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Plan

Processus de Poisson

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1 Counting processes and stochastic integral

$$N_t = \begin{cases} n & \text{if } \ t \in [T_n, T_{n+1}[\\ +\infty & \text{otherwise} \end{cases}$$

or, equivalently

$$N_t = \sum_{n \geq 1} \mathbb{1}_{\{T_n \leq t\}} = \sum_{n \geq 1} n \mathbb{1}_{\{T_n \leq t < T_{n+1}\}} \, .$$

We denote by N_{t-} the left-limit of N_s when $s \to t, s < t$ and by $\Delta N_s = N_s - N_{s-}$ the jump process of N.

The stochastic integral

$$\int_0^t C_s dN_s$$

is defined as

$$(C\star N)_t = \int_0^t C_s dN_s = \int_{]0,t]} C_s dN_s = \sum_{n=1}^{\infty} C_{T_n} \mathbb{1}_{\{T_n \leq t\}} \,.$$

2 Standard Poisson process

2.1 Definition

The standard Poisson process is a counting process such that
- for every s, t, $N_{t+s} - N_t$ is independent of $\mathcal{F}_t^N$,
- for every s, t, the r.v. $N_{t+s} - N_t$ has the same law as N_s.

2.2 First properties

$$E(N_t) = \lambda t, \quad \mathrm{Var}\,(N_t) = \lambda t$$

for every $x > 0$, $t > 0, u, \alpha \in I\!R$

$$E(x^{N_t}) = e^{\lambda t(x-1)}\,;\; E(e^{iuN_t}) = e^{\lambda t(e^{iu}-1)}\,;\; E(e^{\alpha N_t}) = e^{\lambda t(e^{\alpha}-1)}\,. \quad (2.1)$$

2.3 Martingale properties

For each $\alpha \in I\!\!R$, for each bounded Borel function h, the following processes are $\mathbf{F}$-martingales:

$$
\begin{aligned}
&(i) \quad M_t = N_t - \lambda t, \\
&(ii) \quad M_t^2 - \lambda t = (N_t - \lambda t)^2 - \lambda t, \\
&(iii) \quad \exp(\alpha N_t - \lambda t(e^\alpha - 1)), \\
&(iv) \quad \exp\Big[\int_0^t h(s) dN_s - \lambda \int_0^t (e^{h(s)} - 1) ds\Big] .
\end{aligned}
\tag{2.2}
$$

For any $\beta > -1$, any bounded Borel function h, and any bounded Borel function φ valued in $]-1, \infty[$, the processes

$$
\begin{aligned}
&\exp[\ln(1+\beta)N_t - \lambda\beta t] = (1+\beta)^{N_t} e^{-\lambda\beta t}, \\
&\exp\Big[\int_0^t h(s)dM_s + \lambda\int_0^t (1 + h(s) - e^{h(s)})ds\Big], \\
&\exp\Big[\int_0^t \ln(1+\varphi(s))dN_s - \lambda\int_0^t \varphi(s)ds\Big], \\
&\exp\Big[\int_0^t \ln(1+\varphi(s))dM_s + \lambda\int_0^t (\ln(1+\varphi(s)) - \varphi(s)ds)\Big],
\end{aligned}
$$

are martingales.

Let H be an $\mathbf{F}$-predictable bounded process, then the following processes are martingales

$$
\begin{aligned}
&(H\star M)_t = \int_0^t H_s dM_s = \int_0^t H_s dN_s - \lambda \int_0^t H_s ds \\
&((H\star M)_t)^2 - \lambda \int_0^t H_s^2 ds \qquad\qquad (2.3)\\
&\exp\left(\int_0^t H_s dN_s + \lambda \int_0^t (1 - e^{H_s}) ds\right)
\end{aligned}
$$

Property (2.3) does not extend to adapted processes H. For example, from $\int_0^t (N_s - N_{s-}) dM_s = N_t$, it follows that $\displaystyle\int_0^t N_s dM_s$ is not a martingale.

Remark 2.1 Note that (i) and (iii) imply that the process $(M_t^2 - N_t; t \geq 0)$ is a martingale.

The process λt is the predictable quadratic variation $\langle M \rangle$, whereas the process $(N_t, t \geq 0)$ is the optional quadratic variation $[M]$.

For any $\mu \in [0,1]$, the processes $M_t^2 - (\mu N_t + (1-\mu)\lambda t)$ are also martingales.

2.4 Infinitesimal Generator

The Poisson process is a Lévy process, hence a Markov process, its infinitesimal generator $\mathcal{L}$ is defined as

$$\mathcal{L}(f)(x) = \lambda[f(x+1) - f(x)]\,.$$

Therefore, for any bounded Borel function f, the process

$$C_t^f = f(N_t) - f(0) - \int_0^t \mathcal{L}(f)(N_s)ds$$

is a martingale.

Furthermore,

$$C_t^f = \int_0^t [f(N_{s-} + 1) - f(N_{s-})]dM_s\,.$$

Exercise:

Extend the previous formula to functions $f(t,x)$ defined on $\mathbb{R}^+ \times \mathbb{R}$ and C^1 with respect to t, and prove that if

$$L_t = \exp(\log(1+\phi)N_t - \lambda\phi t)$$

then

$$dL_t = L_{t-}\phi dM_t\,.$$

2.4.1 Watanabe's characterization

Let N be a counting process and assume that there exists $\lambda > 0$ such that $M_t = N_t - \lambda t$ is a martingale. Then N is a Poisson process with intensity λ.

2.5 Change of Probability

If N is a Poisson process, then, for $\beta > -1$,

$$L_t = (1+\beta)^{N_t} e^{-\lambda\beta t}$$

is a strictly positive martingale with expectation equal to 1.

Let Q be the probability defined as $\frac{dQ}{dP}|_{\mathcal{F}_t} = L_t$.

The process N is a Q-Poisson process with intensity equal to $(1+\beta)\lambda$.

2.6 Hitting Times

Let $T_x = \inf\{t, N_t \geq x\}$. Then, for $n \leq x < n+1$, T_x is equal to the time of the n^{th}-jump of N, hence has a Gamma (n) law.

3 Inhomogeneous Poisson Processes

3.1 Definition

Let λ be an $\mathbb{R}^+$-valued function satisfying $\int_0^t \lambda(u)du < \infty, \forall t$.

An inhomogeneous Poisson process N with intensity λ is a counting process with independent increments which satisfies for $t > s$

$$P(N_t - N_s = n) = e^{-\Lambda(s,t)} \frac{(\Lambda(s,t))^n}{n!} \tag{3.1}$$

where $\Lambda(s,t) = \Lambda(t) - \Lambda(s) = \int_s^t \lambda(u)du$, and $\Lambda(t) = \int_0^t \lambda(u)du$.

3.2 Martingale Properties

Let N be an inhomogeneous Poisson process with deterministic intensity λ. The process

$$(M_t = N_t - \int_0^t \lambda(s)ds,\ t \geq 0)$$

is an $\mathbf{F}^N$-martingale, and the increasing function $\Lambda(t) = \int_0^t \lambda(s)ds$ is called the compensator of N.

Let ϕ be an $\mathbf{F}^N$- predictable process such that $E(\int_0^t |\phi_s|\lambda(s)ds) < \infty$ for every t. Then, the process $(\int_0^t \phi_s dM_s, t \geq 0)$ is an $\mathbf{F}^N$-martingale.

In particular,

$$E\left(\int_0^t \phi_s\, dN_s\right) = E\left(\int_0^t \phi_s \lambda(s) ds\right) . \tag{3.2}$$

Let H be an $\mathbf{F}^N$- predictable process. The following processes are martingales

$$\begin{aligned}
&(H\star M)_t = \int_0^t H_s dM_s = \int_0^t H_s dN_s - \int_0^t \lambda(s) H_s ds \\
&((H\star M)_t)^2 - \int_0^t \lambda(s) H_s^2 ds \\
&\exp\left(\int_0^t H_s dN_s - \int_0^t \lambda(s)(e^{H_s} - 1) ds\right) .
\end{aligned} \tag{3.3}$$

Proposition 3.1 (Compensation formula.) *For any real numbers u and α, for any t*

$$\begin{aligned} E(e^{iuN_t}) &= \exp((e^{iu}-1)\Lambda(t)) \\ E(e^{\alpha N_t}) &= \exp((e^{\alpha}-1)\Lambda(t))\,. \end{aligned}$$

3.3 Watanabe's Characterization

Proposition 3.2 (Watanabe characterization.) *Let N be a counting process and Λ an increasing, continuous function with zero value at time zero. Let us assume that $M_t = N_t - \Lambda(t)$ is a martingale. Then N is an inhomogeneous Poisson process with compensator Λ.*

3.4 Stochastic calculus

3.4.1 Integration by parts formula

Let $X_t = x + \int_0^t g_s dN_s$ and $Y_t = y + \int_0^t \widetilde{g}_s dN_s$, where g and $\tilde{g}$ are predictable processes.

$$
\begin{aligned}
X_t Y_t &= xy + \sum_{s \le t} \Delta(XY)_s = xy + \sum_{s \le t} Y_{s-} \Delta X_s + \sum_{s \le t} X_{s-} \Delta Y_s + \sum_{s \le t} \Delta X_s \, \Delta Y_s \\
&= xy + \int_0^t X_{s-} dY_s + \int_0^t Y_{s-} dX_s + [X, Y]_t
\end{aligned}
$$

where

$$
[X, Y]_t = \sum_{s \le t} \Delta X_s \, \Delta Y_s = \sum_{s \le t} \widetilde{g}_s g_s \Delta N_s = \int_0^t \widetilde{g}_s g_s dN_s \, .
$$

If $dX_t = h_t dt + g_t dN_t$ and $dY_t = \widetilde{h}_t dt + \widetilde{g}_t dN_t$, one gets

$$X_t Y_t = xy + \int_0^t Y_{s-} dX_s + \int_0^t Y_{s-} dX_s + [X, Y]_t$$

where

$$[X, Y]_t = \int_0^t \widetilde{g}_s g_s dN_s \,.$$

If $dX_t = g_t dM_t$ and $dY_t = \widetilde{g}_t dM_t$, the process $X_t Y_t - [X, Y]_t$ is a martingale.

3.4.2 Itô's Formula

Let N be a Poisson process and f a bounded Borel function. The decomposition

$$f(N_t) = f(N_0) + \sum_{0<s\le t} [f(N_s) - f(N_{s-})]$$

is trivial and corresponds to Itô's formula for a Poisson process.

$$\begin{aligned}\sum_{0<s\le t} [f(N_s) - f(N_{s-})] &= \sum_{0<s\le t} [f(N_{s-} + 1) - f(N_{s-})]\Delta N_s \\ &= \int_0^t [f(N_{s-} + 1) - f(N_{s-})] dN_s \,.\end{aligned}$$

Let

$$X_t = x + \int_0^t g_s dN_s = x + \sum_{T_n \le t} g_{T_n} \ ,$$

with g a predictable process.

The trivial equality

$$F(X_t) = F(X_0) + \sum_{s \le t} \left(F(X_s) - F(X_{s-})\right) \ ,$$

holds for any bounded function F.

Let

$$dX_t = h_t dt + g_t dM_t = (h_t - g_t\lambda(t))dt + g_t dN_t$$

and $F \in C^{1,1}(I\!R^+ \times I\!R)$. Then

$$\begin{aligned} F(t,X_t) &= F(0,X_0) + \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_{s-})(h_s - g_s\lambda(s))ds \\ &\quad + \sum_{s\leq t} F(s,X_s) - F(s,X_{s-}) \qquad (3.4) \\ &= F(0,X_0) + \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_{s-})dX_s \\ &\quad + \sum_{s\leq t} \left[F(s,X_s) - F(s,X_{s-}) - \partial_x F(s,X_{s-})g_s\Delta N_s\right] . \end{aligned}$$

The formula (3.4) can be written as

$$\begin{aligned}
F(t,X_t) \quad - \quad & F(0,X_0) = \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_s)(h_s - g_s\lambda(s))ds \\
& \qquad + \int_0^t [F(s,X_s) - F(s,X_{s-})]dN_s \\
= \quad & \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_{s-})dX_s \\
& \qquad + \int_0^t [F(s,X_s) - F(s,X_{s-}) - \partial_x F(s,X_{s-})g_s]dN_s \\
= \quad & \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_{s-})dX_s \\
& \qquad + \int_0^t [F(s,X_{s-}+g_s) - F(s,X_{s-}) - \partial_x F(s,X_{s-})g_s]dN_s \,.
\end{aligned}$$

3.5 Predictable Representation Property

Proposition 3.3 *Let $\mathbf{F}^N$ be the completion of the canonical filtration of the Poisson process N and $H \in L^2(\mathcal{F}^N_\infty)$, a square integrable random variable. Then, there exists a predictable process h such that*

$$H = E(H) + \int_0^\infty h_s dM_s$$

and $E(\int_0^\infty h_s^2 ds) < \infty$.

3.6 Independent Poisson Processes

Definition 3.4 *A process $(N^1, \cdots, N^d)$ is a d-dimensional* $\mathbf{F}$*-Poisson process if each N^j is a right-continuous adapted process such that $N_0^j = 0$ and if there exists constants λ_j such that for every $t \geq s \geq 0$*

$$P\left[\cap_{j=1}^d (N_t^j - N_s^j = n_j) | \mathcal{F}_s\right] = \prod_{j=1}^d e^{-\lambda_j (t-s)} \frac{(\lambda_j (t-s))^{n_j}}{n_j!} .$$

Proposition 3.5 *An* $\mathbf{F}$*-adapted process N is a d-dimensional* $\mathbf{F}$*-Poisson process if and only if*

(i) each N^j is an $\mathbf{F}$*-Poisson process*

(ii) no two N^j's jump simultaneously.

4 Stochastic intensity processes

4.1 Definition

Definition 4.1 *Let N be a counting process,* $\mathbf{F}$*-adapted and* $(\lambda_t, t \geq 0)$ *a non-negative* $\mathbf{F}$*- progressively measurable process such that for every* t*,* $\Lambda_t = \int_0^t \lambda_s ds < \infty$ P *a.s..*
The process N is an inhomogeneous Poisson process with stochastic intensity λ if for every non-negative $\mathbf{F}$*-predictable process* $(\phi_t, t \geq 0)$ *the following equality is satisfied*

$$E\left(\int_0^\infty \phi_s \, dN_s\right) = E\left(\int_0^\infty \phi_s \lambda_s ds\right) .$$

Therefore $(M_t = N_t - \Lambda_t,\ t \geq 0)$ is an $\mathbf{F}$-local martingale.

If ϕ is a predictable process such that $\forall t$, $E(\int_0^t |\phi_s| \lambda_s ds) < \infty$, then $(\int_0^t \phi_s dM_s,\ t \geq 0)$ is an $\mathbf{F}$-martingale.

An inhomogeneous Poisson process N with stochastic intensity λ_t can be viewed as time changed of $\tilde{N}$, a standard Poisson process $N_t = \tilde{N}_{\Lambda_t}$.

4.2 Itô's formula

The formula obtained in Section 3.4 generalizes to inhomogeneous Poisson process with stochastic intensity.

4.3 Exponential Martingales

Proposition 4.2 *Let N be an inhomogeneous Poisson process with stochastic intensity $(\lambda_t,\, t \geq 0)$, and $(\mu_t,\, t \geq 0)$ a predictable process such that $\int_0^t |\mu_s| \lambda_s \, ds < \infty$. Then, the process L defined by*

$$L_t = \begin{cases} \exp(-\int_0^t \mu_s \lambda_s \, ds) & \text{if} \quad t < T_1 \\ \prod_{n, T_n \leq t} (1 + \mu_{T_n}) \exp(-\int_0^t \mu_s \lambda_s \, ds) & \text{if} \quad t \geq T_1 \end{cases} \tag{4.1}$$

is a local martingale, solution of

$$dL_t = L_{t-} \mu_t dM_t, \quad L_0 = 1 \,. \tag{4.2}$$

Moreover, if μ is such that $\forall s, \mu_s > -1$,

$$L_t = \exp\left[- \int_0^t \mu_s \lambda_s ds + \int_0^t \ln(1 + \mu_s) \, dN_s \right] .$$

The local martingale L is denoted by $\mathcal{E}(\mu\star M)$ and named the Doléans-Dade exponential of the process $\mu\star M$.

This process can also be written

$$L_t = \prod_{0<s\leq t}(1+\mu_s\Delta N_s\,)\exp\left[-\int_0^t \mu_s\,\lambda_s\,ds\right]\,.$$

Moreover, if $\forall t, \mu_t > -1$, then L is a non-negative local martingale, therefore it is a supermartingale and

$$\begin{aligned}
L_t &= \exp\left[-\int_0^t \mu_s\lambda_s ds + \sum_{s\leq t}\ln(1+\mu_s)\Delta N_s\right] \\
&= \exp\left[-\int_0^t \mu_s\lambda_s ds + \int_0^t \ln(1+\mu_s)\,dN_s\right] \\
&= \exp\left[\int_0^t [\ln(1+\mu_s)-\mu_s]\,\lambda_s\,ds + \int_0^t \ln(1+\mu_s)\,dM_s\right]\,.
\end{aligned}$$

The process L is a martingale if $\forall t$, $E(L_t) = 1$.

If μ is not greater than -1, then the process L defined in (4.1) is still a local martingale which satisfies $dL_t = L_{t-}\mu_t dM_t$. However it may be negative.

4.4 Change of Probability

Let μ be a predictable process such that $\mu > -1$ and $\int_0^t \lambda_s |\mu_s| ds < \infty$.

Let L be the positive exponential local martingale solution of

$$dL_t = L_{t-} \mu_t dM_t \, .$$

Assume that L is a martingale and let Q be the probability measure equivalent to P defined on $\mathcal{F}_t$ by $Q|_{\mathcal{F}_t} = L_t \, P|_{\mathcal{F}_t}$.

Under Q, the process

$$(M_t^\mu \stackrel{def}{=} M_t - \int_0^t \mu_s \lambda_s ds = N_t - \int_0^t (\mu_s + 1) \lambda_s \, ds \, , t \geq 0)$$

is a local martingale.

5 An Elementary Model of Prices including Jumps

Suppose that S is a stochastic process with dynamics given by

$$dS_t = S_{t-}(b(t)dt + \phi(t)dM_t), \tag{5.1}$$

where M is the compensated compensated martingale associated with a Poisson process and where b, ϕ are deterministic continuous functions, such that $\phi > -1$. The solution of (5.1) is

$$\begin{aligned} S_t &= S_0 \exp\left[-\int_0^t \phi(s)\lambda(s)ds + \int_0^t b(s)ds\right] \prod_{s\leq t}(1 + \phi(s)\Delta N_s) \\ &= S_0 \exp\left[\int_0^t b(s)ds\right] \exp\left[\int_0^t \ln(1+\phi(s))dN_s - \int_0^t \phi(s)\lambda(s)ds\right] . \end{aligned}$$

Hence $S_t \exp\left(-\int_0^t b(s)ds\right)$ is a strictly positive martingale.

We denote by r the deterministic interest rate and by $R_t = \exp\left(-\int_0^t r(s)ds\right)$ the discounted factor.

Any strictly positive martingale L can be written as $dL_t = L_{t-}\mu_t dM_t$ with $\mu > -1$.

Let $(Y_t = R_t S_t L_t, t \geq 0)$. Itô's calculus yields to

$$\begin{aligned} dY_t &\cong Y_{t-}\left((b(t) - r(t))dt + \phi(t)\mu_t d[M]_t\right) \\ &\cong Y_{t-}\left(b(t) - r(t) + \phi(t)\mu_t \lambda(t)\right) dt\,. \end{aligned}$$

Hence, Y is a local martingale if and only if $\mu_t \lambda(t) = -\dfrac{b(t) - r(t)}{\phi(t)}$.

Assume that $\mu > -1$ and $Q|_{\mathcal{F}_t} = L_t\, P|_{\mathcal{F}_t}$. Under Q, N is a Poisson process with intensity $((\mu(s) + 1)\lambda(s), s \geq 0)$ and

$$dS_t = S_{t_-}(r(t)dt + \phi(t)dM_t^\mu)$$

where $(M^\mu(t) = N_t - \int_0^t (\mu(s) + 1)\lambda(s)\, ds\,,\, t \geq 0)$ is the compensated Q-martingale.

Hence Q is the unique equivalent martingale measure.

5.1 Poisson Bridges

Let N be a Poisson process with constant intensity λ and M its compensated martingale. Let $\mathcal{F}_t^N = \sigma(N_s, s \leq t)$ be its natural filtration and $\mathcal{G}_t = \sigma(N_s, s \leq t; N_T)$ the natural filtration enlarged with the terminal value of the process N.

Proposition 5.1 *The process* $\eta_t = N_t - \displaystyle\int_0^t \frac{N_T - N_s}{T - s} ds, \ t \leq T$ *is a* $\mathbf{G}$*-martingale with predictable bracket* $\Lambda_t = \displaystyle\int_0^t \frac{N_T - N_s}{T - s} ds.$

6 Compound Poisson Processes

6.1 Definition and Properties

Let λ be a positive number and $F(dy)$ be a probability law on $\mathbb{R}$. A (λ, F) compound Poisson process is a process $X = (X_t, t \geq 0)$ of the form

$$X_t = \sum_{k=1}^{N_t} Y_k$$

where N is a Poisson process with intensity $\lambda > 0$ and the $(Y_k, k \in \mathbb{N})$ are i.i.d. square integrable random variables with law $F(dy) = P(Y_1 \in dy)$, independent of N.

Proposition 6.1 *A compound Poisson process has stationary and independent increments; the cumulative function of X_t is*

$$P(X_t \leq x) = e^{-\lambda t} \sum_n \frac{(\lambda t)^n}{n!} F^{*n}(x)\,.$$

If $E(|Y_1|) < \infty$, the process $(Z_t = X_t - t\lambda E(Y_1), t \geq 0)$ is a martingale and in particular, $E(X_t) = \lambda t E(Y_1) = \lambda t \int y F(dy)$.

If $E(Y_1^2) < \infty$, the process $(Z_t^2 - t\lambda E(Y_1^2), t \geq 0)$ is a martingale and Var $(X_t) = \lambda t E(Y_1^2)$.

Corollary 6.2 *Let X be a (λ, F) compound Poisson process independent of W. Let*

$$dS_t = S_{t-}(\mu dt + dX_t)\,.$$

Then,

$$S_t = S_0 e^{\mu t} \prod_{k=1}^{N_t} (1 + Y_k)$$

In particular, if $1 + Y_1 > 0, a.s.$

$$S_t = S_0 \exp(\mu t + \sum_{k=1}^{N_t} \ln(1 + Y_k)\,.$$

The process $(S_t e^{-rt}, t \geq 0)$ is a martingale if and only if $\mu + \lambda E(Y_1) = r$.

6.2 Martingales

We now denote by ν the measure $\nu(dy) = \lambda F(dy)$, a (λ, F) compound Poisson process will be called a (λ, ν) compound Poisson process.

Proposition 6.3 *If X is a (λ, ν) compound Poisson process, for any α such that $\displaystyle\int_{-\infty}^{\infty} e^{\alpha u}\nu(du) < \infty$, the process*

$$Z_t^{(\alpha)} = \exp\left(\alpha X_t + t\left(\int_{-\infty}^{\infty}(1 - e^{\alpha u})\nu(du)\right)\right)$$

is a martingale and

$$E(e^{\alpha X_t}) = \exp\left(-t\left(\int_{-\infty}^{\infty}(1 - e^{\alpha u})\nu(du)\right)\right).$$

Proposition 6.4 *Let X be a (λ, ν) compound Poisson process, and f a bounded Borel function. Then, the process*

$$\exp\left(\sum_{k=1}^{N_t} f(Y_k) + t\int_{-\infty}^{\infty}(1 - e^{f(x)})\nu(dx)\right)$$

is a martingale. In particular

$$E\left(\exp\left(\sum_{k=1}^{N_t} f(Y_k)\right)\right) = \exp\left(-t\int_{-\infty}^{\infty}(1 - e^{f(x)})\nu(dx)\right)$$

For any bounded Borel function f, we denote by $\nu(f) = \int_{-\infty}^{\infty} f(x)\nu(dx)$ the product $\lambda E(f(Y_1))$.

Proposition 6.5 *Let X be a (λ, ν) compound Poisson process. The process*

$$M_t^f = \sum_{s \le t} f(\Delta X_s) 1\!\!1_{\{\Delta X_s \neq 0\}} - t\nu(f)$$

is a martingale. Conversely, suppose that X is a pure jump process and that there exists a finite positive measure σ such that

$$\sum_{s \le t} f(\Delta X_s) 1\!\!1_{\{\Delta X_s \neq 0\}} - t\sigma(f)$$

is a martingale for any f, then X is a $(\sigma(1), \sigma)$ compound Poisson process.

6.3 Hitting Times

Let $X_t = bt + \sum_{k=1}^{N_t} Y_k$.

Assume that the support of F is included in $]-\infty, 0]$.

The process $(\exp(uX_t - t\psi(u)), t \geq 0)$ is a martingale, with

$$\psi(u) = bu - \lambda \int_{-\infty}^{0} (1 - e^{uy}) F(dy) \,.$$

Let $T_x = \inf\{t : X_t > x\}$. Since the process X has no positive jumps, $X_{T_x} = x$.

Hence $E(e^{uX_{t\wedge T_x} - (t\wedge T_x)\psi(u)}) = 1$ and when t goes to infinity, one obtains

$$E(e^{ux - T_x\psi(u)} \mathbb{1}_{\{T_x < \infty\}}) = 1 \,.$$

If ψ admits an inverse $\psi^\sharp$, one gets if T_x is finite

$$E(e^{-\lambda T_x}) = e^{-x\psi^\sharp(\lambda)}\,.$$

Setting $Z_k = -Y_k$, the random variables Z_k can be interpreted as losses for insurance companies. The process $z + bt - \sum_{k=1}^{N_t} Z_k$ is called the Cramer-Lundberg risk process. The time $\tau = \inf\{t : X_t \leq 0\}$ is the bankruptcy time for the company.

One sided exponential law. If $F(dy) = \theta e^{\theta y} \mathbb{1}_{\{y<0\}} dy$, one obtains $\psi(u) = bu - \dfrac{\lambda u}{\theta + u}$, hence inverting ψ,

$$E(e^{-\kappa T_x} \mathbb{1}_{\{T_x<\infty\}}) = e^{-x\psi^\sharp(\kappa)} ,$$

with

$$\psi^\sharp(\kappa) = \frac{\lambda + \kappa - \theta b + \sqrt{(\lambda + \kappa - \theta b)^2 + 4\theta b}}{2b} .$$

6.4 Change of Measure

Let X be a (λ, ν) compound Poisson process, $\widetilde{\lambda} > 0$ and $\widetilde{F}$ a probability measure on $\mathbb{R}$, absolutely continuous w.r.t. F and $\widetilde{\nu}(dx) = \widetilde{\lambda}\widetilde{F}(dx)$. Let

$$L_t = \exp\left(t(\lambda - \widetilde{\lambda}) + \sum_{s \leq t} \ln\left(\frac{d\widetilde{\nu}}{\nu}\right)(\Delta X_s) \right) .$$

Set $dQ|_{\mathcal{F}_t} = L_t dP|_{\mathcal{F}_t}$.

Proposition 6.6 *Under Q, the process X is a $(\widetilde{\lambda}, \widetilde{\nu})$ compound Poisson process.*

6.5 Price Process

Let

$$dS_t = (\alpha S_{t-} + \beta)\, dt + (\gamma S_{t-} + \delta) dX_t \tag{6.1}$$

where X is a (λ, ν) compound Poisson process. The solution of (6.1) is a Markov process with infinitesimal generator

$$\mathcal{L}(f)(x) = \int_{-\infty}^{+\infty} [f(x + \gamma xy + \delta y) - f(x)]\, \nu(dy) + (\alpha x + \beta) f'(x)\,.$$

Let S be the solution of (6.1). The process $e^{-rt} S_t$ is a martingale if and only if

$$\gamma \int y\nu(dy) + \alpha = r, \quad \delta \int y\nu(dy) + \beta = 0\,.$$

Let $\widetilde{F}$ be a probability measure absolutely continuous with respect to F and

$$L_t = \exp\left(t(\lambda - \widetilde{\lambda}) + \sum_{s \leq t} \ln\left(\frac{\widetilde{\lambda}}{\lambda}\frac{d\widetilde{F}}{dF}\right)(\Delta X_s)\right) .$$

Let $dQ|_{\mathcal{F}_t} = L_t dP|_{\mathcal{F}_t}$. The process $(S_t e^{-rt}, t \geq 0)$ is a Q-martingale if and only if

$$\widetilde{\lambda}\gamma \int y\widetilde{F}(dy) + \alpha = r, \quad \widetilde{\lambda}\delta \int y\widetilde{F}(dy) + \beta = 0$$

Hence, there are an infinite number of e.m.m.. One can change the intensity of the Poisson process, and/or the law of the jumps.

7 Marked Point Processes

7.1 Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space, (Z_n) a sequence of random variables taking values in a measurable space $(E, \mathcal{E})$, and (T_n) an increasing sequence of positive random variables. For each Borel set $A \subset E$, we define the process $N_t(A) = \sum_n \mathbb{1}_{\{T_n \leq t\}} \mathbb{1}_{\{Z_n \in A\}}$ and the counting measure $\mu(\omega, ds, dz)$ by

$$\int_{]0,t]} \int_E H(s,z)\mu(ds,dz) = \sum_n H(T_n, Z_n) \mathbb{1}_{\{T_n \leq t\}} = \sum_{n=1}^{N_t} H(T_n, Z_n) \,.$$

The natural filtration of N is

$$\mathcal{F}_t^N = \sigma(N_s(A), s \leq t, A \in \mathcal{E}) \,.$$

In what follows, we assume that $N_t(A)$ admits the $\mathbf{F}$-predictable intensity $\lambda_t(A)$, i.e. there exists a predictable process $(\lambda_t(A), t \geq 0)$ such that

$$N_t(A) - \int_0^t \lambda_s(A) ds$$

is a martingale. Then, if $X_t = \sum_{n=1}^{N_t} H(T_n, Z_n)$ where H is an $\mathbf{F}$ predictable process which satisfies

$$E\left(\int_{]0,t]} \int_E |H(s,z)| \lambda_s(dz) ds\right) < \infty$$

the process

$$X_t - \int_0^t \int_E H(s,z) \lambda_s(dz) ds = \int_{]0,t]} \int_E H(s,z) \left[\mu(ds,dz) - \lambda_s(dz) ds\right]$$

is a martingale and in particular

$$E\left(\int_{]0,t]}\int_E H(s,z)\mu(ds,dz)\right) = E\left(\int_{]0,t]}\int_E H(s,z)\lambda_s(dz)ds\right).$$

The process N is called a marked point process.

This is a generalization of the compound Poisson process: we have introduced a spatial dimension for the size of jumps which are no more i.i.d. random variables.

7.2 Predictable representation property

Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a probability space where bF is the filtration generated by the marked point process q.

Then, any $(P, \mathbf{F})$-martingale M admits the representation

$$M_t = M_0 + \int_0^t \int_E H(s, x)(\mu(ds, dx) - \lambda_s(dx)ds)$$

where H is a predictable process such that

$$E\left(\int_0^t \int_E |H(s, x)|\lambda_s(dx)ds\right) < \infty$$

More generally

Proposition 7.1 *Let W be a Brownian motion and $\mu(ds, dz)$ a marked point process. Let $\mathcal{F}_t = \sigma(W_s, p([0, s], A); s \leq t, A \in \mathcal{E})$ completed. Then, any $(P, \mathbf{F})$ local martingale has the representation*

$$M_t = M_0 + \int_0^t \varphi_s dW_s + \int_0^t \int_E H(s, z)(\mu(ds, dz) - \lambda_s(dz)ds)$$

where φ is a predictable process such that $\int_0^t \varphi_s^2 ds < \infty$

7.3 Random Measure

More generally, one defines a random measure. Let $(E, \mathcal{E})$ be a measurable Polish space, i.e., a topological space endowed with a distance under which the space is complete and separable.

Definition 7.2 *A random measure μ on the space $I\!R^+ \times E$ is a family of nonnegative measures $\mu(\omega, dt, dx), \omega \in \Omega)$ defined on $I\!R^+ \times E)$ satisfying* $\mu(\omega, \{0\} \times E) = 0$

8 Poisson Point Processes

8.1 Poisson Measures

Let $(E, \mathcal{E})$ be a measurable Polish space. A random measure ϕ on E is a Poisson measure with intensity ν, where ν is a σ-finite measure on E, if for every Borel set $B \subset E$ with $\nu(B) < \infty$, $\phi(B)$ has a Poisson distribution with parameter $\nu(B)$ and if $B_i, i \leq n$ are disjoint sets, the variables $\phi(B_i), i \leq n$ are independent.

Example: Let ν be a probability measure, $Y_k, k \in I\!N$ be i.i.d. random variables with law ν and N a Poisson variable independent of Y_k's. The random measure $\sum_{k=1}^{N} \delta_{Y_k}$ is a Poisson measure. Here δ_e is the Dirac measure at point e.

8.2 Point Processes

Let $(E, \mathcal{E})$ be a measurable space and δ is an isolated additional point. We set $E_\delta = E \cup \delta, \mathcal{E}_\delta = \sigma(\mathcal{E}, \{\delta\})$.

Definition 8.1 *Let* $\mathbf{e}$ *be a stochastic process defined on a probability space* $(\Omega, \mathcal{F}, P)$*, taking values in* $(E_\delta, \mathcal{E}_\delta)$*. The process* $\mathbf{e}$ *is a point process if*

(i) the map $(t, \omega) \to \mathbf{e}_t(\omega)$ *is* $\mathcal{B}(]0, \infty[) \otimes \mathcal{F}$*-measurable*
(ii) the set $D_\omega = \{t : \mathbf{e}_t(\omega) \neq \delta\}$ *is a.s. countable.*

For every measurable set B of $]0, \infty[\times E$, we set

$$N^B(\omega) = \sum_{s \geq 0} 1\!\!1_B(s, \mathbf{e}_s(\omega)) \, .$$

In particular, if $B =]0, t] \times \Gamma$, we write

$$N_t^\Gamma = N^B = \mathrm{Card}\{s \leq t \, : \, \mathbf{e}(s) \in \Gamma\} \, .$$

Let the space (Ω, P) be endowed with a filtration $\mathbf{F}$. A point process is $\mathbf{F}$-adapted if, for any $\Gamma \in \mathcal{E}$, the process N^Γ is $\mathbf{F}$-adapted. For any

$\Gamma \in \mathcal{E}_\delta$, we define a point process $\mathbf{e}^\Gamma$ by

$$\begin{aligned} \mathbf{e}^\Gamma_t(\omega) &= \mathbf{e}_t(\omega) \,\mathrm{if}\, \mathbf{e}_t(\omega) \in \Gamma \\ \mathbf{e}^\Gamma_t(\omega) &= \delta \,\mathrm{otherwise} \end{aligned}$$

Definition 8.2 *A point process* $\mathbf{e}$ *is discrete if* $N^E_t < \infty$ *a.s. for every* t*. A point process is* σ*-discrete if there is a sequence* E_n *of sets with* $E = \cup E_n$ *such that each* $\mathbf{e}^{E_n}$ *is discrete.*

8.3 Poisson Point Processes

Definition 8.3 *An* $\mathbf{F}$*-Poisson point process is a* σ*-discrete point process such that*

(i) the process $\mathbf{e}$ *is* $\mathbf{F}$*-adapted*

(ii) for any s *and* t *and any* $\Gamma \in \mathcal{E}$*, the law of* $N^{\Gamma}_{s+t} - N^{\Gamma}_{t}$ *conditioned on* $\mathcal{F}_t$ *is the same as the law of* N^{Γ}_{t}*.*

Therefore, for any disjoint family $(\Gamma_i, i = 1, \ldots, d)$, the process $(N^{\Gamma_i}_t, i = 1, \cdots, d)$ is a d-dimensional Poisson process. Moreover, if N^{Γ} is finite almost surely, then $E(N^{\Gamma}_t) < \infty$ and the quantity $\frac{1}{t}E(N^{\Gamma}_t)$ does not depend on t.

Definition 8.4 *The* σ*-finite measure on* $\mathcal{E}$ *defined by*

$$\mathbf{n}(\Gamma) = \frac{1}{t}E(N^{\Gamma}_t)$$

is called the characteristic measure of $\mathbf{e}$*.*

If $\mathbf{n}(\Gamma) < \infty$, the process $N^{\Gamma}_t - t\mathbf{n}(\Gamma)$ is a martingale.

Proposition 8.5 (Compensation formula.) *Let H be a positive process vanishing at δ, measurable with respect to $\mathcal{P} \times \mathcal{E}_\delta$. Then*

$$E\left[\sum_{s\geq 0} H(s,\omega,\mathbf{e}_s(\omega))\right] = E\left[\int_0^\infty ds \int H(s,\omega,u)\mathbf{n}(du)\right].$$

If, for any t, $E\left[\int_0^t ds \int H(s,\omega,u)\mathbf{n}(du)\right] < \infty$, the process

$$\sum_{s\leq t} H(s,\omega,\mathbf{e}_s(\omega)) - \int_0^t ds \int H(s,\omega,u)\mathbf{n}(du)$$

a martingale.

PROOF: It is enough to prove this formula for $H(s,\omega,u) = K(s,\omega)\mathbb{1}_\Gamma(u)$. In that case, $N_t^\Gamma - t\mathbf{n}(\Gamma)$ is a martingale. $\triangle$

Proposition 8.6 (Exponential formula.) *If f is a $\mathcal{B}\otimes\mathcal{E}$-measurable function such that $\int_0^t ds \int |f(s,u)|\mathbf{n}(du) < \infty$ for every t, then,*

$$E\left[\exp\left(i\sum_{0<s\le t} f(s,\mathbf{e}_s)\right)\right] = \exp\left(\int_0^t ds \int (e^{if(s,u)}-1)n(du)\right)$$

Moreover, if $f \ge 0$,

$$E\left[\exp\left(-\sum_{0<s\le t} f(s,\mathbf{e}_s)\right)\right] = \exp\left(-\int_0^t ds \int (1-e^{-f(s,u)})\mathbf{n}(du)\right).$$

If ν is finite, then the associated counting process is a compound Poisson process.

8.4 The Itô measure of Brownian excursions

Let $(B_t, t \ge 0)$ be a Brownian motion and (τ_s) be the inverse of the local time (L_t) at level 0. The set $\{\cup_{s\ge 0}]\tau_{s-}(\omega), \tau_s(\omega)[\,\}$ is (almost surely) equal to the complement of the zero set $\{u : B_u(\omega) = 0\}$. The excursion process $(\mathbf{e}_s, s \geq 0)$ is defined as

$$\mathbf{e}_s(\omega)(t) = \mathbb{1}_{\{t \leq \tau_s - \tau_{s-}\}} B_{\tau_{s-} + t}, t \geq 0.$$

This is a path-valued process $\mathbf{e} : \mathbb{R}_+ \to \Omega_*$, where

$$\Omega_* = \{\epsilon : \mathbb{R}_+ \to \mathbb{R} : \exists V(\epsilon) < \infty, \text{ with } \epsilon(V(\epsilon) + t) = 0, \forall t \geq 0$$

$$\epsilon(u) \neq 0, \forall\, 0 < u < V(\epsilon), \, \epsilon(0) = 0, \, \epsilon \text{ is continuous}\}.$$

Hence, $V(\epsilon)$ is the lifetime of ϵ.

The excursion process is a Poisson Point Process; its characteristic measure $\mathbf{n}$ evaluated on the set Γ, i.e., $\mathbf{n}(\Gamma)$, is defined as the intensity of the Poisson process

$$N_t^\Gamma \stackrel{def}{=} \sum_{s \leq t} \mathbb{1}_{e_s \in \Gamma}.$$

The quantity $\mathbf{n}(\Gamma)$ is the positive real γ such that $N_t^\Gamma - t\gamma$ is an $(\mathcal{F}_{\tau_t})$-martingale.

From Itô's theorem, the excursion process is a Poisson point process.

Conditionally on $V = v$, the process

$$(|\epsilon_u|, u \leq v)$$

is a BES(3) bridge of length v. Let $M(\epsilon) = \sup_{u \leq v} |\epsilon_u|$. Then,

$$\mathbf{n}(M(\epsilon) \in dm) = \frac{dm}{m^2}$$

and, conditionally on $M = m$, the two processes $\epsilon_u, u \leq T_m$ and $\epsilon_{V-u}, u \leq V - T_m$ are two independent BES(3) processes considered up to their first hitting time of m. The Itô-Williams description of the measure $\mathbf{n}$ is

$$\mathbf{n}(d\epsilon) = \int_0^\infty \mathbf{n}_V(dv) \, \frac{1}{2} (\Pi_+^v + \Pi_-^v) \, (d\epsilon)$$

where $\mathbf{n}_V(dv) = \dfrac{dv}{\sqrt{2\pi v^3}}$ is the law of the lifetime V under $\mathbf{n}$ and Π_+^v (resp. Π_-^v) is the law of the standard Bessel Bridge (resp. the law of its opposite) with dimension 3 and length v.