Density models for credit risk

Nicole El Karoui, Ecole Polytechnique, France

Monique Jeanblanc, Université d'Évry; Institut Europlace de Finance

Ying Jiao, Université Paris VII

Workshop on Recent Advances in Risk Management and Numerical Methods in Finance

Tunis, September 24-25, 2008

Financial support from La Fondation du Risque and Fédération Bancaire Française

Density Hypothesis

Let $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ be a filtered probability space.

A strictly positive and finite random variable τ (the default time) is given.

Our goals are

- to show how the information contained in the reference filtration $\mathbb{F}$ can be used to obtain information on the law of τ,
- to investigate the links between martingales in the different filtrations that will appear.

We assume the following **density hypothesis:** **there exists a non-negative measure η such that, for any time t, there exists an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(\omega, \theta) \to \alpha_t(\theta, \omega)$ which satisfies**

$$\mathbb{P}(\tau \in d\theta | \mathcal{F}_t) = \alpha_t(\theta)\eta(d\theta), \quad \mathbb{P} - a.s.$$

The conditional distribution of τ is characterized by the survival probability defined by

$$S_t(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty \alpha_t(u)\eta(du)$$

Let

$$S_t := S_t(t) = \mathbb{P}(\tau > t | \mathcal{F}_t) = \int_t^\infty \alpha_t(u)\eta(du)$$

Observe that the set $A_t := \{S_t > 0\}$ contains a.s. the event $\{\tau > t\}$.

The family $\alpha_t(.)$ is called the **conditional density** of τ w.r.t. η given $\mathcal{F}_t$.

Note that

- $S_t(\theta) = \mathbb{E}(S_\theta | \mathcal{F}_t)$ for any $\theta \geq t$
- the law of τ is $\mathbb{P}(\tau > \theta) = \int_\theta^\infty \alpha_0(u)\eta(du)$
- for any t, $\int_0^\infty \alpha_t(u)\eta(du) = 1$
- for any bounded Borel function f,

$$\mathbb{E}[f(\tau)|\mathcal{F}_t] = \int_0^\infty f(u)\alpha_t(u)\eta(du)$$

- we exclude $\mathbb{F}$-stopping times

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- we exclude $\mathbb{F}$-stopping times

For an integrable $\mathcal{F}_T \otimes \sigma(\tau)$ r.v. $Y_T(\tau)$, one has, for $t \leq T$:

$$\mathbb{E}(Y_T(\tau)|\mathcal{F}_t) = \mathbb{E}(\int_0^\infty Y_T(u)\alpha_T(u)\eta(du)|\mathcal{F}_t)$$

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The default time τ avoids $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \vartheta) = 0$ for every $\mathbb{F}$-stopping time ϑ.

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By using the density, we adopt an **additive** point of view to represent the conditional probability of τ

$$S_t(\theta) = \int_\theta^\infty \alpha_t(u)\eta(du)$$

In the default framework, the "intensity" point of view is often preferred, and one uses a **multiplicative** representation as

$$S_t(\theta) = \exp(-\int_0^\theta \lambda_t(u)\eta(du))$$

where $\lambda_t(u) = -\partial_u \ln S_t(u)$ is the "forward intensity".

Computation of conditional expectations

Let $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ be the smallest right-continuous filtration such that τ is a $\mathbb{D}$-stopping time, and let $\mathbb{G} = \mathbb{F} \vee \mathbb{D}$.

Any $\mathcal{G}_t$-measurable r.v. $H_t^{\mathbb{G}}$ may be represented as

$$H_t^{\mathbb{G}} = H_t^{\mathbb{F}} 1_{\{\tau > t\}} + H_t(\tau) 1_{\{\tau \leq t\}}$$

where $H_t^{\mathbb{F}}$ is an $\mathcal{F}_t$-measurable random variable and $H_t(\tau)$ is $\mathcal{F}_t \otimes \sigma(\tau)$ measurable. In particular,

$$H_t^{\mathbb{G}} 1_{\{\tau > t\}} = H_t^{\mathbb{F}} 1_{\{\tau > t\}},$$

where the random variable $H_t^{\mathbb{F}}$ is the $\mathcal{F}_t$-conditional expectation of $H_t^{\mathbb{G}}$ given the event $\{\tau > t\}$, i.e.,

$$H_t^{\mathbb{F}} = \frac{\mathbb{E}[H_t^{\mathbb{G}} 1_{\{\tau > t\}} | \mathcal{F}_t]}{\mathbb{P}(\tau > t | \mathcal{F}_t)} = \frac{\mathbb{E}[H_t^{\mathbb{G}} 1_{\{\tau > t\}} | \mathcal{F}_t]}{S_t} \quad a.s. \text{ on } A_t; \quad H_t^{\mathbb{G}} = 0 \quad \text{if not}.$$

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The $\mathcal{G}_t$-conditional expectation of an integrable $\sigma(\tau)$-measurable r.v. is given by

$$\alpha_t^{\mathbb{G}}(f) := \mathbb{E}[f(\tau)|\mathcal{G}_t] = \alpha_t^{\mathrm{bd}}(f)\,1_{\{\tau>t\}} + f(\tau)\,1_{\{\tau\le t\}}$$

where

$$\alpha_t^{\mathrm{bd}}(f) := \frac{\mathbb{E}[f(\tau)1_{\{\tau>t\}}|\mathcal{F}_t]}{\mathbb{P}(\tau > t|\mathcal{F}_t)} \quad a.s.\text{ on } A_t; \quad \alpha_t^{\mathrm{bd}}(f) := 0 \quad \text{if not.}$$

The existence of the density allows us to have the representation

$$\alpha_t^{\mathrm{bd}}(f) = \frac{\int_t^\infty f(u)\alpha_t(u)\eta(du)}{S_t} \quad a.s.\text{ on } A_t\,.$$

The $\mathcal{G}_t$-conditional expectation of an integrable $\mathcal{F}_t \otimes \sigma(\tau)$-measurable r.v. $Y_t(\tau)$, is given by

$$\mathbb{E}[Y_t(\tau)|\mathcal{G}_t] = \alpha_t^{\mathrm{bd}}(Y_t)\, 1_{\{\tau > t\}} + Y_t(\tau)\, 1_{\{\tau \leq t\}}$$

where, on A_t

$$\alpha_t^{\mathrm{bd}}(Y_t) := \frac{\mathbb{E}[Y_t(\tau) 1_{\{\tau > t\}} | \mathcal{F}_t]}{S_t} = \frac{1}{S_t} \int_t^{\infty} Y_t(u) \alpha_t(u) \eta(du) \quad a.s..$$

The $\mathcal{G}_t$-conditional expectation of an integrable $\mathcal{F}_T \otimes \sigma(\tau)$-measurable r.v. $Y_t(\tau)$, is given by

$$E[Y_T(\tau)|\mathcal{G}_t] = Y_t^{\mathrm{bd}} 1_{\{t<\tau\}} + Y_t^{\mathrm{ad}}(T,\tau) 1_{\{\tau \le t\}} \quad d\mathbb{P} - a.s..$$

where

$$\begin{aligned} Y_t^{\mathrm{bd}} &= \frac{\mathbb{E}\left[\int_t^\infty Y_T(u)\alpha_T(u)\eta(du)|\mathcal{F}_t\right]}{S_t} \\ Y_t^{\mathrm{ad}}(T,\theta) &= \frac{\mathbb{E}\left[Y_T(\theta)\alpha_T(\theta)\middle|\mathcal{F}_t\right]}{\alpha_t(\theta)} 1_{\{\alpha_t(\theta)>0\}} \quad d\mathbb{P} - a.s.. \end{aligned}$$

Proof: By definition of $\mathbb{G}$, any $\mathcal{G}_t$-measurable r.v. can be written on the set $\{\tau \leq t\}$ as $H_t(\tau)1_{\{\tau\leq t\}}$. Assume that $H_t(\tau)$ is positive or bounded. Using the density $\alpha_t(\theta)$, we obtain

$$\begin{aligned}
\mathbb{E}[H_t(\tau)1_{\{\tau\leq t\}}Y_T(\tau)] &= \int d\theta\, \mathbb{E}[H_t(\theta)1_{\{\theta\leq t\}}Y_T(\theta)\alpha_T(\theta)] \\
&= \int d\theta\, \mathbb{E}\big[H_t(\theta)1_{\{\theta\leq t\}}\mathbb{E}[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t]\big] \\
&= \int d\theta\, \mathbb{E}\Big[H_t(\theta)1_{\{\theta\leq t\}}Y_t^{\mathrm{ad}}(T,\theta)\alpha_t(\theta)\Big] \\
&= \mathbb{E}\Big[H_t(\tau)1_{\{\tau\leq t\}}Y_t^{\mathrm{ad}}(T,\tau)\Big],
\end{aligned}$$

Immersion property

In the particular case where

$$\alpha_t(\theta) = \alpha_\theta(\theta), \quad \forall \theta \leq t$$

one has

$$S_t = 1 - \int_0^t \alpha_t(\theta)\eta(d\theta) = 1 - \int_0^t \alpha_T(\theta)\eta(d\theta) = \mathbb{P}(\tau > t|\mathcal{F}_T) \; a.s.$$

for any $T \geq t$ and $\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$. This last equality is equivalent to the immersion property.

Conversely, if immersion property holds, then

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$$

hence, the process S is decreasing and the conditional survival functions $S_t(\theta)$ are constant in time on $[\theta, \infty)$, i.e., $S_t(\theta) = S_\theta(\theta)$ for $t > \theta$.

Dynamic point of view and density process

Regular Version of Martingales

One of the major difficulties is to prove the existence of a universal càdlàg martingale version of this family of densities. Fortunately, results of Jacod or Stricker and Yor help us to solve this technical problem.

Results on Enlargement of Filtration

We assume that $S_t := S_t(t)$ is continuous.

If X is an $\mathbb{F}$-martingale

$$Y_t = X_t - \int_0^{t\wedge\tau} \frac{d\langle X, S\rangle_s}{S_s} - \int_{t\wedge\tau}^{t} \frac{d\langle X, f_\cdot(\theta;\cdot)\rangle_s}{f(\theta;s)}\bigg|_{\theta=\tau} \in \mathcal{M}(\mathbb{G}).$$

Proof. We prove that $\widehat{X}$ is a $\mathbb{G}$-martingale. Let us consider a $\mathcal{G}_s$-measurable random variable of the form $F_s h(\tau \wedge s)$ with F_s a bounded $\mathcal{F}_s$-measurable random variable and $h : \mathbb{R}^+ \to \mathbb{R}$ a bounded Borel function. Then,

$$\begin{aligned}
\mathbb{E}\left(F_s h(\tau \wedge s)\left(\widehat{X}_t - \widehat{X}_s\right)\right) &= \mathbb{E}\left(F_s h(\tau) 1_{\tau \le s}\left(\widehat{X}_t - \widehat{X}_s\right)\right) \\
&\quad + \mathbb{E}\left(F_s h(s) 1_{s < \tau}\left(\widehat{X}_t - \widehat{X}_s\right)\right) \\
&= a + b
\end{aligned}$$

and we can compute each part of the right hand side member:

Computation of $a = \mathbb{E}\left(F_s h(\tau) 1_{\tau \le s}\left(\widehat{X}_t - \widehat{X}_s\right)\right), \quad s < t$**.**

On $\{\tau \le s\}$, $t \wedge \tau = s \wedge \tau = \tau$ hence

$$1_{\tau \le s}\left(\int_0^{t\wedge\tau} \frac{d\langle X, S\rangle_u}{S_u} - \int_0^{s\wedge\tau} \frac{d\langle X, S\rangle_u}{S_u}\right) = 0,$$

and it follows that

$$a = \mathbb{E}\left(F_s h(\tau) 1_{\tau \le s}\left(X_t - X_s\right)\right) - \mathbb{E}\left(F_s h(\tau) 1_{\tau \le s}\left(\int_s^t \frac{d\langle X, \alpha_.(\theta)\rangle_u}{\alpha_u(\theta)}\Big|_{\theta=\tau}\right)\right)$$

$\widehat{X}_t = X_t - \int_0^{t\wedge\tau} \frac{d\langle X,S\rangle_s}{S_s} - \int_{t\wedge\tau}^t \frac{d\langle X,\alpha_.(\theta)\rangle_s}{\alpha_s(\theta)}\Big|$

We prove that $a = 0$

$$
\begin{aligned}
\mathbb{E}\left(F_s h(\tau) 1_{\tau \le s} \left(X_t - X_s\right)\right) &= \mathbb{E}\left(F_s \left(X_t - X_s\right) \int_0^s h(\theta)\alpha_t(\theta)\eta(d\theta)\right) \\
&= \int_0^s h(\theta)\mathbb{E}\left(F_s\left(X_t\alpha_t(\theta) - X_s\alpha_s(\theta)\right)\right)\eta(d\theta) \\
&= \int_0^s h(\theta)\mathbb{E}\left(F_s \int_s^t d\left\langle X, \alpha.(\theta)\right\rangle_v\right)\eta(d\theta)
\end{aligned}
$$

where the first equality comes from a conditioning w.r.t. $\mathcal{F}_t$, the second from the martingale property of $\alpha.(\theta)$, and the third from integration by parts and the fact that X and $\alpha.(\theta)$ are martingales.

$$a = \mathbb{E}\left(F_s h(\tau) 1_{\tau \le s}\left(X_t - X_s\right)\right) - \mathbb{E}\left(F_s h(\tau) 1_{\tau \le s}\left(\int_s^t \frac{d\langle X, \alpha.(\theta)\rangle_u}{\alpha_{u}(\theta)}\Big|_{\theta=\tau}\right)\right)$$

Moreover, for $dK_v(\theta) = d\left\langle X, \alpha.(\theta)\right\rangle_v / \alpha_v(\theta)$

$$\begin{aligned}\mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s} \int_s^t dK_v(\tau)\right) &= \mathbb{E}\left(F_s \int_0^s h(\theta) \int_s^t dK_v(\theta)\alpha_t(\theta)\eta(d\theta)\right) \\ &= \int_0^s h(\theta)\mathbb{E}\left(F_s \int_s^t \alpha_v(\theta) dK_v(\theta)\right)\eta(d\theta)\end{aligned}$$

where the first equality comes from (*) applied to the $\mathbb{F}$-predictable process indexed by u $J_t^u = h(u)1_{u \leq s} \int_s^t dK_v(u)$ (F_s is F_t-measurable) and the second from the martingale property of $\alpha.(\theta)$

Hence, $a = 0$.

b : We rewrite b as

$$\begin{aligned} b &= \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_t - X_{t\wedge\tau}\right)\right) + \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_{t\wedge\tau} - X_s\right)\right) \\ & -\mathbb{E}\left(F_s h(s) 1_{s<\tau} \int_s^{t\wedge\tau} \frac{d\left\langle X, S\right\rangle_u}{S_u}\right) - \mathbb{E}\left(F_s h(s) 1_{s<\tau} \int_{t\wedge\tau}^t \frac{d\langle X, \alpha_.(\tau)\rangle_u}{\alpha_u(\tau)}\right). \end{aligned}$$

Using Jeulin's formula before default, we have

$$\begin{aligned} \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_{t\wedge\tau} - X_s\right)\right) &= \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_{t\wedge\tau} - X_{s\wedge\tau}\right)\right) \\ &= \mathbb{E}\left(F_s h(s) 1_{s<\tau} \int_s^{t\wedge\tau} \frac{d\langle X, S\rangle_u}{S_u}\right), \end{aligned}$$

and it follows

$$\begin{aligned} b &= \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_t - X_{t\wedge\tau}\right)\right) - \mathbb{E}\left(F_s h(s) 1_{s<\tau} \int_{t\wedge\tau}^t \frac{d\langle X, \alpha_.(\tau)\rangle_u}{\alpha_u(\tau)}\right) \\ &= \mathbb{E}\left(F_s h(s) 1_{s<\tau\le t}\left(X_t - X_\tau\right)\right) - \mathbb{E}\left(F_s h(s) 1_{s<\tau\le t} \int_\tau^t \frac{d\langle X, \alpha_.(\tau)\rangle_u}{\alpha(\tau, u)}\right). \end{aligned}$$

Moreover, we can write the decomposition:

$$\begin{aligned}\mathbb{E}\left(F_s h(s) 1_{s<\tau\leq t} X_\tau\right) &= \mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} X_v dH_v\right) \\ &= \mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} X_v dA_v\right) \\ &= \mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} X_v \alpha_v(v)\eta(dv)\right)\end{aligned}$$

where the second equality comes from the definition of the predictable dual projection, and the third from the computation of the Doob Meyer decomposition of S.

It follows

$$\begin{aligned} b &= \mathbb{E}\left(F_s h(s) X_t \int_{v\in]s,t]} \alpha_t(v)\eta(dv)\right) - \mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} X_v \alpha_v(v)\eta(dv)\right) \\ &\quad -\mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} \int_{u\in]v,t]} \frac{d\left\langle X, \alpha.(v)\right\rangle_u}{\alpha_u(v)} \alpha_t(v)\eta(dv)\right) \\ &= \mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} \left((X_t\alpha_t(v) - X_v\alpha_v(v)) - \int_{u\in]v,t]} d\left\langle X, \alpha.(v)\right\rangle_u\right)\eta(dv)\right) \end{aligned}$$

where the second equality comes from integration by parts formula.

The proof is done.

$\mathbb{F}$-decompositions of the survival process S

• The **Doob-Meyer decomposition** of the super-martingale S is given by

$$S_t = 1 + M_t^{\mathbb{F}} - \int_0^t \alpha_u(u)\eta(du)$$

where $M^{\mathbb{F}}$ is the càdlàg square-integrable martingale defined as

$$M_t^{\mathbb{F}} = -\int_0^t \big(\alpha_t(u) - \alpha_u(u)\big)\eta(du) = \mathbb{E}[\int_0^\infty \alpha_u(u)\eta(du)|\mathcal{F}_t] - 1.$$

• Let $\zeta^{\mathbb{F}} := \inf\{t : S_{t-} = 0\}$. We define $\lambda_t^{\mathbb{F}} := \frac{\alpha_t(t)}{S_{t-}}$ for any $t \le \zeta^{\mathbb{F}}$ and let $\lambda_t^{\mathbb{F}} = \lambda_{t\wedge\zeta^{\mathbb{F}}}^{\mathbb{F}}$ for any $t > \zeta^{\mathbb{F}}$. The **multiplicative decomposition** of S is given by

$$S_t = L_t^{\mathbb{F}} e^{-\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} \qquad \text{where} \qquad dL_t^{\mathbb{F}} = e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} dM_t^{\mathbb{F}}.$$

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Proof: 1) First notice that $(\int_0^t \alpha_u(u)\eta(du), t \geq 0)$ is an $\mathbb{F}$-adapted continuous increasing process. By the martingale property of $(\alpha_t(\theta), t \geq 0)$, for any fixed t,

$$S_t = \int_t^\infty \alpha_t(u)\eta(du) = \mathbb{E}[\int_t^\infty \alpha_u(u)\eta(du)|\mathcal{F}_t], \ a.s..$$

From the properties of the density, $1 - S_t = \int_0^t \alpha_t(u)\eta(du)$ and

$$M_t^{\mathbb{F}} := -\int_0^t (\alpha_t(u) - \alpha_u(u))\eta(du) = \mathbb{E}[\int_0^\infty \alpha_u(u)\eta(du)|\mathcal{F}_t] - 1.$$

2) By definition of $L_t^{\mathbb{F}}$ and 1), we have

$$dL_t^{\mathbb{F}} = e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} dS_t + e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} \lambda_t^{\mathbb{F}} S_t \eta(dt) = e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} dM_t^{\mathbb{F}},$$

which implies the result. △

Relationship with the $\mathbb{G}$-intensity

Definition: *Let τ be a $\mathbb{G}$-stopping time. The $\mathbb{G}$-compensator $\Lambda^{\mathbb{G}}$ of τ is the $\mathbb{G}$-predictable increasing process such that $(1_{\{\tau\leq t\}} - \Lambda^{\mathbb{G}}_t, t \geq 0)$ is a $\mathbb{G}$-martingale. The $\mathbb{G}$-compensator is stopped at τ, i.e., $\Lambda^{\mathbb{G}}_t = \Lambda^{\mathbb{G}}_{t\wedge\tau}$. $\Lambda^{\mathbb{G}}$ coincides, on the set $\{\tau \geq t\}$, with an $\mathbb{F}$-predictable process $\Lambda^{\mathbb{F}}$, i.e. $\Lambda^{\mathbb{G}}_t 1_{\{\tau\geq t\}} = \Lambda^{\mathbb{F}}_t 1_{\{\tau\geq t\}}$.*

- the $\mathbb{G}$-compensator $\Lambda^{\mathbb{G}}$ of τ admits a density given by

$$\lambda^{\mathbb{G}}_t = 1_{\{\tau>t\}}\lambda^{\mathbb{F}}_t = 1_{\{\tau>t\}}\frac{\alpha_t(t)}{S_{t-}}.$$

In particular, τ is a totally inaccessible $\mathbb{G}$-stopping time.

- For any $t < \zeta^{\mathbb{F}}$ and $T \geq t$, we have $\alpha_t(T) = \mathbb{E}[\lambda^{\mathbb{G}}_T | \mathcal{F}_t]$.

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Definition: *Let* τ *be a* $\mathbb{G}$*-stopping time. The* $\mathbb{G}$*-compensator* $\Lambda^{\mathbb{G}}$ *of* τ *is the* $\mathbb{G}$*-predictable increasing process such that* $(1_{\{\tau\leq t\}} - \Lambda^{\mathbb{G}}_t, t \geq 0)$ *is a* $\mathbb{G}$*-martingale. The* $\mathbb{G}$*-compensator is stopped at* τ*, i.e.,* $\Lambda^{\mathbb{G}}_t = \Lambda^{\mathbb{G}}_{t\wedge\tau}$*.* $\Lambda^{\mathbb{G}}$ *coincides, on the set* $\{\tau \geq t\}$*, with an* $\mathbb{F}$*-predictable process* $\Lambda^{\mathbb{F}}$*, i.e.* $\Lambda^{\mathbb{G}}_t 1_{\{\tau\geq t\}} = \Lambda^{\mathbb{F}}_t 1_{\{\tau\geq t\}}$.

• the $\mathbb{G}$-compensator $\Lambda^{\mathbb{G}}$ of τ admits a density given by

$$\lambda^{\mathbb{G}}_t = 1_{\{\tau>t\}}\lambda^{\mathbb{F}}_t = 1_{\{\tau>t\}}\frac{\alpha_t(t)}{S_{t-}}.$$

In particular, τ is a totally inaccessible $\mathbb{G}$-stopping time.

• For any $t < \zeta^{\mathbb{F}}$ and $T \geq t$, we have $\alpha_t(T) = \mathbb{E}[\lambda^{\mathbb{G}}_T|\mathcal{F}_t]$.

PROOF: 1) We shall prove that $(1_{\{\tau\le t\}} - \int_0^t \lambda_s^{\mathbb{G}} \eta(ds), t \ge 0)$ is a $\mathbb{G}$-martingale, which is equivalent to the $\mathbb{G}$-martingale property for $(1_{\{\tau>t\}} e^{\int_0^t \lambda_s^{\mathbb{G}} \eta(ds)} = 1_{\{\tau>t\}} e^{\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)}, t \ge 0)$.

This follows from

$$\begin{aligned}
\mathbb{E}[1_{\{\tau>t\}} e^{\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)} | \mathcal{G}_s] &= 1_{\{\tau>s\}} \frac{\mathbb{E}[1_{\{\tau>t\}} e^{\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)} | \mathcal{F}_s]}{S_s} \\
&= 1_{\{\tau>s\}} \frac{\mathbb{E}[S_t e^{\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)} | \mathcal{F}_s]}{S_s} = 1_{\{\tau>s\}} \frac{L_s^{\mathbb{F}}}{S_s},
\end{aligned}$$

where the last equality follows from the $\mathbb{F}$-local martingale property of $L^{\mathbb{F}}$. Moreover, the continuity of the compensator $\Lambda^{\mathbb{G}}$ implies that τ is totally inaccessible.

2) By the martingale property of density, for any $T \geq t$, $\alpha_t(T) = \mathbb{E}[\alpha_T(T)|\mathcal{F}_t]$. Applying 1), we obtain

$$\alpha_t(T) = \mathbb{E}\Big[\alpha_T(T)\frac{1_{\{\tau>T\}}}{S_{T-}}|\mathcal{F}_t\Big] = \mathbb{E}[\lambda_T^{\mathbb{G}}|\mathcal{F}_t], \quad \forall t < \zeta^{\mathbb{F}},$$

hence, the value of the density can be partially deduced from the intensity. $\triangle$

• Note that $\zeta^{\mathbb{F}} \geq \tau$.

• In the case where immersion holds, the survival process S is a decreasing continuous process given by $S_t = 1 - \int_0^t \alpha_u(u)\eta(du)$. So the process $\lambda^{\mathbb{F}}$ is in fact the $\mathbb{F}$-intensity of default.

$\mathbb{G}$-martingale characterization

Any $\mathbb{G}$-martingale may be splitted into two martingales, the first one stopped at time τ and the second one starting at time τ, that is

$$Y_t^{\mathbb{G}} = Y_t^{bd,\mathbb{G}} + Y_t^{ad,\mathbb{G}}$$

where $Y_t^{bd,\mathbb{G}} = Y_{t\wedge\tau}^{\mathbb{G}}$ and $Y_t^{ad,\mathbb{G}} = (Y_t^{\mathbb{G}} - Y_\tau^{\mathbb{G}})1_{\{\tau\leq t\}}$. We now study the two types of martingales respectively.

$\mathbb{G}$-martingale stopped at time τ

A $\mathbb{G}$-adapted càdlàg process $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale stopped at time τ if and only if there exist an $\mathbb{F}$-adapted càdlàg process Y defined on $[0,\zeta^{\mathbb{F}})$ and an optional process $Y_{\cdot}(\tau)$ such that

$$Y_t^{\mathbb{G}} = Y_t 1_{\{\tau>t\}} + Y_\tau(\tau) 1_{\{\tau\le t\}}$$

and

$$\left(U_t := Y_t S_t + \int_0^t Y_s(s)\alpha_s(s)\eta(ds),\ t\ge 0\right) \quad \text{is an } \mathbb{F}\text{-local martingale on } [0,\zeta^{\mathbb{F}})$$

The martingale U is the $\mathbb{F}$ projection of $Y^{\mathbb{G}}$.

Equivalently,

$$\left(L_t^{\mathbb{F}}[Y_t + \int_0^t (Y_s(s) - Y_s)\lambda_s^{\mathbb{F}}\eta(ds)],\ t\ge 0\right) \text{ is an } \mathbb{F}\text{-local martingale on } [0,\zeta^{\mathbb{F}}).$$

$\mathbb{G}$-martingale stopped at time τ

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$$(U_t := Y_t S_t + \int_0^t Y_s(s)\alpha_s(s)\eta(ds),\, t \ge 0) \quad \text{is an } \mathbb{F}\text{-local martingale on } [0, \zeta^{\mathbb{F}})$$

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$\mathbb{G}$-martingale stopped at time τ

A $\mathbb{G}$-adapted càdlàg process $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale stopped at time τ if and only if there exist an $\mathbb{F}$-adapted càdlàg process Y defined on $[0, \zeta^{\mathbb{F}})$ and an optional process $Y_{.}(\tau)$ such that

$$Y_t^{\mathbb{G}} = Y_t 1_{\{\tau > t\}} + Y_\tau(\tau) 1_{\{\tau \le t\}}$$

and

$$(U_t := Y_t S_t + \int_0^t Y_s(s)\alpha_s(s)\eta(ds),\, t \ge 0) \quad \text{is an } \mathbb{F}\text{-local martingale on } [0, \zeta^{\mathbb{F}})$$

The martingale U is the $\mathbb{F}$ projection of $Y^{\mathbb{G}}$.

Equivalently,

$$(L_t^{\mathbb{F}}[Y_t + \int_0^t (Y_s(s) - Y_s)\lambda_s^{\mathbb{F}}\eta(ds)],\, t \ge 0) \text{ is an } \mathbb{F}\text{-local martingale on } [0, \zeta^{\mathbb{F}}).$$

$S_t = L_t^{\mathbb{F}} \exp(-\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)),\ \zeta^{\mathbb{F}} = \inf\{t : S_{t-} = 0\}$

PROOF:

$$Y_t^{\mathbb{F}} = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = Y_t S_t + \int_0^t Y_s(s)\alpha_t(s)\eta(ds)$$

Since $(\int_0^t Y_s(s)(\alpha_t(s) - \alpha_s(s))\eta(ds), t \geq 0)$ is an $\mathbb{F}$-local martingale, and so is

$$U_t = Y_t S_t + \int_0^t Y_s(s)\alpha_s(s)\eta(ds)\,.$$

Conversely, if U is an $\mathbb{F}$-local martingale, it is easy to verify that

$$\mathbb{E}[Y_T^{\mathbb{G}} - Y_t^{\mathbb{G}}|\mathcal{G}_t] = 0, \text{ a.s..}$$

Recall that $L_t^{\mathbb{F}} = S_t e^{\Lambda_t^{\mathbb{F}}}$. We have

$$d(Y_t L_t^{\mathbb{F}}) = e^{\Lambda_t} d(Y_t S_t) + e^{\Lambda_t} Y_t S_t \lambda_t^{\mathbb{F}} \eta(dt) = e^{\Lambda_t} dU_t + (Y_t - Y_t(t))\lambda_t^{\mathbb{F}} L_t^{\mathbb{F}} \eta(dt).$$

The local martingale property of U is then equivalent to that of $(Y_t L_t^{\mathbb{F}} - \int_0^t (Y_s - Y_s(s))\lambda_s^{\mathbb{F}} L_s^{\mathbb{F}} \eta(ds), t \geq 0)$, and then to the condition . $\triangle$

With the above notation, a martingale $Y^{\mathbb{G}}$ which is stopped and continuous at τ is characterized by the condition that $(L_t^{\mathbb{F}} Y_t, t \geq 0)$ is an $\mathbb{F}$-local martingale.

To study in more detail the jump impact, we give a decomposition of the $\mathbb{G}$-martingale stopped at τ, i.e., $Y^{bd,\mathbb{G}}$ into a martingale continuous at τ and a pure jump martingale.

There exist two martingales $Y^{c,bd}$ and $Y^{s,bd}$ such that $Y^{bd,\mathbb{G}} = Y^{c,bd} + Y^{s,bd}$, which satisfy the following conditions:

1) $(Y_t^{s,bd} = (Y_\tau(\tau) - Y_\tau)1_{\{\tau \leq t\}} - \int_0^{t\wedge\tau}(Y_s(s) - Y_s)\lambda_s^{\mathbb{F}}\eta(ds), t \geq 0)$ is a pure jump $\mathbb{G}$-martingale;

2) $(Y_t^{c,bd} = \widetilde{Y}_{\tau\wedge t}, t \geq 0)$ where $\widetilde{Y}_t = Y_t + \int_0^t (Y_s(s) - Y_s)\lambda_s^{\mathbb{F}}\eta(ds)$ is a $\mathbb{G}$-martingale which is continuous at τ.

$\mathbb{G}$-martingale starting at τ

Any càdlàg process $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale starting at τ if and only if there exists an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-optional process $Y_t(.)$ such that $Y_t^{\mathbb{G}} = Y_t(\tau)1_{\{\tau \le t\}}$ and that $(Y_t(\theta)\alpha_t(\theta), t \ge \theta)$ is an $\mathbb{F}$-martingale on $[\theta, \zeta^{\theta})$.

Combining the previous characterization results, we obtain:

A càdlàg process $Y^{\mathbb{G}}$ is a **$\mathbb{G}$-martingale** if and only if there exist an $\mathbb{F}$-adapted càdlàg process Y and an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-optional process $Y_t(.)$ such that

$$Y_t^{\mathbb{G}} = Y_t 1_{\{\tau > t\}} + Y_t(\tau) 1_{\{\tau \leq t\}}$$

and that

- $(Y_t S_t + \int_0^t Y_s(s)\alpha_s(s)\eta(ds),\ t \geq 0)$ is an $\mathbb{F}$-local martingale;
- $(Y_t(\theta)\alpha_t(\theta), t \geq \theta)$ is an $\mathbb{F}$-martingale on $[\theta, \zeta^\theta)$.

PROOF: Notice that $Y_t^{ad,\mathbb{G}} = (Y_t(\tau) - Y_\tau(\tau))1_{\{\tau \leq t\}}$. Then the theorem follows directly by applying previous results. $\triangle$

In our paper, our model is based on the knowledge of the density process.

It is also possible to build a model starting from the process S and the family of processes $\alpha_t(u), t \geq u$, with the compatibility conditions $\int_0^\infty \alpha_\infty(u)\eta(du) = 1$ and $S_t - \int_0^t \alpha_u(u)\eta(du)$ is an $\mathbb{F}$-martingale.

With the knowledge of these processes, one can construct $\lambda_t^{\mathbb{F}} = \alpha_t(t)/S_{t-}$ and $\alpha_s(u) = \mathbb{E}(\alpha_u(u)|\mathcal{F}_t)$ for $s < u$. It is not difficult to check that the second compatibility condition implies that $L_t^{\mathbb{F}} := S_t \exp \int_0^t \lambda_s^{\mathbb{F}} \eta(ds)$ is a $\mathbb{F}$-martingale. Once more time, the knowledge of the $\mathbb{F}$-intensity process $\lambda^{\mathbb{F}}$ and of the supermartingale S allow us to define the "predefault" density $\alpha_t(u), t \leq u$, but not the afterdefault density.

Furthermore, starting from an $\mathbb{F}$-super-martingale S, valued in $[0, 1]$ and a family of non-negative martingales $\alpha_t(u), t \geq u$ satisfying the compatibility conditions and constructed on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ it is possible to construct, on an enlarged probability space a random variable τ and a probability $\mathbb{Q}$ such that α is the $\mathbb{Q}$ density of τ and $\mathbb{Q}$ coincides with $\mathbb{P}$ on $\mathbb{F}$. We do not give any details on that construction.

Notation We shall call the family $(S, \alpha_t(u), t \geq u, L^{\mathbb{F}})$, when the compatibility conditions are satisfied, the parameters of the model.

Girsanov theorem

Let $Z_t^{\mathbb{G}} = z_t 1_{\{\tau > t\}} + z_t(\tau) 1_{\{\tau \leq t\}}$ be a positive $\mathbb{G}$-martingale with $Z_0^{\mathbb{G}} = 1$ and let $Z_t^{\mathbb{F}} = z_t S_t + \int_0^t z_t(u) \alpha_t(u) \eta(du)$ be its $\mathbb{F}$ projection.
Let $\mathbb{Q}$ be the probability measure defined on $\mathcal{G}_t$ by $d\mathbb{Q} = Z_t^{\mathbb{G}} d\mathbb{P}$.
Then $(\Omega, \mathbb{Q}, \mathbb{G}, \mathbb{F}, \tau)$ satisfies the density hypothesis and:
(i) the $\mathbb{Q}$-conditional survival process is defined on $[0, \zeta^{\mathbb{F}})$ by $S_t^{\mathbb{Q}} = S_t \dfrac{z_t}{Z_t^{\mathbb{F}}}$

(ii) the $(\mathbb{F}, \mathbb{Q})$-intensity process is $\lambda_t^{\mathbb{F},\mathbb{Q}} = \lambda_t^{\mathbb{F}} \dfrac{z_t(t)}{z_{t-}}$, $\eta(dt)$- a.s.;

(iii) $\alpha_t^{\mathbb{Q}}(\theta) = \alpha_t(\theta) \frac{z_t(\theta)}{Z_t^{\mathbb{F}}}, \quad \forall t \in [\theta, \zeta^{\theta})$;

(iv) $L^{\mathbb{F},\mathbb{Q}}$ is the $(\mathbb{F}, \mathbb{Q})$-local martingale

$$L_t^{\mathbb{F},\mathbb{Q}} = L_t^{\mathbb{F}} \frac{z_t}{Z_t^{\mathbb{F}}} \exp \int_0^t (\lambda_s^{\mathbb{F},\mathbb{Q}} - \lambda_s^{\mathbb{F}}) \eta(ds), t \in [0, \zeta^{\mathbb{F}})$$

Girsanov theorem

Let $Z_t^{\mathbb{G}} = z_t 1_{\{\tau > t\}} + z_t(\tau) 1_{\{\tau \le t\}}$ be a positive $\mathbb{G}$-martingale with $Z_0^{\mathbb{G}} = 1$ and let $Z_t^{\mathbb{F}} = z_t S_t + \int_0^t z_t(u)\alpha_t(u)\eta(du)$ be its $\mathbb{F}$ projection.
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(i) the $\mathbb{Q}$-conditional **survival process** is defined by $S_t^{\mathbb{Q}} = S_t \dfrac{z_t}{Z_t^{\mathbb{F}}}$

(ii) the $(\mathbb{F}, \mathbb{Q})$-intensity process is $\lambda_t^{\mathbb{F},\mathbb{Q}} = \lambda_t^{\mathbb{F}} \dfrac{z_t(t)}{z_{t-}}$, $\eta(dt)$- a.s.;

(iii) (iii) $\alpha_t^{\mathbb{Q}}(\theta) = \alpha_t(\theta) \frac{z_t(\theta)}{Z_t^{\mathbb{F}}}, \quad \forall t \in [\theta, \zeta^\theta)$;

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$$L_t^{\mathbb{F},\mathbb{Q}} = L_t^{\mathbb{F}} \frac{z_t}{Z_t^{\mathbb{F}}} \exp \int_0^t (\lambda_s^{\mathbb{F},\mathbb{Q}} - \lambda_s^{\mathbb{F}}) \eta(ds), t \in [0, \zeta^{\mathbb{F}})$$

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Let $Z_t^{\mathbb{G}} = z_t 1_{\{\tau > t\}} + z_t(\tau) 1_{\{\tau \le t\}}$ be a positive $\mathbb{G}$-martingale with $Z_0^{\mathbb{G}} = 1$ and let $Z_t^{\mathbb{F}} = z_t S_t + \int_0^t z_t(u) \alpha_t(u) \eta(du)$ be its $\mathbb{F}$ projection.

Let $\mathbb{Q}$ be the probability measure defined on $\mathcal{G}_t$ by $d\mathbb{Q} = Z_t^{\mathbb{G}} d\mathbb{P}$.

Then $(\Omega, \mathbb{Q}, \mathbb{G}, \mathbb{F}, \tau)$ satisfies the density hypothesis and:

(i) the $\mathbb{Q}$-conditional **survival process** is defined by $S_t^{\mathbb{Q}} = S_t \dfrac{z_t}{Z_t^{\mathbb{F}}}$

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(iii) $\alpha_t^{\mathbb{Q}}(\theta) = \alpha_t(\theta) \frac{z_t(\theta)}{Z_t^{\mathbb{F}}}, \quad \forall t \in [\theta, \zeta^\theta)$;

(iv) $L^{\mathbb{F},\mathbb{Q}}$ is the $(\mathbb{F}, \mathbb{Q})$-local martingale

$$L_t^{\mathbb{F},\mathbb{Q}} = L_t^{\mathbb{F}} \frac{z_t}{Z_t^{\mathbb{F}}} \exp \int_0^t (\lambda_s^{\mathbb{F},\mathbb{Q}} - \lambda_s^{\mathbb{F}}) \eta(ds), t \in [0, \zeta^{\mathbb{F}})$$

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Let $Z_t^{\mathbb{G}} = z_t 1_{\{\tau > t\}} + z_t(\tau) 1_{\{\tau \le t\}}$ be a positive $\mathbb{G}$-martingale with $Z_0^{\mathbb{G}} = 1$ and let $Z_t^{\mathbb{F}} = z_t S_t + \int_0^t z_t(u)\alpha_t(u)\eta(du)$ be its $\mathbb{F}$ projection.
Let $\mathbb{Q}$ be the probability measure defined on $\mathcal{G}_t$ by $d\mathbb{Q} = Z_t^{\mathbb{G}} d\mathbb{P}$.
Then $(\Omega, \mathbb{Q}, \mathbb{G}, \mathbb{F}, \tau)$ satisfies the density hypothesis and:
(i) the $\mathbb{Q}$-conditional **survival process** is defined by $S_t^{\mathbb{Q}} = S_t \dfrac{z_t}{Z_t^{\mathbb{F}}}$

(ii) the $(\mathbb{F}, \mathbb{Q})$-**intensity process** is $\lambda_t^{\mathbb{F},\mathbb{Q}} = \lambda_t^{\mathbb{F}} \dfrac{z_t(t)}{z_{t-}}$, $\eta(dt)$- a.s.;

(iii) $\alpha_t^{\mathbb{Q}}(\theta) = \alpha_t(\theta) \frac{z_t(\theta)}{Z_t^{\mathbb{F}}}, \quad \forall t \in [\theta, \zeta^\theta)$;

(iv) $L^{\mathbb{F},\mathbb{Q}}$ is the $(\mathbb{F}, \mathbb{Q})$-local martingale

$$L_t^{\mathbb{F},\mathbb{Q}} = L_t^{\mathbb{F}} \frac{z_t}{Z_t^{\mathbb{F}}} \exp \int_0^t (\lambda_s^{\mathbb{F},\mathbb{Q}} - \lambda_s^{\mathbb{F}})\eta(ds), t \in [0, \zeta^{\mathbb{F}})$$

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Let $Z_t^{\mathbb{G}} = z_t 1_{\{\tau > t\}} + z_t(\tau) 1_{\{\tau \le t\}}$ be a positive $\mathbb{G}$-martingale with $Z_0^{\mathbb{G}} = 1$ and let $Z_t^{\mathbb{F}} = z_t S_t + \int_0^t z_t(u)\alpha_t(u)\eta(du)$ be its $\mathbb{F}$ projection.
Let $\mathbb{Q}$ be the probability measure defined on $\mathcal{G}_t$ by $d\mathbb{Q} = Z_t^{\mathbb{G}} d\mathbb{P}$.
Then $(\Omega, \mathbb{Q}, \mathbb{G}, \mathbb{F}, \tau)$ satisfies the density hypothesis and:

(i) the $\mathbb{Q}$-conditional **survival process** is defined by $S_t^{\mathbb{Q}} = S_t \frac{z_t}{Z_t^{\mathbb{F}}}$

(ii) the $(\mathbb{F}, \mathbb{Q})$-**intensity process** is $\lambda_t^{\mathbb{F},\mathbb{Q}} = \lambda_t^{\mathbb{F}} \frac{z_t(t)}{z_{t-}}$, $\eta(dt)$- a.s.;

(iii) $\alpha_t^{\mathbb{Q}}(\theta) = \alpha_t(\theta) \frac{z_t(\theta)}{Z_t^{\mathbb{F}}}, \quad \forall t \in [\theta, \zeta^\theta)$;

(iv) $L^{\mathbb{F},\mathbb{Q}}$ is the $(\mathbb{F}, \mathbb{Q})$-local martingale

$$L_t^{\mathbb{F},\mathbb{Q}} = L_t^{\mathbb{F}} \frac{z_t}{Z_t^{\mathbb{F}}} \exp \int_0^t (\lambda_s^{\mathbb{F},\mathbb{Q}} - \lambda_s^{\mathbb{F}}) \eta(ds), t \in [0, \zeta^{\mathbb{F}})$$

PROOF: For any $t \in [0, \zeta^{\mathbb{F}})$, the $\mathbb{Q}$-conditional probability can be calculated by

$$S_t^{\mathbb{Q}} = \mathbb{Q}(\tau > t | \mathcal{F}_t) = \frac{\mathbb{E}[1_{\{\tau > t\}} Z_t^{\mathbb{G}} | \mathcal{F}_t]}{Z_t^{\mathbb{F}}} = z_t \frac{S_t}{Z_t^{\mathbb{F}}}$$

and, for any $\theta \leq t$,

$$\mathbb{Q}(\tau \leq \theta | \mathcal{F}_t) = \frac{\mathbb{E}^{\mathbb{P}}[1_{\{\tau \leq \theta\}} Z_t^{\mathbb{G}} | \mathcal{F}_t]}{Z_t^{\mathbb{F}}} = \frac{\mathbb{E}^{\mathbb{P}}[1_{\{\tau \leq \theta\}} z_t(\tau) | \mathcal{F}_t]}{Z_t^{\mathbb{F}}} = \frac{\int_0^\theta z_t(u) \alpha_t(u) \eta(du)}{Z_t^{\mathbb{F}}}.$$

The density process is then obtained by taking derivatives.

Finally, we use $\lambda_t^{\mathbb{F},\mathbb{Q}} = \alpha_t^{\mathbb{Q}}(t)/S_{t-}^{\mathbb{Q}}$ and $L_t^{\mathbb{F},\mathbb{Q}} = S_t^{\mathbb{Q}} e^{\int_0^t \lambda_s^{\mathbb{F},\mathbb{Q}} \eta(ds)}$.

Modelling of density process

To model the family of density processes, it's natural to make references to the classical interest rate models.

HJM framework and short rate models

We suppose in what follows that the representation theorem of $\mathbb{F}$-martingales holds. For any $\theta \geq 0$, the $\mathbb{F}$-martingale $(S_t(\theta), t \geq 0)$ can be written as a stochastic integral w.r.t. a basic multi-dimensional (continuous) $\mathbb{F}$-martingale M.

Suppose that for any $\theta \geq 0$, the bounded martingale $(S_t(\theta), t \geq 0)$ satisfies

$$dS_t(\theta) = Z_t(\theta) dM_t$$

where $(Z_t(\theta), t \geq 0)$ is an $\mathbb{F}$-predictable process and M is a multi-dimensional $\mathbb{F}$-martingale. If the process $z_t(\theta)$ such that $Z_t(\theta) = \int_0^\theta z_t(u)\eta(du)$ is bounded by an integrable process, then

1. $d\alpha_t(\theta) = -z_t(\theta) dM_t$.
2. The martingale part in the Doob-Meyer decomposition of S is given by $M_t^{\mathbb{F}} = 1 - \int_0^t Z_s(s) dM_s$.

PROOF: 1) is obvious by definition. 2) is obtained by using previous results and integration by part, in fact,

$$M_t^{\mathbb{F}} = 1 - \int_0^t \eta(du) \int_u^t z_s(u) dM_s = 1 - \int_0^t dM_s \Big(\int_0^s z_s(u)\eta(du) \Big).$$

Observe in addition that $Z_t(0) = 0$ since $S_t(0) = 1$ for any $t \geq 0$, which implies 2)

We can also consider $(S_t(\theta), t \geq 0)$ in the classical HJM models where its dynamics is given in multiplicative form. We also deduce the dynamics of the forward rate, in both forward and backward forms. The density can then be calculated as $\alpha_t(\theta) = \lambda_t(\theta) S_t(\theta)$.

For any $t, \theta \geq 0$, let $\Psi(t,\theta) = \frac{Z_t(\theta)}{S_t(\theta)}$ and define $\psi(t,\theta)$ by $\Psi(t,\theta) = \int_0^\theta \psi(t,u)\eta(du)$. Recall the forward rate $\lambda_t(\theta)$ of τ given by $\lambda_t(\theta) = -\partial_\theta \ln S_t(\theta)$. If M is a continuous martingale, then

1. $S_t(\theta) = S_0(\theta) \exp\left(\int_0^t \Psi(s,\theta) dM_s - \frac{1}{2}\int_0^t |\Psi(s,\theta)|^2 d\langle M\rangle_s\right)$;
2. $\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi(s,\theta) dM_s + \int_0^t \psi(s,\theta)\Psi(s,\theta)^* d\langle M\rangle_s$;
3. $S_t = \exp\left(-\int_0^t \lambda_s^{\mathbb{F}} \eta(ds) + \int_0^t \Psi(s,s) dM_s - \frac{1}{2}\int_0^t |\Psi(s,s)|^2 d\langle M\rangle_s\right)$;
4. $\lambda_t(\theta) = \lambda_\theta^{\mathbb{F}} + \int_t^\theta \psi(s,\theta) dM_s - \int_t^\theta \psi(s,\theta)\Psi(s,\theta)^* d\langle M\rangle_s$.

For any $t, \theta \geq 0$, let $\Psi(t,\theta) = \frac{Z_t(\theta)}{S_t(\theta)}$ and define $\psi(t,\theta)$ by $\Psi(t,\theta) = \int_0^\theta \psi(t,u)\eta(du)$. Recall the forward rate $\lambda_t(\theta)$ of τ given by $\lambda_t(\theta) = -\partial_\theta \ln S_t(\theta)$. If M is a continuous martingale, then

1. $S_t(\theta) = S_0(\theta) \exp\left(\int_0^t \Psi(s,\theta) dM_s - \frac{1}{2} \int_0^t |\Psi(s,\theta)|^2 d\langle M \rangle_s \right)$;
2. $\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi(s,\theta) dM_s + \int_0^t \psi(s,\theta)\Psi(s,\theta)^* d\langle M \rangle_s$;
3. $S_t = \exp\left(-\int_0^t \lambda_s^{\mathbb{F}} \eta(ds) + \int_0^t \Psi(s,s) dM_s - \frac{1}{2} \int_0^t |\Psi(s,s)|^2 d\langle M \rangle_s \right)$;
4. $\lambda_t(\theta) = \lambda_\theta^{\mathbb{F}} + \int_t^\theta \psi(s,\theta) dM_s - \int_t^\theta \psi(s,\theta)\Psi(s,\theta)^* d\langle M \rangle_s$.

For any $t, \theta \geq 0$, let $\Psi(t,\theta) = \frac{Z_t(\theta)}{S_t(\theta)}$ and define $\psi(t,\theta)$ by $\Psi(t,\theta) = \int_0^\theta \psi(t,u)\eta(du)$. Recall the forward rate $\lambda_t(\theta)$ of τ given by $\lambda_t(\theta) = -\partial_\theta \ln S_t(\theta)$. If M is a continuous martingale, then

1. $S_t(\theta) = S_0(\theta) \exp\left(\int_0^t \Psi(s,\theta) dM_s - \frac{1}{2}\int_0^t |\Psi(s,\theta)|^2 d\langle M\rangle_s\right)$;

2. $\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi(s,\theta) dM_s + \int_0^t \psi(s,\theta)\Psi(s,\theta)^* d\langle M\rangle_s$;

3. $S_t = \exp\left(-\int_0^t \lambda_s^{\mathbb{F}} \eta(ds) + \int_0^t \Psi(s,s) dM_s - \frac{1}{2}\int_0^t |\Psi(s,s)|^2 d\langle M\rangle_s\right)$;

4. $\lambda_t(\theta) = \lambda_\theta^{\mathbb{F}} + \int_t^\theta \psi(s,\theta) dM_s - \int_t^\theta \psi(s,\theta)\Psi(s,\theta)^* d\langle M\rangle_s$.

$dS_t(\theta) = Z_t(\theta) dM_t$

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PROOF: The process $S_t(\theta)$ is the solution of the equation

$$\frac{dS_t(\theta)}{S_t(\theta)} = \Psi(t,\theta)dM_t, \qquad \forall\, t,\theta \geq 0.$$

Hence 1), from which we deduce immediately 2) by differentiation w.r.t. θ.

Now, note that by 1),

$$\ln S_t = -\int_0^t \lambda_0(s)\eta(ds) + \int_0^t \Psi(s,t)dM_s - \frac{1}{2}\int_0^t |\Psi(s,t)|^2 d\left\langle M\right\rangle_s$$

Moreover, we have by 2) that

$$\begin{aligned}\int_0^t \lambda_s(s)\eta(ds) &= \int_0^t \lambda_0(s)\eta(ds) - \int_0^t \eta(ds)\int_0^s \psi(u,s)dM_u \\ &\quad + \int_0^t \eta(ds)\int_0^s \psi(u,s)\Psi(u,s)^* d\langle M\rangle_u \\ &= \int_0^t \lambda_0(s)\eta(ds) - \int_0^t (\Psi(u,t)-\Psi(u,u))dM_u \\ &\quad + \frac{1}{2}(\int_0^t |\Psi(u,t)|^2 - \int_0^t |\Psi(u,u)|^2)d\langle M\rangle_u .\end{aligned}$$

Observe in addition that by definition of the forward rate $\lambda_t(\theta)$ and then, we have $\lambda_s(s) = \lambda_s^{\mathbb{F}}$, which implies 3). Finally, 4) is a direct result from 2).

As a conditional survival probability, $S_t(\theta)$ is decreasing on θ, which is equivalent to that $\lambda_t(\theta)$ is positive. This condition is similar as for the zero coupon bond prices.

In the second approach, we can borrow short rate models for the $\mathbb{F}$-intensity $\lambda^{\mathbb{F}}$ of τ, and then obtain the dynamics of conditional probability $S_t(T)$ for $T \geq t$. The monotonicity condition of $S_t(T)$ on T is equivalent to the positivity condition on $\lambda^{\mathbb{F}}$.

Other examples

Example: "Cox-like" construction. Here

- λ is a non-negative $\mathbb{F}$-adapted process, $\Lambda_t = \int_0^t \lambda_s ds$
- Θ is a given r.v. independent of $\mathcal{F}_\infty$ with unit exponential law
- V is a $\mathcal{F}_\infty$ -measurable non-negative random variable
- $\tau = \inf\{t : \Lambda_t \geq \Theta V\}$.

For any θ and t,

$$G_t(\theta) = \mathbb{P}(\tau > \theta|\mathcal{F}_t) = \mathbb{P}(\Lambda_\theta < \Theta V|\mathcal{F}_t) = \mathbb{P}\left(\exp -\frac{\Lambda_\theta}{V} \geq e^{-\Theta}\middle| \mathcal{F}_t\right).$$

Let us denote $\exp(-\Lambda_t/V) = 1 - \int_0^t \psi_s ds$, with

$$\psi_s = (\lambda_s/V)\exp -\int_0^s (\lambda_u/V)\,du,$$

and define $\gamma_t(s) = \mathbb{E}\left(\psi_s|\,\mathcal{F}_t\right)$. Then, $\alpha_t(s) = \gamma_t(s)/\gamma_0(s)$.

Backward construction of the density

Let $\varphi(\cdot, \alpha)$ be a family of densities on $\mathbb{R}^+$, depending of some parameter and $X \in \mathcal{F}_\infty$ a random variable. Then

$$\int_0^\infty \varphi(u, X)du = 1$$

and whe can chose

$$\alpha_t(u) = \mathbb{E}(\alpha_\infty(u)|\mathcal{F}_t) = \mathbb{E}(\varphi(u, X)|\mathcal{F}_t)$$

Multidefaults

Computation of prices in case of multidefaults is now easy

- In a first step, one orders the default
- Computation before the first default are done in the reference filtration
- Between the first and the second default, one takes as new reference filtration the filtration generated by the first default and the previous reference filtration, as explained previously for the "after default" computations
- and we continue till the end

THANK YOU FOR YOUR ATTENTION