Default correlation: a dynamic approach

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We consider, in a financial market

- A default event at some random time τ
- A **promised contingent claim** X, paid at time T if $\tau > T$
- A **recovery process** Z: Z_τ is the recovery payoff at time of default, if it occurs prior to or at the maturity date T.

Let $(\Omega, \mathbf{F}, \mathbb{P})$ be a given filtered probability space, τ a random time and

$$H_t = 1\!\!1_{\tau \leq t}$$

Let $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ where $\mathcal{H}_t = \sigma(H_s, s \leq t)$.

The promised contingent claim X is $\mathcal{F}_T$-measurable, the recovery process Z is $\mathbf{F}$-adapted.
A main problem is to prove that $\mathbf{F}$-martingales remains $\mathbf{G}$-semi-martingales.
We recall that for $\mathbf{F} \subset \mathbf{G}$, the filtration $\mathbf{F}$ is said to be **immersed** in $\mathbf{G}$ is any $\mathbf{F}$-martingale is a $\mathbf{G}$-martingale.

Initial times

For any random time τ, we write

$$G_t^T(\omega) = \mathbb{P}(\tau > T|\mathcal{F}_t)(\omega)$$

the conditional survival process. *The positive random time τ is called an* ***initial time*** *if* $\mathbb{P}(\tau \in ds|\mathcal{F}_t) \ll ds$. Then,

$$G_t^T = \mathbb{P}(\tau > T|\mathcal{F}_t) = \int_T^\infty f(u;t)du\,.$$

From $G_s^T = \mathbb{E}(G_t^T|\mathcal{F}_s)$ for any $s \leq t$, it follows that for any $u \geq 0$, $(f(u;t))_t$ is a non negative $\mathbb{F}$-martingale. The law of τ is $\mathbb{P}(\tau > T) = \int_T^\infty f(u;0)du$.

• Under the condition that the initial time τ avoids the $\mathbb{F}$-stopping times, there is equivalence between $\mathbb{F}$ is immersed in $\mathbb{G}$ and for any $u \geq 0$, the martingale $f(u;\cdot)$ is constant after u.

• Let $(K(u;t))_{t\geq 0}$ be a family of $\mathbb{F}$-predictable processes indexed by $u \geq 0$. Then

$$\mathbb{E}\left(K(\tau;t)|\,\mathcal{F}_t\right) = \int_0^\infty K(u;t)f(u;t)du$$

• If X is an $\mathbb{F}$-martingale, assuming that $G_t = G_t^t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ is continuous

$$Y_t = X_t - \int_0^{t\wedge\tau} \frac{d\left\langle X, G\right\rangle_s}{G_s} - \int_{t\wedge\tau}^{t} \frac{d\left\langle X, f(\theta;\cdot)\right\rangle_s}{f(\theta;s)}\bigg|_{\theta=\tau} \in \mathcal{M}(\mathbb{G}).$$

The process $G_t, t \geq 0$ is a supermartingale, and admits a Doob-Meyer decomposition as

$$G_t = Z_t - A_t$$

where Z is a martingale and A a predictable increasing process.

The process

$$H_t - \int_0^t (1 - H_s) \frac{dA_s}{G_s}$$

is a **G**-martingale.

$G_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$

From the remark that, if $(Y_t, t \geq 0)$ is a **G**-adapted process, there exists an **F**-adapted process $(y_t, t \geq 0)$ such that

$$Y_t \mathbb{1}_{t<\tau} = y_t \mathbb{1}_{t<\tau}$$

we obtain the key formulae:

- For any integrable $\mathcal{F}_T$ measurable r.v. X

$$\mathbb{E}(\mathbb{1}_{\{T<\tau\}} X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}(G_T X \,|\, \mathcal{F}_t)\,.$$

- Let x be an **F**-predictable process. Then,

$$\mathbb{E}(x_\tau \mathbb{1}_{\tau<T} | \mathcal{G}_t) = x_\tau \mathbb{1}_{\{\tau<t\}} - \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t} \mathbb{E}(\int_t^T x_u dG_u | \mathcal{F}_t)$$

- Let φ be a Borel function. Then,

$$\mathbb{E}(\varphi_\tau \mathbb{1}_{\tau<T} | \mathcal{G}_t) = \varphi(\tau) \mathbb{1}_{\{\tau<t\}} + \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t} \mathbb{E}\left(\int_t^T \varphi(u) f(u;u) du \Big| \mathcal{F}_t \right)$$

$G_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$

From Itô-Wentzell formula,

$$G_t = G(t;t) = G(0;0) + \int_0^t g(s;s)\,dW_s + \int_0^t f(s;s)\,ds$$

It follows that

$$H_t - \int_0^t (1 - H_s)\lambda_s ds$$

is a **G**-martingale, where

$$\lambda_t = \frac{f(t;t)}{G_t} = -\frac{\partial_1 G(t;t)}{G(t;t)}$$

The intensity rate λ can be obtained as

$$\lambda_t = \lim_{h \to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t + h | \mathcal{F}_t)}{\mathbb{P}(t < \tau | \mathcal{F}_t)} .$$

A general model of density process

Our aim is to give a large class of examples of density process. We start with an explicit construction of a family $(f(u) = f(u;t), t \geq 0)$ which satisfy:

(a) $f(u;\cdot)$ is a family of martingales

(b) $f(u;t) > 0, \forall t \geq 0, \forall u \geq 0$

(c) $\int_0^\infty f(u;t)du = 1$, $\forall t \geq 0$.

Define $f(u;t) = \lambda_t(u) \exp\left(-\int_0^u \lambda_t(v)dv\right)$ where $d\lambda_t(u) = m_t(u)dt + \sigma_t(u)dW_t$ with

$$m_t(u) = \sigma_t(u) \int_0^u \sigma_t(v)dv, \qquad \forall t, u \geq 0.$$

Then, for any u, the process $f(u;\cdot)$ is a martingale.
Condition b) is satisfied if $\lambda_t(u) \geq 0$

We have $\lambda_t(u) = \lambda_0(u) + \int_0^t m_s(u)ds + \int_0^t \sigma_s(u)dW_s$. The non negativity of λ will be satisfied if

- m is non negative
- for any u, $\lambda_0(u) \geq 0$
- the quantity $X_t = \lambda_0(u) + \int_0^t \sigma_s(u)dW_s$ is a positive

martingale.
This is the case if X is a Doléans Dade exponential: there exists a family $b(u)$ of adapted processes

$$X_t = \lambda_0(u) \exp\left(\int_0^t b_s(u)dW_s - \frac{1}{2}\int_0^t b_s^2(u)ds\right)$$

In that case, $\int_0^t \sigma_s(u)dW_t = \int_0^t b_s(u)X_s dW_s$. Hence, we make the choice of

$$\sigma_t(u) = b_t(u)X_t = b_t(u)\lambda_0(u) \exp\left(\int_0^t b_s(u)dW_s - \frac{1}{2}\int_0^t b_s^2(u)ds\right)$$

and propose a family of density processes.

Let $\lambda_0(u)$ be a family of probability densities on $\mathbb{R}^+$. Assume that $b(u)$ is a given family of non-negative adapted processes. Define

$$\sigma_t(u) = b_t(u)\lambda_0(u)\exp\left(\int_0^t b_s(u)dW_s - \frac{1}{2}\int_0^t b_s^2(u)ds\right)$$

and

$$\begin{aligned} f_t(u) &= \lambda_t(u)\exp\left(-\int_0^t \lambda_t(v)dv\right) \\ \lambda_t(u) &= \lambda_0(u) + \int_0^t m_s(u)ds + \int_0^t \sigma_s(u)dW_s \\ m_t(u) &= \sigma_t(u)\int_0^u \sigma_t(v)dv\,. \end{aligned}$$

Then the family $f(u)$ satisfies the above conditions.

CDS prices

A CDS (**Credit Default Swap**) is a contract:

The holder of the CDS pays to the seller a fee at a constant rate κ up to default time, or up to maturity.

The issuer of the CDS pays, at default time, the amount $\delta(\tau)$ to the holder.

The price of a CDS at initiation time is 0.

The time-t price of a CDS initiated at time 0 is

$$S_t(\kappa) = B_t \mathbb{E}\Big(\mathbb{1}_{\{t<\tau\le T\}} \delta_\tau B_\tau^{-1} \,\Big|\, \mathcal{G}_t \Big) - \mathbb{E}\Big(\mathbb{1}_{\{t<\tau\}} \kappa \int_t^{T\wedge\tau} B_u^{-1} du \,\Big|\, \mathcal{G}_t \Big),$$

Using the key formulae

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \left(\int_t^T \mathbb{E}\big(B_u^{-1} \delta_u f(u;u) \big|\, \mathcal{F}_t\big)\, du - \kappa \int_t^T \mathbb{E}\big(B_u^{-1} G_u \,\big|\, \mathcal{F}_t\big)\, du \right)$$

Let $\widetilde{S}_t(\kappa)$ be the predefault price of the CDS, i.e., $\widetilde{S}_t(\kappa)$ is an **F**-adapted process such that $1\!\!1_{t<\tau}\widetilde{S}_t(\kappa) = 1\!\!1_{t<\tau}S_t(\kappa)$. Then,

$$d\widetilde{S}_t(\kappa) = \frac{1}{G(t;t)}\left[\Big(\kappa G(t;t) - (\delta(t) + \widetilde{S}_t(\kappa))f(t;t)\Big)dt + \sigma(T;t)\left(dW_t - \frac{g(t;t)}{G(t;t)}dt\right)\right]$$

or

$$d\widetilde{S}_t(\kappa) = \left[\Big(\kappa - (\delta(t) + \widetilde{S}_t(\kappa))\lambda_t\Big)dt + \sigma(T;t)\left(dW_t - \frac{g(t;t)}{G(t;t)}dt\right)\right]$$

where

$$\sigma(T;t) = \int_t^T \big(\delta(u)\partial_1 g(u;t) - \kappa g(u;t)\big)\,du + g(t;t)\widetilde{S}_t(\kappa)$$

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$$d\widetilde{S}_t(\kappa) = \frac{1}{G(t;t)}\left[\Big(\kappa G(t;t) - (\delta(t) + \widetilde{S}_t(\kappa))f(t;t)\Big)dt + \sigma(T;t)\,(dW_t - \frac{g(t;t)}{G(t;t)}dt)\right]$$

or

$$d\widetilde{S}_t(\kappa) = \left[\Big(\kappa - (\delta(t) + \widetilde{S}_t(\kappa))\lambda_t\Big)dt + \sigma(T;t)\,(dW_t - \frac{g(t;t)}{G(t;t)}dt)\right]$$

where

$$\sigma(T;t) = \int_t^T \big(\delta(u)\partial_1 g(u;t) - \kappa g(u;t)\big)\,du + g(t;t)\widetilde{S}_t(\kappa)$$

If immersion property holds, $g(t;t) = 0$.

Hedging of Credit Spreads and Default Correlations

In this section, we assume that $\mathbf{F}$ is a Brownian filtration and that the interest rate is null.

Our aim is to obtain the dynamics of a CDS in the simple case where two different credit names are considered.

We assume that we are given two strictly positive random times τ_1 and τ_2.

Joint Survival Process

We introduce the **conditional joint survival process** $G(u,v;t)$

$$G(u,v;t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t).$$

We write

$$\partial_1 G(u,v;t) = \frac{\partial}{\partial u} G(u,v;t), \quad \partial_{12} G(u,v;t) = \frac{\partial^2}{\partial u \partial v} G(u,v;t) = f(u,v;t)$$

so that

$$G(u,v;t) = \int_u^\infty \Big(\int_v^\infty f(x,y;t)\,dy \Big)\,dx$$

where $f(u,v;s)$ is some $\mathbf{F}$-predictable process (in fact an $(\mathbf{F},\mathbb{Q})$-martingale).

For any fixed $(u, v) \in \mathbb{R}^2_+$, the $\mathbf{F}$-martingale $G(u, v; t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t)$ admits the integral representation

$$G(u, v; t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v) + \int_0^t g(u, v; s)\, dW_s$$

Let us introduce the filtrations $\mathbf{H}^i, \mathbf{H}, \mathbf{G}^i$ and $\mathbf{G}$ associated with default times by setting

$$\mathcal{H}_t^i = \sigma(H_s^i;\, s \in [0,t]), \quad \mathcal{H}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2, \quad \mathcal{G}_t^i = \mathcal{F}_t \vee \mathcal{H}_t^i, \quad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t,$$

where $H_t^i = \mathbb{1}_{\{\tau_i \le t\}}$. **In this talk, we assume that immersion hypothesis holds between F and G**. In particular, any $\mathbf{F}$-martingale is also a $\mathbf{G}^i$-martingale for $i = 1, 2$.

In general, there is no reason that any $\mathbf{G}^i$-martingale is a $\mathbf{G}$-martingale.

$$\widehat{M}_t^1 = H_t^1 - \int_0^{t\wedge\tau_1\wedge\tau_2} \widetilde{\lambda}_u^1 \, du - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \lambda^{1|2}(\tau_2; u)\, du,$$

is a **G**-martingale, where

$$\widetilde{\lambda}_t^1 = -\frac{\partial_1 G(t,t;t)}{G(t,t;t)},$$

is the **pre-default intensity** for the 1-th name and

$$\lambda^{1|2}(s;t) = -\frac{f(t,s;t)}{\partial_2 G(t,s;t)}$$

is the post-2default intensity for the 1-th name

Note that $\widetilde{\lambda} = \widetilde{\lambda}^1 + \widetilde{\lambda}^2$ is the **F**-intensity of $\tau_{(1)} = \tau_1 \wedge \tau_2$.

Valuation of Single-Name CDSs

Let us now examine the valuation of single-name CDSs under the assumption that the interest rate equals zero.

We consider the CDS

- with the constant spread κ_1,
- which delivers $\delta_1(\tau_1)$ at time τ_1 if $\tau_1 < T_1$, where δ_1 is a

deterministic function.

The value $S^1(\kappa_1)$ of this CDS, computed in the filtration $\mathbf{G}$, i.e., taking care on the information on the second default contained in that filtration, is computed in two successive steps.

We set $\tau_{(1)} = \tau_1 \wedge \tau_2$.

On the set $t < \tau_{(1)}$, the ex-dividend price of the CDS equals

$$S_t^1(\kappa_1) = \widetilde{S}_t^1(\kappa_1) = \frac{1}{G(t,t;t)} \left(- \int_t^{T_1} \delta_1(u) \partial_1 G(u,t;t)\, du - \kappa_1 \int_t^{T_1} G(u,t;t)\, du \right).$$

On the event $\{\tau_2 \leq t < \tau_1\}$, we have that

$$S_t^1(\kappa_1) = \frac{1}{\partial_2 G(t,\tau_2;t)} \left(- \int_t^{T_1} \delta_1(u) f(u,\tau_2;t)\, du - \kappa_1 \int_t^{T_1} \partial_2 G(u,\tau_2;t)\, du \right).$$

Price Dynamics of Single-Name CDSs

By applying the Itô-Wentzell theorem, we get

$$
\begin{aligned}
G(u,t;t) &= G(u,0;0) + \int_0^t g(u,s;s)\,dW_s + \int_0^t \partial_2 G(u,s;s)\,ds \\
G(t,t;t) &= G(0,0;0) + \int_0^t g(s,s;s)\,dW_s + \int_0^t \left(\partial_1 G(s,s;s) + \partial_2 G(s,s;s)\right) ds.
\end{aligned}
$$

The dynamics of the process $\widetilde{S}^1(\kappa_1)$ are

$$d\widetilde{S}^1_t(\kappa_1) = \Big(- \widetilde{\lambda}^1_t \delta_1(t) + \kappa_1 + \widetilde{\lambda}^1_t \widetilde{S}^1_t(\kappa_1) - \widetilde{\lambda}^2_t S^1_{t|2}(\kappa_1) \Big) dt + \sigma^1(T_1;t)\, dW_t$$

where

$$\begin{aligned}
\sigma^1(T_1;t) &= -\frac{1}{G_(t,t;t)} \left(\int_t^{T_1} \big(\delta_1(u)\, \partial_1 g(u,t;t) + \kappa_1 g(u,t;t)\big)\, du \right) \\
S^1_{t|2}(\kappa_1) &= \frac{1}{\partial_2 G(t,t;t)} \left(- \int_t^{T_1} \delta_1(u) f(u,t;t)\, du - \kappa_1 \int_t^{T_1} \partial_2 G(u,t;t)\, du \right).
\end{aligned}$$

The cumulative price

$$S_t^{c,1}(\kappa_1) = S_t^1(\kappa_1) + B_t \int_{]0,t]} B_u^{-1} \, dD_u$$

where

$$D_t = D_t(\kappa_1, \delta_1, T_1, \tau_1) = \delta_1(\tau_1) \mathbb{1}_{\{\tau_1 \leq t\}} - \kappa_1(t \wedge (T_1 \wedge \tau_1))$$

satisfies, on $[0, T_1 \wedge \tau_{(1)}]$,

$$dS_t^{c,1}(\kappa_1) \quad = \quad (\delta_1(t) - \widetilde{S}_t^1(\kappa_1)) \, d\widehat{M}_t^1 + (S_{t|2}^1(\kappa_1) - \widetilde{S}_t^1(\kappa_1)) \, d\widehat{M}_t^2 + \sigma^1(T_1; t) dW_t \, .$$

On $\tau_1 > t > \tau_2$

$$dS_t^1 = d\widehat{S}_t^1 = \sigma_{1|2}(T_1;t)dW_t + (\delta_1(t)\lambda^{1|2}(\tau_2;t) - \kappa_1 + \widehat{S}_t^1\lambda^{1|2}(\tau_2;t))dt$$

where

$$\begin{aligned}
\sigma_{1|2}(T^1;t) &= -\int_t^T \delta_1(u)\partial_1\partial_2 g(u,\tau_2;t)du - \kappa_1\int_t^{T_1}\partial_2 g(u,\tau_2;t)du \\
\lambda^{1|2}(s;t) &= -\frac{f(t,s;t)}{\partial_2 G(t,s;t)}
\end{aligned}$$

Replication of a First-to-Default Claim

A **first-to-default claim** with maturity T is a claim $(X, A, Z, \tau_{(1)})$ where

- X is an $\mathcal{F}_T$-measurable amount payable at maturity if no default occurs
- $A : [0, T] \to \mathbb{R}$ with $A_0 = 0$ represents the dividend stream up to $\tau_{(1)}$,
- $Z = (Z^1, Z^2, \dots, Z^n)$ is the vector of $\mathbf{F}$-predictable, real-valued processes, where $Z^i_{\tau_{(1)}}$ specifies the recovery received at time $\tau_{(1)}$ if the ith name is the first defaulted name, that is, on the event $\{\tau_i = \tau_{(1)} \leq T\}$.
- We denote by $G_{(1)}(t; t) = G(t, \cdots, t; t)$

The cumulative price S^{cum} is given by

$$dS_t^{cum} = \sum_{i=1}^{n} (Z_t^i - S_{t-})\, d\widehat{M}_t^i + (1 - H_t^{(1)})(G_{(1)}(t;t))^{-1}\, dm_t,$$

where the **F**-martingale m is given by

$$\mathbb{E}_{Q^*}\left(G_{(1)}(T;T)X + \sum_{i=1}^{n} \int_0^T G_{(1)}(u;u) Z_u^i \widetilde{\lambda}_u^i\, du - \int_0^T G_{(1)}(u;u)\, dA_u \,\Big|\, \mathcal{F}_t \right).$$

The pre-default ex-dividend price satisfies

$$d\widetilde{S}_t = \widetilde{\lambda}_t \widetilde{S}_t \, dt - \sum_{i=1}^{n} \widetilde{\lambda}_t^i Z_t^i \, dt + dA_t + (G(t,t;t))^{-1} \, dm_t.$$

Since **F** is generated by a Brownian motion, there exists an **F**-predictable process ζ such that

$$d\widetilde{S}_t = \widetilde{\lambda}_t \widetilde{S}_t \, dt - \sum_{i=1}^{n} \widetilde{\lambda}_t^i Z_t^i \, dt + dA_t + (G(t,t;t))^{-1} \zeta_t \, dW_t.$$

We say that a self-financing strategy $\phi = (\phi^0, \phi^1, \dots, \phi^k)$ **replicates** a first-to-default claim $(X, A, Z, \tau_{(1)})$ if its wealth process $V(\phi)$ satisfies the equality $V_{t\wedge\tau_{(1)}}(\phi) = S_{t\wedge\tau_{(1)}}$ for any $t \in [0, T]$.

We have, for any $t \in [0, T]$,

$$dV_t(\phi) = \sum_{\ell=1}^{k} \phi_t^i \Big(\big(\delta_t^\ell - \widetilde{S}_t^\ell(\kappa_\ell)\big)\, dM_t^\ell + \sum_{j=1\,, j\neq\ell}^{k} \big(S_{t|j}^\ell - \widetilde{S}_t^\ell(\kappa_\ell)\big)\, dM_t^j$$

$$+ (1 - H_t)(G(t, t; t))^{-1}\, dn_t^\ell \Big)$$

where

$$n_t^\ell = \mathbb{E}_{Q^*}\left(\int_0^{T_\ell} G(u, u; u) \Big(\delta_u^\ell \widetilde{\lambda}_u^i + \sum_{j=1, j\neq\ell}^{n} S_{u|j}^\ell \widetilde{\lambda}_u^j \Big)\, du - \kappa_\ell \int_0^{T_\ell} G(u, u; u)\, du \,\Big|\, \mathcal{F}_t \right).$$

Let $\widetilde{\phi}_t = (\widetilde{\phi}_t^1, \widetilde{\phi}_t^2, \ldots, \widetilde{\phi}_t^k)$ be a solution to the following equations

$$\widetilde{\phi}_t^\ell \big(\delta_t^\ell - \widetilde{S}_t^\ell(\kappa_\ell)\big) + \sum_{j=1,\, j\neq\ell}^{n} \widetilde{\phi}_t^j \big(S_{t|\ell}^j(\kappa_j) - \widetilde{S}_t^j(\kappa_j)\big) = Z_t^\ell - \widetilde{S}_t$$

and $\sum_{\ell=1}^k \widetilde{\phi}_t^\ell \zeta_t^\ell = \zeta_t$.

Let us set $\phi_t^\ell = \widetilde{\phi}^\ell(\tau_{(1)} \wedge t)$ for $\ell = 1, 2, \ldots, k$ and $t \in [0, T]$.

Then the self-financing trading strategy $\phi = ((V(\phi) - \phi \cdot S), \ldots, \phi^k)$ replicates the first-to-default claim $(X, A, Z, \tau_{(1)})$.

Replication with Market CDSs

When considering trading strategies involving CDSs issued in the past, one encounters a practical difficulty regarding their liquidity.

Recall that for each maturity T_i by the CDS issued at time t we mean the CDS over $[t, T]$ with the spread $\kappa(t, T_i) = \kappa_i$.

We now define a **market CDS** — which at any time t has similar features as the T_i-maturity CDS issued at this date t, in particular, it has the ex-dividend price equal to zero.

A T_i-**maturity market CDS** has the dividend process equal to

$$^*D^i_t = \int_{]0,t]} B_u \, d(B_u^{-1} S^i_u(\kappa_i)) + D^i_t,$$

where $D^i = D(\kappa_i, \delta^i, T_i, \tau)$ for some fixed spread κ_i.

The ex-dividend price $^*S^i$ **of the** T_i**-maturity market CDS equals zero for any** $t \in [0, T_i]$.

Since market CDSs are traded on the ex-dividend basis, to describe the self-financing trading strategies in the savings account B and the market CDSs with ex-dividend prices ${}^*S^i$.

A strategy $\phi = (\phi^0, \ldots, \phi^k)$ in the savings account B and the market CDSs with dividends ${}^*D^i$ is said to be *self-financing* if its wealth $V_t(\phi) = \phi^0_t B_t$ satisfies $V_t(\phi) = V_0(\phi) + G_t(\phi)$ for every $t \in [0, T]$, where the gains process $G(\phi)$ is defined as follows

$$G_t(\phi) = \int_{]0,t]} \phi^0_u \, dB_u + \sum_{i=1}^{k} \int_{]0,t]} \phi^i_u \, d\,{}^*D^i_u.$$

Let ϕ be a self-financing strategy in the savings account B and ex-dividend prices $S^i(\kappa_i)$, $i = 1, \dots, k$.

Then the strategy $\psi = (\psi^0, \dots, \psi^k)$ where $\psi^i = \phi^i$ for $i = 1, \dots, k$ and $\psi^0_t = B_t^{-1} V_t(\phi)$ is a self-financing strategy in the savings account B and the market CDSs with dividends ${}^*D^i$ and its wealth process satisfies $V(\psi) = V(\phi)$.

The cumulative price of the T_i-maturity market CDS satisfies

$$\begin{aligned} {}^*S_t^{c,i} &= {}^*S_t^i + B_t \int_{]0,t]} B_u^{-1}\, d\,{}^*D_u^i \\ &= \mathbb{1}_{\{t<\tau\}}(\kappa_t^i - \kappa_i)\widetilde{A}(t,T) + B_t \int_{]0,t]} B_u^{-1}\, dD_u^i \end{aligned}$$

where

$$\widetilde{A}(t,T) = \frac{B_t}{G_t}\, \mathbb{E}_{Q^*}\left(\int_t^{T\wedge\tau} B_u^{-1} du \,\Big|\, \mathcal{F}_t \right).$$

Assume that there exist $\mathbf{F}$-predictable processes $\phi^1, \ldots, \phi^k$ satisfying the following conditions, for any $t \in [0, T]$,

$$\sum_{i=1}^{k} \phi_t^i \big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big) = Z_t - \widetilde{S}_t, \quad \sum_{i=1}^{k} \phi_t^i \zeta_t^i = \xi_t.$$

Let the process $V(\phi)$ be given by

$$dV_t(\phi) = \sum_{i=1}^{k} \phi_t^i \Big(\big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big)\, dM_t + (1 - H_t) B_t G_t^{-1} dn_t^i \Big)$$

with the initial condition $V_0(\phi) = Y_0$ and let ϕ^0 be given by, for $t \in [0, T]$,

$$\phi_t^0 = B_t^{-1} V_t(\phi).$$

Then the self-financing trading strategy $\phi = (\phi^0, \ldots, \phi^k)$ in the savings account B and market CDSs with dividends ${}^*D^i$, $i = 1, \ldots, k$ replicates the defaultable claim (X, A, Z, τ).

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Thank you for your attention