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1 Definition

A mixed process is a process X

$$X_t = X_0 + \int_0^t h_s\, ds + \int_0^t f_s dW_s + \int_0^t g_s dM_s\, .$$

The jump times of the process X are those of N, the jump of X is $\Delta X_t = X_t - X_{t-} = g_t \Delta N_t$.

2 Itô's Formula

Let

$$
\begin{aligned}
dX_t &= h_t dt + f_t dW_t + g_t dM_t \,, \\
dY_t &= \tilde{h}_t dt + \tilde{f}_t dW_t + \tilde{g}_t dM_t \,.
\end{aligned}
$$

2.1 Integration by Parts

The integration by parts formula reads

$$d(XY) = X_{-}\star dY + Y_{-}\star dX + d[X,Y]$$

with $d[X,Y]_t = f_t \tilde{f}_t dt + g_t \tilde{g}_t dN_t$.

2.2 Itô's Formula: One Dimensional Case

Let F be a $C^{1,2}$ function, and

$$dX_t = h_t dt + f_t dW_t + g_t dM_t \,.$$

Then,

$$\begin{aligned} F(t, X_t) \quad &= \quad F(0, X_0) + \int_0^t \partial_s F(s, X_s)\, ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\ &\quad + \frac{1}{2} \int_0^t \partial_{xx} F(s, X_s) f_s^2 \, ds \qquad (2.1) \\ &\quad + \sum_{s \le t} [F(s, X_s) - F(s, X_{s^-}) - \partial_x F(s, X_{s-}) \Delta X_s] \,. \end{aligned}$$

3 Predictable Representation Theorem

Let Z be a square integrable $\mathbf{F}$-martingale. There exist two predictable processes (H_1, H_2) such that $Z = z + H_1 \cdot W + H_2 \cdot M$, with

$$\int_0^t (H_1(s))^2 ds < \infty \, , \int_0^t (H_2(s))^2 \lambda(s) ds < \infty, \; a.s.$$

4 Change of probability

4.1 Exponential Martingales

Let γ and ψ be two predictable processes such that $\gamma_t > -1$. The solution of

$$dL_t = L(t-)(\psi_t dW_t + \gamma_t dM_t) \tag{4.1}$$

is the strictly positive exponential local martingale

$$\begin{aligned}
L_t &= L_0 \prod_{s \le t}(1 + \gamma_s \Delta N_s)\, e^{-\int_0^t \gamma_s \lambda(s) ds} \;\; \exp\left(\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right) \\
&= L_0 \exp\left(\int_0^t \ln(1+\gamma_s) dN_s - \int_0^t \lambda(s)\gamma_s ds + \int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right) \\
&= L_0 \exp\left(\int_0^t \ln(1+\gamma_s) dM_s + \int_0^t [\ln(1+\gamma_s) - \gamma_s]\lambda_s ds\right) \\
&\qquad \times \exp\left(\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right) .
\end{aligned}$$

4.2 Girsanov's Theorem

If P and Q are equivalent probabilities, there exist two predictable processes ψ and γ, with $\gamma > -1$ such that the Radon-Nikodym density L is of the form

$$dL_t = L_{t-}(\psi_t dW_t + \gamma_t dM_t)\,.$$

Then, $\tilde{W}$ and $\tilde{M}$ are Q-martingales where

$$\tilde{W}_t = W_t - \int_0^t \psi_s ds,\;\tilde{M}_t = M_t - \int_0^t \lambda(s)\gamma_s ds\,.$$

5 Mixed Processes in Finance

The dynamics of the price are supposed to be given by

$$dS_t = S(t-)(b_t dt + \sigma_t dW_t + \phi_t dM_t) \tag{5.1}$$

or in closed form

$$S_t = S_0 \exp\left(\int_0^t b_s ds\right) \mathcal{E}(\phi\star M)_t \, \mathcal{E}(\sigma\star W)_t \, .$$

$$dS_t = S_{t-}((r-\delta)dt + \sigma dW_t + \phi dM_t)$$

Here $M_t = N_t - \lambda t$ is a Q-martingale.

We can write

$$E(e^{-rt}(K-S_t)^+) = E\left(e^{-\delta t}Z_t(\frac{KS_0}{S_t} - S_0)^+\right) = \tilde{E}\left(e^{-\delta t}(\frac{KS_0}{S_t} - S_0)^+\right) .$$

Setting $\widehat{Q}$ where $d\widehat{Q}|_{\mathcal{F}_t} = Z_t dQ|_{\mathcal{F}_t}$ the process $\widehat{S} = 1/S$ follows

$$d\widehat{S}_t = \widehat{S}_{t-}((\delta - r)dt - \sigma d\tilde{W}_t - \frac{\phi}{1+\phi}d\tilde{M}_t)$$

where $\widehat{W_t} = W_t - \sigma t$ is a $\widehat{Q}$-BM and $\widehat{M_t} = N_t - \lambda(1+\phi)t$ is a $\widehat{Q}$-martingale. Hence, denoting by C_E (resp. P_E) the price of a European call (resp. put)

$$P_E(x, K, r, \delta; \sigma, \phi, \lambda) = C_E(K, x, \delta, r; \sigma, -\frac{\phi}{1+\phi}, \lambda(1+\phi)) .$$

5.1 Hitting Times

Let $S_t = S_0 e^{X_t}$. Let us denote by $T_L(S)$ the first passage time of the process S at level L, for $L > S_0$ as $T_L(S) = \inf\{t \geq 0 : S_t \geq L\}$ and its companion first passage time $T_\ell(X) = T_L(S)$, the first passage time of the process X at level $\ell = \ln(L/S_0)$, for $\ell > 0$ as $T_\ell(X) = \inf\{t \geq 0 : X_t \geq \ell\}$.

The process $Z^{(k)}$ is the martingale

$$Z_t^{(k)} = S_0^k \mathcal{E}(\phi_k M)_t \, \mathcal{E}(\sigma k W)_t = S_0^k \exp(kX_t - tg(k)) \tag{5.2}$$

and $g(k)$ is the so-called Lévy exponent

$$g(k) = bk + \frac{1}{2}\sigma^2 k(k-1) + \lambda[(1+\phi)^k - 1 - k\phi]\,. \tag{5.3}$$

When there are no positive jumps, i.e., $\phi \in]-1, 0[$,

$$E[\exp(-g(k)T_\ell)] = \exp(-k\ell)\,.$$

Inverting the Lévy exponent $g(k)$ we obtain

$$\begin{aligned} E(\exp(-uT_\ell)) &= \exp(-g^{-1}(u)\,\ell), \quad \text{for } S_0 < L; \\ E(\exp(-uT_\ell)) &= 1 \ \text{ otherwise}. \end{aligned} \tag{5.4}$$

Here $g^{-1}(u)$ is the positive root of $g(k) = u$.

If the jump size is positive there is a non zero probability that X_{T_ℓ} is strictly greater than ℓ. In this case, we introduce the so-called overshoot $K(\ell)$

$$K(\ell) = X_{T_\ell} - \ell\,. \tag{5.5}$$

The difficulty is to obtain the law of the overshoot.

5.2 Affine Jump Diffusion Models

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t + dX_t$$

where X is a (λ, ν) compound Poisson process. The infinitesimal generator of S is

$$\mathcal{L}f = \partial_t f + \mu(x)\partial_x f + \frac{1}{2}\mathrm{Tr}(\partial_{xx} f \sigma\sigma^T) + \lambda \int (f(x+z,t) - f(x,t))d\nu(z)$$

for $f \in C_b^2$.

Proposition 5.1 *Suppose that* $\mu(x) = \mu_0 + \mu_1 x\,; \sigma^2(x) = \sigma_0 + \sigma_1 x$ *are affine functions, and that* $\int e^{zy}\nu(dy) < \infty, \forall z$. *Then, for any affine function* $\psi(x) = \psi_0 + \psi_1 x$, *there exist two functions* α *and* β *such that*

$$E(e^{\theta S_T} \exp\left(-\int_t^T \psi(S_s)ds\right) |\mathcal{F}_t) = e^{\alpha(t)+\beta(t)S_t}\,.$$

5.3 Mixed Processes involving Compound Poisson Processes

Proposition 5.2 *Let W be a Brownian motion and X be a (λ, F) compound Poisson process independent of W. Let*

$$dS_t = S_{t-}(\mu dt + \sigma dW_t + dX_t)\,.$$

The process $(S_t e^{-rt}, t \geq 0)$ is a martingale if and only if $\mu + \lambda E(Y_1) = r$.

5.4 General Jump-Diffusion Processes

Let W be a Brownian motion and $p(ds, dz)$ a marked point process. Let $\mathcal{F}_t = \sigma(W_s, p([0, s], A), A \in \mathcal{E};\ s \leq t)$. The solution of

$$dS_t = S_{t-}(\mu_t dt + \sigma_t dW_t + \int_{I\!R} \varphi(t, x) p(dt, dx))$$

can be written in an exponential form as

$$S_t = S_0 \exp\left(\int_0^t \left[\mu_s - \frac{1}{2}\sigma_s^2 \right] ds + \int_0^t \sigma_s dW_s \right) \prod_{n=1}^{N_t} (1 + \varphi(T_n, Z_n))$$

where $N_t = p((0, t], I\!R)$ is the total number of jumps.

6 Incompleteness

6.1 Risk-neutral Probability Measures Set

Assume that

$$d(RS)_t = R(t)S_{t-}([b(t) - r(t)]dt + \sigma(t)dW_t + \phi(t)dM_t) \tag{6.1}$$

The set $\mathcal{Q}$ of e.m.m. is the set of probability measures $P^{\psi,\gamma}$ such that $\left.\frac{dP^{\psi,\gamma}}{dP}\right|_{\mathcal{F}_t} = L_t^{\psi,\gamma}$ where $L_t^{\psi,\gamma} \stackrel{def}{=} L_t^{\psi,W} \, L_t^{\gamma,M}$

$$\left\{\begin{array}{lcll} L_t^{\psi,W} & = & \mathcal{E}(\psi\star W)_t & = \exp\left[\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right] \\ \\ L_t^{\gamma,M} & = & \mathcal{E}(\gamma\star M)_t & = \exp\left[\int_0^t \ln(1+\gamma_s)dN_s - \int_0^t \lambda(s)\gamma_s ds\right] . \end{array}\right.$$

$$b(t) - r(t) + \sigma(t)\psi_t + \lambda(t)\phi(t)\gamma_t = 0 \ \ , \ \ dP \otimes dtp.s. \tag{6.2}$$

6.2 The Range of Prices for European Call Case

We study now the range of viable prices associated with a European call option, that is, the interval $]\inf_{\gamma\in\Gamma} V_t^\gamma\,, \sup_{\gamma\in\Gamma} V_t^\gamma[$, for $B = (S_T - K)^+$. We denote by $\mathcal{BS}$ the Black-Scholes function, that is, the function such that

$$R(t)\mathcal{BS}(x,t) = E(R(T)(X_T - K)^+\,|X_t = x)\;,\;\; \mathcal{BS}(x,T) = (x-K)^+$$

when

$$dX_t = X_t(r(t)dt + \sigma(t)\,dW_t)\,. \tag{6.3}$$

Theorem 6.1 *Let $P^\gamma \in \mathcal{Q}$. Then, the associated viable price is bounded below by the Black-Scholes function, evaluated at the underlying asset value, and bounded above by the underlying asset value, i.e.,*

$$R(t)\mathcal{BS}(S_t,t) \le E^\gamma(R(T)\,(S_T-K)^+|\mathcal{F}_t) \le R(t)\,S_t$$

The range of viable prices $V_t^\gamma = \dfrac{R(T)}{R(t)}E^\gamma((S_T-K)^+|\mathcal{F}_t)$ is exactly the interval $]\mathcal{BS}(S_t,t),\; S_t[$.

7 Complete Markets

7.1 A Two Assets Model

$$
\begin{aligned}
dS_1(t) &= S_1(t-)(b_1(t)dt + \sigma_1(t)dW_t + \phi_1(t)dM_t) \\
dS_2(t) &= S_2(t_-)(b_2(t)dt + \sigma_2(t)dW_t + \phi_2(t)dM_t)\,,
\end{aligned}
$$

Under the conditions

$$
\begin{aligned}
|\sigma_1(t)\phi_2(t) - \sigma_2(t)\phi_1(t)| \geq \epsilon &> 0 \\
\frac{[b_2(t) - \lambda(t)\phi_2(t) - r(t)]\,\sigma_1(t) - [b_1(t) - \lambda(t)\phi_1(t) - r(t)]\,\sigma_2(t)}{\sigma_2(t)\phi_1(t) - \sigma_1(t)\phi_2(t)} &> 0
\end{aligned}
$$

we obtain an arbitrage free complete market. The risk-neutral probability is defined by $Q|_{\mathcal{F}_t} = L_t^{\psi,\gamma}\, P|_{\mathcal{F}_t}$, where

$$
dL_t = L_{t-}[\psi_t dW_t + \gamma_t dM_t]
$$

and

$$
b_i(t) - r(t) + \sigma_i(t)\psi_t + \lambda(t)\phi_i(t)\gamma_t = 0, i = 1, 2
$$

7.2 Structure equation

The equation

$$d[X,X]_t = dt + \beta(t)dX_t \tag{7.1}$$

as a unique solution which is a martingale, for β a deterministic function.

7.2.1 Dritschel and Protter's model

$$dS_t = S_{t-}\sigma dZ_s$$

where Z is a martingale satisfying (7.1) with β constant, $-2 \leq \beta < 0$

7.2.2 Privault's model

Let ϕ and α be two bounded deterministic Borel functions defined on $I\!R^+$. Let

$$\lambda(t) = \begin{cases} \alpha^2(t)/\phi^2(t) & \text{if } \phi(t) \neq 0, \\ 0 & \text{if } \phi(t) = 0,\ t \in I\!R^+. \end{cases}$$

Let B be a standard Brownian motion, and N an inhomogeneous Poisson process with intensity λ. The process defined as $(X_t, t \geq 0)$

$$dX_t = 1\!\!1_{\{\phi(t)=0\}} dB_t + \frac{\phi(t)}{\alpha(t)} \left(dN_t - \lambda(t) dt \right), t \in I\!R^+, X_0 = 0 \tag{7.2}$$

satisfies the structure equation

$$d[X,X]_t = dt + \frac{\phi(t)}{\alpha(t)} dX_t \; .$$