Old results as new tools for Credit risk

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Credit Risk Model

- A reference **filtration** **F** and a **random time** τ are given
- **H** is the filtration generated by the process $H_t = \mathbb{1}_{\tau \leq t}$
- $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$

The filtration **G** enjoys the Minimal assumption: if $A \in \mathcal{G}_t$, there exists $\widetilde{A} \in \mathcal{F}_t$ such that

$$A \cap \{t < \tau\} = \widetilde{A} \cap \{t < \tau\}$$

It follows that, if $(Y_t, t \geq 0)$ is a **G**-adapted process, there exists an **F**-adapted process $(\widetilde{Y}_t, t \geq 0)$ such that

$$Y_t \mathbb{1}_{t<\tau} = \widetilde{Y}_t \mathbb{1}_{t<\tau} \,.$$

The survival hazard process is denoted by S:

$$\mathbb{P}(\tau > t | \mathcal{F}_t) = S_t$$

Assuming that $S_t > 0$, we obtain:

- for any integrable $\mathcal{F}_T$-measurable r.v. X

$$\boxed{\mathbb{E}(\mathbb{1}_{\{T<\tau\}}X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{1}{S_t} \mathbb{E}(S_T X \,|\, \mathcal{F}_t)\,.}$$

- for any positive predictable process h

$$\boxed{E(h_\tau \mathbb{1}_{\{\tau<T\}}|\mathcal{G}_t) = h_\tau \mathbb{1}_{\{\tau<t\}} - \mathbb{1}_{\{\tau>t\}} \frac{1}{S_t} E(\int_t^T h_u dS_u|\mathcal{F}_t)\,.}$$

The process $(S_t,\, t \geq 0)$ is a **F**-supermartingale and admits a Doob-Meyer decomposition

$$S_t = M_t^\tau - A_t^\tau$$

where M^τ is an **F**-martingale, and A^τ is an **F**-predictable increasing process.

The process

$$H_t - \Lambda_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} \frac{dA_s^\tau}{S_{s-}}$$

is a **G**-martingale.

If A is absolutely continuous w.r.t. Lebesgue measure,

$$H_t - \Lambda_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} \frac{a_s^\tau}{S_{s^-}}\, ds$$

and $\lambda_t = \frac{a_t^\tau}{S_{t^-}}$ is called the intensity.
In a general setting, the knowledge of the intensity does not allow to compute the conditional expectations

$$\mathbb{E}(\mathbb{1}_{\{T<\tau\}} X \,|\, \mathcal{G}_t)$$

Martingale density

For any T, the process $S_t(T) \overset{def}{=} \mathbb{P}(\tau > T|\mathcal{F}_t)$ is an **F**-martingale.

We assume that there exists a family of strictly positive processes $(\alpha_t(u), t \geq 0)$ such that

$$S_t(T) = \int_T^\infty \alpha_t(u)du$$

For any u, the process $(\alpha_t(u), t \geq 0)$ is an **F**-martingale, and can be interpreted as

$$\alpha_t(u)du = \mathbb{P}(\tau \in du|\mathcal{F}_t)$$

In the case where **F** is a Brownian filtration generated by W

$$d_t\alpha_t(u) = \psi_t(u)dW_t$$

which, from Ventzell theorem, implies that (if S is continuous)

$$dS_t = \Psi_t dW_t - \alpha_t(t)dt$$

where $\Psi_t = \int_t^\infty \psi_t(u)du$.
This equality gives the Doob-Meyer decomposition of S. In particular, we obtain that

$$H_t - \int_0^{t\wedge\tau} \frac{\alpha_s(s)}{S_s} ds$$

is a **G**-martingale.

In a general setting, if the family of processes α exists, and if S is continuous, the process

$$H_t - \Lambda_{t\wedge\tau}$$

where

$$d\Lambda_t = \frac{\alpha_t(t)}{S_t}dt = \frac{\alpha_t(t)}{\int_t^\infty \alpha_t(u)du}dt\,,$$

is a **G**-martingale.

We can now extend the previous result. Let $X \in \mathcal{F}_T$ and h a Borel function. Then,

$$\begin{aligned}
\mathbb{E}(Xh(T\wedge\tau)|\mathcal{G}_t)\mathbb{1}_{\{\tau>t\}} &= \frac{\mathbb{E}\big[\int_t^\infty Xh(u\wedge T)\alpha_T(u)du|\mathcal{F}_t\big]}{S_t}\mathbb{1}_{\{\tau>t\}} \\
\mathbb{E}[Xh(T\wedge\tau)|\mathcal{G}_t]\mathbb{1}_{\{\tau\le t\}} &= \mathbb{E}\Big[Xh(s)\frac{\alpha_T(s)}{\alpha_t(s)}\big|\mathcal{F}_t\Big]\Big|_{s=\tau}\mathbb{1}_{\{\tau\le t\}}
\end{aligned}$$

Intuitively, with $Y = Xh(T \wedge \tau)$

$$E(Y|\mathcal{G}_t)1\!\!1_{\{\tau \leq t\}} = 1\!\!1_{\{\tau \leq t\}} f_t(\tau)$$

where $f_t(u)$ is a family of **F**-adapted processes. It follows that, for $u < t$

$$\begin{aligned} f_t(u) &= \mathbb{E}(Y|\mathcal{F}_t, \tau = u) = \frac{\mathbb{E}(Y 1\!\!1_{\tau \in du}|\mathcal{F}_t)}{\mathbb{P}(\tau \in du|\mathcal{F}_t)} \\ &= \frac{\mathbb{E}(Xh(u)1\!\!1_{\tau \in du}|\mathcal{F}_t)}{\mathbb{P}(\tau \in du|\mathcal{F}_t} = \frac{\mathbb{E}(Xh(u)\mathbb{P}(\tau \in du|\mathcal{F}_T)|\mathcal{F}_t)}{\mathbb{P}(\tau \in du|\mathcal{F}_t)} \end{aligned}$$

More generally, if $(Y(t,u), t \geq 0)$ is a family of **F**-adapted processes, then

$$\begin{aligned}
\mathbb{E}(Y(T,\tau)|\mathcal{G}_t)1\!\!1_{\{\tau>t\}} &= \frac{\mathbb{E}\big[\int_t^\infty Y(T,u)\alpha_T(u)du|\mathcal{F}_t\big]}{S_t} 1\!\!1_{\{\tau>t\}} \\
\mathbb{E}[Y(T,\tau)|\mathcal{G}_t]1\!\!1_{\{\tau\leq t\}} &= \mathbb{E}\Big[Y(T,T)\frac{\alpha_T(s)}{\alpha_t(s)}\big|\mathcal{F}_t\Big]\Big|_{s=\tau} 1\!\!1_{\{\tau\leq t\}}
\end{aligned}$$

Particular case: Immersion property (or (H) hypothesis)

The filtration **F** is said to be **immersed** in **G** if **F**-martingales are **G**-martingales. This condition is equivalent to

$$S_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty) = \mathbb{P}(\tau > t|\mathcal{F}_s), \forall t, s;\, t \le s\,.$$

In that case, the process S is non-increasing, and $H_t - \Lambda_{t\wedge\tau}$ is a **G**-martingale, where $\Lambda_t = -\ln S_t$.
There is equivalence between **F** is immersed in **G** and for any $u \geq 0$, the martingale $(\alpha_t(u), t \geq 0)$ is constant after u.

Examples

COX process model: Immersion

Let λ be an $\mathbf{F}$-adapted non-negative process, Θ a random variable with exponential law, independent of $\mathcal{F}_\infty$ and

$$\tau \overset{def}{=} \inf\{t : \int_0^t \lambda_s ds \geq \Theta\}$$

In that case,

$$S_t = \exp\left(-\int_0^t \lambda_u du\right) ,$$

and immersion property holds. In particular,

$$\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \exp\left(-\int_0^\theta \lambda_u du\right) = S_\theta, \ \forall \theta < t \, .$$

Imperfect information: Immersion Let

$$\tau = \inf\left\{t, X_t \leq b\left(t\right)\right\}.$$

where $X_t = \Psi\left(B_t, t\right)$ where B is a Brownian motion and $\Psi(\centerdot, t)$ is invertible. The filtration $\mathbf{F}$ is the natural filtration of the observed process is Y

$$\begin{array}{rcl} dY_t & = & \mu_1\left(Y_t, t\right)dt + \sigma\left(Y_t, t\right)dB_t + s\left(Y_t, t\right)dW_t \\ Y_0 & = & y_0\,. \end{array}$$

It can be proved that $\mathbf{F}$ is immersed in $\mathbf{F} \vee \mathbf{H}$ and numerical approximation of the process S are done.

Partial Information: No immersion

Assume that

$$V_t = ve^{\sigma(W_t+\nu t)} = ve^{\sigma X_t}$$

and let, for $\alpha < v$

$$\tau = \inf\{t : V_t \le \alpha\} = \inf\{t : X_t \le a\}$$

The reference filtration $\mathbf{F}$ is the filtration of the observations of V at discrete times $t_1, \cdots t_n$ where $t_n \le t < t_{n+1}$, i.e.,

$$\mathcal{F}_t = \sigma(V_{t_1}, \cdots, V_{t_n}, \, t_i \le t)$$

The process $S_t = P(\tau > t|\mathcal{F}_t)$ is not decreasing.

Two default times: no Immersion

Let $\tau_i, i = 1, 2$ two default times, $H^i_t = 1\!\!1_{\tau_i \le t}$, $\mathbf{H}^i$ the natural filtration of H^i. Let $\mathbf{F} = \mathbf{H}^2$, $\mathbf{G} = \mathbf{F} \vee \mathbf{H}^1$.
Let $G(t, s) = \mathbb{P}(\tau_1 > t, \tau_2 > s)$ and $S^{(1)}_t = \mathbb{P}(\tau_1 > t | \mathcal{F}_t)$.
Then,

$$\begin{aligned} dS^{(1)}_t &= \left(\frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} - \frac{G(t,t)}{G(0,t)} \right) dM^2_t \\ &\quad + \left(H^2_t \partial_1 h^{(1)}(t, \tau_2) + (1 - H^2_t) \frac{\partial_1 G(t,t)}{G(0,t)} \right) dt \end{aligned}$$

where M^2 is the $\mathbf{F}$-martingale $M^2_t = H^2_t - \int_0^{t \wedge \tau_2} \frac{\mathbb{P}(\tau_2 \in ds)}{\mathbb{P}(\tau_2 > s)}$ and $h^{(1)}(t, s) = \frac{\partial_2 G(t,s)}{\partial_2 G(0,s)}$

Generalized Cox Processes

Let Θ be independent of $\mathcal{F}_\infty$ and $V \in \mathcal{F}_\infty$, with $V > 0$.

$$\tau = \inf\{t : \Lambda_t \geq \Theta V\}$$

Then,

$$S_t(T) = \mathbb{P}(\tau > T|\mathcal{F}_t) = \mathbb{P}(\Lambda_T > \Theta V|\mathcal{F}_t) = \mathbb{P}\left(\exp -\frac{\Lambda_T}{V} \geq \exp -\Theta \middle| \mathcal{F}_t\right),$$

Let us denote $\Psi_t = \exp -\Lambda_t/V = \int_t^\infty \psi_s ds$, with

$$\psi_s = \frac{\lambda_s}{V} \exp - \int_0^s \frac{\lambda_u}{V} du,$$

and define, for any pair (s, t)

$$\gamma(s,t) = \mathbb{E}\left(\psi_s \middle| \mathcal{F}_t\right) .$$

We have for any T and t :

$$S_t(T) = \int_T^\infty \gamma(s,t) ds = \int_T^\infty \alpha_t(s) \eta(ds)$$

where $\eta([0,t]) = \mathbb{P}(\tau \leq t)$ is the law of the variable τ and $\alpha_t(s) = \gamma(s,t)/\gamma(s,0)$.

Two default times

Let τ_1, τ_2 two default times and $\sigma_1 < \sigma_2$ the ordered defaults. We suppose that the conditional probability admits a density

$$\mathbb{P}(\sigma_1 > u_1, \sigma_2 > u_2 | \mathcal{F}_t) = \int_{u_1}^{\infty} dv_1 \int_{u_2}^{\infty} dv_2 \, p_t(v_1, v_2)$$

In particular

$$\int_u^{\infty} p_t(u, v) dv = \alpha_t^{(1)}(u)$$

where $\alpha_t^{(1)}(u)$ is the density of $\mathbb{P}(\sigma_1 > u | \mathcal{F}_t)$.

Let $H_t^1 = 1\!\!1_{\sigma_1 \leq t}$ and $\mathbf{H}^1$ its natural filtration. Then

$$\mathbb{P}(\sigma_2 > t | \mathcal{F}_t \vee \mathcal{H}_t^1) = 1\!\!1_{\sigma_1 > t} + 1\!\!1_{\sigma_1 < t} \frac{\int_t^\infty p_t(\sigma_1, v) dv}{\alpha_t^{(1)}(\sigma_1)}$$

The process $\mathbb{P}(\sigma^2 > \theta | \mathcal{F}_t \vee \mathcal{H}_t^1)$ admits a density:

$$\mathbb{P}(\sigma^2 > \theta | \mathcal{F}_t \vee \mathcal{H}_t^1) = \int_\theta^\infty \alpha_t^{(2)}(s) ds$$

where

$$\alpha_t^{(2)}(\theta) = 1\!\!1_{\sigma_1 > t} \frac{\int_t^\infty du\, p_t(u,\theta)}{\int_t^\infty du \int_u^\infty dv p_t(u,v)} + 1\!\!1_{\sigma_1 \leq t} \frac{p_t(\sigma_1,\theta)}{\alpha_t^{(1)}(\sigma_1)}$$

The intensity processes can be computed: the process $\mathbb{1}_{\sigma_2 \le t} - \int_0^{t\wedge\sigma_2} \lambda_s^{(2)} ds$ where

$$\lambda_t^{(2)} = \mathbb{1}_{[\sigma_1,\sigma_2]}(t) \frac{p_t(\sigma_1, t)}{\int_t^\infty p_t(\sigma_1, v) dv}$$

is a martingale

As a general result, we get, for $\mathbf{G} = \mathbf{H} \vee \mathbf{H}^1 \vee \mathbf{H}^2$

$$\mathbb{E}(Xh(\sigma_1,\sigma_2)|\mathcal{G}_t)1\!\!1_{t<\sigma_1} = 1\!\!1_{t<\sigma_1} \frac{\mathbb{E}(\int_t^\infty du_1 \int_{u_1}^\infty du_2 X h(u_1,u_2) p_T(u_1,u_2)|\mathcal{F}_t)}{\int_t^\infty du_1 \int_t^\infty du_2\, p_t(u_1,u_2)}$$

$$\mathbb{E}(Xh(\sigma_1,\sigma_2)|\mathcal{G}_t)1\!\!1_{\sigma_1\le t<\sigma_2} = 1\!\!1_{\sigma_1\le t<\sigma_2} \frac{\mathbb{E}(\int_{u_1\vee t}^\infty du_2 X h(u_1,u_2) p_T(u_1,u_2)|\mathcal{F}_t)}{\int_{t\vee u_1}^\infty du_2\, p_t(u_1,u_2)}\Big|_{u_1=\sigma_1}$$

$$\mathbb{E}(Xh(\sigma_1,\sigma_2)|\mathcal{G}_t)1\!\!1_{\sigma_2\le t} = 1\!\!1_{\sigma_2\le t} \frac{\mathbb{E}(X h(u_1,u_2) p_T(u_1,u_2)|\mathcal{F}_t)}{p_t(u_1,u_2)}\Big|_{u_1=\sigma_1,u_2=\sigma_2}$$

Enlargement of filtration formula

In general, τ is not an honest time. However, it is possible to prove that any **F**-martingale is a **G**-semi-martingale. If X is a **F** martingale, if S admits a density, then

$$\begin{aligned} X_t &= \widetilde{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X, M^\tau\rangle_s}{S_{s-}} + \int_{t\wedge\tau}^t g(\tau, dv)\,, \\ &= \widetilde{X}_t + \int_0^{t\wedge\tau} \int_s^\infty \eta(dv) \frac{d\left\langle X, \alpha^v\right\rangle_s}{S_{s-}} + \int_{t\wedge\tau}^t g(\tau, dv) \end{aligned}$$

where $\widetilde{X}$ is a **G**-martingale and

$$g(u, dv) = \frac{d\langle \alpha.(u), X\rangle_v}{\alpha_v(u)}$$

Multiplicative decomposition

The super-martingale $(S_t, t \geq 0)$ admits a multiplicative decomposition as

$$S_t = L_t D_t$$

where L is a local martingale and D a predictable non-increasing process.

From $S_t = M_t^\tau - A_t^\tau$ and $S_t = L_t D_t$ it follows from Yoeurp theorem that

$$dS_t = L_{t-} dD_t + D_t dL_t$$

hence

$$dD_t = -\frac{1}{L_{t-}} dA_t^\tau = -D_{t-} \frac{dA_t^\tau}{S_{t-}} = -D_{t-}\, d\Lambda_t$$

and $D_t = \mathcal{E}(-\Lambda)_t$. If Λ is continuous $D_t = \exp(-\Lambda_t)$.

Of course, if immersion property holds, $L \equiv 1$.

The price of a DZC with maturity T is (we assume null interest rate)

$$D(t,T) = \mathbb{P}(\tau > T|\mathcal{G}_t) = 1\!\!1_{\tau>t}\frac{1}{S_t}\mathbb{E}(S_T|\mathcal{F}_t) \overset{def}{=} 1\!\!1_{\tau>t}\widetilde{D}(t,T)$$

Using the multiplicative decomposition $S_t = D_t L_t$, and assuming that $(L_t, t \leq T)$ is a martingale, the price of the defaultable payoff $X1\!\!1_{T<\tau}$ is

$$E(X1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{t<\tau}\frac{1}{S_t}\mathbb{E}(XS_T|\mathcal{F}_t) = 1\!\!1_{t<\tau}\frac{1}{D_t}\widehat{\mathbb{E}}(XD_T|\mathcal{F}_t)$$

where $d\widehat{P}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$

To end the talk, the sentence

> *Le livre n'est pas terminé.*
> *La fin n'a pas été écrite, elle n'a jamais été trouvée.*
> *M. Duras, L'été 80.*

has to be changed into

> *Le* travail *n'est pas terminé.*
> *La fin n'a pas été écrite, elle n'a* pas encore *été trouvée.*