministic barrier  $\bar{v}(t) = Ke^{-\gamma(T-t)}, t \in [0, T[$ , for some constant K > 0. As soon as the value of firm's assets crosses this lower threshold, the bondholders take over the firm. Otherwise, default takes place at debt's maturity or not depending on whether  $V_T < L$  or not.

Default time. Let us set

$$v_t = \begin{cases} \bar{v}(t), & \text{for } t < T, \\ L, & \text{for } t = T. \end{cases}$$

The default event occurs at the first time  $t \in [0, T]$  at which the firm's value  $V_t$  falls below the level  $v_t$ , or the default event does not occur at all. The default time equals (inf  $\emptyset = +\infty$ )

$$\tau = \inf \{ t \in [0, T] : V_t \le v_t \}.$$

The recovery process Z and the recovery payoff  $\widetilde{X}$  are proportional to the value process:  $Z \equiv \beta_2 V$ and  $\widetilde{X} = \beta_1 V_T$  for some constants  $\beta_1, \beta_2 \in [0, 1]$ . The case examined by Black and Cox [8] corresponds to  $\beta_1 = \beta_2 = 1$ .

To summarize, we consider the following model:

$$X = L, \ A \equiv 0, \ Z \equiv \beta_2 V, \ X = \beta_1 V_T, \ \tau = \overline{\tau} \wedge \widehat{\tau},$$

where the early default time  $\bar{\tau}$  equals

$$\bar{\tau} = \inf \{ t \in [0, T) : V_t \le \bar{v}(t) \}$$

and  $\hat{\tau}$  stands for Merton's default time:  $\hat{\tau} = T \mathbb{1}_{\{V_T < L\}} + \infty \mathbb{1}_{\{V_T \ge L\}}$ .

**Bond valuation.** Similarly as in Merton's model, it is assumed that the short term interest rate is deterministic and equal to a positive constant r. We postulate, in addition, that  $\bar{v}(t) \leq LB(t,T)$  or, more explicitly,

$$Ke^{-\gamma(T-t)} \le Le^{-r(T-t)}, \quad \forall t \in [0,T],$$

so that, in particular,  $K \leq L$ . This condition ensures that the payoff to the bondholder at the default time  $\tau$  never exceeds the face value of debt, discounted at a risk-free rate.

**PDE approach.** Since the model for the value process V is given in terms of a Markovian diffusion, a suitable partial differential equation can be used to characterize the value process of the corporate bond. Let us write  $D(t,T) = u(V_t,t)$ . Then the pricing function u = u(v,t) of a defaultable bond satisfies the following PDE:

$$u_t(v,t) + (r-\kappa)vu_v(v,t) + \frac{1}{2}\sigma_V^2 v^2 u_{vv}(v,t) - ru(v,t) = 0$$

on the domain

$$\{(v,t) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 < t < T, v > Ke^{-\gamma(T-t)}\}$$

with the boundary condition

$$u(Ke^{-\gamma(T-t)}, t) = \beta_2 Ke^{-\gamma(T-t)}$$

and the terminal condition  $u(v,T) = \min(\beta_1 v, L)$ .

**Probabilistic approach.** For any t < T the price  $D(t,T) = u(V_t,t)$  of a defaultable bond has the following probabilistic representation, on the event  $\{\tau > t\} = \{\bar{\tau} > t\}$ 

$$D(t,T) = \mathbb{E}_{\mathbb{Q}} \left( Le^{-r(T-t)} \mathbb{1}_{\{\bar{\tau} \ge T, V_T \ge L\}} \middle| \mathcal{F}_t \right) \\ + \mathbb{E}_{\mathbb{Q}} \left( \beta_1 V_T e^{-r(T-t)} \mathbb{1}_{\{\bar{\tau} \ge T, V_T < L\}} \middle| \mathcal{F}_t \right) \\ + \mathbb{E}_{\mathbb{Q}} \left( K \beta_2 e^{-\gamma(T-\bar{\tau})} e^{-r(\bar{\tau}-t)} \mathbb{1}_{\{t < \bar{\tau} < T\}} \middle| \mathcal{F}_t \right).$$

After default – that is, on the event  $\{\tau \leq t\} = \{\bar{\tau} \leq t\}$ , we clearly have

$$D(t,T) = \beta_2 \bar{v}(\tau) B^{-1}(\tau,T) B(t,T) = K \beta_2 e^{-\gamma (T-\tau)} e^{r(t-\tau)}.$$

To compute the expected values above, we observe that:

- the first two conditional expectations can be computed by using the formula for the conditional probability  $\mathbb{Q}\{V_s \ge x, \tau \ge s \mid \mathcal{F}_t\}$ ,
- to evaluate the third conditional expectation, it suffices employ the conditional probability law of the first passage time of the process V to the barrier  $\bar{v}(t)$ .

**Black and Cox formula.** Before we state the bond valuation result due to Black and Cox [8], we find it convenient to introduce some notation. We denote

$$\begin{split} \nu &= r - \kappa - \frac{1}{2}\sigma_V^2, \\ m &= \nu - \gamma = r - \kappa - \gamma - \frac{1}{2}\sigma_V^2 \\ b &= m\sigma^{-2}. \end{split}$$

For the sake of brevity, in the statement of Proposition 1.2.2 we shall write  $\sigma$  instead of  $\sigma_V$ . As already mentioned, the probabilistic proof of this result is based on the knowledge of the probability law of the first passage time of the geometric (exponential) Brownian motion to an exponential barrier.

**Proposition 1.2.2** Assume that  $m^2 + 2\sigma^2(r - \gamma) > 0$ . Prior to bond's default, that is: on the event  $\{\tau > t\}$ , the price process  $D(t, T) = u(V_t, t)$  of a defaultable bond equals

$$D(t,T) = LB(t,T) \left( N \left( h_1(V_t,T-t) \right) - Z_t^{2b\sigma^{-2}} N \left( h_2(V_t,T-t) \right) \right) + \beta_1 V_t e^{-\kappa(T-t)} \left( N \left( h_3(V_t,T-t) \right) - N \left( h_4(V_t,T-t) \right) \right) + \beta_1 V_t e^{-\kappa(T-t)} Z_t^{2b+2} \left( N \left( h_5(V_t,T-t) \right) - N \left( h_6(V_t,T-t) \right) \right) + \beta_2 V_t \left( Z_t^{\theta+\zeta} N \left( h_7(V_t,T-t) \right) + Z_t^{\theta-\zeta} N \left( h_8(V_t,T-t) \right) \right),$$

where  $Z_t = \bar{v}(t)/V_t$ ,  $\theta = b + 1$ ,  $\zeta = \sigma^{-2}\sqrt{m^2 + 2\sigma^2(r - \gamma)}$  and

$$\begin{split} h_1(V_t, T - t) &= \frac{\ln{(V_t/L)} + \nu(T - t)}{\sigma\sqrt{T - t}}, \\ h_2(V_t, T - t) &= \frac{\ln{\bar{v}^2(t)} - \ln(LV_t) + \nu(T - t)}{\sigma\sqrt{T - t}}, \\ h_3(V_t, T - t) &= \frac{\ln{(L/V_t)} - (\nu + \sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \\ h_4(V_t, T - t) &= \frac{\ln{(K/V_t)} - (\nu + \sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \end{split}$$

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$$h_5(V_t, T - t) = \frac{\ln \bar{v}^2(t) - \ln(LV_t) + (\nu + \sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

$$h_6(V_t, T - t) = \frac{\ln \bar{v}^2(t) - \ln(KV_t) + (\nu + \sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

$$h_7(V_t, T - t) = \frac{\ln (\bar{v}(t)/V_t) + \zeta\sigma^2(T - t)}{\sigma\sqrt{T - t}},$$

$$h_8(V_t, T - t) = \frac{\ln (\bar{v}(t)/V_t) - \zeta\sigma^2(T - t)}{\sigma\sqrt{T - t}}.$$

**Special cases.** Assume that  $\beta_1 = \beta_2 = 1$  and the barrier function  $\bar{v}$  is such that K = L. Then necessarily  $\gamma \ge r$ . It can be checked that for K = L we have  $D(t,T) = D_1(t,T) + D_3(t,T)$  where:

$$D_1(t,T) = LB(t,T) \big( N \big( h_1(V_t, T-t) \big) - Z_t^{2\hat{a}} N \big( h_2(V_t, T-t) \big) \big)$$
  
$$D_3(t,T) = V_t \big( Z_t^{\theta+\zeta} N \big( h_7(V_t, T-t) \big) + Z_t^{\theta-\zeta} N \big( h_8(V_t, T-t) \big) \big).$$

• Case  $\gamma = r$ . If we also assume that  $\gamma = r$  then  $\zeta = -\sigma^{-2}\hat{\nu}$ , and thus

$$V_t Z_t^{\theta+\zeta} = LB(t,T), \quad V_t Z_t^{\theta-\zeta} = V_t Z_t^{2\hat{a}+1} = LB(t,T) Z_t^{2\hat{a}}.$$

It is also easy to see that in this case

$$h_1(V_t, T - t) = \frac{\ln(V_t/L) + \nu(T - t)}{\sigma\sqrt{T - t}} = -h_7(V_t, T - t),$$

while

$$h_2(V_t, T - t) = \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T - t)}{\sigma\sqrt{T - t}} = h_8(V_t, T - t).$$

We conclude that if  $\bar{v}(t) = Le^{-r(T-t)} = LB(t,T)$  then D(t,T) = LB(t,T). This result is quite intuitive. A corporate bond with a safety covenant represented by the barrier function, which equals the discounted value of the bond's face value, is equivalent to a default-free bond with the same face value and maturity.

• Case  $\gamma > r$ . For K = L and  $\gamma > r$ , it is natural to expect that D(t,T) would be smaller than LB(t,T). It is also possible to show that when  $\gamma$  tends to infinity (all other parameters being fixed), then the Black and Cox price converges to Merton's price.

## 1.2.3 Further Developments

The Black and Cox first-passage-time approach was later developed by, among others: Brennan and Schwartz [13, 14] – an analysis of convertible bonds, Nielsen et al. [55] – a random barrier and random interest rates, Leland [47], Leland and Toft [48] – a study of an optimal capital structure, bankruptcy costs and tax benefits, Longstaff and Schwartz [49] – a constant barrier and random interest rates, Brigo [15].

**Other stopping times.** In general, one can study the bond valuation problem for the default time given as

$$\tau = \inf \{ t \in \mathbb{R}_+ : V_t \le L(t) \},\$$

where L(t) is a deterministic function and V is a geometric Brownian motion. However, there exists few explicit results.

**Moraux's model.** Moraux [54] propose to model the default time as a *Parisian stopping time*. For a continuous process V and a given t > 0, we introduce  $g_t^b(V)$ , the last time before t at which the process V was at level b, that is,

$$g_t^b(V) = \sup \{ 0 \le s \le t : V_s = b \}.$$

The Parisian stopping time is the first time at which the process V is below the level b for a time period of length greater than D, that is,

$$G_D^{-,b}(V) = \inf \{ t \in \mathbb{R}_+ : (t - g_t^b(V)) \mathbb{1}_{\{V_t < b\}} \ge D \}.$$

Clearly, this time is a stopping time. Let  $\tau = G_D^{-,b}(V)$ . In the case of Black-Scholes dynamics, it is possible to find the joint law of  $(\tau, V_{\tau})$ 

Another default time is the first time where the process V has spend more than D time below a level, that is,  $\tau = \inf\{t \in \mathbb{R}_+ : A_t^V > D\}$  where  $A_t^V = \int_0^t \mathbb{1}_{\{V_s > b\}} ds$ . The law of this time is related to *cumulative options*.

**Campi and Sbuelz model.** Campi and Sbuelz [18] assume that the default time is given by a first hitting time of 0 by a CEV process, and they study the difficult problem of pricing an equity default swap. More precisely, they assume that the dynamics of the firm are

$$dS_t = S_{t-} \left( (r-\kappa) \, dt + \sigma S_t^\beta \, dW_t - dM_t \right)$$

where W is a Brownian motion and M the compensated martingale of a Poisson process (i.e.,  $M_t = N_t - \lambda t$ ), and they define

$$\tau = \inf \{ t \in \mathbb{R}_+ : S_t \le 0 \}.$$

In other terms, Campi and Sbuelz [18] set  $\tau = \tau^{\beta} \wedge \tau^{N}$ , where  $\tau^{N}$  is the first jump of the Poisson process and

$$\tau^{\beta} = \inf \left\{ t \in \mathbb{R}_+ : X_t \le 0 \right\}$$

where in turn

$$dX_t = X_{t-} \left( \left( r - \kappa + \lambda \right) dt + \sigma X_t^\beta dW_t \right).$$

Using that the CEV process can be expressed in terms of a time-changed Bessel process, and results on the hitting time of 0 for a Bessel process of dimension smaller than 2, they obtain closed from solutions.

Zhou's model. Zhou [61] studies the case where the dynamics of the firm is

$$dV_t = V_{t-} \left( \left( \mu - \lambda \nu \right) dt + \sigma \, dW_t + dX_t \right)$$

where W is a Brownian motion, X a compound Poisson process, that is,  $X_t = \sum_{1}^{N_t} e^{Y_i} - 1$  where  $\ln Y_i \stackrel{\text{law}}{=} N(a, b^2)$  with  $\nu = \exp(a + b^2/2) - 1$ . Note that for this choice of parameters the process  $V_t e^{-\mu t}$  is a martingale. Zhou first studies Merton's problem in that setting. Next, he gives an approximation for the first passage problem when the default time is  $\tau = \inf \{ t \in \mathbb{R}_+ : V_t \leq L \}$ .

## 1.2.4 Optimal Capital Structure

We consider a firm that has an interest paying bonds outstanding. We assume that it is a consol bond, which pays continuously coupon rate c. Assume that r > 0 and the payout rate  $\kappa$  is equal to zero. This condition can be given a financial interpretation as the restriction on the sale of assets, as opposed to issuing of new equity. Equivalently, we may think about a situation in which the stockholders will make payments to the firm to cover the interest payments. However, they have the right to stop making payments at any time and either turn the firm over to the bondholders or pay them a lump payment of c/r per unit of the bond's notional amount.

Recall that we denote by  $E(V_t)$  ( $D(V_t)$ , resp.) the value at time t of the firm equity (debt, resp.), hence the total value of the firm's assets satisfies  $V_t = E(V_t) + D(V_t)$ .

Black and Cox [8] argue that there is a critical level of the value of the firm, denoted as  $v^*$ , below which no more equity can be sold. The critical value  $v^*$  will be chosen by stockholders, whose aim is to minimize the value of the bonds (equivalently, to maximize the value of the equity). Let us

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observe that  $v^*$  is nothing else than a constant default barrier in the problem under consideration; the optimal default time  $\tau^*$  thus equals  $\tau^* = \inf \{ t \in \mathbb{R}_+ : V_t \leq v^* \}.$ 

To find the value of  $v^*$ , let us first fix the bankruptcy level  $\bar{v}$ . The ODE for the pricing function  $u^{\infty} = u^{\infty}(V)$  of a consol bond takes the following form (recall that  $\sigma = \sigma_V$ )

$$\frac{1}{2}V^2\sigma^2 u_{VV}^\infty + rVu_V^\infty + c - ru^\infty = 0,$$

subject to the lower boundary condition  $u^{\infty}(\bar{v}) = \min(\bar{v}, c/r)$  and the upper boundary condition

$$\lim_{V \to \infty} u_V^\infty(V) = 0.$$

For the last condition, observe that when the firm's value grows to infinity, the possibility of default becomes meaningless, so that the value of the defaultable consol bond tends to the value c/r of the default-free consol bond. The general solution has the following form:

$$u^{\infty}(V) = \frac{c}{r} + K_1 V + K_2 V^{-\alpha},$$

where  $\alpha = 2r/\sigma^2$  and  $K_1, K_2$  are some constants, to be determined from boundary conditions. We find that  $K_1 = 0$ , and

$$K_2 = \begin{cases} \bar{v}^{\alpha+1} - (c/r)\bar{v}^{\alpha}, & \text{if } \bar{v} < c/r, \\ 0, & \text{if } \bar{v} \ge c/r. \end{cases}$$

Hence, if  $\bar{v} < c/r$  then

$$u^{\infty}(V_t) = \frac{c}{r} + \left(\bar{v}^{\alpha+1} - \frac{c}{r}\bar{v}^{\alpha}\right)V_t^{-\alpha}$$

or, equivalently,

$$u^{\infty}(V_t) = \frac{c}{r} \left( 1 - \left(\frac{\bar{v}}{V_t}\right)^{\alpha} \right) + \bar{v} \left(\frac{\bar{v}}{V_t}\right)^{\alpha}.$$

It is in the interest of the stockholders to select the bankruptcy level in such a way that the value of the debt,  $D(V_t) = u^{\infty}(V_t)$ , is minimized, and thus the value of firm's equity

$$E(V_t) = V_t - D(V_t) = V_t - \frac{c}{r}(1 - \bar{q}_t) - \bar{v}\bar{q}_t$$

is maximized. It is easy to check that the optimal level of the barrier does not depend on the current value of the firm, and it equals

$$v^* = \frac{c}{r}\frac{\alpha}{\alpha+1} = \frac{c}{r+\sigma^2/2}.$$

Given the optimal strategy of the stockholders, the price process of the firm's debt (i.e., of a consol bond) takes the form, on the event  $\{\tau^* > t\}$ ,

$$D^*(V_t) = \frac{c}{r} - \frac{1}{\alpha V_t^{\alpha}} \left(\frac{c}{r + \sigma^2/2}\right)^{\alpha + 1}$$

or, equivalently,

$$D^*(V_t) = \frac{c}{r}(1 - q_t^*) + v^* q_t^*,$$

where

$$q_t^* = \left(\frac{v^*}{V_t}\right)^\alpha = \frac{1}{V_t^\alpha} \left(\frac{c}{r+\sigma^2/2}\right)^\alpha.$$

**Further developments.** We end this section by mentioning that other important developments in the area of optimal capital structure were presented in the papers by Leland [47], Leland and Toft [48], Christensen et al. [20]. Chen and Kou [19], Dao [23], Hilberink and Rogers [38], LeCourtois and Quittard-Pinon [46] study the same problem, but they model the firm's value process as a diffusion with jumps. The reason for this extension was to eliminate an undesirable feature of previously examined models, in which short spreads tend to zero when a bond approaches maturity date.

## **1.3** Stochastic Interest Rates

In this section, we assume that the underlying probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , endowed with the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ , supports the short-term interest rate process r and the value process V. The dynamics under the martingale measure  $\mathbb{Q}$  of the firm's value and of the price of a default-free zero-coupon bond B(t, T) are

$$dV_t = V_t ((r_t - \kappa(t)) dt + \sigma(t) dW_t)$$

and

$$dB(t,T) = B(t,T)(r_t dt + b(t,T) dW_t)$$

respectively, where W is a d-dimensional standard Q-Brownian motion. Furthermore,  $\kappa : [0,T] \to \mathbb{R}$ ,  $\sigma : [0,T] \to \mathbb{R}^d$  and  $b(\cdot,T) : [0,T] \to \mathbb{R}^d$  are assumed to be bounded functions. The forward value  $F_V(t,T) = V_t/B(t,T)$  of the firm satisfies under the forward martingale measure  $\mathbb{P}_T$ 

$$dF_V(t,T) = -\kappa(t)F_V(t,T)\,dt + F_V(t,T)\big(\sigma(t) - b(t,T)\big)\,dW_t^T$$

where the process  $W_t^T = W_t - \int_0^t b(u, T) \, du, t \in [0, T]$ , is a *d*-dimensional Brownian motion under  $\mathbb{P}_T$ . For any  $t \in [0, T]$ , we set

$$F_V^{\kappa}(t,T) = F_V(t,T)e^{-\int_t^T \kappa(u) \, du}.$$

Then

$$dF_V^{\kappa}(t,T) = F_V^{\kappa}(t,T) \big(\sigma(t) - b(t,T)\big) \, dW_t^T$$

Furthermore, it is apparent that  $F_V^{\kappa}(T,T) = F_V(T,T) = V_T$ . We consider the following modification of the Black and Cox approach

$$X = L, \ Z_t = \beta_2 V_t, \ \tilde{X} = \beta_1 V_T, \ \tau = \inf \{ t \in [0, T] : V_t < v_t \},$$

where  $\beta_2, \beta_1 \in [0, 1]$  are constants, and the barrier v is given by the formula

$$v_t = \begin{cases} KB(t,T)e^{\int_t^T \kappa(u) \, du} & \text{for } t < T, \\ L & \text{for } t = T, \end{cases}$$

with the constant K satisfying  $0 < K \leq L$ .

Let us denote, for any  $t \leq T$ ,

$$\kappa(t,T) = \int_t^T \kappa(u) \, du, \quad \sigma^2(t,T) = \int_t^T |\sigma(u) - b(u,T)|^2 \, du$$

where  $|\cdot|$  is the Euclidean norm in  $\mathbb{R}^d$ . For brevity, we write  $F_t = F_V^{\kappa}(t,T)$ , and we denote

$$\eta_{+}(t,T) = \kappa(t,T) + \frac{1}{2}\sigma^{2}(t,T), \quad \eta_{-}(t,T) = \kappa(t,T) - \frac{1}{2}\sigma^{2}(t,T).$$

The following result extends Black and Cox valuation formula for a corporate bond to the case of random interest rates.

**Proposition 1.3.1** For any t < T, the forward price of a defaultable bond  $F_D(t,T) = D(t,T)/B(t,T)$  equals on the set  $\{\tau > t\}$ 

$$L(N(\hat{h}_{1}(F_{t},t,T)) - (F_{t}/K)e^{-\kappa(t,T)}N(\hat{h}_{2}(F_{t},t,T))) + \beta_{1}F_{t}e^{-\kappa(t,T)}(N(\hat{h}_{3}(F_{t},t,T)) - N(\hat{h}_{4}(F_{t},t,T))) + \beta_{1}K(N(\hat{h}_{5}(F_{t},t,T)) - N(\hat{h}_{6}(F_{t},t,T))) + \beta_{2}KJ_{+}(F_{t},t,T) + \beta_{2}F_{t}e^{-\kappa(t,T)}J_{-}(F_{t},t,T),$$

where

$$\begin{split} \widehat{h}_{1}(F_{t},t,T) &= \frac{\ln{(F_{t}/L)} - \eta_{+}(t,T)}{\sigma(t,T)}, \\ \widehat{h}_{2}(F_{t},T,t) &= \frac{2\ln{K} - \ln(LF_{t}) + \eta_{-}(t,T)}{\sigma(t,T)}, \\ \widehat{h}_{3}(F_{t},t,T) &= \frac{\ln{(L/F_{t})} + \eta_{-}(t,T)}{\sigma(t,T)}, \\ \widehat{h}_{4}(F_{t},t,T) &= \frac{\ln{(K/F_{t})} + \eta_{-}(t,T)}{\sigma(t,T)}, \\ \widehat{h}_{5}(F_{t},t,T) &= \frac{2\ln{K} - \ln(LF_{t}) + \eta_{+}(t,T)}{\sigma(t,T)}, \\ \widehat{h}_{6}(F_{t},t,T) &= \frac{\ln{(K/F_{t})} + \eta_{+}(t,T)}{\sigma(t,T)}, \end{split}$$

and for any fixed  $0 \le t < T$  and  $F_t > 0$  we set

$$J_{\pm}(F_t, t, T) = \int_t^T e^{\kappa(u, T)} dN \left( \frac{\ln(K/F_t) + \kappa(t, T) \pm \frac{1}{2}\sigma^2(t, u)}{\sigma(t, u)} \right).$$

In the special case when  $\kappa \equiv 0$ , the formula of Proposition 1.3.1 covers as a special case the valuation result established by Briys and de Varenne [16]. In some other recent studies of first passage time models, in which the triggering barrier is assumed to be either a constant or an unspecified stochastic process, typically no closed-form solution for the value of a corporate debt is available, and thus a numerical approach is required (see, for instance, Longstaff and Schwartz [49], Nielsen et al. [55], or Saá-Requejo and Santa-Clara [58]).

## 1.4 Random Barrier

In the case of full information and Brownian filtration, the first hitting time of a deterministic barrier is predictable. This is no longer the case when we deal with incomplete information (as in Duffie and Lando [28]), or when an additional source of randomness is present. We present here a formula for credit spreads arising in a special case of a totally inaccessible time of default. For a more detailed study we refer to Babbs and Bielecki [1]. As we shall see, the method we use here is close to the general method presented in Chapter 3.

We suppose here that the default barrier is a random variable  $\eta$  defined on the underlying probability space  $(\Omega, \mathbb{P})$ . The default occurs at time  $\tau$  where

$$\tau = \inf\{t : V_t \le \eta\},\$$

where V is the value of the firm and, for simplicity,  $V_0 = 1$ . Note that

$$\{\tau > t\} = \{\inf_{u \le t} V_u > \eta\}.$$

We shall denote by  $m_t^V$  the running minimum of V, i.e.  $m_t^V = \inf_{u \le t} V_u$ . With this notation,  $\{\tau > t\} = \{m_t^V > \eta\}$ . Note that  $m^V$  is a decreasing process.

#### 1.4.1 Independent Barrier

In a first step we assume that, under the risk-neutral probability  $\mathbb{Q}$ , a random variable  $\eta$  modelling is independent of the value of the firm. We denote by  $F_{\eta}$  the cumulative distribution function of  $\eta$ , i.e.,  $F_{\eta}(z) = \mathbb{Q}(\eta \leq z)$ . We assume that  $F_{\eta}$  is differentiable and we denote by  $f_{\eta}$  its derivative. **Lemma 1.4.1** Let  $F_t = \mathbb{Q}(\tau \leq t | \mathcal{F}_t)$  and  $\Gamma_t = -\ln(1 - F_t)$ . Then

$$\Gamma_t = -\int_0^t \frac{f_\eta(m_u^V)}{F_\eta(m_u^V)} \, dm_u^V.$$

*Proof.* If  $\eta$  is independent of  $\mathcal{F}_{\infty}$ , then

$$F_t = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{Q}(m_t^V \le \eta \,|\, \mathcal{F}_t) = 1 - F_\eta(m_t^V).$$

The process  $m^V$  is decreasing. It follows that  $\Gamma_t = -\ln F_\eta(m_t^V)$ , hence  $d\Gamma_t = -\frac{f_\eta(m_t^V)}{F_\eta(m_t^V)} dm_t^V$  and

$$\Gamma_t = -\int_0^t \frac{f_\eta(m_u^V)}{F_\eta(m_u^V)} \, dm_u^V$$

as expected.

**Example 1.4.1** Assume that  $\eta$  is uniformly distributed on the interval [0, 1]. Then  $\Gamma_t = -\ln m_t^V$ . The computation of the expected value  $\mathbb{E}_{\mathbb{Q}}(e^{\Gamma_T} f(V_T))$  requires the knowledge of the joint law of the pair  $(V_T, m_T^V)$ .

We postulate now that the value process V is a geometric Brownian motion with a drift, that is, we set  $V_t = e^{\Psi_t}$ , where  $\Psi_t = \mu t + \sigma W_t$ . It is clear that  $\tau = \inf \{ t \in \mathbb{R}_+ : \Psi_t^* \leq \psi \}$ , where  $\Psi^*$  is the running minimum of the process  $\Psi$ :  $\Psi_t^* = \inf \{ \Psi_s : 0 \leq s \leq t \}$ .

We choose the Brownian filtration as the reference filtration, i.e., we set  $\mathbb{F} = \mathbb{F}^W$ . Let us denote by G(z) the cumulative distribution function under  $\mathbb{Q}$  of the barrier  $\psi$ . We assume that G(z) > 0 for z < 0 and that G admits the density g with respect to the Lebesgue measure (note that g(z) = 0 for z > 0). This means that we assume that the value process V (hence also the process  $\Psi$ ) is perfectly observed.

In addition, we postulate that the bond investor can observe the occurrence of the default time. Thus, he can observe the process  $H_t = \mathbb{1}_{\{\tau \leq t\}} = \mathbb{1}_{\{\Psi_t^* \leq \psi\}}$ . We denote by  $\mathbb{H}$  the natural filtration of the process H. The information available to the investor is represented by the (enlarged) filtration  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ .

We assume that the default time  $\tau$  and interest rates are independent under  $\mathbb{Q}$ . It is then is possible to establish the following result (see Giesecke [36] or Babbs and Bielecki [1]). Note that the process  $\Psi^*$  is decreasing, so that the integral with respect to this process is a (pathwise) Stieltjes integral.

**Proposition 1.4.1** Under the assumptions stated above, and additionally assuming  $L = 1, Z \equiv 0$ and  $\tilde{X} = 0$ , we have that for every t < T

$$S(t,T) = -\mathbb{1}_{\{\tau > t\}} \frac{1}{T-t} \ln \mathbb{E}_{\mathbb{Q}} \Big( e^{\int_t^T \frac{f_{\eta}(\Psi_u^*)}{F_{\eta}(\Psi_u^*)} d\Psi_u^*} \, \Big| \, \mathcal{F}_t \Big).$$

Later on, we will introduce the notion of a hazard process of a random time. For the default time  $\tau$  defined above, the F-hazard process  $\Gamma$  exists and is given by the formula

$$\Gamma_t = -\int_0^t \frac{f_\eta(\Psi_u^*)}{F_\eta(\Psi_u^*)} d\Psi_u^*.$$

This process is continuous, and thus the default time  $\tau$  is a totally inaccessible stopping time with respect to the filtration  $\mathbb{G}$ .

## Chapter 2

## Hazard Function Approach

In this chapter, we provide a detailed analysis of the very special case of the *reduced form* methodology, when the flow of information available to an agent reduces to the observations of the random time which models the default event. The focus is on the evaluation of conditional expectations with respect to the filtration generated by a default time with the use of the hazard function.

## 2.1 The Toy Model

We begin with the simple case where a riskless asset, with deterministic interest rate  $(r(s); s \ge 0)$  is the only asset available in the default-free market. The price at time t of a risk-free zero-coupon bond with maturity T equals

$$B(t,T) = \exp\Big(-\int_t^T r(s)\,ds\Big).$$

Default occurs at time  $\tau$ , where  $\tau$  is assumed to be a positive random variable with density f, constructed on a probability space  $(\Omega, \mathcal{G}, \mathbb{Q})$ . We denote by F the cumulative function of the random variable  $\tau$  defined as  $F(t) = \mathbb{Q}(\tau \leq t) = \int_0^t f(s) ds$  and we assume that F(t) < 1 for any t > 0. Otherwise, there would exists a date  $t_0$  for which  $F(t_0) = 1$ , so that the default would occurs before or at  $t_0$  with probability 1.

We emphasize that the random payoff of the form  $\mathbb{1}_{\{T < \tau\}}$  cannot be perfectly hedged with deterministic zero-coupon bonds, which are the only tradeable primary assets in our model. To hedge the risk, we shall later postulate that some defaultable asset is traded, e.g., a defaultable zero-coupon bond or a credit default swap.

It is not difficult to generalize the study presented in what follows to the case where  $\tau$  does not admit a density, by dealing with the right-continuous version of the cumulative function. The case where  $\tau$  is bounded can also be studied along the same method. We leave the details to the reader.

#### 2.1.1 Defaultable Zero-Coupon Bond with Recovery at Maturity

A defaultable zero-coupon bond (DZC in short), or a corporate zero-coupon bond, with maturity T and the rebate (recovery)  $\delta$  paid at maturity, consists of:

- The payment of one monetary unit at time T if default has not occurred before time T, i.e., if  $\tau > T$ ,
- A payment of  $\delta$  monetary units, made at maturity, if  $\tau \leq T$ , where  $0 < \delta < 1$ .

#### Value of the Defaultable Zero-Coupon Bond

The "fair value" of the defaultable zero-coupon bond is defined as the expectation of discounted payoffs

$$D^{(\delta)}(0,T) = B(0,T) \mathbb{E}_{\mathbb{Q}} (\mathbb{1}_{\{T < \tau\}} + \delta \mathbb{1}_{\{\tau \le T\}})$$
  
=  $B(0,T) \mathbb{E}_{\mathbb{Q}} (1 - (1 - \delta) \mathbb{1}_{\{\tau \le T\}})$   
=  $B(0,T) (1 - (1 - \delta) F(T)).$  (2.1)

In fact, this quantity is a net present value and is equal to the value of the default free zero-coupon bond minus the expected loss, computed under the historical probability. Obviously, this value is not a hedging price.

The time-t value depends whether or not default has happened before this time. If default has occurred before time t, the payment of  $\delta$  will be made at time T, and the price of the DZC is  $\delta B(t,T)$ .

If the default has not yet occurred, the holder does not know when it will occur. The value  $D^{(\delta)}(t,T)$  of the DZC is the conditional expectation of the discounted payoff

$$B(t,T) \left( \mathbb{1}_{\{T < \tau\}} + \delta \mathbb{1}_{\{\tau \le T\}} \right)$$

given the information available at time t. We obtain

$$D^{(\delta)}(t,T) = \mathbb{1}_{\{\tau \le t\}} B(t,T)\delta + \mathbb{1}_{\{t < \tau\}} D^{(\delta)}(t,T)$$

where the pre-default value  $\widetilde{D}^{(\delta)}$  is defined as

$$\widetilde{D}^{(\delta)}(t,T) = \mathbb{E}_{\mathbb{Q}}\left(B(t,T)\left(\mathbb{1}_{\{T<\tau\}} + \delta\mathbb{1}_{\{\tau\leq T\}}\right) \mid t<\tau\right) \\
= B(t,T)\left(1 - (1-\delta)\mathbb{Q}(\tau\leq T \mid t<\tau)\right) \\
= B(t,T)\left(1 - (1-\delta)\frac{\mathbb{Q}(t<\tau\leq T)}{\mathbb{Q}(t<\tau)}\right) \\
= B(t,T)\left(1 - (1-\delta)\frac{F(T) - F(t)}{1 - F(t)}\right).$$
(2.2)

Note that the value of the DZC is discontinuous at time  $\tau$ , unless F(T) = 1 (or  $\delta = 1$ ). In the case F(T) = 1, the default appears with probability one before maturity and the DZC is equivalent to a payment of  $\delta$  at maturity. If  $\delta = 1$ , the DZC is simply a default-free zero coupon bond.

Formula (2.2) can be rewritten as follows

$$D^{(\delta)}(t,T) = B(t,T) - \text{EDLGD} \times \text{DP}$$

where the expected discounted loss given default (EDLGD) is defined as  $B(t,T)(1-\delta)$  and the conditional default probability (DP) is defined as follows

$$DP = \frac{\mathbb{Q}(t < \tau \le T)}{\mathbb{Q}(t < \tau)} = \mathbb{Q}(\tau \le T \mid t < \tau) .$$

In case the payment is a function of the default time, say  $\delta(\tau)$ , the value of this defaultable zerocoupon is

$$D^{(\delta)}(0,T) = \mathbb{E}_{\mathbb{Q}} \Big( B(0,T) \, \mathbb{1}_{\{T < \tau\}} + B(0,T) \delta(\tau) \mathbb{1}_{\{\tau \le T\}} \Big) \\ = B(0,T) \Big( \mathbb{Q}(T < \tau) + \int_0^T \delta(s) f(s) \, ds \Big).$$

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If the default has not occurred before t, the pre-default time-t value  $\widetilde{D}^{(\delta)}(t,T)$  satisfies

$$\begin{split} \widetilde{D}^{(\delta)}(t,T) &= B(t,T) \mathbb{E}_{\mathbb{Q}}(\mathbbm{1}_{\{T < \tau\}} + \delta(\tau) \mathbbm{1}_{\{\tau \le T\}} \mid t < \tau) \\ &= B(t,T) \Big( \frac{\mathbb{Q}(T < \tau)}{\mathbb{Q}(t < \tau)} + \frac{1}{\mathbb{Q}(t < \tau)} \int_{t}^{T} \delta(s) f(s) \, ds \Big) \end{split}$$

To summarize, we have

$$D^{(\delta)}(t,T) = \mathbb{1}_{\{t < \tau\}} \widetilde{D}^{(\delta)}(t,T) + \mathbb{1}_{\{\tau \le t\}} \,\delta(\tau) \,B(t,T).$$

## Hazard Function

Let us recall the standing assumption that F(t) < 1 for any  $t \in \mathbb{R}_+$ . We introduce the hazard function  $\Gamma$  by setting

$$\Gamma(t) = -\ln(1 - F(t))$$

for any  $t \in \mathbb{R}_+$ . Since we assumed that F is differentiable, the derivative  $\Gamma'(t) = \gamma(t) = \frac{f(t)}{1 - F(t)}$ , where f(t) = F'(t). This means that

$$1 - F(t) = e^{-\Gamma(t)} = \exp\left(-\int_0^t \gamma(s) \, ds\right) = \mathbb{Q}(\tau > t).$$

The quantity  $\gamma(t)$  is the hazard rate. The interpretation of the hazard rate is the probability that the default occurs in a small interval dt given that the default did not occur before time t

$$\gamma(t) = \lim_{h \to 0} \frac{1}{h} \mathbb{Q}(\tau \le t + h \mid \tau > t).$$

Note that  $\Gamma$  is increasing.

Then formula (2.2) reads

$$\widetilde{D}^{(\delta)}(t,T) = B(t,T) \left( \frac{1-F(T)}{1-F(t)} + \delta \frac{F(T)-F(t)}{1-F(t)} \right) = R_T^{t,d} + \delta \left( B(t,T) - R_T^{t,d} \right),$$

where we denote

$$R_T^{t,d} = \exp\Big(-\int_t^T (r(s) + \gamma(s))\,ds\Big).$$

In particular, for  $\delta = 0$ , we obtain  $\widetilde{D}(t,T) = R_T^{t,d}$ . Hence the spot rate has simply to be adjusted by means of the *credit spread* (equal to  $\gamma$ ) in order to evaluate DZCs with zero recovery.

The dynamics of  $\widetilde{D}^{(\delta)}$  can be easily written in terms of the function  $\gamma$  as

$$d\widetilde{D}^{(\delta)}(t,T) = (r(t) + \gamma(t))\,\widetilde{D}^{(\delta)}(t,T)\,dt - B(t,T)\gamma(t)\delta(t)\,dt.$$

The dynamics of  $D^{(\delta)}(t,T)$  will be derived in the next section.

If  $\gamma$  and  $\delta$  are constant, the credit spread equals

$$\frac{1}{T-t}\ln\frac{B(t,T)}{\widetilde{D}^{(\delta)}(t,T)} = \gamma - \frac{1}{T-t}\ln\left(1 + \delta(e^{\gamma(T-t)} - 1)\right)$$

and it converges to  $\gamma(1-\delta)$  when t goes to T.

For any t < T, the quantity  $\gamma(t,T) = \frac{f(t,T)}{1-F(t,T)}$  where

$$F(t,T) = \mathbb{Q}(\tau \le T \mid \tau > t)$$

and  $f(t,T) dT = \mathbb{Q}(\tau \in dT | \tau > t)$  is called the *conditional hazard rate*. It is easily seen that

$$F(t,T) = 1 - \exp\Big(-\int_t^T \gamma(s,T)\,ds\Big).$$

Note, however, that in the present setting, we have that

$$1 - F(t,T) = \frac{\mathbb{Q}(\tau > T)}{\mathbb{Q}(\tau > t)} = \exp\left(-\int_t^T \gamma(s) \, ds\right)$$

and thus  $\gamma(s,T) = \gamma(s)$ .

**Remark 2.1.1** In the case where  $\tau$  is the first jump of an inhomogeneous Poisson process with deterministic intensity  $(\lambda(t), t \ge 0)$ , we have

$$f(t) = \frac{\mathbb{Q}(\tau \in dt)}{dt} = \lambda(t) \exp\left(-\int_0^t \lambda(s) \, ds\right) = \lambda(t) e^{-\Lambda(t)}$$

where  $\Lambda(t) = \int_0^t \lambda(s) \, ds$  and  $\mathbb{Q}(\tau \leq t) = F(t) = 1 - e^{-\Lambda(t)}$ . Hence the hazard function is equal to the compensator of the Poisson process, that is,  $\Gamma(t) = \Lambda(t)$ . Conversely, if  $\tau$  is a random time with the density f, setting  $\Lambda(t) = -\ln(1 - F(t))$  allows us to interpret  $\tau$  as the first jump time of an inhomogeneous Poisson process with the intensity equal to the derivative of  $\Lambda$ .

## 2.1.2 Defaultable Zero-Coupon with Recovery at Default

By a defaultable zero-coupon bond with maturity T we mean here a security consisting of:

- The payment of one monetary unit at time T if default has not yet occurred,
- The payment of  $\delta(\tau)$  monetary units, where  $\delta$  is a deterministic function, made at time  $\tau$  if  $\tau \leq T$ , that is, if default occurred prior to or at bond's maturity.

#### Value of the Defaultable Zero-Coupon

The value of this defaultable zero-coupon bond is

$$D^{(\delta)}(0,T) = \mathbb{E}_{\mathbb{Q}} \Big( B(0,T) \, \mathbb{1}_{\{T < \tau\}} + B(0,\tau) \delta(\tau) \mathbb{1}_{\{\tau \le T\}} \Big) \\ = \mathbb{Q}(T < \tau) B(0,T) + \int_{0}^{T} B(0,s) \delta(s) \, dF(s) \\ = G(T) B(0,T) - \int_{0}^{T} B(0,s) \delta(s) \, dG(s),$$
(2.3)

where  $G(t) = 1 - F(t) = \mathbb{Q}(t < \tau)$  is the survival probability. Obviously, if the default has occurred before time t, the value of the DZC is null (this was not the case for the recovery payment made at bond's maturity), and  $D^{(\delta)}(t,T) = \mathbb{1}_{\{t < \tau\}} \widetilde{D}^{(\delta)}(t,T)$  where  $\widetilde{D}^{(\delta)}(t,T)$  is a deterministic function (the predefault price). The pre-default time-t value  $\widetilde{D}^{(\delta)}(t,T)$  satisfies

$$\begin{split} B(0,t)\tilde{D}^{(\delta)}(t,T) &= & \mathbb{E}_{\mathbb{Q}}\big(B(0,T)\,\mathbb{1}_{\{T<\tau\}} + B(0,\tau)\delta(\tau)\mathbb{1}_{\{\tau\leq T\}}\,|\,t<\tau\big) \\ &= & \frac{\mathbb{Q}(T<\tau)}{\mathbb{Q}(t<\tau)}\,B(0,T) + \frac{1}{\mathbb{Q}(t<\tau)}\int_{t}^{T}B(0,s)\delta(s)\,dF(s). \end{split}$$

Hence

$$R(t)G(t)\widetilde{D}^{(\delta)}(t,T) = G(T)B(0,T) - \int_t^T B(0,s)\delta(s) \, dG(s)$$

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In terms of the hazard function  $\Gamma$ , we get

$$\widetilde{D}^{(\delta)}(0,T) = e^{-\Gamma(T)}B(0,T) + \int_0^T B(0,s)e^{-\Gamma(s)}\delta(s)\,d\Gamma(s).$$
(2.4)

The time-t value  $\widetilde{D}^{(\delta)}(t,T)$  satisfies

$$B(0,t)e^{-\Gamma(t)}\widetilde{D}^{(\delta)}(t,T) = e^{-\Gamma(T)}B(0,T) + \int_{t}^{T} B(0,s)e^{-\Gamma(s)}\delta(s) \,d\Gamma(s).$$

Note that the process  $t \to D^{(\delta)}(t,T)$  admits a discontinuity at time  $\tau$ .

#### A Particular Case

If F is differentiable then the function  $\gamma = \Gamma'$  satisfies  $f(t) = \gamma(t)e^{-\Gamma(t)}$ . Then

$$\widetilde{D}^{(\delta)}(0,T) = e^{-\Gamma(T)}B(0,T) + \int_{0}^{T} B(0,s)\gamma(s)e^{-\Gamma(s)}\delta(s) \, ds, \qquad (2.5)$$
$$= R^{d}(T) + \int_{0}^{T} R^{d}(s)\gamma(s)\delta(s) \, ds,$$

and

$$R^{d}(t)\widetilde{D}^{(\delta)}(t,T) = R^{d}(T) + \int_{t}^{T} R^{d}(s)\gamma(s)\delta(s) \, ds$$

with

$$R^{d}(t) = \exp\Big(-\int_{0}^{t} (r(s) + \gamma(s)) \, ds\Big).$$

The 'defaultable interest rate' is  $r + \gamma$  and is, as expected, greater than r (the value of a DZC with  $\delta = 0$  is smaller than the value of a default-free zero-coupon). The dynamics of  $\widetilde{D}^{(\delta)}(t,T)$  are

$$d\widetilde{D}^{(\delta)}(t,T) = \left( (r(t) + \gamma(t))\widetilde{D}^{(\delta)}(t,T) - \delta(t)\gamma(t)) \right) dt.$$

The dynamics of  $D^{(\delta)}(t,T)$  include a jump at time  $\tau$  (see the next section).

#### Fractional Recovery of Treasury Value

This case corresponds to the following recovery  $\delta(t) = \delta B(t,T)$  at the moment of default. Under this convention, we have that

$$D^{(\delta)}(t,T) = \mathbb{1}_{\{t < \tau\}} \left( e^{-\int_t^T (r(s) + \gamma(s) \, ds} + \delta B(t,T) \int_t^T \gamma(s) e^{\int_t^s \gamma(u) \, du} \, ds \right).$$

#### Fractional Recovery of Market Value

~ . .

Let us assume here that the recovery is  $\delta(t) = \delta \widetilde{D}^{(\delta)}(t,T)$  where  $\delta$  is a constant, that is, the recovery is  $\delta D^{(\delta)}(\tau -, T)$ . The dynamics of  $\widetilde{D}^{(\delta)}$  are

$$d\widetilde{D}^{(\delta)}(t,T) = \left(r(t) + \gamma(t)(1 - \delta(t))\right)\widetilde{D}^{(\delta)}(t,T) dt,$$

hence

$$\widetilde{D}^{(\delta)}(t,T) = \exp\left(-\int_t^T r(s)ds - \int_t^T \gamma(s)(1-\delta(s))\,ds\right).$$

## 2.1.3 Implied Default Probabilities

If defaultable zero-coupon bonds with zero recovery are traded in the market at price  $D^{(\delta,*)}(t,T)$ , the implied survival probability is  $\mathbb{Q}^*$  such that

$$\mathbb{Q}^*(\tau > T \mid \tau > t) = \frac{D^{(\delta,*)}(t,T)}{B(t,T)}.$$

Of course, this probability may differ from the historical probability. The implied hazard rate is the function  $\lambda(t, T)$  such that

$$\lambda(t,T) = -\frac{\partial}{\partial T} \ln \frac{D^{(\delta,*)}(t,T)}{B(t,T)} = \gamma^*(T).$$

In the toy model, the implied hazard rate is not very interesting. The aim is to obtain

$$\widetilde{D}^{(\delta,*)}(t,T) = B(t,T) \exp\Big(-\int_t^T \lambda(t,s)\,ds\Big).$$

This approach will be useful when the pre-default price is stochastic, rather than deterministic.

### 2.1.4 Credit Spreads

A term structure of credit spreads associated with the zero-coupon bonds S(t,T) is defined as

$$S(t,T) = -\frac{1}{T-t} \ln \frac{D^{(\delta,*)}(t,T)}{B(t,T)}.$$

In our setting, on the set  $\{\tau > t\}$ 

$$S(t,T) = -\frac{1}{T-t} \ln \mathbb{Q}^*(\tau > T \,|\, \tau > t),$$

whereas  $S(t,T) = \infty$  on the set  $\{\tau \leq t\}$ .

## 2.2 Martingale Approach

We shall now present the results of the previous section in a different form, following rather closely Dellacherie ([24], page 122). We keep the standing assumption that F(t) < 1 for any  $t \in \mathbb{R}_+$ , but we do impose any further assumptions on the c.d.f. F of  $\tau$  under  $\mathbb{Q}$  at this stage.

**Definition 2.2.1** The hazard function  $\Gamma$  by setting

$$\Gamma(t) = -\ln(1 - F(t))$$

for any  $t \in \mathbb{R}_+$ .

We denote by  $(H_t, t \ge 0)$  the right-continuous increasing process  $H_t = \mathbb{1}_{\{t \ge \tau\}}$  and by  $(\mathcal{H}_t)$  its natural filtration. The filtration  $\mathbb{H}$  is the smallest filtration which makes  $\tau$  a stopping time. The  $\sigma$ -algebra  $\mathcal{H}_t$  is generated by the sets  $\{\tau \le s\}$  for  $s \le t$ . The key point is that any integrable  $\mathcal{H}_t$ -measurable r.v. H has the form

$$H = h(\tau) \mathbb{1}_{\{\tau \le t\}} + h(t) \mathbb{1}_{\{t < \tau\}}$$

where h is a Borel function.

We now give some elementary formula for the computation of a conditional expectation with respect to  $\mathcal{H}_t$ , as presented, for instance, in Brémaud [11], Dellacherie [24], or Elliott [30].

**Remark 2.2.1** Note that if the cumulative distribution function F is continuous then  $\tau$  is known to be a  $\mathbb{H}$ -totally inaccessible stopping time (see Dellacherie and Meyer [27] IV, Page 107). We will not use this property explicitly.

### 2.2.1 Key Lemma

**Lemma 2.2.1** For any integrable,  $\mathcal{G}$ -measurable r.v. X we have that

$$\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{H}_s) \mathbb{1}_{\{s < \tau\}} = \mathbb{1}_{\{s < \tau\}} \frac{\mathbb{E}_{\mathbb{Q}}(X \mathbb{1}_{\{s < \tau\}})}{\mathbb{Q}(s < \tau)}.$$
(2.6)

*Proof.* The conditional expectation  $\mathbb{E}_{\mathbb{Q}}(X | \mathcal{H}_s)$  is clearly  $\mathcal{H}_s$ -measurable. Therefore, it can be written in the form

$$\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{H}_s) = h(\tau) \mathbb{1}_{\{s \ge \tau\}} + h(s) \mathbb{1}_{\{s < \tau\}}$$

for some Borel function h. By multiplying both members by  $\mathbbm{1}_{\{s<\tau\}},$  and taking the expectation, we obtain

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{s<\tau\}}\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{H}_s)] = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}}X \mid \mathcal{H}_s)] = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}}X)$$
  
$$= \mathbb{E}_{\mathbb{Q}}(h(s)\mathbb{1}_{\{s<\tau\}}) = h(s)\mathbb{Q}(s<\tau).$$

Hence  $h(s) = \frac{\mathbb{E}_{\mathbb{Q}}(X \mathbb{1}_{\{s < \tau\}})}{\mathbb{Q}(s < \tau)}$ , which yields the desired result.

**Corollary 2.2.1** Assume that Y is  $\mathcal{H}_{\infty}$ -measurable, so that  $Y = h(\tau)$  for some Borel measurable function  $h : \mathbb{R}_+ \to \mathbb{R}$ . If the hazard function  $\Gamma$  of  $\tau$  is continuous then

$$\mathbb{E}_{\mathbb{Q}}(Y \mid \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{t < \tau\}} \int_t^\infty h(u) e^{\Gamma(t) - \Gamma(u)} d\Gamma(u).$$
(2.7)

If  $\tau$  admits the intensity function  $\gamma$  then

$$\mathbb{E}_{\mathbb{Q}}(Y \mid \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{t < \tau\}} \int_t^\infty h(u) \gamma(u) e^{-\int_t^u \gamma(v) \, dv} \, du.$$

In particular, for any  $t \leq s$  we have

$$\mathbb{Q}(\tau > s \,|\, \mathcal{H}_t) = \mathbb{1}_{\{t < \tau\}} e^{-\int_t^s \gamma(v) \,dv}$$

and

$$\mathbb{Q}(t < \tau < s \,|\, \mathcal{H}_t) = \mathbb{1}_{\{t < \tau\}} \Big( 1 - e^{-\int_t^s \gamma(v) \,dv} \Big).$$

## 2.2.2 Martingales Associated with Default Time

**Proposition 2.2.1** The process  $(M_t, t \ge 0)$  defined as

$$M_t = H_t - \int_0^{\tau \wedge t} \frac{dF(s)}{1 - F(s)} = H_t - \int_0^t (1 - H_{s-}) \frac{dF(s)}{1 - F(s)}$$

is an  $\mathbb H\text{-}martingale.$ 

*Proof.* Let s < t. Then:

$$\mathbb{E}_{\mathbb{Q}}(H_t - H_s \,|\, \mathcal{H}_s) = \mathbb{1}_{\{s < \tau\}} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s < \tau \le t\}} \,|\, \mathcal{H}_s) = \mathbb{1}_{\{s < \tau\}} \frac{F(t) - F(s)}{1 - F(s)},\tag{2.8}$$

which follows from (2.6) with  $X = \mathbb{1}_{\{\tau \le t\}}$ .

On the other hand, the quantity

$$C \stackrel{\text{def}}{=} \mathbb{E}_{\mathbb{Q}} \left[ \int_{s}^{t} (1 - H_{u-}) \frac{dF(u)}{1 - F(u)} \, \big| \, \mathcal{H}_{s} \right],$$

is equal to

$$C = \int_{s}^{t} \frac{dF(u)}{1 - F(u)} \mathbb{E}_{\mathbb{Q}} \left[ \mathbb{1}_{\{\tau > u\}} \middle| \mathcal{H}_{s} \right]$$
  
$$= \mathbb{1}_{\{\tau > s\}} \int_{s}^{t} \frac{dF(u)}{1 - F(u)} \left( 1 - \frac{F(u) - F(s)}{1 - F(s)} \right)$$
  
$$= \mathbb{1}_{\{\tau > s\}} \left( \frac{F(t) - F(s)}{1 - F(s)} \right)$$

which, in view of (2.8), proves the result.

The function

$$\int_{0}^{t} \frac{dF(s)}{1 - F(s)} = -\ln(1 - F(t)) = \Gamma(t)$$

is the hazard function.

From Proposition 2.2.1, we obtain the Doob-Meyer decomposition of the submartingale  $H_t$  as  $M_t + \Gamma(t \wedge \tau)$ . The predictable process  $A_t = \Gamma(t \wedge \tau)$  is called the *compensator* of H.

In particular, if F is differentiable, the process

$$M_t = H_t - \int_0^{\tau \wedge t} \gamma(s) \, ds = H_t - \int_0^t \gamma(s) (1 - H_s) \, ds$$

is a martingale, where  $\gamma(s) = \frac{f(s)}{1 - F(s)}$  is a deterministic, non-negative function, called *the intensity* of  $\tau$ .

**Proposition 2.2.2** Assume that F (and thus also  $\Gamma$ ) is a continuous function. Then the process  $M_t = H_t - \Gamma(t \wedge \tau)$  follows a  $\mathbb{H}$ -martingale.

We can now write the dynamics of a defaultable zero-coupon bond with recovery  $\delta$  paid at hit, assuming that M is a martingale under the risk-neutral probability.

Proposition 2.2.3 The risk-neutral dynamics of a DZC with recovery paid at hit is

$$dD^{(\delta)}(t,T) = \left(r(t)D^{(\delta)}(t,T) - \delta(t)\gamma(t)(1-H_t)\right)dt - \widetilde{D}^{(\delta)}(t,T)\,dM_t \tag{2.9}$$

where M is the risk-neutral martingale  $M_t = H_t - \int_0^t (1 - H_s) \gamma_s \, ds.$ 

*Proof.* Combining the equality

$$D^{(\delta)}(t,T) = \mathbb{1}_{t < \tau} \widetilde{D}^{(\delta)}(t,T) = (1 - H_t) \widetilde{D}^{(\delta)}(t,T)$$

with the dynamics of  $\widetilde{D}^{(\delta)}(t,T)$ , we obtain

$$dD^{(\delta)}(t,T) = (1-H_t)d\tilde{D}^{(\delta)}(t,T) - \tilde{D}^{(\delta)}(t,T) dH_t$$
  
=  $(1-H_t) \left( (r(t)+\gamma(t))\tilde{D}^{(\delta)}(t,T) - \delta(t)\gamma(t) \right) dt - \tilde{D}^{(\delta)}(t,T) ) dH_t$   
=  $\left( r(t)D^{(\delta)}(t,T) - \delta(t)\gamma(t)(1-H_t) \right) dt - \tilde{D}^{(\delta)}(t,T) dM_t.$ 

We emphasize that we work here under a risk-neutral probability. We shall see further on how to compute the risk-neutral default intensity from historical one, using a suitable Radon-Nikodým density process.  $\hfill \Box$ 

**Proposition 2.2.4** The process  $L_t \stackrel{\text{def}}{=} \mathbb{1}_{\{\tau > t\}} \exp\left(\int_0^t \gamma(s) ds\right)$  is an  $\mathbb{H}$ -martingale and it satisfies

$$L_t = 1 - \int_{]0,t]} L_{u-} \, dM_u. \tag{2.10}$$

In particular, for  $t \in [0, T]$ ,

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>T\}} \mid \mathcal{H}_t) = \mathbb{1}_{\{\tau>t\}} \exp\left(-\int_t^T \gamma(s) ds\right)$$

*Proof.* Let us first show that L is an  $\mathbb{H}$ -martingale. Since the function  $\gamma$  is deterministic, for t > s

$$\mathbb{E}_{\mathbb{Q}}(L_t \mid \mathcal{H}_s) = \exp\left(\int_0^t \gamma(u) du\right) \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t < \tau\}} \mid \mathcal{H}_s).$$

From the equality (2.6)

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t < \tau\}} | \mathcal{H}_s) = \mathbb{1}_{\{\tau > s\}} \frac{1 - F(t)}{1 - F(s)} = \mathbb{1}_{\{\tau > s\}} \exp\left(-\Gamma(t) + \Gamma(s)\right)$$

Hence

$$\mathbb{E}_{\mathbb{Q}}(L_t \mid \mathcal{H}_s) = \mathbb{1}_{\{\tau > s\}} \exp\left(\int_0^s \gamma(u) \, du\right) = L_s$$

To establish (2.10), it suffices to apply the integration by parts formula to the process

$$L_t = (1 - H_t) \exp\left(\int_0^t \gamma(s) \, ds\right).$$

We obtain

$$dL_t = -\exp\left(\int_0^t \gamma(s) \, ds\right) dH_t + \gamma(t) \exp\left(\int_0^t \gamma(s) \, ds\right) (1 - H_t) \, dt$$
$$= -\exp\left(\int_0^t \gamma(s) \, ds\right) \, dM_t.$$

An alternative method is to show that L is the exponential martingale of M, i.e., L is the unique solution of the SDE

$$dL_t = -L_{t-} dM_t, \quad L_0 = 1.$$

This equation can be solved pathwise.

**Proposition 2.2.5** Assume that  $\Gamma$  is a continuous function. Then for any (bounded) Borel measurable function  $h : \mathbb{R}_+ \to \mathbb{R}$ , the process

$$M_t^h = \mathbb{1}_{\{\tau \le t\}} h(\tau) - \int_0^{t \wedge \tau} h(u) \, d\Gamma(u)$$
(2.11)

is an  $\mathbb{H}$ -martingale.

*Proof.* The proof given below provides an alternative proof of Proposition 2.2.2. We wish to establish through direct calculations the martingale property of the process  $M^h$  given by formula (2.11). To this end, notice that formula (2.7) in Corollary 2.2.1 gives

$$E(h(\tau)\mathbb{1}_{\{t<\tau\leq s\}} \mid \mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}} e^{\Gamma(t)} \int_t^s h(u) e^{-\Gamma(u)} d\Gamma(u).$$

On the other hand, using the same formula, we get

$$J \stackrel{\text{def}}{=} E\Big(\int_{t\wedge\tau}^{s\wedge\tau} h(u) \, d\Gamma(u)\Big) = E\big(\tilde{h}(\tau)\mathbb{1}_{\{t<\tau\leq s\}} + \tilde{h}(s)\mathbb{1}_{\{\tau>s\}} \,|\, \mathcal{H}_t\big)$$

where we set  $\tilde{h}(s) = \int_t^s h(u) \, d\Gamma(u)$ . Consequently,

$$J = \mathbb{1}_{\{t < \tau\}} e^{\Gamma(t)} \left( \int_t^s \tilde{h}(u) e^{-\Gamma(u)} \, d\Gamma(u) + e^{-\Gamma(s)} \tilde{h}(s) \right)$$

To conclude the proof, it is enough to observe that Fubini's theorem yields

$$\begin{split} &\int_{t}^{s} e^{-\Gamma(u)} \int_{t}^{u} h(v) \, d\Gamma(v) \, d\Gamma(u) + e^{-\Gamma(s)} \tilde{h}(s) \\ &= \int_{t}^{s} h(u) \int_{u}^{s} e^{-\Gamma(v)} \, d\Gamma(v) \, d\Gamma(u) + e^{-\Gamma(s)} \int_{t}^{s} h(u) \, d\Gamma(u) \\ &= \int_{t}^{s} h(u) e^{-\Gamma(u)} \, d\Gamma(u), \end{split}$$

as expected.

**Corollary 2.2.2** Let  $h : \mathbb{R}_+ \to \mathbb{R}$  be a (bounded) Borel measurable function. Then the process

$$\widetilde{M}_t^h = \exp\left(\mathbb{1}_{\{\tau \le t\}} h(\tau)\right) - \int_0^{t \wedge \tau} (e^{h(u)} - 1) \, d\Gamma(u) \tag{2.12}$$

is an  $\mathbb{H}$ -martingale.

*Proof.* It is enough to observe that

$$\exp\left(\mathbb{1}_{\{\tau \le t\}}h(\tau)\right) = \mathbb{1}_{\{\tau \le t\}}e^{h(\tau)} + \mathbb{1}_{\{t \ge \tau\}} = \mathbb{1}_{\{\tau \le t\}}(e^{h(\tau)} - 1) + 1$$

and to apply the preceding result to  $e^h - 1$ .

**Proposition 2.2.6** Assume that  $\Gamma$  is a continuous function. Let  $h : \mathbb{R}_+ \to \mathbb{R}$  be a non-negative Borel measurable function such that the random variable  $h(\tau)$  is integrable. Then the process

$$\widehat{M}_t = (1 + \mathbb{1}_{\{\tau \le t\}} h(\tau)) \exp\left(-\int_0^{t \wedge \tau} h(u) \, d\Gamma(u)\right)$$
(2.13)

is an  $\mathbb H\text{-}martingale.$ 

Proof. Observe that

$$\widehat{M}_{t} = \exp\left(-\int_{0}^{t} (1-H_{u})h(u) \, d\Gamma(u)\right) + \mathbb{1}_{\{\tau \leq t\}}h(\tau) \exp\left(-\int_{0}^{\tau} (1-H_{u})h(u) \, d\Gamma(u)\right)$$
$$= \exp\left(-\int_{0}^{t} (1-H_{u})h(u) \, d\Gamma(u)\right) + \int_{0}^{t} h(u) \exp\left(-\int_{0}^{u} (1-H_{s})h(s) \, d\Gamma(s)\right) dH_{u}$$

From Itô's calculus,

$$d\widehat{M}_t = \exp\left(-\int_0^t (1-H_u)h(u)\,d\Gamma(u)\right)(-(1-H_t)h(t)\,d\Gamma(t) + h(t)\,dH_t)$$
$$= h(t)\exp\left(-\int_0^t (1-H_u)h(u)\,d\Gamma(u)\right)dM_t.$$

It is instructive to compare this result with the Doléans-Dade exponential of the process hM.

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**Example 2.2.1** In the case where N is an inhomogeneous Poisson process with deterministic intensity  $\lambda$  and  $\tau$  is the moment of the first jump of N, let  $H_t = N_{t \wedge \tau}$ . It is well known that  $N_t - \int_0^t \lambda(s) \, ds$  is a martingale. Therefore, the process stopped at time  $\tau$  is also a martingale, i.e.,  $H_t - \int_0^{t \wedge \tau} \lambda(s) \, ds$  is a martingale. Furthermore, we have seen in Remark 2.1.1 that we can reduce our attention to this case, since any random time can be viewed as the first time where an inhomogeneous Poisson process jumps.

**Exercise 2.2.1** Assume that F is only right-continuous, and let F(t-) be the left-hand side limit of F at t. Show that the process  $(M_t, t \ge 0)$  defined as

$$M_t = H_t - \int_0^{\tau \wedge t} \frac{dF(s)}{1 - F(s)} = H_t - \int_0^t (1 - H_{s-}) \frac{dF(s)}{1 - F(s)}$$

is an  $\mathbb{H}$ -martingale.

### 2.2.3 Representation Theorem

The next result furnishes a suitable version of representation theorem for  $\mathbb{H}$ -martingales.

**Proposition 2.2.7** Let h be a (bounded) Borel function. Then the martingale  $M_t^h = \mathbb{E}_{\mathbb{Q}}(h(\tau) | \mathcal{H}_t)$  admits the representation

$$\mathbb{E}_{\mathbb{Q}}(h(\tau) | \mathcal{H}_t) = \mathbb{E}_{\mathbb{Q}}(h(\tau)) - \int_0^{t \wedge \tau} (g(s) - h(s)) \, dM_s,$$

where  $M_t = H_t - \Gamma(t \wedge \tau)$  and

$$g(t) = -\frac{1}{G(t)} \int_{t}^{\infty} h(u) \, dG(u) = \frac{1}{G(t)} \mathbb{E}_{\mathbb{Q}}(h(\tau) \mathbb{1}_{\tau > t}).$$
(2.14)

Note that  $g(t) = M_t^h$  on  $\{t < \tau\}$ . In particular, any square-integrable  $\mathbb{H}$ -martingale  $(X_t, t \ge 0)$  can be written as  $X_t = X_0 + \int_0^t \zeta_s dM_s$  where  $(\zeta_t, t \ge 0)$  is an  $\mathbb{H}$ -predictable process.

*Proof.* We give below two different proofs. a) From Lemma 2.2.1

$$M_t^h = h(\tau) \mathbb{1}_{\{\tau \le t\}} + \mathbb{1}_{\{t < \tau\}} \frac{\mathbb{E}_{\mathbb{Q}}(h(\tau) \mathbb{1}_{\{t < \tau\}})}{\mathbb{Q}(t < \tau)}$$
  
=  $h(\tau) \mathbb{1}_{\{\tau \le t\}} + \mathbb{1}_{\{t < \tau\}} e^{\Gamma(t)} \mathbb{E}_{\mathbb{Q}}(h(\tau) \mathbb{1}_{\{t < \tau\}}).$ 

An integration by parts leads to

$$e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}} \left( h(\tau) \mathbb{1}_{\{t < \tau\}} \right) = e^{\Gamma_t} \int_t^\infty h(s) dF(s) = g(t)$$
$$= \int_0^\infty h(s) dF(s) - \int_0^t e^{\Gamma(s)} h(s) dF(s) + \int_0^t \mathbb{E}_{\mathbb{Q}} (h(\tau) \mathbb{1}_{\{s < \tau\}}) e^{\Gamma(s)} d\Gamma(s)$$

Therefore, since  $\mathbb{E}_{\mathbb{Q}}(h(\tau)) = \int_{0}^{\infty} h(s) dF(s)$  and  $M_{s}^{h} = e^{\Gamma(s)} \mathbb{E}_{\mathbb{Q}}(h(\tau) \mathbb{1}_{\{s < \tau\}}) = g(s)$  on  $\{s < \tau\}$ , the following equality holds on the event  $\{t < \tau\}$ :

$$e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}} \big( h(\tau) \mathbb{1}_{\{t < \tau\}} \big) = \mathbb{E}_{\mathbb{Q}} (h(\tau)) - \int_0^t e^{\Gamma(s)} h(s) dF(s) + \int_0^t g(s) d\Gamma(s).$$

Hence

$$\mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}(h(\tau) \mid \mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}} \left( \mathbb{E}_{\mathbb{Q}}(h(\tau)) + \int_0^{t\wedge\tau} (g(s) - h(s)) \frac{dF(s)}{1 - F(s)} \right) \\
= \mathbb{1}_{\{t<\tau\}} \left( \mathbb{E}_{\mathbb{Q}}(h(\tau)) - \int_0^{t\wedge\tau} (g(s) - h(s)) (dH_s - d\Gamma(s)) \right)$$

where the last equality is due to  $\mathbb{1}_{\{t < \tau\}} \int_0^{t \wedge \tau} (g(s) - h(s)) dH_s = 0.$ 

On the complementary set  $\{t \ge \tau\}$ , we have seen that  $\mathbb{E}_{\mathbb{Q}}(h(\tau) \mid \mathcal{H}_t) = h(\tau)$ , whereas

$$\int_{0}^{t\wedge\tau} (g(s) - h(s))(dH_s - d\Gamma(s)) = \int_{]0,\tau]} (g(s) - h(s))(dH_s - d\Gamma(s))$$
$$= \int_{]0,\tau[} (g(s) - h(s))(dH_s - d\Gamma(s)) + (g(\tau^-) - h(\tau)).$$

Therefore,

$$\mathbb{E}_{\mathbb{Q}}(h(\tau)) - \int_{0}^{t\wedge\tau} (g(s) - h(s))(dH_s - d\Gamma(s)) = M_{\tau^-}^H - (M_{\tau^-}^H - h(\tau)) = h(\tau).$$

The predictable representation theorem follows immediately.

b) An alternative proof consists in computing the conditional expectation

$$M_t^h = \mathbb{E}_{\mathbb{Q}}(h(\tau) \mid \mathcal{H}_t) = h(\tau) \mathbb{1}_{\{\tau < t\}} + \mathbb{1}_{\{\tau > t\}} e^{-\Gamma(t)} \int_t^\infty h(u) dF(u)$$
  
=  $\int_0^t h(s) dH_s + (1 - H_t) e^{-\Gamma(t)} \int_t^\infty h(u) dF(u) = \int_0^t h(s) dH_s + (1 - H_t) g(t)$ 

and to use Itô's formula and that  $dM_t = dH_t - \gamma(t)(1 - H_t) dt$ . Using that

$$dF(t) = e^{\Gamma(t)} d\Gamma(t) = e^{\Gamma(t)} \gamma(t) dt = -dG(t)$$

we obtain

$$dM_t^h = h(t) dH_t + (1 - H_t)h(t)\gamma(t)dt - g(t) dH_t - (1 - H_t)g(t)\gamma(t) dt$$
  
=  $(h(t) - g(t)) dH_t + (1 - H_t)(h(t) - g(t))\gamma(t) dt = (h(t) - g(t)) dM_t$ 

and thus the proof is completed.

**Exercise 2.2.2** Assume that the function  $\Gamma$  is right-continuous. Establish the following representation formula

$$\mathbb{E}_{\mathbb{Q}}(h(\tau) \mid \mathcal{H}_t) = \mathbb{E}_{\mathbb{Q}}(h(\tau)) - \int_0^{t \wedge \tau} e^{\Delta \Gamma(s)}(g(s) - h(s)) \, dM_s.$$

## 2.2.4 Change of a Probability Measure

Let  $\mathbb{Q}$  be an arbitrary probability measure on  $(\Omega, \mathcal{H}_{\infty})$ , which is absolutely continuous with respect to  $\mathbb{P}$ . We denote by F the c.d.f. of  $\tau$  under  $\mathbb{P}$ . Let  $\eta$  stand for the  $\mathcal{H}_{\infty}$ -measurable density of  $\mathbb{Q}$  with respect to  $\mathbb{P}$ 

$$\eta \stackrel{\text{def}}{=} \frac{d\mathbb{Q}}{d\mathbb{P}} = h(\tau) \ge 0, \quad \mathbb{P}\text{-a.s.},$$
(2.15)

where  $h: \mathbb{R} \to \mathbb{R}_+$  is a Borel measurable function satisfying

$$\mathbb{E}_{\mathbb{P}}(h(\tau)) = \int_0^\infty h(u) \, dF(u) = 1.$$

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We can use the general version of Girsanov's theorem. Nevertheless, we find it preferable to establish a simple version of this theorem in our particular setting. Of course, the probability measure  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  if and only if the inequality in (2.15) is strict  $\mathbb{P}$ -a.s. Furthermore, we shall assume that  $\mathbb{Q}(\tau = 0) = 0$  and  $\mathbb{Q}(\tau > t) > 0$  for any  $t \in \mathbb{R}_+$ . Actually the first condition is satisfied for any  $\mathbb{Q}$  absolutely continuous with respect to  $\mathbb{P}$ . For the second condition to hold, it is sufficient and necessary to assume that for every t

$$\mathbb{Q}(\tau > t) = 1 - F^*(t) = \int_{]t,\infty[} h(u) \, dF(u) > 0,$$

where  $F^*$  is the c.d.f. of  $\tau$  under  $\mathbb{Q}$ 

$$F^*(t) \stackrel{\text{def}}{=} \mathbb{Q}(\tau \le t) = \int_{[0,t]} h(u) \, dF(u). \tag{2.16}$$

Put another way, we assume that

$$g(t) \stackrel{\text{def}}{=} e^{\Gamma(t)} \mathbb{E} \left( \mathbbm{1}_{\{\tau > t\}} h(\tau) \right) = e^{\Gamma(t)} \int_{]t,\infty[} h(u) \, dF(u) = e^{\Gamma(t)} \, \mathbb{Q}(\tau > t) > 0.$$

We assume throughout that this is the case, so that the hazard function  $\Gamma^*$  of  $\tau$  with respect to  $\mathbb{Q}$  is well defined. Our goal is to examine relationships between hazard functions  $\Gamma^*$  and  $\Gamma$ . It is easily seen that in general we have

$$\frac{\Gamma^*(t)}{\Gamma(t)} = \frac{\ln\left(\int_{]t,\infty[} h(u) \, dF(u)\right)}{\ln(1 - F(t))},\tag{2.17}$$

since by definition  $\Gamma^*(t) = -\ln(1 - F^*(t)).$ 

Assume first that F is an absolutely continuous function, so that the intensity function  $\gamma$  of  $\tau$  under  $\mathbb{P}$  is well defined. Recall that  $\gamma$  is given by the formula

$$\gamma(t) = \frac{f(t)}{1 - F(t)}.$$

On the other hand, the c.d.f.  $F^*$  of  $\tau$  under  $\mathbb{Q}$  now equals

$$F^*(t) \stackrel{\text{def}}{=} \mathbb{Q}(\tau \le t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \le t\}}h(\tau)) = \int_0^t h(u)f(u) \, du.$$

so that  $F^*$  follows an absolutely continuous function. Therefore, the intensity function  $\gamma^*$  of the random time  $\tau$  under  $\mathbb{Q}$  exists, and it is given by the formula

$$\gamma^*(t) = \frac{h(t)f(t)}{1 - F^*(t)} = \frac{h(t)f(t)}{1 - \int_0^t h(u)f(u)\,du}.$$

To derive a more straightforward relationship between the intensities  $\gamma$  and  $\gamma^*$ , let us introduce an auxiliary function  $h^* : \mathbb{R}_+ \to \mathbb{R}$ , given by the formula  $h^*(t) = h(t)/g(t)$ .

Notice that

$$\gamma^*(t) = \frac{h(t)f(t)}{1 - \int_0^t h(u)f(u)\,du} = \frac{h(t)f(t)}{\int_t^\infty h(u)f(u)\,du} = \frac{h(t)f(t)}{e^{-\Gamma(t)}g(t)} = h^*(t)\,\frac{f(t)}{1 - F(t)} = h^*(t)\gamma(t).$$

This means also that  $d\Gamma^*(t) = h^*(t) d\Gamma(t)$ . It appears that the last equality holds true if F is merely a continuous function. Indeed, if F (and thus  $F^*$ ) is continuous, we get

$$d\Gamma^*(t) = \frac{dF^*(t)}{1 - F^*(t)} = \frac{d(1 - e^{-\Gamma(t)}g(t))}{e^{-\Gamma(t)}g(t)} = \frac{g(t)d\Gamma(t) - dg(t)}{g(t)} = h^*(t)\,d\Gamma(t).$$

To summarize, if the hazard function  $\Gamma$  is continuous then  $\Gamma^*$  is also continuous and  $d\Gamma^*(t) = h^*(t) d\Gamma(t)$ .

To understand better the origin of the function  $h^*$ , let us introduce the following non-negative  $\mathbb{P}$ -martingale (which is strictly positive when the probability measures  $\mathbb{Q}$  and  $\mathbb{P}$  are equivalent)

$$\eta_t \stackrel{\text{def}}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathcal{H}_t} = \mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{H}_t) = \mathbb{E}_{\mathbb{P}}(h(\tau) \,|\, \mathcal{H}_t), \tag{2.18}$$

so that  $\eta_t = M_t^h$ . The general formula for  $\eta_t$  reads (cf. (2.2.1))

$$\eta_t = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\tau > t} e^{\Gamma(t)} \int_{]t,\infty[} h(u) \, dF(u) = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} g(t).$$

Assume now that F is a continuous function. Then

$$\eta_t = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} \int_t^\infty h(u) e^{\Gamma(t) - \Gamma(u)} d\Gamma(u).$$

On the other hand, using the representation theorem, we get

$$M_t^h = M_0^h + \int_{]0,t]} M_{u-}^h(h^*(u) - 1) \, dM_u$$

where  $h^*(u) = h(u)/g(u)$ . We conclude that

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-}(h^*(u) - 1) \, dM_u. \tag{2.19}$$

It is thus easily seen that

$$\eta_t = \left(1 + \mathbb{1}_{\{\tau \le t\}} v(\tau)\right) \exp\left(-\int_0^{t \wedge \tau} v(u) \, d\Gamma(u)\right),\tag{2.20}$$

where we write  $v(t) = h^*(t) - 1$ . Therefore, the martingale property of the process  $\eta$ , which is obvious from (2.18), is also a consequence of Proposition 2.2.6.

**Remark 2.2.2** In view of (2.19), we have

$$\eta_t = \mathcal{E}_t \Big( \int_0^{\cdot} (h^*(u) - 1) \, dM_u \Big),$$

where  $\mathcal{E}$  stands for the Doléans exponential. Representation (2.20) for the random variable  $\eta_t$  can thus be obtained from the general formula for the Doléans exponential.

We are in the position to formulate the following result (all statements were already established above).

**Proposition 2.2.8** Let  $\mathbb{Q}$  be any probability measure on  $(\Omega, \mathcal{H}_{\infty})$  absolutely continuous with respect to  $\mathbb{P}$ , so that (2.15) holds for some function h. Assume that  $\mathbb{Q}(\tau > t) > 0$  for every  $t \in \mathbb{R}_+$ . Then

$$\frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathcal{H}_t} = \mathcal{E}_t \Big( \int_0^{\cdot} (h^*(u) - 1) \, dM_u \Big), \tag{2.21}$$

where

$$h^{*}(t) = h(t)/g(t), \quad g(t) = e^{\Gamma(t)} \int_{t}^{\infty} h(u) \, dF(u),$$

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and  $\Gamma^*(t) = g^*(t)\Gamma(t)$  with

$$g^{*}(t) = \frac{\ln\left(\int_{]t,\infty[} h(u) \, dF(u)\right)}{\ln(1 - F(t))}.$$
(2.22)

If, in addition, the random time  $\tau$  admits the intensity function  $\gamma$  under  $\mathbb{P}$ , then the intensity function  $\gamma^*$  of  $\tau$  under  $\mathbb{Q}$  satisfies  $\gamma^*(t) = h^*(t)\gamma(t)$  a.e. on  $\mathbb{R}_+$ . More generally, if the hazard function  $\Gamma$  of  $\tau$  under  $\mathbb{P}$  is continuous, then the hazard function  $\Gamma^*$  of  $\tau$  under  $\mathbb{Q}$  is also continuous, and it satisfies  $d\Gamma^*(t) = h^*(t) d\Gamma(t)$ .

**Corollary 2.2.3** If F is continuous then  $M_t^* = H_t - \Gamma^*(t \wedge \tau)$  is an  $\mathbb{H}$ -martingale under  $\mathbb{Q}$ .

Proof. In view Proposition 2.2.2, the corollary is an immediate consequence of the continuity of  $\Gamma^*$ . Alternatively, we may check directly that the product  $U_t = \eta_t M_t^* = \eta_t (H_t - \Gamma^*(t \wedge \tau))$  follows a  $\mathbb{H}$ -martingale under  $\mathbb{P}$ . To this end, observe that the integration by parts formula for functions of finite variation yields

$$U_{t} = \int_{]0,t]} \eta_{t-} dM_{t}^{*} + \int_{]0,t]} M_{t}^{*} d\eta_{t}$$
  
= 
$$\int_{]0,t]} \eta_{t-} dM_{t}^{*} + \int_{]0,t]} M_{t-}^{*} d\eta_{t} + \sum_{u \le t} \Delta M_{u}^{*} \Delta \eta_{u}$$
  
= 
$$\int_{]0,t]} \eta_{t-} dM_{t}^{*} + \int_{]0,t]} M_{t-}^{*} d\eta_{t} + \mathbb{1}_{\{\tau \le t\}} (\eta_{\tau} - \eta_{\tau-}).$$

Using (2.19), we obtain

$$U_{t} = \int_{]0,t]} \eta_{t-} dM_{t}^{*} + \int_{]0,t]} M_{t-}^{*} d\eta_{t} + \eta_{\tau-} \mathbb{1}_{\{\tau \leq t\}} (h^{*}(\tau) - 1)$$
  
$$= \int_{]0,t]} \eta_{t-} d(\Gamma(t \wedge \tau) - \Gamma^{*}(t \wedge \tau) + \mathbb{1}_{\{\tau \leq t\}} (h^{*}(\tau) - 1)) + N_{t},$$

where the process N, which equals

$$N_t = \int_{]0,t]} \eta_{t-} \, dM_t + \int_{]0,t]} M_{t-}^* \, d\eta_t$$

is manifestly an  $\mathbb{H}$ -martingale with respect to  $\mathbb{P}$ . It remains to show that the process

$$N_t^* \stackrel{\text{def}}{=} \Gamma(t \wedge \tau) - \Gamma^*(t \wedge \tau) + \mathbb{1}_{\{\tau \le t\}}(h^*(\tau) - 1)$$

follows an  $\mathbb{H}$ -martingale with respect to  $\mathbb{P}$ . By virtue of Proposition 2.2.5, the process

$$\mathbb{1}_{\{\tau \le t\}}(h^*(\tau) - 1) + \Gamma(t \land \tau) - \int_0^{t \land \tau} h^*(u) \, d\Gamma(u)$$

is an  $\mathbb{H}$ -martingale. Therefore, to conclude the proof it is enough to notice that

$$\int_0^{t\wedge\tau} h^*(u) \, d\Gamma(u) - \Gamma^*(t\wedge\tau) = \int_0^{t\wedge\tau} (h^*(u) \, d\Gamma(u) - d\Gamma^*(u)) = 0,$$

where the last equality is a consequence of the relationship  $d\Gamma^*(t) = h^*(t) d\Gamma(t)$  established in Proposition 2.2.8.

### 2.2.5 Incompleteness of the Toy Model

In order to study the completeness of the financial market, we first need to specify the class of primary traded assets. If the market consists only of the risk-free zero-coupon bond then there exists infinitely many equivalent martingale measures (EMMs). The discounted asset prices are constant; hence the class  $\mathcal{Q}$  of all EMMs is the set of all probability measures equivalent to the historical probability. For any  $\mathbb{Q} \in \mathcal{Q}$ , we denote by  $F_Q$  the cumulative distribution function of  $\tau$  under  $\mathbb{Q}$ , i.e.,

$$F_O(t) = \mathbb{Q}(\tau \le t).$$

The range of prices is defined as the set of prices which do not induce arbitrage opportunities. For a DZC with a constant rebate  $\delta$  paid at maturity, the range of prices is thus equal to the set

$$\{\mathbb{E}_{\mathbb{Q}}(R_T(\mathbb{1}_{\{T<\tau\}}+\delta\mathbb{1}_{\{\tau$$

This set is exactly the interval  $]\delta R_T, R_T[$ . Indeed, it is obvious that the range of prices is included in the interval  $]\delta R_T, R_T[$ . Now, in the set Q, one can select a sequence of probabilities  $\mathbb{Q}_n$  that converges weakly to the Dirac measure at point 0 (resp. at point T). The bounds are obtained as limit cases: the default appears at time 0<sup>+</sup> or it never occurs. Obviously, this range of arbitrage prices is too wide for any practical purposes.

#### 2.2.6 Risk-Neutral Probability Measures

It is usual to interpret the absence of arbitrage opportunities as the existence of an EMM. If defaultable zero-coupon bonds (DZCs) are traded, their prices are given by the market. Therefore, the equivalent martingale measure  $\mathbb{Q}$  to be used for pricing purposes is chosen by the market. Specifically, the pricing measure  $\mathbb{Q}$  is such that, on the event  $\{t < \tau\}$ ,

$$D^{(\delta)}(t,T) = B(t,T)\mathbb{E}_{\mathbb{Q}}(\left[\mathbb{1}_{\{T < \tau\}} + \delta\mathbb{1}_{\{t < \tau \le T\}}\right] | t < \tau).$$

Therefore, we can derive the cumulative function of  $\tau$  under  $\mathbb{Q}$  from the market prices of the DZC as shown below.

#### Case of Zero Recovery

If a DZC with zero recovery of maturity T is traded at some price  $D^{(\delta)}(t,T)$  belonging to the interval ]0, B(t,T)[ then, under any risk-neutral probability  $\mathbb{Q}$ , the process  $B(0,t)D^{(\delta)}(t,T)$  is a martingale. At this stage, we do not know whether the market model is complete, so we do not claim that an EMM is unique. The following equalities thus hold

$$D^{(\delta)}(t,T)B(0,t) = \mathbb{E}_{\mathbb{Q}}(B(0,T)\mathbb{1}_{\{T<\tau\}} \mid \mathcal{H}_t) = B(0,T)\mathbb{1}_{\{t<\tau\}} \exp\left(-\int_t^T \lambda^Q(s) \, ds\right)$$

where  $\lambda^Q(s) = \frac{dF_Q(s)/ds}{1 - F_Q(s)}$ . It is easily seen that if  $D^{(\delta)}(0,T)$  belongs to the range of viable prices

[0, B(0, T)[ for any T then the function  $\lambda^Q$  is strictly positive and the converse implication holds as well. The process  $\lambda^Q$  is the *implied risk-neutral default intensity*, that is, the unique  $\mathbb{Q}$ -intensity of  $\tau$  that is consistent with the market data for DZCs. More precisely, the value of the integral  $\int_t^T \lambda^Q(s) ds$  is known for any t as soon as there DZC bonds will all maturities are traded at time 0. The unique risk-neutral intensity can be obtained from the prices of DZCs by differentiation with respect to maturity

$$r(t) + \lambda^Q(t) = -\partial_T \ln D^{(\delta)}(t,T) \mid_{T=t}.$$

**Remark 2.2.3** It is important to stress that in the present set-up there is no specific relationship between the risk-neutral default intensity and the historical one. The risk-neutral default intensity

can be greater or smaller than the historical one. The historical default intensity can be deduced from observation of default time whereas the risk-neutral one is obtained from the prices of traded defaultable claims.

### Fixed Recovery at Maturity

If the prices of DZCs with different maturities are known then we deduce from (2.1)

$$F_Q(T) = \frac{B(0,T) - D^{(\delta)}(0,T)}{B(0,T)(1-\delta)}$$

where  $F_Q(t) = \mathbb{Q}(\tau \leq t)$ . Hence the probability distribution law of  $\tau$  under the 'market' EMM is known. However, as observed by Hull and White [39], extracting risk-neutral default probabilities from bond prices is in practice, usually more complicated since recovery rate is usually non-zero and most corporate bonds are coupon-bearing bonds.

# **Recovery at Default**

In this case, the cumulative distribution function can be obtained using the derivative of the defaultable zero-coupon price with respect to the maturity. Indeed, denoting by  $\partial_T D^{(\delta)}(0,T)$  the derivative of the value of the DZC at time 0 with respect to the maturity and assuming that G = 1 - F is differentiable, we obtain from (2.3)

$$\partial_T D^{(\delta)}(0,T) = g(T)B(0,T) - G(T)B(0,T)r(T) - \delta(T)g(T)B(0,T),$$

where g(t) = G'(t). Solving this equation leads to

$$\mathbb{Q}(\tau > t) = G(t) = \Delta(t) \left[ 1 + \int_0^t \partial_T D^{(\delta)}(0, s) \frac{1}{B(0, s)(1 - \delta(s))} (\Delta(s))^{-1} ds \right],$$

where we denote  $\Delta(t) = \exp\left(\int_0^t \frac{r(u)}{1-\delta(u)} \, du\right)$ .

# 2.3 Pricing and Trading Defaultable Claims

This section gives a summary of basic results concerning the valuation and trading of generic defaultable claims. We start by analyzing the valuation of recovery payoffs.

# 2.3.1 Recovery at Maturity

Let S be the price of an asset which delivers only a recovery  $Z_{\tau}$  at time T. We know already that the process

$$M_t = H_t - \int_0^t (1 - H_s) \gamma_s \, ds$$

is an  $\mathbb{H}$ -martingale. Recall that  $\gamma(t) = f(t)/G(t)$ , where f is the probability density function of  $\tau$  and  $G(t) = \mathbb{Q}(\tau > t)$ . Observe that

$$e^{-rt}S_t = \mathbb{E}_{\mathbb{Q}}(Z_{\tau}e^{-rT} | \mathcal{G}_t) = e^{-rT} \mathbb{1}_{\{\tau \le t\}} Z_{\tau} + e^{-rT} \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{Q}}(Z_{\tau}\mathbb{1}_{\{t < \tau < T\}})}{G(t)}$$
$$= e^{-rT} \int_0^t Z_u \, dH_u + e^{-rT} \mathbb{1}_{\{\tau > t\}} \widetilde{S}_t$$

where  $\widetilde{S}_t$  is the pre-default price, which is given here by the deterministic function

$$\widetilde{S}_t = \frac{\mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{t < \tau < T\}})}{G(t)} = \frac{\int_t^T Z_u f_u \, du}{G(t)} \,.$$

Hence

$$d\widetilde{S}_t = f(t) \frac{\int_t^T Z_u f_u du}{G^2(t)} dt - \frac{Z_t f_t}{G(t)} dt = \widetilde{S}_t \frac{f(t)}{G(t)} dt - \frac{Z_t f_t}{G(t)} dt \,.$$

It follows that

$$d(e^{-rt}S_t) = e^{-rT} \left( Z_t \, dH_t + (1 - H_t) \frac{f(t)}{G(t)} (\widetilde{S}_t - Z_t) dt - \widetilde{S}_{t-} \, dH_t \right)$$
  
=  $\left( e^{-rT} Z_t - e^{-rt} S_{t-} \right) \left( dH_t - (1 - H_t) \gamma_t \, dt \right)$   
=  $e^{-rt} \left( e^{-r(T-t)} Z_t - S_{t-} \right) dM_t.$ 

In that case, the discounted price is a martingale under the risk-neutral probability  $\mathbb{Q}$  and the price S does not vanish (so long as  $\delta$  does not)

# 2.3.2 Recovery at Default

Assume now that the recovery is paid at default time. Then the price of the derivative is obviously equal to 0 after the default time and

$$e^{-rt}S_t = \mathbb{E}_{\mathbb{Q}}(Z_{\tau}e^{-r\tau}\mathbb{1}_{\{t < \tau \le T\}} \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{Q}}(e^{-r\tau}Z_{\tau}\mathbb{1}_{\{t < \tau < T\}})}{G(t)} = \mathbb{1}_{\{\tau > t\}}\widetilde{S}_t$$

where the pre-default price is the deterministic function

$$\widetilde{S}_t = \frac{1}{G(t)} \int_t^T Z_u e^{-ru} f(u) \, du.$$

Consequently,

$$d\widetilde{S}_t = -Z_t e^{-rt} \frac{f(t)}{G(t)} dt + f(t) \frac{\int_t^T Z_u e^{-ru} f(u) du}{(\mathbb{Q}(\tau > t)^2)} dt$$
$$= -Z_t e^{-rt} \frac{f(t)}{G(t)} dt + \widetilde{S}_t \frac{f(t)}{G(t)} dt$$
$$= \frac{f(t)}{G(t)} \left( -Z_t e^{-rt} + \widetilde{S}_t \right) dt$$

and thus

$$d(e^{-rt}S_t) = (1 - H_t)\frac{f(t)}{G(t)}(-Z_t e^{-rt} + \widetilde{S}_t) dt - \widetilde{S}_t dH_t$$
  
=  $-\widetilde{S}_t(dH_t - (1 - H_t)\gamma_t dt) = (Z_t e^{-rt} - \widetilde{S}_t) dM_t - Z_t e^{-rt}(1 - H_t)\gamma_t dt$   
=  $e^{-rt}(Z_t - S_{t-}) dM_t - Z_t e^{-rt}(1 - H_t)\gamma_t dt.$ 

In that case, the discounted process is not an  $\mathbb H\text{-}\mathrm{martingale}$  under the risk-neutral probability. By contrast, the process

$$S_t e^{-rt} + \int_0^t Z_s e^{-rs} (1 - H_s) \gamma_s \, ds$$

follows an  $\mathbb{H}$ -martingale. The recovery can be formally interpreted as a dividend process paid at the rate  $Z\gamma$  up to time  $\tau$ .

# 2.3.3 Generic Defaultable Claims

Let us first recall the notation. A strictly positive random variable  $\tau$ , defined on a probability space  $(\Omega, \mathcal{G}, \mathbb{Q})$ , is termed a *random time*. In view of its interpretation, it will be later referred to as a *default time*. We introduce the jump process  $H_t = \mathbb{1}_{\{\tau \leq t\}}$  associated with  $\tau$ , and we denote by  $\mathbb{H}$  the filtration generated by this process. We assume that we are given, in addition, some auxiliary filtration  $\mathbb{F}$ , and we write  $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ , meaning that we have  $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$  for every  $t \in \mathbb{R}_+$ .

**Definition 2.3.1** By a defaultable claim maturing at T we mean the quadruple  $(X, A, Z, \tau)$ , where X is an  $\mathcal{F}_T$ -measurable random variable, A is an  $\mathbb{F}$ -adapted process of finite variation, Z is an  $\mathbb{F}$ -predictable process, and  $\tau$  is a random time.

The financial interpretation of the components of a defaultable claim becomes clear from the following definition of the dividend process D, which describes all cash flows associated with a defaultable claim over the lifespan ]0,T], that is, after the contract was initiated at time 0. Of course, the choice of 0 as the date of inception is arbitrary.

**Definition 2.3.2** The dividend process D of a defaultable claim maturing at T equals, for every  $t \in [0,T]$ ,

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]} (1 - H_u) \, dA_u + \int_{]0,t]} Z_u \, dH_u.$$

The financial interpretation of the definition above justifies the following terminology: X is the promised payoff, A represents the process of promised dividends, and the process Z, termed the recovery process, specifies the recovery payoff at default. It is worth stressing that, according to our convention, the cash payment (premium) at time 0 is not included in the dividend process D associated with a defaultable claim.

When dealing with a credit default swap, it is natural to assume that the premium paid at time 0 equals zero, and the process A represents the fee (annuity) paid in instalments up to maturity date or default, whichever comes first. For instance, if  $A_t = -\kappa t$  for some constant  $\kappa > 0$ , then the 'price' of a stylized credit default swap is formally represented by this constant, referred to as the continuously paid *credit default rate*.

If the other covenants of the contract are known (i.e., the payoffs X and Z are given), the valuation of a swap is equivalent to finding the level of the rate  $\kappa$  that makes the swap valueless at inception. Typically, in a credit default swap we have X = 0, and Z is determined in reference to recovery rate of a reference credit-risky entity. In a more realistic approach, the process A is discontinuous, with jumps occurring at the premium payment dates. In this note, we shall only deal with a stylized CDS with a continuously paid premium.

Let us return to the general set-up. It is clear that the dividend process D follows a process of finite variation on [0, T]. Since

$$\int_{]0,t]} (1 - H_u) \, dA_u = \int_{]0,t]} \mathbb{1}_{\{\tau > u\}} \, dA_u = A_{\tau -} \mathbb{1}_{\{\tau \le t\}} + A_t \mathbb{1}_{\{\tau > t\}}$$

it is also apparent that if default occurs at some date t, the 'promised dividend'  $A_t - A_{t-}$  that is due to be received or paid at this date is disregarded. If we denote  $\tau \wedge t = \min(\tau, t)$  then we have

$$\int_{]0,t]} Z_u \, dH_u = Z_{\tau \wedge t} \mathbb{1}_{\{\tau \le t\}} = Z_\tau \mathbb{1}_{\{\tau \le t\}}$$

Let us stress that the process  $D_u - D_t$ ,  $u \in [t, T]$ , represents all cash flows from a defaultable claim received by an investor who purchases it at time t. Of course, the process  $D_u - D_t$  may depend on the past behavior of the claim (e.g., through some intrinsic parameters, such as credit spreads) as well as on the history of the market prior to t. The past dividends are not valued by the market, however, so that the current market value at time t of a claim (i.e., the price at which it trades at time t) depends only on future dividends to be paid or received over the time interval [t, T].

Suppose that our underlying financial market model is arbitrage-free, in the sense that there exists a *spot martingale measure*  $\mathbb{Q}$  (also referred to as a *risk-neutral probability*), meaning that  $\mathbb{Q}$  is equivalent to  $\mathbb{Q}$  on  $(\Omega, \mathcal{G}_T)$ , and the price process of any tradeable security, paying no coupons or dividends, follows a  $\mathbb{G}$ -martingale under  $\mathbb{Q}$ , when discounted by the *savings account* B, given by

$$B_t = \exp\left(\int_0^t r_u \, du\right), \quad \forall t \in \mathbb{R}_+.$$
(2.23)

# 2.3.4 Buy-and-Hold Strategy

We write  $S^i$ , i = 1, ..., k to denote the price processes of k primary securities in an arbitrage-free financial model. We make the standard assumption that the processes  $S^i$ , i = 1, ..., k - 1 follow semimartingales. In addition, we set  $S_t^k = B_t$  so that  $S^k$  represents the value process of the savings account. The last assumption is not necessary, however. We can assume, for instance, that  $S^k$  is the price of a *T*-maturity risk-free zero-coupon bond, or choose any other strictly positive price process as as numéraire.

For the sake of convenience, we assume that  $S^i$ , i = 1, ..., k - 1 are non-dividend-paying assets, and we introduce the discounted price processes  $S^{i*}$  by setting  $S_t^{i*} = S_t^i/B_t$ . All processes are assumed to be given on a filtered probability space  $(\Omega, \mathbb{G}, \mathbb{Q})$ , where  $\mathbb{Q}$  is interpreted as the real-life (i.e., statistical) probability measure.

Let us now assume that we have an additional traded security that pays dividends during its lifespan, assumed to be the time interval [0, T], according to a process of finite variation D, with  $D_0 = 0$ . Let S denote a (yet unspecified) price process of this security. In particular, we do not postulate a priori that S follows a semimartingale. It is not necessary to interpret S as a price process of a defaultable claim, though we have here this particular interpretation in mind.

Let a  $\mathbb{G}$ -predictable,  $\mathbb{R}^{k+1}$ -valued process  $\phi = (\phi^0, \phi^1, \dots, \phi^k)$  represent a generic trading strategy, where  $\phi_t^j$  represents the number of shares of the  $j^{\text{th}}$  asset held at time t. We identify here  $S^0$ with S, so that S is the  $0^{\text{th}}$  asset. In order to derive a pricing formula for this asset, it suffices to examine a simple trading strategy involving S, namely, the buy-and-hold strategy.

Suppose that one unit of the 0<sup>th</sup> asset was purchased at time 0, at the initial price  $S_0$ , and it was hold until time T. We assume all the proceeds from dividends are re-invested in the savings account B. More specifically, we consider a *buy-and-hold* strategy  $\psi = (1, 0, \ldots, 0, \psi^k)$ , where  $\psi^k$  is a  $\mathbb{G}$ -predictable process. The associated *wealth process*  $V(\psi)$  equals

$$V_t(\psi) = S_t + \psi_t^k B_t, \quad \forall t \in [0, T],$$

$$(2.24)$$

so that its initial value equals  $V_0(\psi) = S_0 + \psi_0^k$ .

**Definition 2.3.3** We say that a strategy  $\psi = (1, 0, \dots, 0, \psi^k)$  is self-financing if

$$dV_t(\psi) = dS_t + dD_t + \psi_t^k \, dB_t,$$

or more explicitly, for every  $t \in [0, T]$ ,

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + D_t + \int_{]0,t]} \psi_u^k \, dB_u.$$
(2.25)

We assume from now on that the process  $\psi^k$  is chosen in such a way (with respect to S, D and B) that a buy-and-hold strategy  $\psi$  is self-financing. Also, we make a standing assumption that the random variable  $Y = \int_{[0,T]} B_u^{-1} dD_u$  is Q-integrable.

**Lemma 2.3.1** The discounted wealth  $V_t^*(\psi) = B_t^{-1}V_t(\psi)$  of any self-financing buy-and-hold trading strategy  $\psi$  satisfies, for every  $t \in [0, T]$ ,

$$V_t^*(\psi) = V_0^*(\psi) + S_t^* - S_0^* + \int_{]0,t]} B_u^{-1} dD_u.$$
(2.26)

Hence we have, for every  $t \in [0, T]$ ,

$$V_T^*(\psi) - V_t^*(\psi) = S_T^* - S_t^* + \int_{]t,T]} B_u^{-1} \, dD_u.$$
(2.27)

*Proof.* We define an auxiliary process  $\widehat{V}(\psi)$  by setting  $\widehat{V}_t(\psi) = V_t(\psi) - S_t = \psi_t^k B_t$  for  $t \in [0, T]$ . In view of (2.25), we have

$$\widehat{V}_t(\psi) = \widehat{V}_0(\psi) + D_t + \int_{]0,t]} \psi_u^k \, dB_u,$$

and so the process  $\widehat{V}(\psi)$  follows a semimartingale. An application of Itô's product rule yields

$$d(B_t^{-1}\hat{V}_t(\psi)) = B_t^{-1}d\hat{V}_t(\psi) + \hat{V}_t(\psi) dB_t^{-1} = B_t^{-1}dD_t + \psi_t^k B_t^{-1}dB_t + \psi_t^k B_t dB_t^{-1} = B_t^{-1}dD_t,$$

where we have used the obvious identity:  $B_t^{-1}dB_t + B_t dB_t^{-1} = 0$ . Integrating the last equality, we obtain

$$B_t^{-1}(V_t(\psi) - S_t) = B_0^{-1}(V_0(\psi) - S_0) + \int_{]0,t]} B_u^{-1} dD_u,$$

and this immediately yields (2.26).

It is worth noting that Lemma 2.3.1 remains valid if the assumption that  $S^k$  represents the savings account B is relaxed. It suffices to assume that the price process  $S^k$  is a numéraire, that is, a strictly positive continuous semimartingale. For the sake of brevity, let us write  $S^k = \beta$ . We say that  $\psi = (1, 0, \ldots, 0, \psi^k)$  is self-financing it the wealth process

$$V_t(\psi) = S_t + \psi_t^k \beta_t, \quad \forall t \in [0, T],$$

satisfies, for every  $t \in [0, T]$ ,

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + D_t + \int_{]0,t]} \psi_u^k d\beta_u$$

**Lemma 2.3.2** The relative wealth  $V_t^*(\psi) = \beta_t^{-1} V_t(\psi)$  of a self-financing trading strategy  $\psi$  satisfies, for every  $t \in [0, T]$ ,

$$V_t^*(\psi) = V_0^*(\psi) + S_t^* - S_0^* + \int_{]0,t]} \beta_u^{-1} dD_u,$$

where  $S^* = \beta_t^{-1} S_t$ .

*Proof.* The proof proceeds along the same lines as before, noting that  $\beta^1 d\beta + \beta d\beta^1 + d\langle \beta, \beta^1 \rangle = 0$ .  $\Box$ 

## 2.3.5 Spot Martingale Measure

Our next goal is to derive the risk-neutral valuation formula for the ex-dividend price  $S_t$ . To this end, we assume that our market model is arbitrage-free, meaning that it admits a (not necessarily unique) martingale measure  $\mathbb{Q}$ , equivalent to  $\mathbb{Q}$ , which is associated with the choice of B as a numéraire.

**Definition 2.3.4** We say that  $\mathbb{Q}$  is a spot martingale measure if the discounted price  $S^{i*}$  of any non-dividend paying traded security follows a  $\mathbb{Q}$ -martingale with respect to  $\mathbb{G}$ .

It is well known that the discounted wealth process  $V^*(\phi)$  of any self-financing trading strategy  $\phi = (0, \phi^1, \phi^2, \dots, \phi^k)$  is a local martingale under  $\mathbb{Q}$ . In what follows, we shall only consider *admissible* trading strategies, that is, strategies for which the discounted wealth process  $V^*(\phi)$  is a martingale under  $\mathbb{Q}$ . A market model in which only admissible trading strategies are allowed is *arbitrage-free*, that is, there are no arbitrage opportunities in this model.

Following this line of arguments, we postulate that the trading strategy  $\psi$  introduced in Section 2.3.4 is also *admissible*, so that its discounted wealth process  $V^*(\psi)$  follows a martingale under  $\mathbb{Q}$  with respect to  $\mathbb{G}$ . This assumption is quite natural if we wish to prevent arbitrage opportunities to appear in the extended model of the financial market. Indeed, since we postulate that S is traded, the wealth process  $V(\psi)$  can be formally seen as an additional non-dividend paying tradeable security.

To derive a pricing formula for a defaultable claim, we make a natural assumption that the market value at time t of the 0<sup>th</sup> security comes exclusively from the future dividends stream, that is, from the cash flows occurring in the open interval ]t, T[. Since the lifespan of S is [0,T], this amounts to postulate that  $S_T = S_T^* = 0$ . To emphasize this property, we shall refer to S as the *ex-dividend price* of the 0<sup>th</sup> asset.

**Definition 2.3.5** A process S with  $S_T = 0$  is the ex-dividend price of the 0<sup>th</sup> asset if the discounted wealth process  $V^*(\psi)$  of any self-financing buy-and-hold strategy  $\psi$  follows a  $\mathbb{G}$ -martingale under  $\mathbb{Q}$ .

As a special case, we obtain the ex-dividend price a defaultable claim with maturity T.

**Proposition 2.3.1** The ex-dividend price process S associated with the dividend process D satisfies, for every  $t \in [0, T]$ ,

$$S_t = B_t \mathbb{E}_{\mathbb{Q}} \Big( \int_{]t,T]} B_u^{-1} dD_u \, \Big| \, \mathcal{G}_t \Big).$$
(2.28)

*Proof.* The postulated martingale property of the discounted wealth process  $V^*(\psi)$  yields, for every  $t \in [0, T]$ ,

$$\mathbb{E}_{\mathbb{Q}}\left(V_T^*(\psi) - V_t^*(\psi) \,\middle|\, \mathcal{G}_t\right) = 0.$$

Taking into account (2.27), we thus obtain

$$S_t^* = \mathbb{E}_{\mathbb{Q}}\left(S_T^* + \int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t\right).$$

Since, by virtue of the definition of the ex-dividend price we have  $S_T = S_T^* = 0$ , the last formula yields (2.28).

It is not difficult to show that the ex-dividend price S satisfies, for every  $t \in [0, T]$ ,

$$S_t = \mathbb{1}_{\{t < \tau\}} \widetilde{S}_t, \tag{2.29}$$

where the process  $\widetilde{S}$  represents the *ex-dividend pre-default price* of a defaultable claim.

The cum-dividend price process  $\bar{S}$  associated with the dividend process D is given by the formula, for every  $t \in [0, T]$ ,

$$\bar{S}_t = B_t \mathbb{E}_{\mathbb{Q}} \Big( \int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big).$$
(2.30)

The corresponding discounted cum-dividend price process,  $\widehat{S} \stackrel{\text{def}}{=} B^{-1}\overline{S}$ , is a  $\mathbb{G}$ -martingale under  $\mathbb{Q}$ .

The savings account B can be replaced by an arbitrary numéraire  $\beta$ . The corresponding valuation formula becomes, for every  $t \in [0, T]$ ,

$$S_t = \beta_t \mathbb{E}_{\mathbb{Q}^\beta} \Big( \int_{]t,T]} \beta_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big), \tag{2.31}$$

where  $\mathbb{Q}^{\beta}$  is a martingale measure on  $(\Omega, \mathcal{G}_T)$  associated with a numéraire  $\beta$ , that is, a probability measure on  $(\Omega, \mathcal{G}_T)$  given by the formula

$$\frac{d\mathbb{Q}^{\beta}}{d\mathbb{Q}} = \frac{\beta_T}{\beta_0 B_T}, \quad \mathbb{Q}\text{-a.s.}$$

# 2.3.6 Self-Financing Trading Strategies

Let us now examine a general trading strategy  $\phi = (\phi^0, \phi^1, \dots, \phi^k)$  with G-predictable components. The associated *wealth process*  $V(\phi)$  equals  $V_t(\phi) = \sum_{i=0}^k \phi_t^i S_t^i$ , where, as before  $S^0 = S$ . A strategy  $\phi$  is said to be *self-financing* if  $V_t(\phi) = V_0(\phi) + G_t(\phi)$  for every  $t \in [0, T]$ , where the *gains process*  $G(\phi)$  is defined as follows:

$$G_t(\phi) = \int_{]0,t]} \phi_u^0 \, dD_u + \sum_{i=0}^k \int_{]0,t]} \phi_u^i \, dS_u^i.$$

**Corollary 2.3.1** Let  $S^k = B$ . Then for any self-financing trading strategy  $\phi$ , the discounted wealth process  $V^*(\phi) = B_t^{-1}V_t(\phi)$  follows a martingale under  $\mathbb{Q}$ .

*Proof.* Since B is a continuous process of finite variation, Itô's product rule gives

$$dS_t^{i*} = S_t^i \, dB_t^{-1} + B_t^{-1} dS_t^i$$

for i = 0, 1, ..., k, and so

$$\begin{split} dV_t^*(\phi) &= V_t(\phi) \, dB_t^{-1} + B_t^{-1} dV_t(\phi) \\ &= V_t(\phi) \, dB_t^{-1} + B_t^{-1} \Big( \sum_{i=0}^k \phi_t^i \, dS_t^i + \phi_t^0 \, dD_t \Big) \\ &= \sum_{i=0}^k \phi_t^i \Big( S_t^i \, dB_t^{-1} + B_t^{-1} dS_t^i \Big) + \phi_t^0 B_t^{-1} dD_t \\ &= \sum_{i=1}^{k-1} \phi_t^i \, dS_t^{i*} + \phi_t^0 \Big( dS_t^* + B_t^{-1} dD_t \Big) = \sum_{i=1}^{k-1} \phi_t^i \, dS_t^{i*} + \phi_t^0 \, d\widehat{S}_t, \end{split}$$

where the auxiliary process  $\hat{S}$  is given by the following expression:

$$\widehat{S}_t = S_t^* + \int_{]0,t]} B_u^{-1} \, dD_u.$$

To conclude, it suffices to observe that in view of (2.28) the process  $\hat{S}$  satisfies

$$\widehat{S}_{t} = \mathbb{E}_{\mathbb{Q}} \Big( \int_{]0,T]} B_{u}^{-1} \, dD_{u} \, \Big| \, \mathcal{G}_{t} \Big), \tag{2.32}$$

and thus it follows a martingale under  $\mathbb{Q}$ .

It is worth noting that  $\hat{S}_t$ , given by formula (2.32), represents the discounted *cum-dividend price* at time t of the 0<sup>th</sup> asset, that is, the arbitrage price at time t of all past and future dividends

associated with the 0<sup>th</sup> asset over its lifespan. To check this, let us consider a buy-and-hold strategy such that  $\psi_0^k = 0$ . Then, in view of (2.27), the terminal wealth at time T of this strategy equals

$$V_T(\psi) = B_T \int_{[0,T]} B_u^{-1} \, dD_u.$$
(2.33)

It is clear that  $V_T(\psi)$  represents all dividends from S in the form of a single payoff at time T. The arbitrage price  $\pi_t(\widehat{Y})$  at time t < T of a claim  $\widehat{Y} = V_T(\psi)$  equals (under the assumption that this claim is attainable)

$$\pi_t(\widehat{Y}) = B_t \mathbb{E}_{\mathbb{Q}} \Big( \int_{]0,T]} B_u^{-1} dD_u \, \Big| \, \mathcal{G}_t \Big)$$

and thus  $\widehat{S}_t = B_t^{-1} \pi_t(\widehat{Y})$ . It is clear that discounted cum-dividend price follows a martingale under  $\mathbb{Q}$  (under the standard integrability assumption).

**Remarks 2.3.1** (i) Under the assumption of uniqueness of a spot martingale measure  $\mathbb{Q}$ , any  $\mathbb{Q}$ -integrable contingent claim is attainable, and the valuation formula established above can be justified by means of replication.

(ii) Otherwise – that is, when a martingale probability measure  $\mathbb{Q}$  is not uniquely determined by the model  $(S^1, S^2, \ldots, S^k)$  – the right-hand side of (2.28) may depend on the choice of a particular martingale probability, in general. In this case, a process defined by (2.28) for an arbitrarily chosen spot martingale measure  $\mathbb{Q}$  can be taken as the no-arbitrage price process of a defaultable claim. In some cases, a market model can be completed by postulating that S is also a traded asset.

# 2.3.7 Martingale Properties of Prices of Defaultable Claims

In the next result, we summarize the martingale properties of prices of a generic defaultable claim.

**Corollary 2.3.2** The discounted cum-dividend price  $\widehat{S}_t$ ,  $t \in [0,T]$ , of a defaultable claim is a  $\mathbb{Q}$ -martingale with respect to  $\mathbb{G}$ . The discounted ex-dividend price  $S_t^*$ ,  $t \in [0,T]$ , satisfies

$$S_t^* = \widehat{S}_t - \int_{]0,t]} B_u^{-1} \, dD_u, \quad \forall t \in [0,T],$$

and thus it follows a supermartingale under  $\mathbb{Q}$  if and only if the dividend process D is increasing.

In some practical applications, the finite variation process A is interpreted as the positive premium paid in instalments by the claim-holder to the counterparty in exchange for a positive recovery (received by the claim-holder either at maturity or at default). It is thus natural to assume that Ais a decreasing process, and all other components of the dividend process are increasing processes (that is, we postulate that  $X \ge 0$ , and  $Z \ge 0$ ). It is rather clear that, under these assumptions, the discounted ex-dividend price  $S^*$  is neither a super- or submartingale under  $\mathbb{Q}$ , in general.

Assume now that  $A \equiv 0$ , so that the premium for a defaultable claim is paid upfront at time 0, and it is not accounted for in the dividend process D. We postulate, as before, that  $X \ge 0$ , and  $Z \ge 0$ . In this case, the dividend process D is manifestly increasing, and thus the discounted ex-dividend price  $S^*$  is a supermartingale under  $\mathbb{Q}$ . This feature is quite natural since the discounted expected value of future dividends decreases when time elapses.

The final conclusion is that the martingale properties of the price of a defaultable claim depend on the specification of a claim and conventions regarding the prices (ex-dividend price or cum-dividend price). This point will be illustrated below by means of a detailed analysis of prices of credit default swaps.

# Chapter 3

# Hazard Process Approach

In the general *reduced-form approach*, we deal with two kinds of information: the information from the assets prices, denoted as  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$ , and the information from the default time, that is, the knowledge of the time where the default occurred in the past, if the default has indeed already appeared. As we already know, the latter information is modeled by the filtration  $\mathbb{H}$  generated by the default process H.

At the intuitive level, the *reference filtration*  $\mathbb{F}$  is generated by prices of some assets, or by other economic factors (such as, e.g., interest rates). This filtration can also be a subfiltration of the prices. The case where  $\mathbb{F}$  is the trivial filtration is exactly what we have studied in the toy example. Though in typical examples  $\mathbb{F}$  is chosen to be the Brownian filtration, most theoretical results do not rely on such a specification of the filtration  $\mathbb{F}$ . We denote by  $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$  the *full filtration* (also known as the *enlarged filtration*).

Special attention will be paid in this chapter to the so-called hypothesis (H), which, in the present context, postulates the invariance of the martingale property with respect to the enlargement of  $\mathbb{F}$  by the observations of a default time. In order to examine the exact meaning of market completeness in a defaultable world and to deduce the hedging strategies for credit derivatives, we shall establish a suitable version of a representation theorem. Most results from this chapter can be found, for instance, in survey papers by Jeanblanc and Rutkowski [42, 43].

# **3.1** General Case

The concepts introduced in the previous chapter will now be extended to a more general set-up, when allowance for a larger flow of information – formally represented hereafter by some reference filtration  $\mathbb{F}$  – is made.

We denote by  $\tau$  a non-negative random variable on a probability space  $(\Omega, \mathcal{G}, \mathbb{Q})$ , satisfying:  $\mathbb{Q}\{\tau = 0\} = 0$  and  $\mathbb{Q}\{\tau > t\} > 0$  for any  $t \in \mathbb{R}_+$ . We introduce a right-continuous process Hby setting  $H_t = \mathbb{1}_{\{\tau \leq t\}}$  and we denote by  $\mathbb{H}$  the associated filtration:  $\mathcal{H}_t = \sigma(H_u : u \leq t)$ . Let  $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$  be an arbitrary filtration on  $(\Omega, \mathcal{G}, \mathbb{Q})$ . All filtrations considered in what follows are implicitly assumed to satisfy the 'usual conditions' of right-continuity and completeness. For each  $t \in \mathbb{R}_+$ , the total information available at time t is captured by the  $\sigma$ -field  $\mathcal{G}_t$ .

We shall focus on the case described in the following assumption. We assume that we are given an auxiliary filtration  $\mathbb{F}$  such that  $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ ; that is,  $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$  for any  $t \in \mathbb{R}_+$ . For the sake of simplicity, we assume that the  $\sigma$ -field  $\mathcal{F}_0$  is trivial (so that  $\mathcal{G}_0$  is a trivial  $\sigma$ -field as well).

The process H is obviously  $\mathbb{G}$ -adapted, but it is not necessarily  $\mathbb{F}$ -adapted. In other words, the random time  $\tau$  is a  $\mathbb{G}$ -stopping time, but it may fail to be an  $\mathbb{F}$ -stopping time.

**Lemma 3.1.1** Assume that the filtration  $\mathbb{G}$  satisfies  $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ . Then  $\mathbb{G} \subseteq \mathbb{G}^*$ , where  $\mathbb{G}^* = (\mathcal{G}_t^*)_{t \ge 0}$  with

$$\mathcal{G}_t^* \stackrel{\text{def}}{=} \left\{ A \in \mathcal{G} \, : \, \exists B \in \mathcal{F}_t, \ A \cap \{\tau > t\} = B \cap \{\tau > t\} \right\}$$

*Proof.* It is rather clear that the class  $\mathcal{G}_t^*$  is a sub- $\sigma$ -field of  $\mathcal{G}$ . Therefore, it is enough to check that  $\mathcal{H}_t \subseteq \mathcal{G}_t^*$  and  $\mathcal{F}_t \subseteq \mathcal{G}_t^*$  for every  $t \in \mathbb{R}_+$ . Put another way, we need to verify that if either  $A = \{\tau \leq u\}$  for some  $u \leq t$  or  $A \in \mathcal{F}_t$ , then there exists an event  $B \in \mathcal{F}_t$  such that  $A \cap \{\tau > t\} = B \cap \{\tau > t\}$ . In the former case we may take  $B = \emptyset$ , and in the latter B = A.

For any  $t \in \mathbb{R}_+$ , we write  $F_t = \mathbb{Q}\{\tau \leq t \mid \mathcal{F}_t\}$ , and we denote by G the  $\mathbb{F}$ -survival process of  $\tau$  with respect to the filtration  $\mathbb{F}$ , given as:

$$G_t \stackrel{\text{def}}{=} 1 - F_t = \mathbb{Q}\{\tau > t \mid \mathcal{F}_t\}, \quad \forall t \in \mathbb{R}_+.$$

Notice that for any  $0 \le t \le s$  we have  $\{\tau \le t\} \subseteq \{\tau \le s\}$ , and so

$$\mathbb{E}_{\mathbb{Q}}(F_s \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\mathbb{Q}\{\tau \le s \mid \mathcal{F}_s\} \mid \mathcal{F}_t) = \mathbb{Q}\{\tau \le s \mid \mathcal{F}_t\} \ge \mathbb{Q}\{\tau \le t \mid \mathcal{F}_t\} = F_t.$$

This shows that the process F (G, resp.) follows a bounded, non-negative  $\mathbb{F}$ -submartingale ( $\mathbb{F}$ -supermartingale, resp.) under  $\mathbb{Q}$ . We may thus deal with the right-continuous modification of F (of G) with finite left-hand limits. The next definition is a rather straightforward generalization of the concept of the hazard function (see Definition 2.2.1).

**Definition 3.1.1** Assume that  $F_t < 1$  for  $t \in \mathbb{R}_+$ . The  $\mathbb{F}$ -hazard process of  $\tau$  under  $\mathbb{Q}$ , denoted by  $\Gamma$ , is defined through the formula  $1 - F_t = e^{-\Gamma_t}$ . Equivalently,  $\Gamma_t = -\ln G_t = -\ln (1 - F_t)$  for every  $t \in \mathbb{R}_+$ .

Since  $G_0 = 1$ , it is clear that  $\Gamma_0 = 0$ . For the sake of conciseness, we shall refer briefly to  $\Gamma$  as the  $\mathbb{F}$ -hazard process, rather than the  $\mathbb{F}$ -hazard process under  $\mathbb{Q}$ , unless there is a danger of confusion.

Throughout this chapter, we will work under the standing assumption that the inequality  $F_t < 1$  holds for every  $t \in \mathbb{R}_+$ , so that the F-hazard process  $\Gamma$  is well defined. Hence the case when  $\tau$  is an F-stopping time (that is, the case when  $\mathbb{F} = \mathbb{G}$ ) is not dealt with here.

# 3.1.1 Key Lemma

We shall first focus on the conditional expectation  $\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}}Y | \mathcal{G}_t)$ , where Y is a  $\mathbb{Q}$ -integrable random variable. We start by the following result, which is a direct counterpart of Lemma 2.2.1.

**Lemma 3.1.2** For any  $\mathcal{G}$ -measurable, integrable random variable Y and any  $t \in \mathbb{R}_+$  we have

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}}Y \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}}\mathbb{E}_{\mathbb{Q}}(Y \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}}\frac{\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}}Y \mid \mathcal{F}_t)}{\mathbb{Q}\{\tau > t \mid \mathcal{F}_t\}}.$$
(3.1)

In particular, for any  $t \leq s$ 

$$\mathbb{Q}\{t < \tau \le s \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \,\frac{\mathbb{Q}\{t < \tau \le s \,|\, \mathcal{F}_t\}}{\mathbb{Q}\{\tau > t \,|\, \mathcal{F}_t\}}.$$
(3.2)

*Proof.* Let us denote  $C = \{\tau > t\}$ . We need to verify that (recall that  $\mathcal{F}_t \subseteq \mathcal{G}_t$ )

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{C}Y\mathbb{Q}(C \mid \mathcal{F}_{t}) \mid \mathcal{G}_{t}) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{C}\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{C}Y \mid \mathcal{F}_{t}) \mid \mathcal{G}_{t}).$$

Put another way, we need to show that for any  $A \in \mathcal{G}_t$  we have

$$\int_{A} \mathbb{1}_{C} Y \mathbb{Q}(C \mid \mathcal{F}_{t}) d\mathbb{Q} = \int_{A} \mathbb{1}_{C} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{C} Y \mid \mathcal{F}_{t}) d\mathbb{Q}.$$

In view of Lemma 3.1.1, for any  $A \in \mathcal{G}_t$  we have  $A \cap C = B \cap C$  for some event  $B \in \mathcal{F}_t$ , and so

$$\begin{split} \int_{A} \mathbf{1}_{C} Y \mathbb{Q}(C \mid \mathcal{F}_{t}) \, d\mathbb{Q} &= \int_{A \cap C} Y \mathbb{Q}(C \mid \mathcal{F}_{t}) \, d\mathbb{Q} = \int_{B \cap C} Y \mathbb{Q}(C \mid \mathcal{F}_{t}) \, d\mathbb{Q} \\ &= \int_{B} \mathbf{1}_{C} Y \mathbb{Q}(C \mid \mathcal{F}_{t}) \, d\mathbb{Q} = \int_{B} \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{C} Y \mid \mathcal{F}_{t}) \mathbb{Q}(C \mid \mathcal{F}_{t}) \, d\mathbb{Q} \\ &= \int_{B} \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{C} \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{C} Y \mid \mathcal{F}_{t}) \mid \mathcal{F}_{t}) \, d\mathbb{Q} = \int_{B \cap C} \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{C} Y \mid \mathcal{F}_{t}) \, d\mathbb{Q} \\ &= \int_{A \cap C} \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{C} Y \mid \mathcal{F}_{t}) \, d\mathbb{Q} = \int_{A} \mathbf{1}_{C} \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{C} Y \mid \mathcal{F}_{t}) \, d\mathbb{Q}. \end{split}$$

This ends the proof.

The following corollary is straightforward.

Corollary 3.1.1 Let Y be an  $\mathcal{G}_T$ -measurable, integrable random variable. Then

$$\mathbb{E}_{\mathbb{Q}}(Y\mathbb{1}_{\{T<\tau\}} \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{Q}}(Y\mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t)}{\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}}(Y e^{-\Gamma_T} \mid \mathcal{F}_t).$$
(3.3)

**Lemma 3.1.3** Let h be an  $\mathbb{F}$ -predictable process. Then

$$\mathbb{E}_{\mathbb{Q}}(h_{\tau}\mathbb{1}_{\{\tau < T\}} \mid \mathcal{G}_t) = h_{\tau}\mathbb{1}_{\{\tau \le t\}} + \mathbb{1}_{\{\tau > t\}}e^{\Gamma_t}\mathbb{E}_{\mathbb{Q}}\left(\int_t^T h_u \, dF_u \mid \mathcal{F}_t\right)$$
(3.4)

We are not interested in  $\mathbb{G}$ -predictable processes, mainly because any  $\mathbb{G}$ -predictable process is equal, on the event  $\{t \leq \tau\}$ , to an  $\mathbb{F}$ -predictable process. As we shall see, this elementary result will allow us to compute the value of credit derivatives, as soon as some elementary defaultable assets are priced by the market.

# 3.1.2 Martingales

**Proposition 3.1.1** (i) The process  $L_t = (1 - H_t)e^{\Gamma(t)}$  is a G-martingale. (ii) If X is an F-martingale then XL is a G-martingale. (iii) If the process  $\Gamma$  is increasing and continuous, then the process  $M_t = H_t - \Gamma(t \wedge \tau)$  is a G-martingale.

*Proof.* (i) From Lemma 3.1.2, for any t > s,

$$\mathbb{E}_{\mathbb{Q}}(L_t \mid \mathcal{G}_s) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mid \mathcal{G}_s) = \mathbb{1}_{\{\tau > s\}} e^{\Gamma_s} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mid \mathcal{F}_s) = \mathbb{1}_{\{\tau > s\}} e^{\Gamma_s} = L_s$$

since

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{F}_s) = \mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)e^{\Gamma_t}|\mathcal{F}_s) = 1$$

(ii) From Lemma 3.1.2,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(L_{t}X_{t} \mid \mathcal{G}_{s}) &= \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}}L_{t}X_{t} \mid \mathcal{G}_{s}) \\ &= \mathbb{1}_{\{\tau > s\}}e^{\Gamma_{s}}\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}}e^{-\Gamma_{t}}X_{t} \mid \mathcal{F}_{s}) \\ &= \mathbb{1}_{\{\tau > s\}}e^{\Gamma_{s}}\mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}} \mid \mathcal{F}_{t})e^{-\Gamma_{t}}X_{t} \mid \mathcal{F}_{s}) \\ &= L_{s}X_{s}. \end{aligned}$$

(iii) From integration by parts formula (H is a finite variation process, and  $\Gamma$  an increasing continuous process):

$$dL_t = (1 - H_t)e^{\Gamma_t}d\Gamma_t - e^{\Gamma_t}dH_t$$

and the process  $M_t = H_t - \Gamma(t \wedge \tau)$  can be written

$$M_t \equiv \int_{]0,t]} dH_u - \int_{]0,t]} (1 - H_u) d\Gamma_u = -\int_{]0,t]} e^{-\Gamma_u} dL_u$$

and is a G-local martingale since L is G-martingale. It should be noted that, if  $\Gamma$  is not increasing, the differential of  $e^{\Gamma}$  is more complicated.

# 3.1.3 Interpretation of the Intensity

The submartingale property of F implies, from the Doob-Meyer decomposition, that F = Z + A where Z is a F-martingale and A a F-predictable increasing process.

Lemma 3.1.4 We have

$$\mathbb{E}_{\mathbb{Q}}(h_{\tau}\mathbb{1}_{\{\tau < T\}} \mid \mathcal{G}_t) = h_{\tau}\mathbb{1}_{\{\tau < t\}} + \mathbb{1}_{\{\tau > t\}}e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T h_u \, dA_u \, \Big| \, \mathcal{F}_t\Big).$$

In this general setting, the process  $\Gamma$  is not with finite variation. Hence, part (iii) in Proposition 3.1.1 does not yield the Doob-Meyer decomposition of H. We shall assume, for simplicity, that F is continuous.

**Proposition 3.1.2** Assume that F is a continuous process. Then the process

$$M_t = H_t - \int_0^{t \wedge \tau} \frac{dA_u}{1 - F_u}, \quad \forall t \in \mathbb{R}_+,$$

is a G-martingale.

*Proof.* Let s < t. We give the proof in two steps, using the Doob-Meyer decomposition F = Z + A of F.

First step. We shall prove that

$$\mathbb{E}_{\mathbb{Q}}(H_t \mid \mathcal{G}_s) = H_s + \mathbb{1}_{\{s < \tau\}} \frac{1}{1 - F_s} \mathbb{E}_{\mathbb{Q}}(A_t - A_s \mid \mathcal{F}_s)$$

Indeed,

$$\begin{split} \mathbb{E}_{\mathbb{Q}}(H_t \mid \mathcal{G}_s) &= 1 - \mathbb{Q}(t < \tau \mid \mathcal{G}_s) = 1 - \mathbb{1}_{\{s < \tau\}} \frac{1}{1 - F_s} \mathbb{E}_{\mathbb{Q}}(1 - F_t \mid \mathcal{F}_s) \\ &= 1 - \mathbb{1}_{\{s < \tau\}} \frac{1}{1 - F_s} \mathbb{E}_{\mathbb{Q}}(1 - Z_t - A_t \mid \mathcal{F}_s) \\ &= 1 - \mathbb{1}_{\{s < \tau\}} \frac{1}{1 - F_s} (1 - Z_s - A_s - \mathbb{E}_{\mathbb{Q}}(A_t - A_s \mid \mathcal{F}_s)) \\ &= 1 - \mathbb{1}_{\{s < \tau\}} \frac{1}{1 - F_s} (1 - F_s - \mathbb{E}_{\mathbb{Q}}(A_t - A_s \mid \mathcal{F}_s)) \\ &= \mathbb{1}_{\{\tau \le s\}} + \mathbb{1}_{\{s < \tau\}} \frac{1}{1 - F_s} \mathbb{E}_{\mathbb{Q}}(A_t - A_s \mid \mathcal{F}_s) \end{split}$$

Second step. Let us

$$\Lambda_t = \int_0^t (1 - H_s) \frac{dA_s}{1 - F_s}.$$

We shall prove that

$$\mathbb{E}_{\mathbb{Q}}(\Lambda_{t\wedge\tau} \mid \mathcal{G}_s) = \Lambda_{s\wedge\tau} + \mathbb{1}_{\{s<\tau\}} \frac{1}{1-F_s} \mathbb{E}_{\mathbb{Q}}(A_t - A_s \mid \mathcal{F}_s).$$

From the key lemma, we obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\Lambda_{t\wedge\tau} \mid \mathcal{G}_{s}) &= \Lambda_{s\wedge\tau} \mathbb{1}_{\{\tau \leq s\}} + \mathbb{1}_{\{s < \tau\}} \frac{1}{1 - F_{s}} \mathbb{E}_{\mathbb{Q}}\left(\int_{s}^{\infty} \Lambda_{t\wedge u} \, dF_{u} \mid \mathcal{F}_{s}\right) \\ &= \Lambda_{s\wedge\tau} \mathbb{1}_{\{\tau \leq s\}} + \mathbb{1}_{\{s < \tau\}} \frac{1}{1 - F_{s}} \mathbb{E}_{\mathbb{Q}}\left(\int_{s}^{t} \Lambda_{u} \, dF_{u} + \int_{t}^{\infty} \Lambda_{t} \, dF_{u} \mid \mathcal{F}_{s}\right) \\ &= \Lambda_{s\wedge\tau} \mathbb{1}_{\{\tau \leq s\}} + \mathbb{1}_{\{s < \tau\}} \frac{1}{1 - F_{s}} \mathbb{E}_{\mathbb{Q}}\left(\int_{s}^{t} \Lambda_{u} \, dF_{u} + \Lambda_{t}(1 - F_{t}) \mid \mathcal{F}_{s}\right). \end{aligned}$$

Using the integration by parts formula and the fact that  $\Lambda$  is of bounded variation and continuous, we obtain

$$d(\lambda_t(1-F_t)) = -\Lambda_t \, dF_t + (1-F_t) d\Lambda_t = -\Lambda_t \, dF_t + dA_t.$$

Hence

$$\int_{s}^{t} \Lambda_{u} \, dF_{u} + \Lambda_{t}(1 - F_{t}) = -\Lambda_{t}(1 - F_{t}) + \Lambda_{s}(1 - F_{s}) + A_{t} - A_{s} + \Lambda_{t}(1 - F_{t}) = \Lambda_{s}(1 - F_{s}) + A_{t} - A_{s}.$$

It follows that

$$\mathbb{E}_{\mathbb{Q}}(\Lambda_{t\wedge\tau} \mid \mathcal{G}_s) = \Lambda_{s\wedge\tau} \mathbb{1}_{\{\tau \leq s\}} + \mathbb{1}_{\{s<\tau\}} \frac{1}{1-F_s} \mathbb{E}_{\mathbb{Q}}\left(\Lambda_s(1-F_s) + A_t - A_s \mid \mathcal{F}_s\right)$$
$$= \Lambda_{s\wedge\tau} + \mathbb{1}_{\{s<\tau\}} \frac{1}{1-F_s} \mathbb{E}_{\mathbb{Q}}\left(A_t - A_s \mid \mathcal{F}_s\right).$$

This completes the proof.

Let us assume that A is absolutely continuous with respect to the Lebesgue measure and let us denote by a its derivative. We have proved the existence of a  $\mathbb{F}$ -adapted process  $\gamma$ , called the intensity, such that the process

$$H_t - \int_0^{t \wedge \tau} \gamma_u \, du = H_t - \int_0^t (1 - H_u) \gamma_u \, du$$

is a G-martingale. More precisely,  $\gamma_t = \frac{a_t}{1-F_t}$  for  $t \in \mathbb{R}_+$ .

**Lemma 3.1.5** The intensity process  $\gamma$  satisfies

$$\gamma_t = \lim_{h \to 0} \frac{1}{h} \frac{\mathbb{Q}(t < \tau < t + h \,|\, \mathcal{F}_t)}{\mathbb{Q}(t < \tau \,|\, \mathcal{F}_t)} \,.$$

*Proof.* The martingale property of M implies that

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t < \tau < t+h\}} | \mathcal{G}_t) - \int_t^{t+h} \mathbb{E}_{\mathbb{Q}}((1-H_s)\lambda_s | \mathcal{G}_t) \, ds = 0.$$

It follows that, by the projection on  $\mathcal{F}_t$ ,

$$\mathbb{Q}(t < \tau < t + h \,|\, \mathcal{F}_t) = \int_t^{t+h} \lambda_s \,\mathbb{Q}(s < \tau \,|\, \mathcal{F}_t) \,ds.$$

# 3.1.4 Reduction of the Reference Filtration

Suppose from now on that  $\widetilde{\mathcal{F}}_t \subset \mathcal{F}_t$  and define  $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$ . The associated hazard process is given by  $\widetilde{\Gamma}_t = -\ln(\widetilde{G}_t)$  with  $\widetilde{G}_t = \mathbb{Q}(t < \tau | \widetilde{\mathcal{F}}_t) = \mathbb{E}_{\mathbb{Q}}(G_t | \widetilde{\mathcal{F}}_t)$ . Then the key lemma implies that

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}}Y \,|\, \widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{\tau > t\}}e^{\Gamma_t} \,\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > t\}}Y \,|\, \widetilde{\mathcal{F}}_t)$$

If Y is a  $\widetilde{\mathcal{F}}_T$ -measurable variable, then

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>T\}}Y \,|\, \widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{\tau>t\}} e^{\widetilde{\Gamma}_t} \,\mathbb{E}_{\mathbb{Q}}(\widetilde{G}_TY \,|\, \widetilde{\mathcal{F}}_t).$$

From the equality

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>T\}}Y\,|\,\widetilde{\mathcal{G}}_t) = \mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>T\}}Y\,|\,\mathcal{G}_t)\,|\,\widetilde{\mathcal{G}}_t),$$

we deduce that

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>T\}}Y|\widetilde{\mathcal{G}}_{t}) = \mathbb{E}_{\mathbb{Q}}\left(e^{\Gamma_{t}}\mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}(G_{T}Y|\mathcal{F}_{t})|\widetilde{\mathcal{G}}_{t}\right) \\
= \mathbb{1}_{\{\tau>t\}}e^{\widetilde{\Gamma}_{t}}\mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_{t}}\mathbb{E}_{\mathbb{Q}}(G_{T}Y|\mathcal{F}_{t})|\widetilde{\mathcal{F}}_{t}\right).$$

From the uniqueness of the pre-default  $\mathbb{F}$ -adapted value, we obtain, for any t,

$$\mathbb{E}_{\mathbb{Q}}(\widetilde{G}_{T}Y|\widetilde{\mathcal{F}}_{t}) = \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\tau > t\}}e^{\Gamma_{t}}\mathbb{E}_{\mathbb{Q}}(G_{T}Y|\mathcal{F}_{t}) \mid \widetilde{\mathcal{F}}_{t}\right)$$

As a check, a simple computation shows

$$\begin{split} \mathbb{E}_{\mathbb{Q}}\left(\mathbbm{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}(G_{T}Y|\mathcal{F}_{t})e^{\Gamma_{t}} \,\middle|\, \widetilde{\mathcal{F}}_{t}\right) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(\mathbbm{1}_{\{\tau>t\}}|\mathcal{F}_{t})e^{\Gamma_{t}}\mathbb{E}_{\mathbb{Q}}(G_{T}Y|\mathcal{F}_{t})\,\middle|\, \widetilde{\mathcal{F}}_{t}\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(G_{T}Y|\mathcal{F}_{t})\,\middle|\, \widetilde{\mathcal{F}}_{t}\right) = \mathbb{E}_{\mathbb{Q}}(G_{T}Y|\widetilde{\mathcal{F}}_{t}) \\ &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(G_{T}|\widetilde{\mathcal{F}}_{T})Y|\widetilde{\mathcal{F}}_{t}\right) = \mathbb{E}_{\mathbb{Q}}(\widetilde{G}_{T}Y|\widetilde{\mathcal{F}}_{t}) \end{split}$$

since Y we assumed that is  $\widetilde{\mathcal{F}}_T$ -measurable.

Let F = Z + A be the Doob-Meyer decomposition of the submartingale F with respect to  $\mathbb{F}$  and let us assume that A is differentiable with respect to t, that is,  $A_t = \int_0^t a_s \, ds$ . Then the process  $\widetilde{A}_t = \mathbb{E}_{\mathbb{Q}}(A_t | \widetilde{\mathcal{F}}_t)$  is a submartingale with respect to  $\widetilde{\mathbb{F}}$  with the Doob-Meyer decomposition  $\widetilde{A} = \widetilde{z} + \widetilde{\alpha}$ . Hence. setting  $\widetilde{Z}_t = \mathbb{E}_{\mathbb{Q}}(Z_t | \widetilde{\mathcal{F}}_t)$ , the submartingale

$$\widetilde{F}_t = \mathbb{Q}(t \ge \tau \,|\, \widetilde{\mathcal{F}}_t) = \mathbb{E}_{\mathbb{Q}}(F_t \,|\, \widetilde{\mathcal{F}}_t)$$

admits the Doob-Meyer decomposition  $\widetilde{F} = \widetilde{Z} + \widetilde{z} + \widetilde{\alpha}$ . The next lemma provide a link between  $\widetilde{\alpha}$  and a.

**Lemma 3.1.6** The compensator of  $\widetilde{F}$  equals

$$\widetilde{\alpha}_t = \int_0^t \mathbb{E}_{\mathbb{Q}}(a_s \,|\, \widetilde{\mathcal{F}}_s) \, ds.$$

*Proof.* Let us show that the process

$$M_t^F = \mathbb{E}_{\mathbb{Q}}(F_t \,|\, \widetilde{\mathcal{F}}_t) - \int_0^t \mathbb{E}_{\mathbb{Q}}(a_s \,|\, \widetilde{\mathcal{F}}_s) \,ds$$

is an  $\widetilde{\mathbb{F}}\text{-martingale}.$  Clearly, it is integrable and  $\widetilde{\mathbb{F}}\text{-adapted}.$  Moreover,

$$\begin{split} \mathbb{E}_{\mathbb{Q}}(M_{T}^{F} \mid \widetilde{\mathcal{F}}_{t}) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(F_{T} \mid \widetilde{\mathcal{F}}_{T}) - \int_{0}^{T} \mathbb{E}_{\mathbb{Q}}(a_{s} \mid \widetilde{\mathcal{F}}_{s}) \, ds \mid \widetilde{\mathcal{F}}_{t}\right) \\ &= \mathbb{E}_{\mathbb{Q}}(F_{T} \mid \widetilde{\mathcal{F}}_{t}) - \mathbb{E}_{\mathbb{Q}}\left(\int_{0}^{t} \mathbb{E}_{\mathbb{Q}}(a_{s} \mid \widetilde{\mathcal{F}}_{s}) \, ds \mid \widetilde{\mathcal{F}}_{t}\right) - \mathbb{E}_{\mathbb{Q}}\left(\int_{t}^{T} \mathbb{E}_{\mathbb{Q}}(a_{s} \mid \widetilde{\mathcal{F}}_{s}) \, ds \mid \widetilde{\mathcal{F}}_{t}\right) \\ &= \widetilde{Z}_{t} + \mathbb{E}_{\mathbb{Q}}\left(\int_{0}^{t} a_{s} \, ds \mid \widetilde{\mathcal{F}}_{t}\right) + \mathbb{E}_{\mathbb{Q}}\left(\int_{t}^{T} a_{s} \, ds \mid \widetilde{\mathcal{F}}_{t}\right) \end{split}$$

#### 3.1. GENERAL CASE

$$-\mathbb{E}_{\mathbb{Q}}\left(\int_{0}^{t} \mathbb{E}_{\mathbb{Q}}(a_{s} \,|\, \widetilde{\mathcal{F}}_{s}) \,ds \,\Big|\, \widetilde{\mathcal{F}}_{t}\right) - \mathbb{E}_{\mathbb{Q}}\left(\int_{t}^{T} \mathbb{E}_{\mathbb{Q}}(a_{s} \,|\, \widetilde{\mathcal{F}}_{s}) \,ds \,\Big|\, \widetilde{\mathcal{F}}_{t}\right)$$

$$= M_{t}^{F} + \mathbb{E}_{\mathbb{Q}}\left(\int_{t}^{T} f_{s} \,ds \,\Big|\, \widetilde{\mathcal{F}}_{t}\right) - \mathbb{E}_{\mathbb{Q}}\left(\int_{t}^{T} \mathbb{E}_{\mathbb{Q}}(f_{s} \,|\, \widetilde{\mathcal{F}}_{s}) \,ds \,\Big|\, \widetilde{\mathcal{F}}_{t}\right)$$

$$= M_{t}^{F} + \int_{t}^{T} \mathbb{E}_{\mathbb{Q}}(f_{s} \,|\, \widetilde{\mathcal{F}}_{t}) \,ds - \int_{t}^{T} \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(f_{s} \,|\, \widetilde{\mathcal{F}}_{s}) \,\Big|\, \widetilde{\mathcal{F}}_{t}\right) \,ds$$

$$= M_{t}^{F} + \int_{t}^{T} \mathbb{E}_{\mathbb{Q}}(a_{s} \,|\, \widetilde{\mathcal{F}}_{t}) \,ds - \int_{t}^{T} \mathbb{E}_{\mathbb{Q}}(a_{s} \,|\, \widetilde{\mathcal{F}}_{t}) \,ds = M_{t}^{F}.$$

Hence the process

$$\left(\widetilde{F}_t - \int_0^t \mathbb{E}_{\mathbb{Q}}(a_s \,|\, \widetilde{\mathcal{F}}_s) \, ds, \, t \ge 0\right)$$

is a  $\widetilde{\mathbb{F}}$ -martingale and the process  $\int_0 \mathbb{E}_{\mathbb{Q}}(a_s | \widetilde{\mathcal{F}}_s) ds$  is predictable. The uniqueness of the Doob-Meyer decomposition implies that

$$\widetilde{\alpha}_t = \int_0^t \mathbb{E}_{\mathbb{Q}}(a_s \,|\, \widetilde{\mathcal{F}}_s) \, ds,$$

as required.

Remark 3.1.1 It follows that

$$H_t - \int_0^{t \wedge \tau} \frac{\widetilde{f}_s}{1 - \widetilde{F}_s} \, ds$$

is a  $\widetilde{\mathbb{G}}$ -martingale and that the  $\widetilde{\mathbb{F}}$ -intensity of  $\tau$  is equal to  $\mathbb{E}_{\mathbb{Q}}(a_s | \widetilde{\mathcal{F}}_s)/\widetilde{G}_s$ , and not, as one might have expected, to  $\mathbb{E}_{\mathbb{Q}}(a_s/G_s | \widetilde{\mathcal{F}}_s)$ . Note that even if the hypothesis (H) holds between  $\widetilde{\mathbb{F}}$  and  $\mathbb{F}$ , this proof cannot be simplified, since the process  $\widetilde{F}_t$  is increasing but not  $\widetilde{\mathbb{F}}$ -predictable (there is no reason for  $\widetilde{F}$  to admit an intensity).

This result can also be proved directly thanks to the following result, due to Brémaud [11]:

$$H_t - \int_0^{t \wedge \tau} \lambda_s \, ds$$

is a **G**-martingale and thus

$$H_t - \int_0^{t \wedge \tau} \mathbb{E}_{\mathbb{Q}}(\lambda_s \,|\, \widetilde{\mathcal{G}}_s) \, ds$$

is a  $\widetilde{\mathbb{G}}\text{-martingale}.$  Note that

$$\int_0^{t\wedge\tau} \mathbb{E}_{\mathbb{Q}}(\lambda_s \,|\, \widetilde{\mathcal{G}}_s) \, ds = \int_0^t \mathbb{1}_{\{s \le \tau\}} \mathbb{E}_{\mathbb{Q}}(\lambda_s \,|\, \widetilde{\mathcal{G}}_s) \, ds = \int_0^t \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s \le \tau\}} \lambda_s \,|\, \widetilde{\mathcal{G}}_s) \, ds$$

and

$$\begin{split} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s \leq \tau\}}\lambda_s \,|\, \widetilde{\mathcal{G}}_s) &= \frac{\mathbb{1}_{\{s \leq \tau\}}}{\widetilde{G}_s} \,\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s \leq \tau\}}\lambda_s \,|\, \widetilde{\mathcal{F}}_s) \\ &= \frac{\mathbb{1}_{\{s \leq \tau\}}}{\widetilde{G}_s} \,\mathbb{E}_{\mathbb{Q}}(G_s\lambda_s \,|\, \widetilde{\mathcal{F}}_s) = \frac{\mathbb{1}_{\{s \leq \tau\}}}{\widetilde{G}_s} \,\mathbb{E}_{\mathbb{Q}}(a_s \,|\, \widetilde{\mathcal{F}}_s). \end{split}$$

We thus conclude that

$$H_t - \int_0^{t \wedge \tau} \frac{\mathbb{E}_{\mathbb{Q}}(a_s \,|\, \widetilde{\mathcal{F}}_s)}{\widetilde{G}_s} \, ds$$

is a  $\widetilde{\mathbb{G}}\text{-martingale},$  which is the desired result.

# 3.1.5 Enlargement of Filtration

We may work directly with the filtration  $\mathbb{G}$ , provided that the decomposition of any  $\mathbb{F}$ -martingale in this filtration is known up to time  $\tau$ . For example, if B is an  $\mathbb{F}$ -Brownian motion, its decomposition in the  $\mathbb{G}$  filtration up to time  $\tau$  is

$$B_{t\wedge\tau} = \beta_{t\wedge\tau} + \int_0^{t\wedge\tau} \frac{d\langle B,G\rangle_s}{G_{s-1}},$$

where  $(\beta_{t \wedge \tau}, t \ge 0)$  is a continuous G-martingale with the increasing process  $t \wedge \tau$ . If the dynamics of an asset S are given by

$$dS_t = S_t (r_t \, dt + \sigma_t \, dB_t)$$

in a default-free framework, where B is a Brownian motion, then its dynamics are

$$dS_t = S_t \left( r_t \, dt + \sigma_t \frac{d \langle B, G \rangle_t}{G_{t-}} + \sigma_t \, d\beta_t \right)$$

in the default filtration, if we restrict our attention to time before default. Therefore, the default will act as a change of drift term on the prices.

# 3.2 Hypothesis (H)

In a general setting,  $\mathbb{F}$  martingales do not remains  $\mathbb{G}$ -martingales. We study here a specific case.

# 3.2.1 Equivalent Formulations

We shall now examine the hypothesis (H) which reads:

(H) Every  $\mathbb{F}$ -local martingale is a  $\mathbb{G}$ -local martingale.

This hypothesis implies, for instance, that any  $\mathbb{F}$ -Brownian motion remains a Brownian motion in the enlarged filtration  $\mathbb{G}$ . It was studied by Brémaud and Yor [12], Mazziotto and Szpirglas [51], and for financial purpose by Kusuoka [44]. This can be written in any of the equivalent forms (see, e.g., Dellacherie and Meyer [25]).

**Lemma 3.2.1** Assume that  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ , where  $\mathbb{F}$  is an arbitrary filtration and  $\mathbb{H}$  is generated by the process  $H_t = \mathbb{1}_{\{\tau \leq t\}}$ . Then the following conditions are equivalent to the hypothesis (H). (i) For any  $t, h \in \mathbb{R}_+$ , we have

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_{t+h}). \tag{3.5}$$

(i') For any  $t \in \mathbb{R}_+$ , we have

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_\infty). \tag{3.6}$$

(ii) For any  $t \in \mathbb{R}_+$ , the  $\sigma$ -fields  $\mathcal{F}_{\infty}$  and  $\mathcal{G}_t$  are conditionally independent given  $\mathcal{F}_t$  under  $\mathbb{Q}$ , that is,

$$\mathbb{E}_{\mathbb{Q}}(\xi \eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\xi \,|\, \mathcal{F}_t) \,\mathbb{E}_{\mathbb{Q}}(\eta \,|\, \mathcal{F}_t)$$

for any bounded,  $\mathcal{F}_{\infty}$ -measurable random variable  $\xi$  and bounded,  $\mathcal{G}_t$ -measurable random variable  $\eta$ . (iii) For any  $t \in \mathbb{R}_+$ , and any  $u \ge t$  the  $\sigma$ -fields  $\mathcal{F}_u$  and  $\mathcal{G}_t$  are conditionally independent given  $\mathcal{F}_t$ . (iv) For any  $t \in \mathbb{R}_+$  and any bounded,  $\mathcal{F}_{\infty}$ -measurable random variable  $\xi$ :  $\mathbb{E}_{\mathbb{Q}}(\xi | \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(\xi | \mathcal{F}_t)$ . (v) For any  $t \in \mathbb{R}_+$ , and any bounded,  $\mathcal{G}_t$ -measurable random variable  $\eta$ :  $\mathbb{E}_{\mathbb{Q}}(\eta | \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\eta | \mathcal{F}_{\infty})$ . *Proof.* If the hypothesis (H) holds then (3.6) is valid as well. If (3.6) holds, the fact that  $\mathcal{H}_t$  is generated by the sets  $\{\tau \leq s\}, s \leq t$  proves that  $\mathcal{F}_{\infty}$  and  $\mathcal{H}_t$  are conditionally independent given  $\mathcal{F}_t$ . The desired property now follows. This result can be also found in [26]. The equivalence between (3.6) and (3.5) is left to the reader.

Using the monotone class theorem, it can be shown that conditions (i) and (i') are equivalent. The proof of equivalence of conditions (i')-(v) can be found, for instance, in Section 6.1.1 of Bielecki and Rutkowski [7] (for related results, see Elliott et al. [31]). Hence we shall only show that condition (iv) and the hypothesis (H) are equivalent.

Assume first that the hypothesis (H) holds. Consider any bounded,  $\mathcal{F}_{\infty}$ -measurable random variable  $\xi$ . Let  $M_t = \mathbb{E}_{\mathbb{Q}}(\xi | \mathcal{F}_t)$  be the martingale associated with  $\xi$ . Of course, M is a local martingale with respect to  $\mathbb{F}$ . Then the hypothesis (H) implies that M is also a local martingale with respect to  $\mathbb{G}$ , and thus a  $\mathbb{G}$ -martingale, since M is bounded (recall that any bounded local martingale is a martingale). We conclude that  $M_t = \mathbb{E}_{\mathbb{Q}}(\xi | \mathcal{G}_t)$  and thus (iv) holds.

Suppose now that (iv) holds. First, we note that the standard truncation argument shows that the boundedness of  $\xi$  in (iv) can be replaced by the assumption that  $\xi$  is Q-integrable. Hence, any  $\mathbb{F}$ -martingale M is an  $\mathbb{G}$ -martingale, since M is clearly  $\mathbb{G}$ -adapted and we have, for every  $t \leq s$ ,

$$M_t = \mathbb{E}_{\mathbb{O}}(M_s \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{O}}(M_s \,|\, \mathcal{G}_t),$$

where the second equality follows from (iv). Suppose now that M is an  $\mathbb{F}$ -local martingale. Then there exists an increasing sequence of  $\mathbb{F}$ -stopping times  $\tau_n$  such that  $\lim_{n\to\infty} \tau_n = \infty$ , for any n the stopped process  $M^{\tau_n}$  follows a uniformly integrable  $\mathbb{F}$ -martingale. Hence  $M^{\tau_n}$  is also a uniformly integrable  $\mathbb{G}$ -martingale, and this means that M is a  $\mathbb{G}$ -local martingale.  $\Box$ 

**Remarks 3.2.1** (i) Equality (3.6) appears in several papers on default risk, typically without any reference to the hypothesis (H). For example, in Madan and Unal [50], the main theorem follows from the fact that (3.6) holds (see the proof of B9 in the appendix of [50]). This is also the case for Wong's model [60].

(ii) If  $\tau$  is  $\mathcal{F}_{\infty}$ -measurable and (3.6) holds then  $\tau$  is an  $\mathbb{F}$ -stopping time. If  $\tau$  is an  $\mathbb{F}$ -stopping time then equality (3.5) holds. If  $\mathbb{F}$  is the Brownian filtration, then  $\tau$  is predictable and  $\Lambda = H$ .

(iii) Though condition (H) does not necessarily hold true, in general, it is satisfied when  $\tau$  is constructed through the so-called canonical approach (or for Cox processes). This hypothesis is quite natural under the historical probability and it is stable under some changes of a probability measure. However, Kusuoka [44] provides an example where (H) holds under the historical probability, but it fails hold after an equivalent change of a probability measure. This counter-example is linked to modeling of dependent defaults.

(iv) Hypothesis (H) holds, in particular, if  $\tau$  is independent from  $\mathcal{F}_{\infty}$  (see Greenfield [37]).

(v) If hypothesis (H) holds then from the condition

 $\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty), \quad \forall \, t \in \mathbb{R}_+,$ 

we deduce easily that F is an increasing process.

**Comments 3.2.1** See Elliott et al. [31] for more comments. The property that F is increasing is equivalent to the fact that any  $\mathbb{F}$ -martingale stopped at time  $\tau$  is a  $\mathbb{G}$  martingale. Nikeghbali and Yor [56] proved that this is equivalent to  $\mathbb{E}_{\mathbb{Q}}(M_{\tau}) = M_0$  for any bounded  $\mathbb{F}$ -martingale M. The hypothesis (H) was also studied by Florens and Fougere [34], who coined the term *noncausality*.

**Proposition 3.2.1** Assume that the hypothesis (H) holds. If X is an  $\mathbb{F}$ -martingale then the processes XL and [L, X] are  $\mathbb{G}$ -local martingales.

*Proof.* We have seen in Proposition 3.1.1 that the process XL is a  $\mathbb{G}$ -martingale. Since  $[L, X] = LX - \int L_- dX - \int X_- dL$  and X is an  $\mathbb{F}$ -martingale (and thus also a  $\mathbb{G}$ -martingale), the process [L, X] is manifestly a  $\mathbb{G}$ -martingale as the sum of three  $\mathbb{G}$ -martingales.  $\Box$ 

# 3.2.2 Canonical Construction of a Default Time

We shall now briefly describe the most commonly used construction of a default time associated with a given hazard process  $\Gamma$ . It should be stressed that the random time obtained through this particular method – which will be called the *canonical construction* in what follows – has certain specific features that are not necessarily shared by all random times with a given  $\mathbb{F}$ -hazard process  $\Gamma$ . We assume that we are given an  $\mathbb{F}$ -adapted, right-continuous, increasing process  $\Gamma$  defined on a filtered probability space  $(\Omega, \mathbb{F}, \mathbb{Q})$ . As usual, we assume that  $\Gamma_0 = 0$  and  $\Gamma_{\infty} = +\infty$ . In many instances,  $\Gamma$  is given by the equality

$$\Gamma_t = \int_0^t \gamma_u \, du, \quad \forall \, t \in \mathbb{R}_+,$$

for some non-negative,  $\mathbb{F}$ -progressively measurable intensity process  $\gamma$ .

To construct a random time  $\tau$ , we shall postulate that the underlying probability space  $(\Omega, \mathbb{F}, \mathbb{Q})$  is sufficiently rich to support a random variable  $\xi$ , which is uniformly distributed on the interval [0, 1] and independent of the filtration  $\mathbb{F}$  under  $\mathbb{Q}$ . In this version of the canonical construction,  $\Gamma$  represents the  $\mathbb{F}$ -hazard process of  $\tau$  under  $\mathbb{Q}$ .

We define the random time  $\tau: \Omega \to \mathbb{R}_+$  by setting

$$\tau = \inf\left\{t \in \mathbb{R}_+ : e^{-\Gamma_t} \le \xi\right\} = \inf\left\{t \in \mathbb{R}_+ : \Gamma_t \ge \eta\right\},\tag{3.7}$$

where the random variable  $\eta = -\ln \xi$  has a unit exponential law under  $\mathbb{Q}$ . It is not difficult to find the process  $F_t = \mathbb{Q}(\tau \leq t | \mathcal{F}_t)$ . Indeed, since clearly  $\{\tau > t\} = \{\xi < e^{-\Gamma_t}\}$  and the random variable  $\Gamma_t$  is  $\mathcal{F}_{\infty}$ -measurable, we obtain

$$\mathbb{Q}(\tau > t \,|\, \mathcal{F}_{\infty}) = \mathbb{Q}(\xi < e^{-\Gamma_t} \,|\, \mathcal{F}_{\infty}) = \mathbb{Q}(\xi < e^{-x})_{x = \Gamma_t} = e^{-\Gamma_t}.$$
(3.8)

Consequently, we have

$$1 - F_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\mathbb{Q}(\tau > t \,|\, \mathcal{F}_\infty) \,|\, \mathcal{F}_t) = e^{-\Gamma_t},\tag{3.9}$$

and thus F is an  $\mathbb{F}$ -adapted, right-continuous, increasing process. It is also clear that the process  $\Gamma$  represents the  $\mathbb{F}$ -hazard process of  $\tau$  under  $\mathbb{Q}$ . As an immediate consequence of (3.8) and (3.9), we obtain the following property of the canonical construction of the default time (cf. (3.6))

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_{\infty}) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t), \quad \forall t \in \mathbb{R}_+.$$
(3.10)

To summarize, we have that

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_{\infty}) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_{u}) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_{t}) = e^{-\Gamma_{t}}$$
(3.11)

for any two dates  $0 \le t \le u$ .

# 3.2.3 Stochastic Barrier

Suppose that

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where  $\Gamma$  is a continuous, strictly increasing,  $\mathbb{F}$ -adapted process. Our goal is to show that there exists a random variable  $\Theta$ , independent of  $\mathcal{F}_{\infty}$ , with the exponential law of parameter 1, such that  $\tau = \inf \{t \ge 0 : \Gamma_t > \Theta\}$ . Let us set  $\Theta \stackrel{\text{def}}{=} \Gamma_{\tau}$ . Then

$$\{t < \Theta\} = \{t < \Gamma_{\tau}\} = \{C_t < \tau\},\$$

where C is the right inverse of  $\Gamma$ , so that  $\Gamma_{C_t} = t$ . Therefore,

$$\mathbb{Q}(\Theta > u \,|\, \mathcal{F}_{\infty}) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established the required properties, namely, the probability distribution of  $\Theta$  and its independence of the  $\sigma$ -field  $\mathcal{F}_{\infty}$ . Furthermore,  $\tau = \inf\{t : \Gamma_t > \Gamma_\tau\} = \inf\{t : \Gamma_t > \Theta\}$ .

# 3.2.4 Change of a Probability Measure

Kusuoka [44] shows, by means of a counter-example, that the hypothesis (H) is not invariant with respect to an equivalent change of the underlying probability measure, in general. It is worth noting that his counter-example is based on two filtrations,  $\mathbb{H}^1$  and  $\mathbb{H}^2$ , generated by the two random times  $\tau^1$  and  $\tau^2$ , and he chooses  $\mathbb{H}^1$  to play the role of the reference filtration  $\mathbb{F}$ . We shall argue that in the case where  $\mathbb{F}$  is generated by a Brownian motion, the above-mentioned invariance property is valid under mild technical assumptions.

# Girsanov's Theorem

From Proposition 3.1.2 we know that the process  $M_t = H_t - \Gamma_{t \wedge \tau}$  is a  $\mathbb{G}$ -martingale. We fix T > 0. For a probability measure  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  on  $(\Omega, \mathcal{G}_T)$  we introduce the  $\mathbb{G}$ -martingale  $\eta_t, t \leq T$ , by setting

$$\eta_t \stackrel{\text{def}}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathcal{G}_t} = \mathbb{E}_{\mathbb{P}}(X \mid \mathcal{G}_t), \quad \mathbb{P}\text{-a.s.},$$
(3.12)

where X is a  $\mathcal{G}_T$ -measurable integrable random variable, such that  $\mathbb{P}(X > 0) = 1$ .

The Radon-Nikodým density process  $\eta$  admits the following representation

$$\eta_t = 1 + \int_0^t \xi_u \, dW_u + \int_{]0,t]} \zeta_u \, dM_u$$

where  $\xi$  and  $\zeta$  are G-predictable stochastic processes. Since  $\eta$  is a strictly positive process, we get

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-} \left( \beta_u \, dW_u + \kappa_u \, dM_u \right) \tag{3.13}$$

where  $\beta$  and  $\kappa$  are  $\mathbb{G}$ -predictable processes, with  $\kappa > -1$ .

**Proposition 3.2.2** Let  $\mathbb{Q}$  be a probability measure on  $(\Omega, \mathcal{G}_T)$  equivalent to  $\mathbb{P}$ . If the Radon-Nikodým density of  $\mathbb{Q}$  with respect to  $\mathbb{P}$  is given by (3.12) with  $\eta$  satisfying (3.13), then the process

$$W_t^* = W_t - \int_0^t \beta_u \, du, \quad \forall t \in [0, T],$$
(3.14)

follows a Brownian motion with respect to  $\mathbb{G}$  under  $\mathbb{Q}$ , and the process

$$M_t^* \stackrel{\text{def}}{=} M_t - \int_{]0, t \wedge \tau]} \kappa_u \, d\Gamma_u = H_t - \int_{]0, t \wedge \tau]} (1 + \kappa_u) \, d\Gamma_u, \quad \forall t \in [0, T], \tag{3.15}$$

is a  $\mathbb{G}$ -martingale orthogonal to  $W^*$ .

*Proof.* Notice first that for  $t \leq T$  we have

$$d(\eta_t W_t^*) = W_t^* d\eta_t + \eta_{t-} dW_t^* + d[W^*, \eta]_t$$
  
=  $W_t^* d\eta_t + \eta_{t-} dW_t - \eta_{t-}\beta_t dt + \eta_{t-}\beta_t d[W, W]_t$   
=  $W_t^* d\eta_t + \eta_{t-} dW_t.$ 

This shows that  $W^*$  is a  $\mathbb{G}$ -martingale under  $\mathbb{Q}$ . Since the quadratic variation of  $W^*$  under  $\mathbb{Q}$  equals  $[W^*, W^*]_t = t$  and  $W^*$  is continuous, by virtue of Lévy's theorem it is clear that  $W^*$  follows a Brownian motion under  $\mathbb{Q}$ . Similarly, for  $t \leq T$ 

$$\begin{aligned} d(\eta_t M_t^*) &= M_t^* \, d\eta_t + \eta_{t-} \, dM_t^* + d[M^*, \eta]_t \\ &= M_t^* \, d\eta_t + \eta_{t-} \, dM_t - \eta_{t-} \kappa_t \, d\Gamma_{t\wedge\tau} + \eta_{t-} \kappa_t \, dH_t \\ &= M_t^* \, d\eta_t + \eta_{t-} (1+\kappa_t) \, dM_t. \end{aligned}$$

We conclude that  $M^*$  is a  $\mathbb{G}$ -martingale under  $\mathbb{Q}$ . To conclude it is enough to observe that  $W^*$  is a continuous process and  $M^*$  follows a process of finite variation.

**Corollary 3.2.1** Let Y be a  $\mathbb{G}$ -martingale with respect to  $\mathbb{Q}$ . Then Y admits the following decomposition

$$Y_t = Y_0 + \int_0^t \xi_u^* \, dW_u^* + \int_{]0,t]} \zeta_u^* \, dM_u^*, \tag{3.16}$$

where  $\xi^*$  and  $\zeta^*$  are  $\mathbb{G}$ -predictable stochastic processes.

*Proof.* Consider the process  $\widetilde{Y}$  given by the formula

$$\widetilde{Y}_t = \int_{]0,t]} \eta_{u-}^{-1} d(\eta_u Y_u) - \int_{]0,t]} \eta_{u-}^{-1} Y_{u-} d\eta_u$$

It is clear that  $\widetilde{Y}$  is a G-martingale under  $\mathbb{P}$ . Notice also that Itô's formula yields

$$\eta_{u-}^{-1} d(\eta_u Y_u) = dY_u + \eta_{u-}^{-1} Y_{u-} d\eta_u + \eta_{u-}^{-1} d[Y,\eta]_u,$$

and thus

$$Y_t = Y_0 + \tilde{Y}_t - \int_{]0,t]} \eta_{u-}^{-1} d[Y,\eta]_u.$$
(3.17)

From the predictable representation theorem, we know that

$$\widetilde{Y}_t = Y_0 + \int_0^t \widetilde{\xi}_u \, dW_u + \int_{]0,t]} \widetilde{\zeta}_u \, dM_u \tag{3.18}$$

for some  $\mathbb{G}$ -predictable processes  $\tilde{\xi}$  and  $\tilde{\zeta}$ . Therefore

$$dY_t = \widetilde{\xi}_t \, dW_t + \widetilde{\zeta}_t \, dM_t - \eta_{t-}^{-1} \, d[Y,\eta]_t$$
$$= \widetilde{\xi}_t \, dW_t^* + \widetilde{\zeta}_t (1+\kappa_t)^{-1} \, dM_t^*$$

since (3.13) combined with (3.17)-(3.18) yield

$$\eta_{t-}^{-1} d[Y,\eta]_t = \widetilde{\xi}_t \beta_t \, dt + \widetilde{\zeta}_t \kappa_t (1+\kappa_t)^{-1} \, dH_t.$$

To derive the last equality we observe, in particular, that in view of (3.17) we have (we take into account continuity of  $\Gamma$ )

$$\Delta[Y,\eta]_t = \eta_{t-}\widetilde{\zeta}_t \kappa_t \, dH_t - \kappa_t \Delta[Y,\eta]_t.$$
(3.16) with  $\xi^* = \widetilde{\xi}$  and  $\zeta^* = \widetilde{\zeta}(1+\kappa)^{-1}.$ 

# **Preliminary Lemma**

We conclude that Y satisfies

Let us first examine a general set-up in which  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ , where  $\mathbb{F}$  is an arbitrary filtration and  $\mathbb{H}$  is generated by the default process H. We say that  $\mathbb{Q}$  is locally equivalent to  $\mathbb{P}$  if  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  on  $(\Omega, \mathcal{G}_t)$  for every  $t \in \mathbb{R}_+$ . Then there exists the Radon-Nikodým density process  $\eta$  such that

$$d\mathbb{Q} \mid_{\mathcal{G}_t} = \eta_t \, d\mathbb{P} \mid_{\mathcal{G}_t}, \quad \forall t \in \mathbb{R}_+.$$

$$(3.19)$$

Part (i) in the next lemma is well known (see Jamshidian [40]). We assume that the hypothesis (H) holds under  $\mathbb{P}$ .

**Lemma 3.2.2** (i) Let  $\mathbb{Q}$  be a probability measure equivalent to  $\mathbb{P}$  on  $(\Omega, \mathcal{G}_t)$  for every  $t \in \mathbb{R}_+$ , with the associated Radon-Nikodým density process  $\eta$ . If the density process  $\eta$  is  $\mathbb{F}$ -adapted then we have  $\mathbb{Q}(\tau \leq t | \mathcal{F}_t) = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$  for every  $t \in \mathbb{R}_+$ . Hence, the hypothesis (H) is also valid under  $\mathbb{Q}$ and the  $\mathbb{F}$ -intensities of  $\tau$  under  $\mathbb{Q}$  and under  $\mathbb{P}$  coincide.

(ii) Assume that  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  on  $(\Omega, \mathcal{G})$  and  $d\mathbb{Q} = \eta_{\infty} d\mathbb{P}$ , so that  $\eta_t = \mathbb{E}_{\mathbb{P}}(\eta_{\infty} | \mathcal{G}_t)$ . Then the hypothesis (H) is valid under  $\mathbb{Q}$  whenever we have, for every  $t \in \mathbb{R}_+$ ,

$$\frac{\mathbb{E}_{\mathbb{P}}(\eta_{\infty}H_t \mid \mathcal{F}_{\infty})}{\mathbb{E}_{\mathbb{P}}(\eta_{\infty} \mid \mathcal{F}_{\infty})} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t H_t \mid \mathcal{F}_{\infty})}{\mathbb{E}_{\mathbb{P}}(\eta_t \mid \mathcal{F}_{\infty})}.$$
(3.20)

### 3.2. HYPOTHESIS (H)

*Proof.* To prove (i), assume that the density process  $\eta$  is  $\mathbb{F}$ -adapted. We have for each  $t \leq s \in \mathbb{R}_+$ 

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_s) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_s),$$

where the last equality follows by another application of the Bayes formula. The assertion now follows from part (i) in Lemma 3.2.1.

To prove part (ii), it suffices to establish the equality

$$\widehat{F}_t \stackrel{\text{def}}{=} \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_\infty), \quad \forall t \in \mathbb{R}_+.$$
(3.21)

Note that since the random variables  $\eta_t \mathbb{1}_{\{\tau \leq t\}}$  and  $\eta_t$  are  $\mathbb{P}$ -integrable and  $\mathcal{G}_t$ -measurable, using the Bayes formula, part (v) in Lemma 3.2.1, and assumed equality (3.20), we obtain the following chain of equalities

$$\begin{aligned} \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_\infty)} \\ &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_\infty \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{F}_\infty)} = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty). \end{aligned}$$

We conclude that the hypothesis (H) holds under  $\mathbb{Q}$  if and only if (3.20) is valid.

Unfortunately, straightforward verification of condition (3.20) is rather cumbersome. For this reason, we shall provide alternative sufficient conditions for the preservation of the hypothesis (H) under a locally equivalent probability measure.

# Case of the Brownian Filtration

Let W be a Brownian motion under  $\mathbb{P}$  and  $\mathbb{F}$  its natural filtration. Since we work under the hypothesis (H), the process W is also a  $\mathbb{G}$ -martingale, where  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ . Hence, W is a Brownian motion with respect to  $\mathbb{G}$  under  $\mathbb{P}$ . Our goal is to show that the hypothesis (H) is still valid under  $\mathbb{Q} \in \mathcal{Q}$  for a large class  $\mathcal{Q}$  of (locally) equivalent probability measures on  $(\Omega, \mathcal{G})$ .

Let  $\mathbb{Q}$  be an arbitrary probability measure locally equivalent to  $\mathbb{P}$  on  $(\Omega, \mathcal{G})$ . Kusuoka [44] (see also Section 5.2.2 in Bielecki and Rutkowski [7]) proved that, under the hypothesis (H), any  $\mathbb{G}$ -martingale under  $\mathbb{P}$  can be represented as the sum of stochastic integrals with respect to the Brownian motion W and the jump martingale M. In our set-up, Kusuoka's representation theorem implies that there exist  $\mathbb{G}$ -predictable processes  $\theta$  and  $\zeta > -1$ , such that the Radon-Nikodým density  $\eta$  of  $\mathbb{Q}$  with respect to  $\mathbb{P}$  satisfies the following SDE

$$d\eta_t = \eta_{t-} \left( \theta_t \, dW_t + \zeta_t \, dM_t \right) \tag{3.22}$$

with the initial value  $\eta_0 = 1$ . More explicitly, the process  $\eta$  equals

$$\eta_t = \mathcal{E}_t \left( \int_0^{\cdot} \theta_u \, dW_u \right) \mathcal{E}_t \left( \int_0^{\cdot} \zeta_u \, dM_u \right) = \eta_t^{(1)} \eta_t^{(2)}, \tag{3.23}$$

where we write

$$\eta_t^{(1)} = \mathcal{E}_t \left( \int_0^t \theta_u \, dW_u \right) = \exp\left( \int_0^t \theta_u \, dW_u - \frac{1}{2} \int_0^t \theta_u^2 \, du \right), \tag{3.24}$$

and

$$\eta_t^{(2)} = \mathcal{E}_t \left( \int_0^t \zeta_u \, dM_u \right) = \exp\left( \int_0^t \ln(1+\zeta_u) \, dH_u - \int_0^{t\wedge\tau} \zeta_u \gamma_u \, du \right). \tag{3.25}$$

Moreover, by virtue of a suitable version of Girsanov's theorem, the following processes  $\widehat{W}$  and  $\widehat{M}$  are  $\mathbb{G}$ -martingales under  $\mathbb{Q}$ 

$$\widehat{W}_t = W_t - \int_0^t \theta_u \, du, \quad \widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u < \tau\}} \gamma_u \zeta_u \, du. \tag{3.26}$$

**Proposition 3.2.3** Assume that the hypothesis (H) holds under  $\mathbb{P}$ . Let  $\mathbb{Q}$  be a probability measure locally equivalent to  $\mathbb{P}$  with the associated Radon-Nikodým density process  $\eta$  given by formula (3.23). If the process  $\theta$  is  $\mathbb{F}$ -adapted then the hypothesis (H) is valid under  $\mathbb{Q}$  and the  $\mathbb{F}$ -intensity of  $\tau$  under  $\mathbb{Q}$  equals  $\widehat{\gamma}_t = (1 + \widetilde{\zeta}_t)\gamma_t$ , where  $\widetilde{\zeta}$  is the unique  $\mathbb{F}$ -predictable process such that the equality  $\widetilde{\zeta}_t \mathbb{1}_{\{t \leq \tau\}} = \zeta_t \mathbb{1}_{\{t \leq \tau\}}$  holds for every  $t \in \mathbb{R}_+$ .

*Proof.* Let  $\widetilde{\mathbb{P}}$  be the probability measure locally equivalent to  $\mathbb{P}$  on  $(\Omega, \mathcal{G})$ , given by

$$d\widetilde{\mathbb{P}} \mid_{\mathcal{G}_t} = \mathcal{E}_t \left( \int_0^{\cdot} \zeta_u \, dM_u \right) \, d\mathbb{P} \mid_{\mathcal{G}_t} = \eta_t^{(2)} \, d\mathbb{P} \mid_{\mathcal{G}_t}.$$
(3.27)

We claim that the hypothesis (H) holds under  $\widetilde{\mathbb{P}}$ . From Girsanov's theorem, the process W follows a Brownian motion under  $\widetilde{\mathbb{P}}$  with respect to both  $\mathbb{F}$  and  $\mathbb{G}$ . Moreover, from the predictable representation property of W under  $\widetilde{\mathbb{P}}$ , we deduce that any  $\mathbb{F}$ -local martingale L under  $\widetilde{\mathbb{P}}$  can be written as a stochastic integral with respect to W. Specifically, there exists an  $\mathbb{F}$ -predictable process  $\xi$  such that

$$L_t = L_0 + \int_0^t \xi_u \, dW_u.$$

This shows that L is also a G-local martingale, and thus the hypothesis (H) holds under  $\widetilde{\mathbb{P}}$ . Since

$$d\mathbb{Q} \mid_{\mathcal{G}_t} = \mathcal{E}_t \left( \int_0^{\cdot} \theta_u \, dW_u \right) \, d\widetilde{\mathbb{P}} \mid_{\mathcal{G}_t},$$

by virtue of part (i) in Lemma 3.2.2, the hypothesis (H) is valid under  $\mathbb{Q}$  as well. The last claim in the statement of the lemma can be deduced from the fact that the hypothesis (H) holds under  $\mathbb{Q}$  and, by Girsanov's theorem, the process

$$\widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u < \tau\}} \gamma_u \zeta_u \, du = H_t - \int_0^t \mathbb{1}_{\{u < \tau\}} (1 + \widetilde{\zeta}_u) \gamma_u \, du$$

is a  $\mathbb{Q}$ -martingale.

We claim that the equality  $\widetilde{\mathbb{P}} = \mathbb{P}$  holds on the filtration  $\mathbb{F}$ . Indeed, we have  $d\widetilde{\mathbb{P}} |_{\mathcal{F}_t} = \widetilde{\eta}_t d\mathbb{P} |_{\mathcal{F}_t}$ , where we write  $\widetilde{\eta}_t = \mathbb{E}_{\mathbb{P}}(\eta_t^{(2)} | \mathcal{F}_t)$ , and

$$\mathbb{E}_{\mathbb{P}}(\eta_t^{(2)} \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^{\cdot} \zeta_u \, dM_u\right) \mid \mathcal{F}_\infty\right) = 1, \quad \forall t \in \mathbb{R}_+,$$
(3.28)

where the first equality follows from part (v) in Lemma 3.2.1.

To establish the second equality in (3.28), we first note that since the process M is stopped at  $\tau$ , we may assume, without loss of generality, that  $\zeta = \tilde{\zeta}$  where the process  $\tilde{\zeta}$  is  $\mathbb{F}$ -predictable. Moreover,the conditional cumulative distribution function of  $\tau$  given  $\mathcal{F}_{\infty}$  has the form  $1 - \exp(-\Gamma_t(\omega))$ . Hence, for arbitrarily selected sample paths of processes  $\zeta$  and  $\Gamma$ , the claimed equality can be seen as a consequence of the martingale property of the Doléans exponential.

Formally, it can be proved by following elementary calculations, where the first equality is a consequence of (3.25)),

$$\mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_{t}\left(\int_{0}^{t}\widetilde{\zeta}_{u}\,dM_{u}\right)\,\Big|\,\mathcal{F}_{\infty}\right) = \mathbb{E}_{\mathbb{P}}\left(\left(1+\mathbb{1}_{\{t\geq\tau\}}\widetilde{\zeta}_{\tau}\right)\exp\left(-\int_{0}^{t\wedge\tau}\widetilde{\zeta}_{u}\gamma_{u}\,du\right)\,\Big|\,\mathcal{F}_{\infty}\right)$$
$$= \mathbb{E}_{\mathbb{P}}\left(\int_{0}^{\infty}\left(1+\mathbb{1}_{\{t\geq u\}}\widetilde{\zeta}_{u}\right)\exp\left(-\int_{0}^{t\wedge u}\widetilde{\zeta}_{v}\gamma_{v}\,dv\right)\gamma_{u}e^{-\int_{0}^{u}\gamma_{v}\,dv}\,du\,\Big|\,\mathcal{F}_{\infty}\right)$$
$$= \mathbb{E}_{\mathbb{P}}\left(\int_{0}^{t}\left(1+\widetilde{\zeta}_{u}\right)\gamma_{u}\exp\left(-\int_{0}^{u}(1+\widetilde{\zeta}_{v})\gamma_{v}\,dv\right)du\,\Big|\,\mathcal{F}_{\infty}\right)$$

$$+ \exp\left(-\int_{0}^{t} \widetilde{\zeta}_{v} \gamma_{v} dv\right) \mathbb{E}_{\mathbb{P}}\left(\int_{t}^{\infty} \gamma_{u} e^{-\int_{0}^{u} \gamma_{v} dv} du \left| \mathcal{F}_{\infty}\right) \right)$$

$$= \int_{0}^{t} (1 + \widetilde{\zeta}_{u}) \gamma_{u} \exp\left(-\int_{0}^{u} (1 + \widetilde{\zeta}_{v}) \gamma_{v} dv\right) du$$

$$+ \exp\left(-\int_{0}^{t} \widetilde{\zeta}_{v} \gamma_{v} dv\right) \int_{t}^{\infty} \gamma_{u} e^{-\int_{0}^{u} \gamma_{v} dv} du$$

$$= 1 - \exp\left(-\int_{0}^{t} (1 + \widetilde{\zeta}_{v}) \gamma_{v} dv\right) + \exp\left(-\int_{0}^{t} \widetilde{\zeta}_{v} \gamma_{v} dv\right) \exp\left(-\int_{0}^{t} \gamma_{v} dv\right) = 1,$$

where the second last equality follows by an application of the chain rule.

#### **Extension to Orthogonal Martingales**

Equality (3.28) suggests that Proposition 3.2.3 can be extended to the case of arbitrary orthogonal local martingales. Such a generalization is convenient, if we wish to cover the situation considered in Kusuoka's counterexample.

Let N be a local martingale under  $\mathbb{P}$  with respect to the filtration  $\mathbb{F}$ . It is also a  $\mathbb{G}$ -local martingale, since we maintain the assumption that the hypothesis (H) holds under  $\mathbb{P}$ . Let  $\mathbb{Q}$  be an arbitrary probability measure locally equivalent to  $\mathbb{P}$  on  $(\Omega, \mathcal{G})$ . We assume that the Radon-Nikodým density process  $\eta$  of  $\mathbb{Q}$  with respect to  $\mathbb{P}$  equals

$$d\eta_t = \eta_{t-} \left( \theta_t \, dN_t + \zeta_t \, dM_t \right) \tag{3.29}$$

for some G-predictable processes  $\theta$  and  $\zeta > -1$  (the properties of the process  $\theta$  depend, of course, on the choice of the local martingale N). The next result covers the case where N and M are orthogonal G-local martingales under P, so that the product MN follows a G-local martingale.

**Proposition 3.2.4** Assume that the following conditions hold:

(a) N and M are orthogonal  $\mathbb{G}$ -local martingales under  $\mathbb{P}$ ,

(b) N has the predictable representation property under  $\mathbb{P}$  with respect to  $\mathbb{F}$ , in the sense that any  $\mathbb{F}$ -local martingale L under  $\mathbb{P}$  can be written as

$$L_t = L_0 + \int_0^t \xi_u \, dN_u, \quad \forall t \in \mathbb{R}_+,$$

for some  $\mathbb{F}$ -predictable process  $\xi$ , (c)  $\widetilde{\mathbb{P}}$  is a probability measure on  $(\Omega, \mathcal{G})$  such that (3.27) holds. Then we have: (i) the hypothesis (H) is valid under  $\widetilde{\mathbb{P}}$ , (ii) if the process  $\theta$  is  $\mathbb{F}$ -adapted then the hypothesis (H) is valid under  $\mathbb{Q}$ .

The proof of the proposition hinges on the following simple lemma.

Lemma 3.2.3 Under the assumptions of Proposition 3.2.4, we have:
(i) N is a G-local martingale under P,
(ii) N has the predictable representation property for F-local martingales under P.

*Proof.* In view of (c), we have  $d\widetilde{\mathbb{P}} |_{\mathcal{G}_t} = \eta_t^{(2)} d\mathbb{P} |_{\mathcal{G}_t}$ , where the density process  $\eta^{(2)}$  is given by (3.25), so that  $d\eta_t^{(2)} = \eta_{t-}^{(2)} \zeta_t dM_t$ . From the assumed orthogonality of N and M, it follows that N and  $\eta^{(2)}$  are orthogonal G-local martingales under  $\mathbb{P}$ , and thus  $N\eta^{(2)}$  is a G-local martingale under  $\mathbb{P}$  as well. This means that N is a G-local martingale under  $\widetilde{\mathbb{P}}$ , so that (i) holds.

To establish part (ii) in the lemma, we first define the auxiliary process  $\tilde{\eta}$  by setting  $\tilde{\eta}_t = \mathbb{E}_{\mathbb{P}}(\eta_t^{(2)} | \mathcal{F}_t)$ . Then manifestly  $d\tilde{\mathbb{P}} |_{\mathcal{F}_t} = \tilde{\eta}_t d\mathbb{P} |_{\mathcal{F}_t}$ , and thus in order to show that any  $\mathbb{F}$ -local martingale under  $\tilde{\mathbb{P}}$  follows an  $\mathbb{F}$ -local martingale under  $\mathbb{P}$ , it suffices to check that  $\tilde{\eta}_t = 1$  for every  $t \in \mathbb{R}_+$ , so that  $\tilde{\mathbb{P}} = \mathbb{P}$  on  $\mathbb{F}$ . To this end, we note that

$$\mathbb{E}_{\mathbb{P}}(\eta_t^{(2)} | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^{\cdot} \zeta_u \, dM_u\right) \, \Big| \, \mathcal{F}_\infty\right) = 1, \quad \forall t \in \mathbb{R}_+,$$

where the first equality follows from part (v) in Lemma 3.2.1, and the second one can established similarly as the second equality in (3.28).

We are in a position to prove (ii). Let L be an  $\mathbb{F}$ -local martingale under  $\widetilde{\mathbb{P}}$ . Then it follows also an  $\mathbb{F}$ -local martingale under  $\mathbb{P}$  and thus, by virtue of (b), it admits an integral representation with respect to N under  $\mathbb{P}$  and  $\widetilde{\mathbb{P}}$ . This shows that N has the predictable representation property with respect to  $\mathbb{F}$  under  $\widetilde{\mathbb{P}}$ .

We now proceed to the proof of Proposition 3.2.4.

Proof of Proposition 3.2.4. We shall argue along the similar lines as in the proof of Proposition 3.2.3. To prove (i), note that by part (ii) in Lemma 3.2.3 we know that any  $\mathbb{F}$ -local martingale under  $\widetilde{\mathbb{P}}$  admits the integral representation with respect to N. But, by part (i) in Lemma 3.2.3, N is a  $\mathbb{G}$ -local martingale under  $\widetilde{\mathbb{P}}$ . We conclude that L is a  $\mathbb{G}$ -local martingale under  $\widetilde{\mathbb{P}}$ , and thus the hypothesis (H) is valid under  $\widetilde{\mathbb{P}}$ . Assertion (ii) now follows from part (i) in Lemma 3.2.2.

**Remark 3.2.1** It should be stressed that Proposition 3.2.4 is not directly employed in what follows. We decided to present it here, since it sheds some light on specific technical problems arising in the context of modeling dependent default times through an equivalent change of a probability measure (see Kusuoka [44]).

**Example 3.2.1** Kusuoka [44] presents a counter-example based on the two independent random times  $\tau_1$  and  $\tau_2$  given on some probability space  $(\Omega, \mathcal{G}, \mathbb{P})$ . We write  $M_t^i = H_t^i - \int_0^{t \wedge \tau_i} \gamma_i(u) \, du$ , where  $H_t^i = \mathbb{1}_{\{t \geq \tau_i\}}$  and  $\gamma_i$  is the deterministic intensity function of  $\tau_i$  under  $\mathbb{P}$ . Let us set  $d\mathbb{Q} \mid \mathcal{G}_t = \eta_t \, d\mathbb{P} \mid \mathcal{G}_t$ , where  $\eta_t = \eta_t^{(1)} \eta_t^{(2)}$  and, for i = 1, 2 and every  $t \in \mathbb{R}_+$ ,

$$\eta_t^{(i)} = 1 + \int_0^t \eta_{u-}^{(i)} \zeta_u^{(i)} \, dM_u^i = \mathcal{E}_t \left( \int_0^\cdot \zeta_u^{(i)} \, dM_u^i \right)$$

for some  $\mathbb{G}$ -predictable processes  $\zeta^{(i)}$ , i = 1, 2, where  $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$ . We set  $\mathbb{F} = \mathbb{H}^1$  and  $\mathbb{H} = \mathbb{H}^2$ . Manifestly, the hypothesis (H) holds under  $\mathbb{P}$ . Moreover, in view of Proposition 3.2.4, it is still valid under the equivalent probability measure  $\widetilde{\mathbb{P}}$  given by

$$d\widetilde{\mathbb{P}} \,|\,_{\mathcal{G}_t} = \mathcal{E}_t \left( \int_0^{\cdot} \zeta_u^{(2)} \, dM_u^2 \right) \, d\mathbb{P} \,|\,_{\mathcal{G}_t}.$$

It is clear that  $\widetilde{\mathbb{P}} = \mathbb{P}$  on  $\mathbb{F}$ , since

$$\mathbb{E}_{\mathbb{P}}(\eta_t^{(2)} \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^{\cdot} \zeta_u^{(2)} \, dM_u^2\right) \mid \mathcal{H}_t^1\right) = 1, \quad \forall t \in \mathbb{R}_+.$$

However, the hypothesis (H) is not necessarily valid under  $\mathbb{Q}$  if the process  $\zeta^{(1)}$  fails to be  $\mathbb{F}$ -adapted. In Kusuoka's counter-example, the process  $\zeta^{(1)}$  was chosen to be explicitly dependent on both random times, and it was shown that the hypothesis (H) does not hold under  $\mathbb{Q}$ . For an alternative approach to Kusuoka's example, through an absolutely continuous change of a probability measure, the interested reader may consult Collin-Dufresne et al. [21].

# 3.3 Kusuoka's Representation Theorem

Kusuoka [44] established the following representation theorem.

**Theorem 3.1** Assume that the hypothesis (H) holds. Then any  $\mathbb{G}$ -square integrable martingale admits a representation as the sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale M associated with  $\tau$ .

We assume, for simplicity, that F is continuous and  $F_t < 1$  for every  $t \in \mathbb{R}^+$ . Since the hypothesis (H) holds, F is an increasing process. Then

$$dF_t = e^{-\Gamma_t} d\Gamma_t$$

and

$$d(e^{\Gamma_t}) = e^{\Gamma_t} d\Gamma_t = e^{\Gamma_t} \frac{dF_t}{1 - F_t} \,. \tag{3.30}$$

**Proposition 3.3.1** Suppose that hypothesis (H) holds under  $\mathbb{Q}$  and that any  $\mathbb{F}$ -martingale is continuous. Then the martingale  $M_t^h = \mathbb{E}_{\mathbb{Q}}(h_\tau | \mathcal{G}_t)$ , where h is an  $\mathbb{F}$ -predictable process such that  $\mathbb{E}_{\mathbb{Q}}(h_\tau) < \infty$ , admits the following decomposition in the sum of a continuous martingale and a discontinuous martingale

$$M_t^h = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u} dm_u^h + \int_{]0,t\wedge\tau]} (h_u - J_u) \, dM_u, \qquad (3.31)$$

where  $m^h$  is the continuous  $\mathbb{F}$ -martingale

$$m_t^h = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^\infty h_u dF_u \mid \mathcal{F}_t\Big),$$

J is the process

$$J_t = e^{\Gamma_t} \left( m_t^h - \int_0^t h_u dF_u \right)$$

and M is the discontinuous  $\mathbb{G}$ -martingale  $M_t = H_t - \Gamma_{t \wedge \tau}$  where  $d\Gamma_u = \frac{dF_u}{1 - F_u}$ .

*Proof.* We know that

$$M_{t}^{h} = \mathbb{E}_{\mathbb{Q}}(h_{\tau} \mid \mathcal{G}_{t}) = \mathbb{1}_{\{\tau \leq t\}}h_{\tau} + \mathbb{1}_{\{\tau > t\}}e^{\Gamma_{t}} \mathbb{E}_{\mathbb{Q}}\left(\int_{t}^{\infty}h_{u} \, dF_{u} \mid \mathcal{F}_{t}\right)$$
(3.32)  
$$= \mathbb{1}_{\{\tau \leq t\}}h_{\tau} + \mathbb{1}_{\{\tau > t\}}e^{\Gamma_{t}}\left(m_{t}^{h} - \int_{0}^{t}h_{u} \, dF_{u}\right).$$

We will now sketch two different proofs of (3.31).

First proof. Noting that  $\Gamma$  is an increasing process and  $m^h$  a continuous martingale, and using the integration by parts formula, we deduce that

$$dJ_t = e^{\Gamma_t} dm_t^h + \left(m_t^h - \int_0^t h_u dF_u\right) \gamma_t e^{\Gamma_t} dt - e^{\Gamma_t} h_t dF_t$$
$$= e^{\Gamma_t} dm_t^h + J_t \gamma_t e^{\Gamma_t} dt - e^{\Gamma_t} h_t dF_t.$$

Therefore, from (3.30)

$$dJ_t = e^{\Gamma_t} dm_t^h + (J_t - h_t) \frac{dF_t}{1 - F_t}$$

or, in an integrated form,

$$J_t = m_0 + \int_0^t e^{\Gamma_u} \, dm_u^h + \int_0^t (J_u - h_u) \, d\Gamma_u.$$

Note that  $J_u = M_u^h$  for  $u < \tau$ . Therefore, on the event  $\{t < \tau\}$ ,

$$M_t^h = m_0^h + \int_0^{t \wedge \tau} e^{\Gamma_u} \, dm_u^h + \int_0^{t \wedge \tau} (J_u - h_u) \, d\Gamma_u.$$

From (3.32), the jump of  $M^h$  at time  $\tau$  is  $h_{\tau} - J_{\tau} = h_{\tau} - M^h_{\tau-}$ . Then (3.31) follows. Second proof. The equality (3.32) can be re-written as

$$M_{t}^{h} = \int_{0}^{t} h_{u} \, dH_{u} + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_{t}} \left( m_{t}^{h} - \int_{0}^{t} h_{u} \, dF_{u} \right).$$

Hence the result can be obtained by the integration by parts formula.

**Remark 3.3.1** Since the hypothesis (H) holds and  $\Gamma$  is  $\mathbb{F}$ -adapted, the processes  $(m_t, t \ge 0)$  and  $(\int_0^{t\wedge\tau} e^{\Gamma_u} dm_u, t \ge 0)$  are also  $\mathbb{G}$ -martingales.

# Chapter 4

# Hedging of Defaultable Claims

In this chapter, we shall study hedging strategies for credit derivatives under assumption that some primary defaultable (as well as non-defaultable) assets are traded, and thus they can be used in replication of non-traded contingent claims. We follow here the papers by Bielecki et al. [5, 4].

# 4.1 Semimartingale Model with a Common Default

In what follows, we fix a finite horizon date T > 0. For the purpose of this chapter, it is enough to formally define a generic defaultable claim through the following definition.

**Definition 4.1.1** A *defaultable claim* with maturity date T is represented by a triplet  $(X, Z, \tau)$ , where:

(i) the default time  $\tau$  specifies the random time of default, and thus also the default events  $\{\tau \leq t\}$  for every  $t \in [0, T]$ ,

(ii) the promised payoff  $X \in \mathcal{F}_T$  represents the random payoff received by the owner of the claim at time T, provided that there was no default prior to or at time T; the actual payoff at time Tassociated with X thus equals  $X \mathbb{1}_{\{T < \tau\}}$ ,

(iii) the  $\mathbb{F}$ -adapted *recovery process* Z specifies the recovery payoff  $Z_{\tau}$  received by the owner of a claim at time of default (or at maturity), provided that the default occurred prior to or at maturity date T.

In practice, hedging of a credit derivative after default time is usually of minor interest. Also, in a model with a single default time, hedging after default reduces to replication of a non-defaultable claim. It is thus natural to define the replication of a defaultable claim in the following way.

**Definition 4.1.2** We say that a self-financing strategy  $\phi$  replicates a defaultable claim  $(X, Z, \tau)$  if its wealth process  $V(\phi)$  satisfies  $V_T(\phi)\mathbb{1}_{\{T \leq \tau\}} = X\mathbb{1}_{\{T < \tau\}}$  and  $V_\tau(\phi)\mathbb{1}_{\{T \geq \tau\}} = Z_\tau\mathbb{1}_{\{T \geq \tau\}}$ .

When dealing with replicating strategies, in the sense of Definition 4.1.2, we will always assume, without loss of generality, that the components of the process  $\phi$  are  $\mathbb{F}$ -predictable processes.

# 4.1.1 Dynamics of Asset Prices

We assume that we are given a probability space  $(\Omega, \mathcal{G}, \mathbb{P})$  endowed with a (possibly multi-dimensional) standard Brownian motion W and a random time  $\tau$  admitting an  $\mathbb{F}$ -intensity  $\gamma$  under  $\mathbb{P}$ , where  $\mathbb{F}$  is the filtration generated by W. In addition, we assume that  $\tau$  satisfies (3.6), so that the hypothesis (H) is valid under  $\mathbb{P}$  for filtrations  $\mathbb{F}$  and  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ . Since the default time admits an  $\mathbb{F}$ -intensity, it is not an  $\mathbb{F}$ -stopping time. Indeed, any stopping time with respect to a Brownian filtration is known to be predictable.

We interpret  $\tau$  as the common default time for all defaultable assets in our model. For simplicity, we assume that only three primary assets are traded in the market, and the dynamics under the historical probability  $\mathbb{P}$  of their prices are, for i = 1, 2, 3 and  $t \in [0, T]$ ,

$$dY_t^i = Y_{t-}^i (\mu_{i,t} \, dt + \sigma_{i,t} \, dW_t + \kappa_{i,t} \, dM_t), \tag{4.1}$$

where  $M_t = H_t - \int_0^t (1 - H_s) \gamma_s ds$  is a martingale, or equivalently,

$$dY_t^i = Y_{t-}^i \left( (\mu_{i,t} - \kappa_{i,t} \gamma_t \mathbb{1}_{\{t < \tau\}}) \, dt + \sigma_{i,t} \, dW_t + \kappa_{i,t} \, dH_t \right). \tag{4.2}$$

The processes  $(\mu_i, \sigma_i, \kappa_i) = (\mu_{i,t}, \sigma_{i,t}, \kappa_{i,t}, t \ge 0)$ , i = 1, 2, 3, are assumed to be  $\mathbb{G}$ -adapted, where  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ . In addition, we assume that  $\kappa_i \ge -1$  for any i = 1, 2, 3, so that  $Y^i$  are nonnegative processes, and they are strictly positive prior to  $\tau$ . Note that, in the case of constant coefficients we have that

$$Y_t^i = Y_0^i e^{\mu_i t} e^{\sigma_i W_t - \sigma_i^2 t/2} e^{-\kappa_i \gamma_i (t \wedge \tau)} (1 + \kappa_i)^{H_t}$$

According to Definition 4.1.2, replication refers to the behavior of the wealth process  $V(\phi)$  on the random interval  $[0, \tau \wedge T]$  only. Hence, for the purpose of replication of defaultable claims of the form  $(X, Z, \tau)$ , it is sufficient to consider prices of primary assets stopped at  $\tau \wedge T$ . This implies that instead of dealing with  $\mathbb{G}$ -adapted coefficients in (4.1), it suffices to focus on  $\mathbb{F}$ -adapted coefficients of stopped price processes. However, for the sake of completeness, we shall also deal with T-maturity claims of the form  $Y = G(Y_T^1, Y_T^2, Y_T^3, H_T)$  (see Section 4.3 below).

### **Pre-Default Values**

As will become clear in what follows, when dealing with defaultable claims of the form  $(X, Z, \tau)$ , we will be mainly concerned with the so-called pre-default prices. The *pre-default price*  $\tilde{Y}^i$  of the *i*th asset is an  $\mathbb{F}$ -adapted, continuous process, given by the equation, for i = 1, 2, 3 and  $t \in [0, T]$ ,

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \left( \left( \mu_{i,t} - \kappa_{i,t} \gamma_t \right) dt + \sigma_{i,t} dW_t \right)$$

$$\tag{4.3}$$

with  $\widetilde{Y}_0^i = Y_0^i$ . Put another way,  $\widetilde{Y}^i$  is the unique  $\mathbb{F}$ -predictable process such that  $\widetilde{Y}_t^i \mathbb{1}_{\{t \leq \tau\}} = Y_t^i \mathbb{1}_{\{t \leq \tau\}}$  for  $t \in \mathbb{R}_+$ . When dealing with the pre-default prices, we may and do assume, without loss of generality, that the processes  $\mu_i, \sigma_i$  and  $\kappa_i$  are  $\mathbb{F}$ -predictable.

It is worth stressing that the historically observed drift coefficient equals  $\mu_{i,t} - \kappa_{i,t}\gamma_t$ , rather than  $\mu_{i,t}$ . The drift coefficient denoted by  $\mu_{i,t}$  is already credit-risk adjusted in the sense of our model, and it is not directly observed. This convention was chosen here for the sake of simplicity of notation. It also lends itself to the following intuitive interpretation: if  $\phi^i$  is the number of units of the *i*th asset held in our portfolio at time *t* then the gains/losses from trades in this asset, prior to default time, can be represented by the differential

$$\phi_t^i \, d\widetilde{Y}_t^i = \phi_t^i \widetilde{Y}_t^i \big( \mu_{i,t} \, dt + \sigma_{i,t} \, dW_t \big) - \phi_t^i \widetilde{Y}_t^i \kappa_{i,t} \gamma_t \, dt.$$

The last term may be here separated, and formally treated as an effect of continuously paid dividends at the dividend rate  $\kappa_{i,t}\gamma_t$ . However, this interpretation may be misleading, since this quantity is not directly observed. In fact, the mere estimation of the drift coefficient in dynamics (4.3) is not practical.

Still, if this formal interpretation is adopted, it is sometimes possible make use of the standard results concerning the valuation of derivatives of dividend-paying assets. It is, of course, a delicate issue how to separate in practice both components of the drift coefficient. We shall argue below that although the dividend-based approach is formally correct, a more pertinent and simpler way of dealing with hedging relies on the assumption that only the effective drift  $\mu_{i,t} - \kappa_{i,t}\gamma_t$  is observable. In practical approach to hedging, the values of drift coefficients in dynamics of asset prices play no essential role, so that they are considered as market observables.

### 4.1. MARTINGALE APPROACH

### Market Observables

To summarize, we assume throughout that the *market observables* are: the pre-default market prices of primary assets, their volatilities and correlations, as well as the jump coefficients  $\kappa_{i,t}$  (the financial interpretation of jump coefficients is examined in the next subsection). To summarize we postulate that under the statistical probability  $\mathbb{P}$  we have

$$dY_t^i = Y_{t-}^i \left( \widetilde{\mu}_{i,t} \, dt + \sigma_{i,t} \, dW_t + \kappa_{i,t} \, dH_t \right) \tag{4.4}$$

where the drift terms  $\tilde{\mu}_{i,t}$  are not observable, but we can observe the volatilities  $\sigma_{i,t}$  (and thus the assets correlations), and we have an a priori assessment of jump coefficients  $\kappa_{i,t}$ . In this general set-up, the most natural assumption is that the dimension of a driving Brownian motion W equals the number of tradable assets. However, for the sake of simplicity of presentation, we shall frequently assume that W is one-dimensional. One of our goals will be to derive closed-form solutions for replicating strategies for derivative securities in terms of market observables only (whenever replication of a given claim is actually feasible). To achieve this goal, we shall combine a general theory of hedging defaultable claims within a continuous semimartingale set-up, with a judicious specification of particular models with deterministic volatilities and correlations.

# **Recovery Schemes**

It is clear that the sample paths of price processes  $Y^i$  are continuous, except for a possible discontinuity at time  $\tau$ . Specifically, we have that

$$\Delta Y^i_\tau := Y^i_\tau - Y^i_{\tau-} = \kappa_{i,\tau} Y^i_{\tau-}$$

so that  $Y^i_{\tau} = Y^i_{\tau-}(1+\kappa_{i,\tau}) = \widetilde{Y}^i_{\tau-}(1+\kappa_{i,\tau}).$ 

A primary asset  $Y^i$  is termed a *default-free asset* (*defaultable asset*, respectively) if  $\kappa_i = 0$  ( $\kappa_i \neq 0$ , respectively). In the special case when  $\kappa_i = -1$ , we say that a defaultable asset  $Y^i$  is subject to a *total default*, since its price drops to zero at time  $\tau$  and stays there forever. Such an asset ceases to exist after default, in the sense that it is no longer traded after default. This feature makes the case of a total default quite different from other cases, as we shall see in our study below.

In market practice, it is common for a credit derivative to deliver a positive recovery (for instance, a *protection payment*) in case of default. Formally, the value of this recovery at default is determined as the value of some underlying process, that is, it is equal to the value at time  $\tau$  of some  $\mathbb{F}$ -adapted recovery process Z.

For example, the process Z can be equal to  $\delta$ , where  $\delta$  is a constant, or to  $g(t, \delta Y_t)$  where g is a deterministic function and  $(Y_t, t \ge 0)$  is the price process of some default-free asset. Typically, the recovery is paid at default time, but it may also happen that it is postponed to the maturity date.

Let us observe that the case where a defaultable asset  $Y^i$  pays a pre-determined recovery at default is covered by our set-up defined in (4.1). For instance, the case of a constant recovery payoff  $\delta_i \geq 0$  at default time  $\tau$  corresponds to the process  $\kappa_{i,t} = \delta_i (Y_{t-}^i)^{-1} - 1$ . Under this convention, the price  $Y^i$  is governed under  $\mathbb{P}$  by the SDE

$$dY_t^i = Y_{t-}^i \left( \mu_{i,t} \, dt + \sigma_{i,t} \, dW_t + \left( \delta_i (Y_{t-}^i)^{-1} - 1 \right) dM_t \right). \tag{4.5}$$

If the recovery is proportional to the pre-default value  $Y_{\tau-}^i$ , and is paid at default time  $\tau$  (this scheme is known as the *fractional recovery of market value*), we have  $\kappa_{i,t} = \delta_i - 1$  and

$$dY_t^i = Y_{t-}^i \left( \mu_{i,t} \, dt + \sigma_{i,t} \, dW_t + (\delta_i - 1) \, dM_t \right). \tag{4.6}$$

# 4.2 Martingale Approach to Valuation and Hedging

Our goal is to derive quasi-explicit conditions for replicating strategies for a defaultable claim in a fairly general set-up introduced in Section 4.1.1. In this section, we only deal with trading strategies based on the reference filtration  $\mathbb{F}$ , and the underlying price processes (that is, prices of default-free assets and pre-default values of defaultable assets) are assumed to be continuous.

To simplify the presentation, we make a standing assumption that all coefficient processes are such that the SDEs appearing below admit unique strong solutions, and all stochastic exponentials (used as Radon-Nikodým derivatives) are true martingales under respective probabilities.

# 4.2.1 Defaultable Asset with Total Default

In this section, we shall examine in some detail a particular model where the two assets,  $Y^1$  and  $Y^2$ , are default-free and satisfy

$$dY_t^i = Y_t^i (\mu_{i,t} \, dt + \sigma_{i,t} \, dW_t), \quad i = 1, 2,$$

where W is a one-dimensional Brownian motion. The third asset is a defaultable asset with total default, so that

$$dY_t^3 = Y_{t-}^3 \left( \mu_{3,t} \, dt + \sigma_{3,t} \, dW_t - dM_t \right).$$

Since we will be interested in replicating strategies in the sense of Definition 4.1.2, we may and do assume, without loss of generality, that the coefficients  $\mu_{i,t}$ ,  $\sigma_{i,t}$ , i = 1, 2, are  $\mathbb{F}$ -predictable, rather than  $\mathbb{G}$ -predictable. Recall that, in general, there exist  $\mathbb{F}$ -predictable processes  $\tilde{\mu}_3$  and  $\tilde{\sigma}_3$  such that

$$\widetilde{\mu}_{3,t} \mathbb{1}_{\{t \le \tau\}} = \mu_{3,t} \mathbb{1}_{\{t \le \tau\}}, \quad \widetilde{\sigma}_{3,t} \mathbb{1}_{\{t \le \tau\}} = \sigma_{3,t} \mathbb{1}_{\{t \le \tau\}}.$$

$$(4.7)$$

We assume throughout that  $Y_0^i > 0$  for every *i*, so that the price processes  $Y^1$ ,  $Y^2$  are strictly positive, and the process  $Y^3$  is nonnegative, and has strictly positive pre-default value.

# **Default-Free Market**

It is natural to postulate that the default-free market with the two traded assets,  $Y^1$  and  $Y^2$ , is arbitrage-free. More precisely, we choose  $Y^1$  as a numéraire, and we require that there exists a probability measure  $\mathbb{P}^1$ , equivalent to  $\mathbb{P}$  on  $(\Omega, \mathcal{F}_T)$ , and such that the process  $Y^{2,1}$  is a  $\mathbb{P}^1$ -martingale. The dynamics of processes  $(Y^1)^{-1}$  and  $Y^{2,1}$  are

$$d(Y_t^1)^{-1} = (Y_t^1)^{-1} \left( (\sigma_{1,t}^2 - \mu_{1,t}) \, dt - \sigma_{1,t} \, dW_t \right), \tag{4.8}$$

and

$$dY_t^{2,1} = Y_t^{2,1} \left( \left( \mu_{2,t} - \mu_{1,t} + \sigma_{1,t} (\sigma_{1,t} - \sigma_{2,t}) \right) dt + \left( \sigma_{2,t} - \sigma_{1,t} \right) dW_t \right),$$

respectively. Hence, the necessary condition for the existence of an EMM  $\mathbb{P}^1$  is the inclusion  $A \subseteq B$ , where  $A = \{(t, \omega) \in [0, T] \times \Omega : \sigma_{1,t}(\omega) = \sigma_{2,t}(\omega)\}$  and  $B = \{(t, \omega) \in [0, T] \times \Omega : \mu_{1,t}(\omega) = \mu_{2,t}(\omega)\}$ . The necessary and sufficient condition for the existence and uniqueness of an EMM  $\mathbb{P}^1$  reads

$$\mathbb{E}_{\mathbb{P}}\left\{\mathcal{E}_{T}\left(\int_{0}^{\cdot}\theta_{u}\,dW_{u}\right)\right\} = 1\tag{4.9}$$

where the process  $\theta$  is given by the formula (by convention, 0/0 = 0)

$$\theta_t = \sigma_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}, \quad \forall t \in [0,T].$$
(4.10)

Note that in the case of constant coefficients, if  $\sigma_1 = \sigma_2$  then the model is arbitrage-free only in the trivial case when  $\mu_2 = \mu_1$ .

## 4.2. MARTINGALE APPROACH

**Remark 4.2.1** Since the martingale measure  $\mathbb{P}^1$  is unique, the default-free model  $(Y^1, Y^2)$  is complete. However, this is not a necessary assumption and thus it can be relaxed. As we shall see in what follows, it is typically more natural to assume that the driving Brownian motion W is multi-dimensional.

# Arbitrage-Free Property

Let us now consider also a defaultable asset  $Y^3$ . Our goal is now to find a martingale measure  $\mathbb{Q}^1$  (if it exists) for relative prices  $Y^{2,1}$  and  $Y^{3,1}$ . Recall that we postulate that the hypothesis (H) holds under  $\mathbb{P}$  for filtrations  $\mathbb{F}$  and  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ . The dynamics of  $Y^{3,1}$  under  $\mathbb{P}$  are

$$dY_t^{3,1} = Y_{t-}^{3,1} \Big\{ \big( \mu_{3,t} - \mu_{1,t} + \sigma_{1,t} (\sigma_{1,t} - \sigma_{3,t}) \big) dt + (\sigma_{3,t} - \sigma_{1,t}) \, dW_t - dM_t \Big\}.$$

Let  $\mathbb{Q}^1$  be any probability measure equivalent to  $\mathbb{P}$  on  $(\Omega, \mathcal{G}_T)$ , and let  $\eta$  be the associated Radon-Nikodým density process, so that

$$d\mathbb{Q}^1 |_{\mathcal{G}_t} = \eta_t \, d\mathbb{P} |_{\mathcal{G}_t},\tag{4.11}$$

where the process  $\eta$  satisfies

$$d\eta_t = \eta_{t-}(\theta_t \, dW_t + \zeta_t \, dM_t) \tag{4.12}$$

for some  $\mathbb{G}$ -predictable processes  $\theta$  and  $\zeta$ , and  $\eta$  is a  $\mathbb{G}$ -martingale under  $\mathbb{P}$ .

From Girsanov's theorem, the processes  $\widehat{W}$  and  $\widehat{M}$ , given by

$$\widehat{W}_t = W_t - \int_0^t \theta_u \, du, \quad \widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u < \tau\}} \gamma_u \zeta_u \, du, \tag{4.13}$$

are G-martingales under  $\mathbb{Q}^1$ . To ensure that  $Y^{2,1}$  is a  $\mathbb{Q}^1$ -martingale, we postulate that (4.9) and (4.10) are valid. Consequently, for the process  $Y^{3,1}$  to be a  $\mathbb{Q}^1$ -martingale, it is necessary and sufficient that  $\zeta$  satisfies

$$\gamma_t \zeta_t = \mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}} (\sigma_{3,t} - \sigma_{1,t}).$$

To ensure that  $\mathbb{Q}^1$  is a probability measure equivalent to  $\mathbb{P}$ , we require that  $\zeta_t > -1$ . The unique martingale measure  $\mathbb{Q}^1$  is then given by the formula (4.11) where  $\eta$  solves (4.12), so that

$$\eta_t = \mathcal{E}_t \left( \int_0^{\cdot} \theta_u \, dW_u \right) \mathcal{E}_t \left( \int_0^{\cdot} \zeta_u \, dM_u \right).$$

We are in a position to formulate the following result.

**Proposition 4.2.1** Assume that the process  $\theta$  given by (4.10) satisfies (4.9), and

$$\zeta_t = \frac{1}{\gamma_t} \left( \mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}} (\sigma_{3,t} - \sigma_{1,t}) \right) > -1.$$
(4.14)

Then the model  $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$  is arbitrage-free and complete. The dynamics of relative prices under the unique martingale measure  $\mathbb{Q}^1$  are

$$dY_t^{2,1} = Y_t^{2,1}(\sigma_{2,t} - \sigma_{1,t}) \, d\widehat{W}_t, dY_t^{3,1} = Y_{t-}^{3,1} \left( (\sigma_{3,t} - \sigma_{1,t}) \, d\widehat{W}_t - d\widehat{M}_t \right).$$

Since the coefficients  $\mu_{i,t}$ ,  $\sigma_{i,t}$ , i = 1, 2, are  $\mathbb{F}$ -adapted, the process  $\widehat{W}$  is an  $\mathbb{F}$ -martingale (hence, a Brownian motion) under  $\mathbb{Q}^1$ . Hence, by virtue of Proposition 3.2.3, the hypothesis (H) holds under  $\mathbb{Q}^1$ , and the  $\mathbb{F}$ -intensity of default under  $\mathbb{Q}^1$  equals

$$\widehat{\gamma}_t = \gamma_t (1 + \zeta_t) = \gamma_t + \left( \mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}} (\sigma_{3,t} - \sigma_{1,t}) \right).$$

**Example 4.2.1** We present an example where the condition (4.14) does not hold, and thus arbitrage opportunities arise. Assume the coefficients are constant and satisfy:  $\mu_1 = \mu_2 = \sigma_1 = 0$ ,  $\mu_3 < -\gamma$  for a constant default intensity  $\gamma > 0$ . Then

$$Y_t^3 = \mathbb{1}_{\{t < \tau\}} Y_0^3 \exp\left(\sigma_3 W_t - \frac{1}{2}\sigma_3^2 t + (\mu_3 + \gamma)t\right) \le Y_0^3 \exp\left(\sigma_3 W_t - \frac{1}{2}\sigma_3^2 t\right) = V_t(\phi),$$

where  $V(\phi)$  represents the wealth of a self-financing strategy  $(\phi^1, \phi^2, 0)$  with  $\phi^2 = \frac{\sigma_3}{\sigma_2}$ . Hence, the arbitrage strategy would be to sell the asset  $Y^3$ , and to follow the strategy  $\phi$ .

**Remark 4.2.2** Let us stress once again, that the existence of an EMM is a necessary condition for viability of a financial model, but the uniqueness of an EMM is not always a convenient condition to impose on a model. In fact, when constructing a model, we should be mostly concerned with its flexibility and ability to reflect the pertinent risk factors, rather than with its mathematical completeness. In the present context, it is natural to postulate that the dimension of the underlying Brownian motion equals the number of tradeable risky assets. In addition, each particular model should be tailored to provide intuitive and handy solutions for a predetermined family of contingent claims that will be priced and hedged within its framework.

# 4.2.2 Two Defaultable Assets with Total Default

Assume now that we have only two assets and both are defaultable assets with total default. Then we have, for i = 1, 2,

$$dY_t^i = Y_{t-}^i (\mu_{i,t} \, dt + \sigma_{i,t} \, dW_t - dM_t), \tag{4.15}$$

where W is a one-dimensional Brownian motion. Hence

$$Y_t^1 = \mathbb{1}_{\{t < \tau\}} \widetilde{Y}_t^1, \quad Y_t^2 = \mathbb{1}_{\{t < \tau\}} \widetilde{Y}_t^2,$$

with the pre-default prices governed by the SDEs

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \big( (\mu_{i,t} + \gamma_t) \, dt + \sigma_{i,t} \, dW_t \big). \tag{4.16}$$

# 4.3 PDE Approach to Valuation and Hedging

In the remaining part of this chapter, in which we follow Bielecki et al. [4] (see also Rutkowski and Yousiph [57]), we will with a Markovian set-up. We assume that trading occurs on the time interval [0, T] and our goal is to replicate a contingent claim of the form

$$Y = \mathbb{1}_{\{T \ge \tau\}} g_1(Y_T^1, Y_T^2, Y_T^3) + \mathbb{1}_{\{T < \tau\}} g_0(Y_T^1, Y_T^2, Y_T^3) = G(Y_T^1, Y_T^2, Y_T^3, H_T),$$

which settles at time T. We do not need to assume here that the coefficients in dynamics of primary assets are  $\mathbb{F}$ -predictable. Since our goal is to develop the PDE approach, it will be essential, however, to postulate a Markovian character of a model. For the sake of simplicity, we assume that the coefficients are constant, so that

$$dY_t^i = Y_{t-}^i (\mu_i \, dt + \sigma_i \, dW_t + \kappa_i \, dM_t), \quad i = 1, 2, 3.$$

The assumption of constancy of coefficients is rarely, if ever, satisfied in practically relevant models of credit risk. It is thus important to note that it was postulated here mainly for the sake of notational convenience, and the general results established in this section can be easily extended to a non-homogeneous Markov case in which  $\mu_{i,t} = \mu_i(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3, H_{t-}), \sigma_{i,t} = \sigma_i(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3, H_{t-}),$  etc.

# 4.3.1 Defaultable Asset with Total Default

We first assume that  $Y^1$  and  $Y^2$  are default-free, so that  $\kappa_1 = \kappa_2 = 0$ , and the third asset is subject to total default, i.e.  $\kappa_3 = -1$ ,

$$dY_t^3 = Y_{t-}^3 (\mu_3 \, dt + \sigma_3 \, dW_t - dM_t).$$

We work throughout under the assumptions of Proposition 4.2.1. This means that any  $\mathbb{Q}^1$ -integrable contingent claim  $Y = G(Y_T^1, Y_T^2, Y_T^3; H_T)$  is attainable, and its arbitrage price equals

$$\pi_t(Y) = Y_t^1 \mathbb{E}_{\mathbb{O}^1}(Y(Y_T^1)^{-1} | \mathcal{G}_t), \quad \forall t \in [0, T].$$
(4.17)

The following auxiliary result is thus rather obvious.

**Lemma 4.3.1** The process  $(Y^1, Y^2, Y^3, H)$  has the Markov property with respect to the filtration  $\mathbb{G}$ under the martingale measure  $\mathbb{Q}^1$ . For any attainable claim  $Y = G(Y_T^1, Y_T^2, Y_T^3; H_T)$  there exists a function  $v : [0, T] \times \mathbb{R}^3 \times \{0, 1\} \to \mathbb{R}$  such that  $\pi_t(Y) = v(t, Y_t^1, Y_t^2, Y_t^3; H_t)$ .

We find it convenient to introduce the *pre-default* pricing function  $v(\cdot; 0) = v(t, y_1, y_2, y_3; 0)$  and the *post-default* pricing function  $v(\cdot; 1) = v(t, y_1, y_2, y_3; 1)$ . In fact, since  $Y_t^3 = 0$  if  $H_t = 1$ , it suffices to study the post-default function  $v(t, y_1, y_2; 1) = v(t, y_1, y_2, 0; 1)$ . Also, we write

$$\alpha_i = \mu_i - \sigma_i \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}, \quad b = (\mu_3 - \mu_1)(\sigma_1 - \sigma_2) - (\mu_1 - \mu_3)(\sigma_1 - \sigma_3).$$

Let  $\gamma > 0$  be the constant default intensity under  $\mathbb{P}$ , and let  $\zeta > -1$  be given by formula (4.14).

**Proposition 4.3.1** Assume that the functions  $v(\cdot; 0)$  and  $v(\cdot; 1)$  belong to the class  $C^{1,2}([0,T] \times \mathbb{R}^3_+, \mathbb{R})$ . Then  $v(t, y_1, y_2, y_3; 0)$  satisfies the PDE

$$\partial_t v(\cdot; 0) + \sum_{i=1}^2 \alpha_i y_i \partial_i v(\cdot; 0) + (\alpha_3 + \zeta) y_3 \partial_3 v(\cdot; 0) + \frac{1}{2} \sum_{i,j=1}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} v(\cdot; 0) \\ - \alpha_1 v(\cdot; 0) + \left(\gamma - \frac{b}{\sigma_1 - \sigma_2}\right) \left[v(t, y_1, y_2; 1) - v(t, y_1, y_2, y_3; 0)\right] = 0$$

subject to the terminal condition  $v(T, y_1, y_2, y_3; 0) = G(y_1, y_2, y_3; 0)$ , and  $v(t, y_1, y_2; 1)$  satisfies the PDE

$$\partial_t v(\cdot;1) + \sum_{i=1}^2 \alpha_i y_i \partial_i v(\cdot;1) + \frac{1}{2} \sum_{i,j=1}^2 \sigma_i \sigma_j y_i y_j \partial_{ij} v(\cdot;1) - \alpha_1 v(\cdot;1) = 0$$

subject to the terminal condition  $v(T, y_1, y_2; 1) = G(y_1, y_2, 0; 1)$ .

*Proof.* For simplicity, we write  $C_t = \pi_t(Y)$ . Let us define

$$\Delta v(t, y_1, y_2, y_3) = v(t, y_1, y_2; 1) - v(t, y_1, y_2, y_3; 0).$$

Then the jump  $\Delta C_t = C_t - C_{t-}$  can be represented as follows:

$$\Delta C_t = \mathbb{1}_{\{\tau=t\}} \left( v(t, Y_t^1, Y_t^2; 1) - v(t, Y_t^1, Y_t^2, Y_{t-}^3; 0) \right) = \mathbb{1}_{\{\tau=t\}} \Delta v(t, Y_t^1, Y_t^2, Y_{t-}^3).$$

We write  $\partial_i$  to denote the partial derivative with respect to the variable  $y_i$ , and we typically omit the variables  $(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3, H_{t-})$  in expressions  $\partial_t v$ ,  $\partial_i v$ ,  $\Delta v$ , etc. We shall also make use of the fact that for any Borel measurable function g we have

$$\int_0^t g(u, Y_u^2, Y_{u-}^3) \, du = \int_0^t g(u, Y_u^2, Y_u^3) \, du$$

since  $Y_u^3$  and  $Y_{u-}^3$  differ only for at most one value of u (for each  $\omega$ ). Let  $\xi_t = \mathbb{1}_{\{t < \tau\}} \gamma$ . An application of Itô's formula yields

$$dC_t = \partial_t v \, dt + \sum_{i=1}^3 \partial_i v \, dY_t^i + \frac{1}{2} \sum_{i,j=1}^3 \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v \, dt + \left( \Delta v + Y_{t-}^3 \partial_3 v \right) dH_t = \partial_t v \, dt + \sum_{i=1}^3 \partial_i v \, dY_t^i + \frac{1}{2} \sum_{i,j=1}^3 \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v \, dt + \left( \Delta v + Y_{t-}^3 \partial_3 v \right) \left( dM_t + \xi_t \, dt \right),$$

and this in turn implies that

$$dC_t = \partial_t v \, dt + \sum_{i=1}^3 Y_{t-}^i \partial_i v \left(\mu_i \, dt + \sigma_i \, dW_t\right) + \frac{1}{2} \sum_{i,j=1}^3 \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v \, dt$$
  
+  $\Delta v \, dM_t + \left(\Delta v + Y_{t-}^3 \partial_3 v\right) \xi_t \, dt$   
=  $\left\{ \partial_t v + \sum_{i=1}^3 \mu_i Y_{t-}^i \partial_i v + \frac{1}{2} \sum_{i,j=1}^3 \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v + \left(\Delta v + Y_{t-}^3 \partial_3 v\right) \xi_t \right\} dt$   
+  $\left(\sum_{i=1}^3 \sigma_i Y_{t-}^i \partial_i v\right) dW_t + \Delta v \, dM_t.$ 

We now use the integration by parts formula together with (4.8) to derive dynamics of the relative price  $\hat{C}_t = C_t(Y_t^1)^{-1}$ . We find that

$$\begin{aligned} d\widehat{C}_{t} &= \widehat{C}_{t-} \left( \left( -\mu_{1} + \sigma_{1}^{2} \right) dt - \sigma_{1} \, dW_{t} \right) \\ &+ \left( Y_{t-}^{1} \right)^{-1} \left\{ \partial_{t} v + \sum_{i=1}^{3} \mu_{i} Y_{t-}^{i} \partial_{i} v + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_{i} \sigma_{j} Y_{t-}^{i} Y_{t-}^{j} \partial_{ij} v + \left( \Delta v + Y_{t-}^{3} \partial_{3} v \right) \xi_{t} \right\} dt \\ &+ \left( Y_{t-}^{1} \right)^{-1} \sum_{i=1}^{3} \sigma_{i} Y_{t-}^{i} \partial_{i} v \, dW_{t} + \left( Y_{t-}^{1} \right)^{-1} \Delta v \, dM_{t} - \left( Y_{t-}^{1} \right)^{-1} \sigma_{1} \sum_{i=1}^{3} \sigma_{i} Y_{t-}^{i} \partial_{i} v \, dt. \end{aligned}$$

Hence, using (4.13), we obtain

$$\begin{split} d\widehat{C}_{t} &= \widehat{C}_{t-} \left( -\mu_{1} + \sigma_{1}^{2} \right) dt + \widehat{C}_{t-} \left( -\sigma_{1} \, d\widehat{W}_{t} - \sigma_{1} \theta \, dt \right) \\ &+ (Y_{t-}^{1})^{-1} \left\{ \partial_{t} v + \sum_{i=1}^{3} \mu_{i} Y_{t-}^{i} \partial_{i} v + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_{i} \sigma_{j} Y_{t-}^{i} Y_{t-}^{j} \partial_{ij} v + \left( \Delta v + Y_{t-}^{3} \partial_{3} v \right) \xi_{t} \right\} dt \\ &+ (Y_{t-}^{1})^{-1} \sum_{i=1}^{3} \sigma_{i} Y_{t-}^{i} \partial_{i} v \, d\widehat{W}_{t} + (Y_{t-}^{1})^{-1} \sum_{i=1}^{3} \sigma_{i} Y_{t-}^{i} \theta \partial_{i} v \, dt \\ &+ (Y_{t-}^{1})^{-1} \Delta v \, d\widehat{M}_{t} + (Y_{t-}^{1})^{-1} \zeta \xi_{t} \Delta v \, dt - (Y_{t-}^{1})^{-1} \sigma_{1} \sum_{i=1}^{3} \sigma^{i} Y_{t-}^{i} \partial_{i} v \, dt. \end{split}$$

This means that the process  $\widehat{C}$  admits the following decomposition under  $\mathbb{Q}^1$ 

$$d\widehat{C}_t = \widehat{C}_{t-} \left( -\mu_1 + \sigma_1^2 - \sigma_1 \theta \right) dt$$

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$$+ (Y_{t-}^{1})^{-1} \left\{ \partial_{t}v + \sum_{i=1}^{3} \mu_{i}Y_{t-}^{i}\partial_{i}v + \frac{1}{2}\sum_{i,j=1}^{3} \sigma_{i}\sigma_{j}Y_{t-}^{i}Y_{t-}^{j}\partial_{ij}v + \left(\Delta v + Y_{t-}^{3}\partial_{3}v\right)\xi_{t} \right\} dt \\ + (Y_{t-}^{1})^{-1}\sum_{i=1}^{3} \sigma_{i}Y_{t-}^{i}\theta\partial_{i}v \, dt + (Y_{t-}^{1})^{-1}\zeta\xi_{t}\Delta v \, dt \\ - (Y_{t-}^{1})^{-1}\sigma_{1}\sum_{i=1}^{3} \sigma_{i}Y_{t-}^{i}\partial_{i}v \, dt + a \mathbb{Q}^{1} \text{-martingale.}$$

From (4.17), it follows that the process  $\widehat{C}$  is a martingale under  $\mathbb{Q}^1$ . Therefore, the continuous finite variation part in the above decomposition necessarily vanishes, and thus we get

$$0 = C_{t-}(Y_{t-}^{1})^{-1} \left\{ -\mu_{1} + \sigma_{1}^{2} - \sigma_{1}\theta \right\}$$
  
+  $(Y_{t-}^{1})^{-1} \left\{ \partial_{t}v + \sum_{i=1}^{3} \mu_{i}Y_{t-}^{i}\partial_{i}v + \frac{1}{2}\sum_{i,j=1}^{3} \sigma_{i}\sigma_{j}Y_{t-}^{i}Y_{t-}^{j}\partial_{ij}v + (\Delta v + Y_{t-}^{3}\partial_{3}v)\xi_{t} \right\}$   
+  $(Y_{t-}^{1})^{-1}\sum_{i=1}^{3} \sigma_{i}Y_{t-}^{i}\theta\partial_{i}v + (Y_{t-}^{1})^{-1}\zeta\xi_{t}\Delta v - (Y_{t-}^{1})^{-1}\sigma_{1}\sum_{i=1}^{3} \sigma_{i}Y_{t-}^{i}\partial_{i}v.$ 

Consequently, we have that

$$0 = C_{t-} \left( -\mu_1 + \sigma_1^2 - \sigma_1 \theta \right)$$
  
+  $\partial_t v + \sum_{i=1}^3 \mu_i Y_{t-}^i \partial_i v + \frac{1}{2} \sum_{i,j=1}^3 \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v + \left( \Delta v + Y_{t-}^3 \partial_3 v \right) \xi_t$   
+  $\sum_{i=1}^3 \sigma_i Y_{t-}^i \theta \partial_i v + \zeta \xi_t \Delta v - \sigma_1 \sum_{i=1}^3 \sigma_i Y_{t-}^i \partial_i v.$ 

Finally, we conclude that

$$\begin{split} \partial_t v + \sum_{i=1}^2 \alpha_i Y_{t-}^i \partial_i v + (\alpha_3 + \xi_t) Y_{t-}^3 \partial_3 v + \frac{1}{2} \sum_{i,j=1}^3 \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v \\ &- \alpha_1 C_{t-} + (1+\zeta) \xi_t \Delta v = 0. \end{split}$$

Recall that  $\xi_t = \mathbb{1}_{\{t < \tau\}} \gamma$ . It is thus clear that the pricing functions  $v(\cdot, 0)$  and  $v(\cdot; 1)$  satisfy the PDEs given in the statement of the proposition.

The next result deals with a replicating strategy for Y.

**Proposition 4.3.2** The replicating strategy  $\phi$  for the claim Y is given by formulae

$$\begin{split} \phi_t^3 Y_{t-}^3 &= -\Delta v(t, Y_t^1, Y_t^2, Y_{t-}^3) = v(t, Y_t^1, Y_t^2, Y_{t-}^3; 0) - v(t, Y_t^1, Y_t^2; 1), \\ \phi_t^2 Y_t^2(\sigma_2 - \sigma_1) &= -(\sigma_1 - \sigma_3)\Delta v - \sigma_1 v + \sum_{i=1}^3 Y_{t-}^i \sigma_i \partial_i v, \\ \phi_t^1 Y_t^1 &= v - \phi_t^2 Y_t^2 - \phi_t^3 Y_t^3. \end{split}$$

Proof. As a by-product of our computations, we obtain

$$d\widehat{C}_t = -(Y_t^1)^{-1} \sigma_1 v \, d\widehat{W}_t + (Y_t^1)^{-1} \sum_{i=1}^3 \sigma_i Y_{t-}^i \partial_i v \, d\widehat{W}_t + (Y_t^1)^{-1} \Delta v \, d\widehat{M}_t.$$

The self-financing strategy that replicates Y is determined by two components  $\phi^2, \phi^3$  and the following relationship:

$$d\widehat{C}_t = \phi_t^2 \, dY_t^{2,1} + \phi_t^3 \, dY_t^{3,1} = \phi_t^2 Y_t^{2,1}(\sigma_2 - \sigma_1) \, d\widehat{W}_t + \phi_t^3 Y_{t-}^{3,1}\left((\sigma_3 - \sigma_1) \, d\widehat{W}_t - d\widehat{M}_t\right).$$

By identification, we obtain  $\phi_t^3 Y_{t-}^{3,1} = (Y_t^1)^{-1} \Delta v$  and

$$\phi_t^2 Y_t^2(\sigma_2 - \sigma_1) - (\sigma_3 - \sigma_1) \Delta v = -\sigma_1 C_t + \sum_{i=1}^3 Y_{t-}^i \sigma_i \partial_i v.$$

This yields the claimed formulae.

**Corollary 4.3.1** In the case of a total default claim, the hedging strategy satisfies the balance condition.

*Proof.* A total default corresponds to the assumption that  $G(y_1, y_2, y_3, 1) = 0$ . We now have  $v(t, y_1, y_2; 1) = 0$ , and thus  $\phi_t^3 Y_{t-}^3 = v(t, Y_t^1, Y_t^2, Y_{t-}^3; 0)$  for every  $t \in [0, T]$ . Hence, the equality  $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0$  holds for every  $t \in [0, T]$ . The last equality is the balance condition for Z = 0. Recall that it ensures that the wealth of a replicating portfolio jumps to zero at default time.  $\Box$ 

# Hedging with the Savings Account

Let us now study the particular case where  $Y^1$  is the savings account, i.e.,

$$dY_t^1 = rY_t^1 dt, \quad Y_0^1 = 1,$$

which corresponds to  $\mu_1 = r$  and  $\sigma_1 = 0$ . Let us write  $\hat{r} = r + \hat{\gamma}$ , where

$$\widehat{\gamma} = \gamma(1+\zeta) = \gamma + \mu_3 - r + \frac{\sigma_3}{\sigma_2}(r-\mu_2)$$

stands for the intensity of default under  $\mathbb{Q}^1$ . The quantity  $\hat{r}$  has a natural interpretation as the riskneutral *credit-risk adjusted* short-term interest rate. Straightforward calculations yield the following corollary to Proposition 4.3.1.

**Corollary 4.3.2** Assume that  $\sigma_2 \neq 0$  and

$$dY_t^1 = rY_t^1 dt, dY_t^2 = Y_t^2 (\mu_2 dt + \sigma_2 dW_t), dY_t^3 = Y_{t-}^3 (\mu_3 dt + \sigma_3 dW_t - dM_t).$$

Then the function  $v(\cdot; 0)$  satisfies

$$\begin{aligned} \partial_t v(t, y_2, y_3; 0) &+ ry_2 \partial_2 v(t, y_2, y_3; 0) + \hat{r}y_3 \partial_3 v(t, y_2, y_3; 0) - \hat{r}v(t, y_2, y_3; 0) \\ &+ \frac{1}{2} \sum_{i,j=2}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} v(t, y_2, y_3; 0) + \hat{\gamma}v(t, y_2; 1) = 0 \end{aligned}$$

with  $v(T, y_2, y_3; 0) = G(y_2, y_3; 0)$ , and the function  $v(\cdot; 1)$  satisfies

$$\partial_t v(t, y_2; 1) + ry_2 \partial_2 v(t, y_2; 1) + \frac{1}{2} \sigma_2^2 y_2^2 \partial_{22} v(t, y_2; 1) - rv(t, y_2; 1) = 0$$

with  $v(T, y_2; 1) = G(y_2, 0; 1)$ .

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In the special case of a survival claim, the function  $v(\cdot; 1)$  vanishes identically, and thus the following result can be easily established.

**Corollary 4.3.3** The pre-default pricing function  $v(\cdot; 0)$  of a survival claim  $Y = \mathbb{1}_{\{T < \tau\}} G(Y_T^2, Y_T^3)$  is a solution of the following PDE:

$$\partial_t v(t, y_2, y_3; 0) + ry_2 \partial_2 v(t, y_2, y_3; 0) + \hat{r}y_3 \partial_3 v(t, y_2, y_3; 0) \\ + \frac{1}{2} \sum_{i,j=2}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} v(t, y_2, y_3; 0) - \hat{r}v(t, y_2, y_3; 0) = 0$$

with the terminal condition  $v(T, y_2, y_3; 0) = G(y_2, y_3)$ . The components  $\phi^2$  and  $\phi^3$  of the replicating strategy satisfy

$$\begin{split} \phi_t^2 \sigma_2 Y_t^2 &= \sum_{i=2}^3 \sigma_i Y_{t-}^i \partial_i v(t,Y_t^2,Y_{t-}^3;0) + \sigma_3 v(t,Y_t^2,Y_{t-}^3;0), \\ \phi_t^3 Y_{t-}^3 &= v(t,Y_t^2,Y_{t-}^3;0). \end{split}$$

**Example 4.3.1** Consider a survival claim  $Y = \mathbb{1}_{\{T < \tau\}} g(Y_T^2)$ , that is, a vulnerable claim with default-free underlying asset. Its pre-default pricing function  $v(\cdot; 0)$  does not depend on  $y_3$ , and satisfies the PDE (y stands here for  $y_2$  and  $\sigma$  for  $\sigma_2$ )

$$\partial_t v(t,y;0) + ry \partial_2 v(t,y;0) + \frac{1}{2} \sigma^2 y^2 \partial_{22} v(t,y;0) - \hat{r} v(t,y;0) = 0$$
(4.18)

with the terminal condition  $v(T, y; 0) = \mathbb{1}_{\{t < \tau\}} g(y)$ . The solution to (4.18) is

$$v(t,y) = e^{(\hat{r}-r)(t-T)} v^{r,g,2}(t,y) = e^{\hat{\gamma}(t-T)} v^{r,g,2}(t,y),$$

where the function  $v^{r,g,2}$  is the Black-Scholes price of  $g(Y_T)$  in a Black-Scholes model for  $Y_t$  with interest rate r and volatility  $\sigma_2$ .

# 4.3.2 Defaultable Asset with Non-Zero Recovery

We now assume that

$$dY_t^3 = Y_{t-}^3(\mu_3 \, dt + \sigma_3 \, dW_t + \kappa_3 \, dM_t)$$

with  $\kappa_3 > -1$  and  $\kappa_3 \neq 0$ . We assume that  $Y_0^3 > 0$ , so that  $Y_t^3 > 0$  for every  $t \in \mathbb{R}_+$ . We shall briefly describe the same steps as in the case of a defaultable asset with total default.

# **Pricing PDE and Replicating Strategy**

We are in a position to derive the pricing PDEs. For the sake of simplicity, we assume that  $Y^1$  is the savings account, so that Proposition 4.3.3 is a counterpart of Corollary 4.3.2. For the proof of Proposition 4.3.3, the interested reader is referred to Bielecki et al. [4].

**Proposition 4.3.3** Let  $\sigma_2 \neq 0$  and let  $Y^1, Y^2, Y^3$  satisfy

$$dY_t^1 = rY_t^1 dt, dY_t^2 = Y_t^2 (\mu_2 dt + \sigma_2 dW_t), dY_t^3 = Y_{t-}^3 (\mu_3 dt + \sigma_3 dW_t + \kappa_3 dM_t).$$

Assume, in addition, that  $\sigma_2(r - \mu_3) = \sigma_3(r - \mu_2)$  and  $\kappa_3 \neq 0, \kappa_3 > -1$ . Then the price of a contingent claim  $Y = G(Y_T^2, Y_T^3, H_T)$  can be represented as  $\pi_t(Y) = v(t, Y_t^2, Y_t^3, H_t)$ , where the pricing functions  $v(\cdot; 0)$  and  $v(\cdot; 1)$  satisfy the following PDEs

$$\partial_t v(t, y_2, y_3; 0) + ry_2 \partial_2 v(t, y_2, y_3; 0) + y_3 (r - \kappa_3 \gamma) \partial_3 v(t, y_2, y_3; 0) - rv(t, y_2, y_3; 0) \\ + \frac{1}{2} \sum_{i,j=2}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} v(t, y_2, y_3; 0) + \gamma \left( v(t, y_2, y_3(1 + \kappa_3); 1) - v(t, y_2, y_3; 0) \right) = 0$$

and

$$\begin{aligned} \partial_t v(t, y_2, y_3; 1) + ry_2 \partial_2 v(t, y_2, y_3; 1) + ry_3 \partial_3 v(t, y_2, y_3; 1) - rv(t, y_2, y_3; 1) \\ &+ \frac{1}{2} \sum_{i,j=2}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} v(t, y_2, y_3; 1) = 0 \end{aligned}$$

subject to the terminal conditions

$$v(T, y_2, y_3; 0) = G(y_2, y_3; 0), \quad v(T, y_2, y_3; 1) = G(y_2, y_3; 1).$$

The replicating strategy  $\phi$  equals

$$\begin{split} \phi_t^2 &= \frac{1}{\sigma_2 Y_t^2} \sum_{i=2}^3 \sigma_i y_i \partial_i v(t, Y_t^2, Y_{t-}^3, H_{t-}) \\ &- \frac{\sigma_3}{\sigma_2 \kappa_3 Y_t^2} \Big( v(t, Y_t^2, Y_{t-}^3(1+\kappa_3); 1) - v(t, Y_t^2, Y_{t-}^3; 0) \Big), \\ \phi_t^3 &= \frac{1}{\kappa_3 Y_{t-}^3} \Big( v(t, Y_t^2, Y_{t-}^3(1+\kappa_3); 1) - v(t, Y_t^2, Y_{t-}^3; 0) \Big), \end{split}$$

and  $\phi_t^1$  is given by  $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 = C_t$ .

# Hedging of a Survival Claim

We shall illustrate Proposition 4.3.3 by means of examples. First, consider a survival claim of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T < \tau\}} g(Y_T^3)$$

Then the post-default pricing function  $v^{g}(\cdot; 1)$  vanishes identically, and the pre-default pricing function  $v^{g}(\cdot; 0)$  solves the PDE

$$\partial_t v^g(\cdot; 0) + ry_2 \partial_2 v^g(\cdot; 0) + y_3 (r - \kappa_3 \gamma) \partial_3 v^g(\cdot; 0) + \frac{1}{2} \sum_{i,j=2}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} v^g(\cdot; 0) - (r + \gamma) v^g(\cdot; 0) = 0$$

with the terminal condition  $v^g(T, y_2, y_3; 0) = g(y_3)$ . Denote  $\alpha = r - \kappa_3 \gamma$  and  $\beta = \gamma(1 + \kappa_3)$ .

It is not difficult to check that  $v^g(t, y_2, y_3; 0) = e^{\beta(T-t)}v^{\alpha, g, 3}(t, y_3)$  is a solution of the above equation, where the function  $w(t, y) = v^{\alpha, g, 3}(t, y)$  is the solution of the standard Black-Scholes PDE equation

$$\partial_t w + y\alpha \partial_y w + \frac{1}{2}\sigma_3^2 y^2 \partial_{yy} w - \alpha w = 0$$

with the terminal condition w(T, y) = g(y), that is, the price of the contingent claim  $g(Y_T)$  in the Black-Scholes framework with the interest rate  $\alpha$  and the volatility parameter equal to  $\sigma_3$ .

Let  $C_t$  be the current value of the contingent claim Y, so that

$$C_t = \mathbb{1}_{\{t < \tau\}} e^{\beta(T-t)} v^{\alpha,g,3}(t, Y_t^3).$$

### 4.3. PDE APPROACH

The hedging strategy of the survival claim is, on the event  $\{t < \tau\}$ ,

$$\begin{split} \phi_t^3 Y_t^3 &= -\frac{1}{\kappa_3} e^{-\beta(T-t)} v^{\alpha,g,3}(t,Y_t^3) = -\frac{1}{\kappa_3} C_t, \\ \phi_t^2 Y_t^2 &= \frac{\sigma_3}{\sigma_2} \left( Y_t^3 e^{-\beta(T-t)} \partial_y v^{\alpha,g,3}(t,Y_t^3) - \phi_t^3 Y_t^3 \right). \end{split}$$

### Hedging of a Recovery Payoff

As another illustration of Proposition 4.3.3, we shall now consider the contingent claim  $G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T \geq \tau\}} g(Y_T^2)$ , that is, we assume that recovery is paid at maturity and equals  $g(Y_T^2)$ . Let  $v^g$  be the pricing function of this claim. The post-default pricing function  $v^g(\cdot; 1)$  does not depend on  $y_3$ . Indeed, the equation (we write here  $y_2 = y$ )

$$\partial_t v^g(\cdot;1) + ry \partial_y v^g(\cdot;1) + \frac{1}{2} \sigma_2^2 y^2 \partial_{yy} v^g(\cdot;1) - rv^g(\cdot;1) = 0,$$

with  $v^g(T, y; 1) = g(y)$ , admits a unique solution  $v^{r,g,2}$ , which is the price of  $g(Y_T)$  in the Black-Scholes model with interest rate r and volatility  $\sigma_2$ .

Prior to default, the price of the claim can be found by solving the following PDE

$$\partial_t v^g(\cdot; 0) + ry_2 \partial_2 v^g(\cdot; 0) + y_3 \left(r - \kappa_3 \gamma\right) \partial_3 v^g(\cdot; 0) + \frac{1}{2} \sum_{i,j=2}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} v^g(\cdot; 0) - (r + \gamma) v^g(\cdot; 0) = -\gamma v^g(t, y_2; 1)$$

with  $v^g(T, y_2, y_3; 0) = 0$ . It is not difficult to check that

$$v^{g}(t, y_{2}, y_{3}; 0) = (1 - e^{\gamma(t-T)})v^{r,g,2}(t, y_{2}).$$

The reader can compare this result with the one of Example 4.3.1. e now assume that

$$dY_t^3 = Y_{t-}^3 (\mu_3 \, dt + \sigma_3 \, dW_t + \kappa_3 \, dM_t)$$

with  $\kappa_3 > -1$  and  $\kappa_3 \neq 0$ . We assume that  $Y_0^3 > 0$ , so that  $Y_t^3 > 0$  for every  $t \in \mathbb{R}_+$ . We shall briefly describe the same steps as in the case of a defaultable asset with total default.

# Arbitrage-Free Property

As usual, we need first to impose specific constraints on model coefficients, so that the model is arbitrage-free. Indeed, an EMM  $\mathbb{Q}^1$  exists if there exists a pair  $(\theta, \zeta)$  such that

$$\theta_t(\sigma_i - \sigma_1) + \zeta_t \xi_t \frac{\kappa_i - \kappa_1}{1 + \kappa_1} = \mu_1 - \mu_i + \sigma_1(\sigma_i - \sigma_1) + \xi_t(\kappa_i - \kappa_1) \frac{\kappa_1}{1 + \kappa_1}, \quad i = 2, 3.$$

To ensure the existence of a solution  $(\theta, \zeta)$  on the set  $\tau < t$ , we impose the condition

$$\sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3},$$

that is,

$$\mu_1(\sigma_3 - \sigma_2) + \mu_2(\sigma_1 - \sigma_3) + \mu_3(\sigma_2 - \sigma_1) = 0.$$

Now, on the event  $\tau \geq t$ , we have to solve the two equations

$$\theta_t(\sigma_2 - \sigma_1) = \mu_1 - \mu_2 + \sigma_1(\sigma_2 - \sigma_1),$$
  
$$\theta_t(\sigma_3 - \sigma_1) + \zeta_t \gamma \kappa_3 = \mu_1 - \mu_3 + \sigma_1(\sigma_3 - \sigma_1).$$

If, in addition,  $(\sigma_2 - \sigma_1)\kappa_3 \neq 0$ , we obtain the unique solution

$$\theta = \sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3}, \zeta = 0 > -1,$$

so that the martingale measure  $\mathbb{Q}^1$  exists and is unique.

# 4.3.3 Two Defaultable Assets with Total Default

We shall now assume that we have only two assets, and both are defaultable assets with total default. We shall briefly outline the analysis of this case, leaving the details and the study of other relevant cases to the reader. We postulate that

$$dY_t^i = Y_{t-}^i \left( \mu_i \, dt + \sigma_i \, dW_t - dM_t \right), \ i = 1, 2, \tag{4.19}$$

so that

$$Y_t^1 = \mathbb{1}_{\{t < \tau\}} \widetilde{Y}_t^1, \quad Y_t^2 = \mathbb{1}_{\{t < \tau\}} \widetilde{Y}_t^2,$$

with the pre-default prices governed by the SDEs

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \big( (\mu_i + \gamma) \, dt + \sigma_i \, dW_t \big), \ i = 1, 2.$$

In the case where the promised payoff X is path-independent, so that

$$X1\!\!1_{\{T<\tau\}} = G(Y_T^1, Y_T^2)1\!\!1_{\{T<\tau\}} = G(\tilde{Y}_T^1, \tilde{Y}_T^2)1\!\!1_{\{T<\tau\}}$$

for some function G, it is possible to use the PDE approach in order to value and replicate survival claims prior to default (needless to say that the valuation and hedging after default are trivial here).

We know already from the martingale approach that hedging of a survival claim  $X \mathbb{1}_{\{T < \tau\}}$  is formally equivalent to replicating the promised payoff X using the pre-default values of tradeable assets

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i ((\mu_i + \gamma) dt + \sigma_i dW_t), \quad i = 1, 2.$$

We need not to worry here about the balance condition, since in case of default the wealth of the portfolio will drop to zero, as it should in view of the equality Z = 0.

We shall find the pre-default pricing function  $v(t, y_1, y_2)$ , which is required to satisfy the terminal condition  $v(T, y_1, y_2) = G(y_1, y_2)$ , as well as the hedging strategy  $(\phi^1, \phi^2)$ . The replicating strategy  $\phi$  is such that for the pre-default value  $\tilde{C}$  of our claim we have  $\tilde{C}_t := v(t, \tilde{Y}_t^1, \tilde{Y}_t^2) = \phi_t^1 \tilde{Y}_t^1 + \phi_t^2 \tilde{Y}_t^2$ , and

$$d\widetilde{C}_t = \phi_t^1 \, d\widetilde{Y}_t^1 + \phi_t^2 \, d\widetilde{Y}_t^2. \tag{4.20}$$

**Proposition 4.3.4** Assume that  $\sigma_1 \neq \sigma_2$ . Then the pre-default pricing function v satisfies the PDE

$$\partial_t v + y_1 \left( \mu_1 + \gamma - \sigma_1 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) \partial_1 v + y_2 \left( \mu_2 + \gamma - \sigma_2 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) \partial_2 v \\ + \frac{1}{2} \left( y_1^2 \sigma_1^2 \partial_{11} v + y_2^2 \sigma_2^2 \partial_{22} v + 2y_1 y_2 \sigma_1 \sigma_2 \partial_{12} v \right) = \left( \mu_1 + \gamma - \sigma_1 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) v$$

with the terminal condition  $v(T, y_1, y_2) = G(y_1, y_2)$ .

*Proof.* We shall merely sketch the proof. By applying Itô's formula to  $v(t, \tilde{Y}_t^1, \tilde{Y}_t^2)$ , and comparing the diffusion terms in (4.20) and in the Itô differential  $dv(t, \tilde{Y}_t^1, \tilde{Y}_t^2)$ , we find that

$$y_1 \sigma_1 \partial_1 v + y_2 \sigma_2 \partial_2 v = \phi^1 y_1 \sigma_1 + \phi^2 y_2 \sigma_2, \tag{4.21}$$

where  $\phi^{i} = \phi^{i}(t, y_{1}, y_{2})$ . Since  $\phi^{1}y_{1} = v(t, y_{1}, y_{2}) - \phi^{2}y_{2}$ , we deduce from (4.21) that

$$y_1\sigma_1\partial_1v + y_2\sigma_2\partial_2v = v\sigma_1 + \phi^2 y_2(\sigma_2 - \sigma_1),$$

and thus

$$\phi^2 y_2 = \frac{y_1 \sigma_1 \partial_1 v + y_2 \sigma_2 \partial_2 v - v \sigma_1}{\sigma_2 - \sigma_1}.$$

### 4.3. PDE APPROACH

On the other hand, by identification of drift terms in (4.21), we obtain

$$\partial_t v + y_1(\mu_1 + \gamma)\partial_1 v + y_2(\mu_2 + \gamma)\partial_2 v + \frac{1}{2} \Big( y_1^2 \sigma_1^2 \partial_{11} v + y_2^2 \sigma_2^2 \partial_{22} v + 2y_1 y_2 \sigma_1 \sigma_2 \partial_{12} v \Big) = \phi^1 y_1(\mu_1 + \gamma) + \phi^2 y_2(\mu_2 + \gamma).$$

Upon elimination of  $\phi^1$  and  $\phi^2$ , we arrive at the stated PDE.

Recall that the historically observed drift terms are  $\hat{\mu}_i = \mu_i + \gamma$ , rather than  $\mu_i$ . The pricing PDE can thus be simplified as follows:

$$\begin{aligned} \partial_t v + y_1 \left( \hat{\mu}_1 - \sigma_1 \frac{\hat{\mu}_2 - \hat{\mu}_1}{\sigma_2 - \sigma_1} \right) \partial_1 v + y_2 \left( \hat{\mu}_2 - \sigma_2 \frac{\hat{\mu}_2 - \hat{\mu}_1}{\sigma_2 - \sigma_1} \right) \partial_2 v \\ &+ \frac{1}{2} \Big( y_1^2 \sigma_1^2 \partial_{11} v + y_2^2 \sigma_2^2 \partial_{22} v + 2y_1 y_2 \sigma_1 \sigma_2 \partial_{12} v \Big) = v \left( \hat{\mu}_1 - \sigma_1 \frac{\hat{\mu}_2 - \hat{\mu}_1}{\sigma_2 - \sigma_1} \right). \end{aligned}$$

The pre-default pricing function v depends on the market observables (drift coefficients, volatilities, and pre-default prices), but not on the (deterministic) default intensity.

To make one more simplifying step, we make an additional assumption about the payoff function. Suppose, in addition, that the payoff function is such that  $G(y_1, y_2) = y_1 g(y_2/y_1)$  for some function  $g : \mathbb{R}_+ \to \mathbb{R}$  (or equivalently,  $G(y_1, y_2) = y_2 h(y_1/y_2)$  for some function  $h : \mathbb{R}_+ \to \mathbb{R}$ ). Then we may focus on relative pre-default prices  $\widehat{C}_t = \widetilde{C}_t(\widetilde{Y}_t^1)^{-1}$  and  $\widetilde{Y}^{2,1} = \widetilde{Y}_t^2(\widetilde{Y}_t^1)^{-1}$ . The corresponding pre-default pricing function  $\widehat{v}(t, z)$ , such that  $\widehat{C}_t = \widehat{v}(t, Y_t^{2,1})$  will satisfy the PDE

$$\partial_t \widehat{v} + \frac{1}{2} (\sigma_2 - \sigma_1)^2 z^2 \partial_{zz} \widehat{v} = 0$$

with terminal condition  $\hat{v}(T, z) = g(z)$ . If the price processes  $Y^1$  and  $Y^2$  in (4.15) are driven by the correlated Brownian motions W and  $\widehat{W}$  with the constant instantaneous correlation coefficient  $\rho$ , then the PDE becomes

$$\partial_t \widehat{v} + \frac{1}{2} (\sigma_2^2 + \sigma_1^2 - 2\rho\sigma_1\sigma_2) z^2 \partial_{zz} \widehat{v} = 0$$

Consequently, the pre-default price  $\tilde{C}_t = \tilde{Y}_t^1 \hat{v}(t, \tilde{Y}_t^{2,1})$  will not depend directly on the drift coefficients  $\hat{\mu}_1$  and  $\hat{\mu}_2$ , and thus, in principle, we should be able to derive an expression the price of the claim in terms of market observables: the prices of the underlying assets, their volatilities and the correlation coefficient. Put another way, neither the default intensity nor the drift coefficients of the underlying assets appear as independent parameters in the pre-default pricing function.

# Bibliography

- [1] S. Babbs and T.R. Bielecki. A note on short spreads. Working paper, 2003.
- [2] T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. Hedging of defaultable claims. In: R.A. Carmona, editor, *Paris-Princeton Lecture on Mathematical Finance 2003*, Lecture Notes in Mathematics 1847, pages 1–132. Springer, 2004.
- [3] T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. Modelling and valuation of credit risk. In: M. Frittelli and W. Runggaldier, editors, *CIME-EMS Summer School on Stochastic Methods in Finance, Bressanone*, Lecture Notes in Mathematics 1856, pages 27–126. Springer, 2004.
- [4] T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. PDE approach to valuation and hedging of credit derivatives. *Quantitative Finance*, 5:257–270, 2005.
- [5] T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. Hedging of credit derivatives in models with totally unexpected default. In: J. Akahori, editor, *Stochastic Processes and Applications to Mathematical Finance*, pages 35–100. World Scientific, 2006.
- [6] T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. Replication of contingent claims in a reducedform credit risk model with discontinuous asset prices. *Stochastic Models*, 22:661–687, 2006.
- [7] T.R. Bielecki and M. Rutkowski. Credit Risk: Modelling, Valuation and Hedging. Springer, 2002.
- [8] F. Black and J.C. Cox. Valuing corporate securities: Some effects of bond indenture provisions. Journal of Finance, 31:351–367, 1976.
- F. Black and M. Scholes. The pricing of options and corporate liabilities. Journal of Political Economy, 81:637–654, 1973.
- [10] C. Bluhm, L. Overbeck, and C. Wagner. An Introduction to Credit Risk Modeling. Chapman & Hall, 2004.
- [11] P. Brémaud. Point Processes and Queues. Martingale Dynamics. Springer, 1981.
- [12] P. Brémaud and M. Yor. Changes of filtration and of probability measures. Z. Wahr. Verw. Gebiete, 45:269–295, 1978.
- [13] M.J. Brennan and E.S. Schwartz. Convertible bonds: Valuation and optimal strategies for call and conversion. *Journal of Finance*, 32:1699–1715, 1977.
- [14] M.J. Brennan and E.S. Schwartz. Analyzing convertible bonds. Journal of Financial and Quantitative Analysis, 15:907–929, 1980.
- [15] D. Brigo. Constant maturity credit default pricing and market models. Working paper, 2004.
- [16] E. Briys and F. de Varenne. Valuing risky fixed rate debt: An extension. Journal of Financial and Quantitative Analysis, 32:239–248, 1997.

- [17] R. Bruyère. Les produits dérivés de crédit. Economica, 2005.
- [18] L. Campi and A. Sbuelz. Closed form pricing of benchmark equity default swaps under the CEV assumption. Working paper, 2005.
- [19] N. Chen and S. Kou. Credit spreads, optimal capital structure, and implied volatility with endogenous default and jump risk. Working paper, 2005.
- [20] P.O. Christensen, C.R. Flor, D. Lando, and K.R. Miltersen. Dynamic capital structure with callable debt and debt renegotiations. Working paper, 2002.
- [21] P. Collin-Dufresne, R.S. Goldstein, and J.-N. Hugonnier. A general formula for valuing defaultable securities. *Econometrica*, 72:1377–1407, 2004.
- [22] D. Cossin and H. Pirotte. Advanced Credit Risk Analysis. J. Wiley, 2001.
- [23] B. Dao. Approche structurelle du risque de crédit avec des processus mixtes diffusion-sauts. PhD thesis, 2005.
- [24] C. Dellacherie. Capacités et processus stochastiques. Springer, 1972.
- [25] C. Dellacherie and P.A. Meyer. Probabilités et Potentiel, chapitres I-IV. Hermann, Paris, 1975. English translation: Probabilities and Potentiel, Chapters I-IV, North-Holland, 1978.
- [26] C. Dellacherie and P.A. Meyer. A propos du travail de Yor sur les grossissements des tribus. In: C. Dellacherie, P.A. Meyer, and M. Weil, editors, *Séminaire de Probabilités XII*, Lecture Notes in Mathematics 649, pages 69–78. Springer, 1978.
- [27] C. Dellacherie and P.A. Meyer. Probabilités et Potentiel, chapitres V-VIII. Hermann, Paris, 1980. English translation: Probabilities and Potentiel, Chapters V-VIII, North-Holland, 1982.
- [28] D. Duffie and D. Lando. Term structure of credit spreads with incomplete accounting information. *Econometrica*, 69:633–664, 2000.
- [29] D. Duffie and K. Singleton. Credit Risk: Pricing, Measurement and Management. Princeton University Press, 2003.
- [30] R. Elliott. Stochastic Calculus and Applications. Springer, 1982.
- [31] R.J. Elliott, M. Jeanblanc, and M. Yor. On models of default risk. Math. Finance, 10:179–196, 2000.
- [32] Y. Elouerkhaoui. Etude des problèmes de corrélation et d'incomplétude dans les marchés de crédit. PhD thesis, 2006.
- [33] P. Embrechts, F. Lindskog, and A.J. McNeil. Modelling dependence with copulas and applications to risk management. In: S. Rachev, editor, *Handbook of Heavy Tailed Distributions in Finance*, pages 329–384. Elsevier Noth Holland, 2003.
- [34] J-P. Florens and D. Fougere. Noncausality in continuous time. *Econometrica*, 64:1195–1212, 1996.
- [35] R. Frey, A.J. McNeil, and P. Embrechts. Risk Management. Princeton University Press, 2006.
- [36] K. Giesecke. Default compensator, incomplete information, and the term structure of credit spreads. Working paper, 2002.
- [37] Y.M. Greenfield. *Hedging of the credit risk embedded in derivative transactions*. PhD thesis, 2000.
- [38] B. Hilberink and L.C.G. Rogers. Optimal capital structure and endogenous default. *Finance and Stochastics*, 6:237–263, 2002.

- [39] J. Hull and A. White. Valuing credit default swaps (i): no counterparty default risk. Journal of Derivatives, 8:29–40, 2000.
- [40] F. Jamshidian. Valuation of credit default swap and swaptions. *Finance and Stochastics*, 8:343–371, 2004.
- [41] M. Jeanblanc and Y. LeCam. Intensity versus hazard process approach. Working paper, 2005.
- [42] M. Jeanblanc and M. Rutkowski. Modeling default risk: An overview. In: Mathematical Finance: Theory and Practice, pages 171–269. High Education Press. Beijing, 2000.
- [43] M. Jeanblanc and M. Rutkowski. Modeling default risk: Mathematical tools. Working paper, 2000.
- [44] S. Kusuoka. A remark on default risk models. Advances in Mathematical Economics, 1:69–82, 1999.
- [45] D. Lando. Credit Risk Modeling. Princeton University Press, 2004.
- [46] O. Le Courtois and F. Quittard-Pinon. The capital structure from the point of view of investors and managers: An analysis with jump processes. Working paper, 2004.
- [47] H. Leland. Corporate debt value, bond covenants, and optimal capital structure. Journal of Finance, 49:1213–1252, 1994.
- [48] H. Leland and K. Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *Journal of Finance*, 51:987–1019, 1996.
- [49] F.A. Longstaff and E.S. Schwartz. A simple approach to valuing risky fixed and floating rate debt. *Journal of Finance*, 50:789–819, 1995.
- [50] D. Madan and H. Unal. Pricing the risks of default. Review of Derivatives Research, 2:121–160, 1998.
- [51] G. Mazziotto and J. Szpirglas. Modèle général de filtrage non linéaire et équations différentielles stochastiques associées. Ann. Inst. Henri Poincaré, 15:147–173, 1979.
- [52] A.J. McNeil, R. Frey, and P. Embrechts. Quantitative Risk Management: Concepts, Techniques, and Tools. Princeton University Press, Princeton, 2005.
- [53] R. Merton. On the pricing of corporate debt: The risk structure of interest rates. Journal of Finance, 3:449–470, 1974.
- [54] F. Moraux. On cumulative Parisian options. *Finance*, 23:127–132, 2002.
- [55] T.N. Nielsen, J. Saa-Requejo, and P. Santa-Clara. Default risk and interest rate risk: The term structure of default spreads. Working paper, 1993.
- [56] A. Nikeghbali and M. Yor. A definition and some properties of pseudo-stopping times. Annals of Probability, 33:1804–1824, 2005.
- [57] M. Rutkowski and K. Yousiph. PDE approach to valuation and hedging of basket credit derivatives. Forthcoming in *International Journal of Theoretical and Applied Finance*.
- [58] J. Saá-Requejo and P. Santa-Clara. Bond pricing with default risk. Working paper, 1999.
- [59] P.J. Schönbucher. Credit Derivatives Pricing Models. Wiley Finance, 2003.
- [60] D. Wong. A unifying credit model. Technical report, Capital Markets Group, 1998.
- [61] C. Zhou. The term structure of credit spreads with jumps risk. *Journal of Banking and Finance*, 25:2015–2040, 2001.