
Enlargement of filtration and Credit risk

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Credit Risk Model

- A reference **filtration** \mathbf{F} and **random time** τ are given
- \mathbf{H} is the filtration generated by the process $H_t = \mathbb{1}_{\tau \leq t}$
- $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$

The filtration \mathbf{G} enjoys the Minimal assumption: if $A \in \mathcal{G}_t$, there exists $\tilde{A} \in \mathcal{F}_t$ such that

$$A \cap \{t < \tau\} = \tilde{A} \cap \{t < \tau\}$$

It follows that, if $(Y_t, t \geq 0)$ is a \mathbf{G} -adapted process, there exists an \mathbf{F} -adapted process $(\tilde{Y}_t, t \geq 0)$ such that

$$Y_t \mathbf{1}_{t < \tau} = \tilde{Y}_t \mathbf{1}_{t < \tau}.$$

The survival hazard process is denoted by S :

$$\mathbb{P}(\tau > t | \mathcal{F}_t) = S_t$$

We obtain (we assume $S_t > 0$):

for any integrable \mathcal{F}_T measurable r.v. X

$$\mathbb{E}(\mathbf{1}_{\{T < \tau\}} X | \mathcal{G}_t) = \mathbf{1}_{\{t < \tau\}} \frac{1}{S_t} \mathbb{E}(S_T X | \mathcal{F}_t).$$

Immersion property

A particular case is when \mathbf{F} is immersed in \mathbf{G} that is when \mathbf{F} -martingales are \mathbf{G} -martingales. This condition is equivalent to

$$\mathbb{P}(\tau \leq t | \mathcal{F}_t) = \mathbb{P}(\tau \leq t | \mathcal{F}_\infty) = \mathbb{P}(\tau \leq t | \mathcal{F}_s), \forall t, \forall s, t \leq s.$$

In that case, the process S is non-increasing

Examples

- **COX process model** Let λ be an \mathbf{F} -adapted non-negative process, Θ a random variable with exponential law, independent of \mathcal{F}_∞ and

$$\tau \stackrel{def}{=} \inf\left\{t : \int_0^t \lambda_s ds \geq \Theta\right\}$$

In that case,

$$S_t = \exp\left(-\int_0^t \lambda_u du\right),$$

and immersion property holds. In particular,

$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \exp\left(-\int_0^s \lambda_u du\right) = S_t, \quad \forall s < t.$$

Imperfect information

Partial information Assume that

$$dV_t = V_t(\mu dt + \sigma dW_t), \quad V_0 = v$$

and let, for $\alpha < v$

$$\tau = \inf\{t : V_t \leq \alpha\}$$

The reference filtration \mathbf{F} is the filtration of the observations of V at discrete times t_1, \dots, t_n where $t_n \leq t < t_{n+1}$, i.e.,

$$\mathcal{F}_t = \sigma(V_{t_1}, \dots, V_{t_n}, t_i \leq t)$$

The process $S_t = P(\tau > t | \mathcal{F}_t)$ is **continuous and decreasing in**
 $[t_i, t_{i+1}[$

For $t_1 < t < t_2$ and $X_{t_1} > a$

$$S_t = \Phi(t - t_1, a - X_{t_1}) \left[1 - \exp \left(-\frac{2a}{t_1} (a - X_{t_1}) \right) \right]$$

where, setting $\nu = \frac{1}{\sigma} \left(r - \frac{\sigma^2}{2} \right)$,

$$\begin{aligned} \Phi(t, z) &= \mathcal{N} \left(\frac{\nu t - z}{\sqrt{t}} \right) - e^{2\nu z} \mathcal{N} \left(\frac{z + \nu t}{\sqrt{t}} \right), & \text{for } z < 0, t > 0, \\ &= 0, & \text{for } z \geq 0, t \geq 0, \end{aligned}$$

$$\Phi(0, z) = 1, \quad \text{for } z < 0$$

However, the process S is **not a decreasing process**: the jump at time t_1 is

$$\Delta S_{t_1} = -\exp \left[-\frac{2}{t_1} a(a - X_{t_1}) \right] - 1 + \Phi(t_1, a).$$

Example of Imperfect information Let

$$\tau = \inf \{t, X_t \leq b(t)\}.$$

where $X_t = \Psi(B_t, t)$ where B is a Brownian motion and $\Psi(\cdot, t)$ is invertible. The filtration \mathbf{F} is the natural filtration of the observed process is Y

$$dY_t = \mu_1(Y_t, t) dt + \sigma(Y_t, t) dB_t + s(Y_t, t) dW_t$$

$$Y_0 = y_0.$$

It can be proved that \mathbf{F} is immersed in $\mathbf{F} \vee \mathbf{H}$ and numerical approximation of the process S are done.

Martingale density

The process $(S_t, t \geq 0)$ is a \mathbf{F} -submartingale and admits a Doob-Meyer decomposition

$$S_t = M_t^\tau - A_t^\tau$$

where M^τ is an \mathbf{F} -martingale, and A^τ is an \mathbf{F} -predictable increasing process.

Moreover, the process

$$H_t - \int_0^{t \wedge \tau} \frac{dA_s^\tau}{1 - S_{s-}}$$

is a \mathbf{G} -martingale.

If \mathbf{F} is \mathbb{P} -immersed in \mathbf{G} , then the process S is decreasing, and $M_t^\tau \equiv 1$.

In that case

$$H_t - \Lambda_{t \wedge \tau}$$

is a martingale where $\Lambda_t = -\ln S_t$.

For any T , the process $S_t(T) \stackrel{def}{=} \mathbb{P}(\tau > T | \mathcal{F}_t)$ is an \mathbf{F} -martingale. We assume that there exists a family of processes $(\alpha_t(u), t \geq 0)$ such that

$$S_t(T) = \int_T^\infty \alpha_t(u) du$$

For any u , the process $(\alpha_t(u), t \geq 0)$ is an \mathbf{F} -martingale, and can be interpreted as

$$\alpha_t(u) du = \mathbb{P}(\tau \in du | \mathcal{F}_t)$$

In the case where \mathbf{F} is a Brownian filtration generated by W

$$d_t\alpha_t(u) = \psi_t(u)dW_t$$

which, from Ventzell theorem, implies that

$$dS_t = \Psi_t dW_t - \alpha_t(t)dt$$

where $\Psi_t = \int_t^\infty \psi_t(u)du$.

This equality gives the Doob-Meyer decomposition of S . In particular, we obtain that

$$H_t - \int_0^{t \wedge \tau} \frac{\alpha_s(s)}{S_s} ds$$

is a \mathbf{G} -martingale.

In fact, this last result is a general one.

We can now extend the previous result. Let $X \in \mathcal{F}_T$ and $T < t$. Then,

$$\mathbb{E}(Xh(T \wedge \tau)|\mathcal{G}_t)\mathbb{1}_{\{\tau>t\}} = \frac{\mathbb{E}\left[\int_t^\infty Xh(u \wedge T)\alpha_T(u)du|\mathcal{F}_t\right]}{S_t}\mathbb{1}_{\{\tau>t\}}$$

$$\mathbb{E}[Xh(T \wedge \tau)|\mathcal{G}_t]\mathbb{1}_{\{\tau\leq t\}} = \mathbb{E}\left[Xh(s)\frac{\alpha_T(s)}{\alpha_t(s)}|\mathcal{F}_t\right]\Big|_{s=\tau}\mathbb{1}_{\{\tau\leq t\}}$$

Intuitively, with $Y = Xh(T \wedge \tau)$

$$E(Y|\mathcal{G}_t)\mathbf{1}_{\{\tau \leq t\}} = \mathbf{1}_{\{\tau \leq t\}}f_t(\tau)$$

where $f_t(u)$ is a family of \mathbf{F} adapted processes. It follows that, for $u < t$

$$\begin{aligned} f_t(u) &= \mathbb{E}(Y|\mathcal{F}_t, \tau = u) = \frac{\mathbb{E}(Y \mathbf{1}_{\tau \in du}|\mathcal{F}_t)}{\mathbb{P}(\tau \in du|\mathcal{F}_t)} \\ &= \frac{\mathbb{E}(Xh(u) \mathbf{1}_{\tau \in du}|\mathcal{F}_t)}{\mathbb{P}(\tau \in du|\mathcal{F}_t)} = \frac{\mathbb{E}(Xh(u)\mathbb{P}(\tau \in du|\mathcal{F}_T)|\mathcal{F}_t)}{\mathbb{P}(\tau \in du|\mathcal{F}_t)} \end{aligned}$$

More generally, if $Y(t, u)$ is a family of \mathbf{F} -adapted processes, then

$$\mathbb{E}(Y(T, \tau) | \mathcal{G}_t) \mathbb{1}_{\{\tau > t\}} = \frac{\mathbb{E} \left[\int_t^\infty Y(T, u) \alpha_T(u) du | \mathcal{F}_t \right]}{S_t} \mathbb{1}_{\{\tau > t\}}$$

$$\mathbb{E}[Y(T, \tau) | \mathcal{G}_t] \mathbb{1}_{\{\tau \leq t\}} = \mathbb{E} \left[Y(T, T) \frac{\alpha_T(s)}{\alpha_t(s)} \middle| \mathcal{F}_t \right] \Big|_{s=\tau} \mathbb{1}_{\{\tau \leq t\}}$$

Immersion property

If \mathbf{F} is immersed in \mathbf{G} , from

$$\mathbb{P}(\tau \leq t | \mathcal{F}_t) = \mathbb{P}(\tau \leq t | \mathcal{F}_s), \forall t, \forall s, t \leq s$$

we obtain $\alpha_t(s) = \alpha_t(t)$, $s \leq t$ and S is a decreasing process. Let $Y(T, \tau)$ be a random variable as above. Then we have

$$\mathbb{E}[Y(T, \tau) | \mathcal{G}_t] \mathbb{1}_{\{\tau \leq t\}} = \mathbb{E}[Y(T, s) | \mathcal{F}_t] \Big|_{s=\tau} \mathbb{1}_{\{\tau \leq t\}}.$$

Examples

- **COX process model**

In the Cox process case, from $\mathbb{P}(\tau > s | \mathcal{F}_t) = \exp - \int_0^s \lambda_u du$, for $s < t$, we obtain

$$\alpha_t(s) = \lambda_s \exp \left(- \int_0^s \lambda_u du \right), \text{ for } s < t$$

For $T > t$,

$$\mathbb{E}(\exp(-\int_t^T \lambda_u du) | \mathcal{F}_t) = \exp(-\int_t^T \delta(t, s) ds)$$

where $\delta(t, s), t \geq 0$ is a family of \mathbf{F} adapted processes defined for $s > t$ as

$$\mathbb{E}(\lambda_s \exp(-\int_t^s \lambda_u du) | \mathcal{F}_t) = \delta(t, s) \mathbb{E}(\exp(-\int_t^s \lambda_u du) | \mathcal{F}_t)$$

It follows that

$$\alpha_t(s) = \delta(t, s) \exp(-\int_0^t \lambda_u du - \int_t^s \delta(t, u) du), s > t$$

Information at discrete times Assume that

$$dV_t = V_t(\mu dt + \sigma dW_t), \quad V_0 = v$$

i.e., $V_t = ve^{\sigma(W_t + \nu t)} = ve^{\sigma X_t}$. The default time is assumed to be the first hitting time of α with $\alpha < v$, i.e.,

$$\tau = \inf\{t : V_t \leq \alpha\} = \inf\{t : X_t \leq a\}$$

where $a = \sigma^{-1} \ln(\alpha/v)$.

Here, \mathbf{F} is the filtration of the observations of V at discrete times t_1, \dots, t_n where $t_n \leq t < t_{n+1}$, i.e.,

$$\mathcal{F}_t = \sigma(V_{t_1}, \dots, V_{t_n}, t_i \leq t)$$

The process $F_t = P(\tau \leq t | \mathcal{F}_t)$ is **continuous and increasing in $[t_i, t_{i+1}[$** but is **not increasing**.

The Doob-Meyer decomposition of F is

$$F_t = \zeta_t + (F_t - \zeta_t),$$

where

$$\zeta_t = \sum_{i, t_i \leq t} \Delta F_{t_i}.$$

is an \mathbf{F} -martingale and $F_t - \zeta_t$ is a predictable increasing process.

From

$$P(\inf_{s \leq t} X_s > z) = \Phi(\nu, t, z),$$

where

$$\begin{aligned} \Phi(\nu, t, z) &= \mathcal{N}\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z} \mathcal{N}\left(\frac{z + \nu t}{\sqrt{t}}\right), & \text{for } z < 0, t > 0, \\ &= 0, & \text{for } z \geq 0, t \geq 0, \\ \Phi(\nu, 0, z) &= 1, & \text{for } z < 0 \end{aligned}$$

we obtain (we skip the parameter ν in the definition of Φ) for $t_1 < t < t_2$ and $X_{t_1} > a$

$$S_t = \Phi(t - t_1, a - X_{t_1}) \left[1 - \exp\left(-\frac{2a}{t_1} (a - X_{t_1})\right) \right].$$

The case $X_{t_1} \leq a$ corresponds to default: for $X_{t_1} \leq a$, $F_t = 1$.

Multiplicative decomposition

The submartingale $(S_t, t \geq 0)$ admits a multiplicative decomposition as

$$S_t = L_t D_t$$

where L is a martingale and D a predictable non-increasing process.

From $S_t = M_t^\tau - A_t^\tau$ and $S_t = L_t D_t$ it follows that

$$dS_t = L_{t-} dD_t + D_t dL_t$$

hence

$$dD_t = -\frac{1}{L_{t-}} dA_t^\tau = -D_{t-} \frac{dA_t^\tau}{S_{t-}} = -D_{t-} d\Lambda_t$$

and $D_t = \mathcal{E}(-\Lambda)_t$. If Λ is continuous $D_t = \exp(-\Lambda_t)$.

Of course, if immersion property holds, $L \equiv 1$.

HJM

The price of a DZC with maturity T is (we assume null interest rate)

$$D(t, T) = \mathbb{P}(\tau > T | \mathcal{G}_t) = \mathbf{1}_{\tau > t} \frac{1}{S_t} \mathbb{E}(S_T | \mathcal{F}_t) \stackrel{def}{=} \mathbf{1}_{\tau > t} \tilde{D}(t, T)$$

Note that $(\mathbf{1}_{\tau > t} \tilde{D}(t, T), t \geq 0)$ is a \mathbf{G} martingale.

Using the multiplicative decomposition $S_t = D_t L_t$,

$$D(t, T) = \mathbf{1}_{\tau > t} \frac{1}{D_t} \hat{\mathbb{E}}(D_T | \mathcal{F}_t) = \mathbf{1}_{\tau > t} \tilde{D}(t, T)$$

where $d\hat{\mathbb{P}}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$.

The process $D_t \tilde{D}(t, T)$ is a $\hat{\mathbb{Q}}, \mathbf{F}$ -martingale

If immersion property holds, $\hat{\mathbb{P}} = \mathbb{P}$.

We assume that there exists a family of \mathbf{F} -adapted processes $\delta(t, s), t \geq 0$ such that

$$\tilde{D}(t, T) = \exp - \int_t^T \delta(t, s) ds$$

Then $\exp \left(- \int_0^t \lambda_s ds - \int_t^T \delta(t, s) ds \right)$ is an \mathbf{F} martingale

If \mathbf{F} is a Brownian filtration (or a filtration which enjoys the PRT)

The price of the defaultable payoff $X \mathbb{1}_{T < \tau}$ is

$$E(X \mathbb{1}_{T < \tau} | \mathcal{G}_t) = \mathbb{1}_{t < \tau} \frac{1}{S_t} \mathbb{E}(X S_T | \mathcal{F}_t) = \mathbb{1}_{t < \tau} \frac{1}{D_t} \widehat{E}(X D_T | \mathcal{F}_t)$$

Let $\widehat{Q}_T | \mathcal{F}_t = D_t \frac{\widetilde{D}(t, T)}{\widetilde{D}(0, T)} \widehat{Q} | \mathcal{F}_t$. Then

$$E(X \mathbb{1}_{T < \tau} | \mathcal{G}_t) = \mathbb{1}_{t < \tau} \frac{1}{\widetilde{D}(t, T)} \mathbb{E}_{\widetilde{Q}_T}(X | \mathcal{F}_t)$$

An example with immersion

Assume that \mathbf{F} is \mathbb{P} -immersed in \mathbf{G} and that the increasing process S_t (hence $\Lambda = -\ln S$) is absolutely continuous w.r.t. Lebesgue measure.

Then, for $s < t$,

$$\mathbb{P}(\tau > s | \mathcal{F}_t) = S_s = \exp \left(- \int_0^s \lambda_u du \right).$$

With the notation of the previous section, for any $T > t$, we have

$$S(t; T) := \mathbb{P}(\tau > T | \mathcal{F}_t) = \exp \left(- \int_0^T (\lambda_s \mathbb{1}_{\{s \leq t\}} + \delta(t, s) \mathbb{1}_{\{s > t\}}) ds \right).$$

The conditional probability $\mathbb{P}(\tau > \theta | \mathcal{F}_t)$ admits a density $\alpha_t(\theta)$ given by

$$\alpha_t(\theta) = \begin{cases} \lambda_\theta \exp \left(- \int_0^\theta \lambda_u du \right), & \theta \leq t \\ \delta(t, \theta) \exp \left(- \int_0^t \lambda_u du - \int_t^\theta \delta(t, u) du \right), & \theta > t \end{cases}$$

such that $\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty \alpha_t(u) du$. The density $\alpha_t(\theta)$ does not

depend on t for $t \geq \theta$ when the (H)-hypothesis holds, which simplifies the calculation.

Using the density $\alpha_t(\theta)$ allows us to calculate the \mathbb{G} -conditional expectations.

Two default times

Let τ_1, τ_2 two default times and $\sigma_1 < \sigma_2$ the ordered defaults. We suppose that the conditional probability

$$\mathbb{P}(\sigma_1 > u_1, \sigma_2 > u_2 | \mathcal{F}_t) = \int_{u_1}^{\infty} dv_1 \int_{u_2}^{\infty} dv_2 p_t(v_1, v_2)$$

In particular

$$\int_u^{\infty} p_t(u, v) dv = \alpha_t^1(u)$$

where $\alpha_t^1(u)$ is the density of $\mathbb{P}(\sigma^1 > u | \mathcal{F}_t)$

Let $H_t^1 = \mathbb{1}_{\sigma^1 \leq t}$ and \mathbf{H}^1 its natural filtration. Set

$S_t^{2,1} = \mathbb{P}(\sigma_2 > t | \mathcal{F}_t \vee \mathcal{H}_t^1)$ Then

$$S_t^{2,1} = \mathbb{1}_{\sigma_1 > t} + \mathbb{1}_{\sigma_1 < t} \frac{\int_t^{\infty} p_t(\sigma_1, v) dv}{\alpha_t^1(\sigma^1)}$$

The process $\mathbb{P}(\sigma^2 > \theta | \mathcal{F}_t \vee \mathcal{H}_t^1)$ admits a density

$$\alpha_t^2(\theta) = \mathbb{1}_{\sigma_1 > t} \frac{\int_t^\infty du p_t(u, \theta)}{\int_t^\infty du \int_u^\infty dv p_t(u, v)} + \mathbb{1}_{\sigma_1 \leq t} \frac{p_t(\sigma_1, \theta)}{\alpha_t^1(\sigma_1)}$$

The process $H_t^2 - \Lambda_{t \wedge \sigma_2}^2$ where $\lambda_t^2 = \mathbb{1}_{[\sigma^1, \sigma^2]}(t) \frac{p_t(\sigma_1, t)}{\int_t^\infty p_t(\sigma_1, v) dv}$ is a martingale