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1 Infinitely Divisible Random Variables

1.1 Definition

A random variable X taking values in $\mathbb{R}^d$ is infinitely divisible if its characteristic function

$$\hat{\mu}(u) = E(e^{i(u\cdot X)}) = (\hat{\mu}_n)^n$$

where $\hat{\mu}_n$ is a vcharacteristic function.

Example 1.1 A Gaussian variable, a Cauchy variable, a Poisson variable and the hitting time of the level a for a Brownian motion are examples of infinitely divisible random variables.

A Lévy measure ν is a positive measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{\mathbb{R}^d\setminus\{0\}} \min(1, \|x\|^2)\nu(dx) < \infty\,.$$

Proposition 1.2 (Lévy-Khintchine representation.)
If X is an infinitely divisible random variable, there exists a triple (m, A, ν) where $m \in \mathbb{R}^d$, A is a non-negative quadratic form and ν is a Lévy measure such that

$$\hat{\mu}(u) = \exp\left(i(u{\cdot}m) - (u{\cdot}Au) + \int_{\mathbb{R}^d} (e^{i(u\cdot x)} - 1 - i(u{\cdot}x)\mathbb{1}_{\{|x|\leq 1\}})\nu(dx)\right) .$$

Example 1.3 Gaussian laws. The Gaussian law $\mathcal{N}(m, \sigma^2)$ has the characteristic function $\exp(ium - u^2\sigma^2/2)$. Its characteristic triple is $(m, \sigma, 0)$.

Cauchy laws. The standard Cauchy law has the characteristic function

$$\exp(-c|u|) = \exp\left(\frac{c}{\pi}\int_{-\infty}^{\infty}(e^{iux} - 1)x^{-2}dx\right).$$

Its characteristic triple is (in terms of m_0) $(0, 0, \pi^{-1}x^{-2}dx)$.

Gamma laws. The Gamma law $\Gamma(a, \nu)$ has the characteristic function

$$(1 - iu/\nu)^{-a} = \exp\left(a\int_{0}^{\infty}(e^{iux} - 1)e^{-\nu x}\frac{dx}{x}\right).$$

Its characteristic triple is (in terms of m_0) $(0, 0, \mathbb{1}_{\{x>0\}}\, ax^{-1}e^{-\nu x}dx)$. The exponential Lévy process corresponds to the case $a = 1$.

1.2 Stable Random Variables

A random variable is stable if for any $a > 0$, there exist $b > 0$ and $c \in I\!R$ such that $[\hat{\mu}(u)]^a = \hat{\mu}(bu)\, e^{icu}$.

Proposition 1.4 *The characteristic function of a stable law can be written*

$$\hat{\mu}(u) = \begin{cases} \exp(ibu - \frac{1}{2}\sigma^2 u^2), & \text{for}\, \alpha = 2 \\ \exp\left(-\gamma|u|^\alpha[1 - i\beta\, \mathrm{sgn}(u) \tan(\pi\alpha/2)]\right), & \text{for}\, \alpha \neq 1, \neq 2 \\ \exp\left(\gamma|u|(1 - i\beta v \ln|u|)\right), & \alpha = 1 \end{cases},$$

where $\beta \in [-1, 1]$. For $\alpha \neq 2$, the Lévy measure of a stable law is absolutely continuous with respect to the Lebesgue measure, with density

$$\nu(dx) = \begin{cases} c^+ x^{-\alpha-1} dx & \text{if } x > 0 \\ c^-|x|^{-\alpha-1} dx & \text{if } x < 0\,. \end{cases}$$

Here $c^\pm$ are non-negative real numbers, such that $\beta = (c^+ - c^-)/(c^+ + c^-)$.

More precisely,

$$\begin{aligned} c^+ &= \frac{1}{2}(1+\beta)\frac{\alpha\gamma}{\Gamma(1-\alpha)\cos(\alpha\pi/2)}\,, \\ c^- &= \frac{1}{2}(1-\beta)\frac{\alpha\gamma}{\Gamma(1-\alpha)\cos(\alpha\pi/2)}\,. \end{aligned}$$

The associated Lévy process is called a stable Lévy process with index α and skewness β.

Example 1.5 A Gaussian variable is stable with $\alpha = 2$. The Cauchy law is stable with $\alpha = 1$.

2 Lévy Processes

2.1 Definition and Main Properties

A $\mathbb{R}^d$-valued process X such that $X_0 = 0$ is a Lévy process if

a- for every $s, t, 0 \leq s \leq t < \infty, X_t - X_s$ is independent of $\mathcal{F}_s^X$

b- for every s, t the r.vs $X_{t+s} - X_t$ and X_s have the same law.

c- X is continuous in probability, i.e., $P(|X_t - X_s| > \epsilon) \rightarrow 0$ when $s \rightarrow t$ for every $\epsilon > 0$.

Brownian motion, Poisson process and compound Poisson processes are examples of Lévy processes.

Proposition 2.1 *Let X be a Lévy process. Then, for any fixed t,*

$$(X_u, u \leq t) \stackrel{law}{=} (X_t - X_{t-u}, u \leq t)$$

Consequently, $(X_t, \inf_{u \leq t} X_u) \stackrel{law}{=} (X_t, X_t - \sup_{u \leq t} X_u)$
and for any $\alpha \in \mathbb{R}$,

$$\int_0^t du\, e^{\alpha X_u} \stackrel{law}{=} e^{\alpha X_t} \int_0^t du\, e^{-\alpha X_u} .$$

2.2 Poisson Point Process, Lévy Measure

For every Borel set $\Lambda \in I\!R^d$, such that $0 \notin \bar{\Lambda}$, where $\bar{\Lambda}$ is the closure of Λ, we define

$$N_t^{\Lambda} = \sum_{0<s\leq t} 1\!\!1_{\Lambda}(\Delta X_s),$$

to be the number of jumps before time t which take values in Λ.

Definition 2.2 *The σ-additive measure ν defined on $I\!R^d - \{0\}$ by*

$$\nu(\Lambda) = E(N_1^{\Lambda})$$

is called the Lévy measure of the process X.

Proposition 2.3 *Assume $\nu(1) < \infty$. Then, the process*

$$N_t^{\Lambda} = \sum_{0<s\leq t} 1\!\!1_{\Lambda}(\Delta X_s)$$

is a Poisson process with intensity $\nu(\Lambda)$. The processes N^{Λ} and N^{Γ} are independent if $\nu(\Gamma \cap \Lambda) = 0$, in particular if Λ and Γ are disjoint. Hence, the jump process of a Lévy process is a Poisson point process.

Let Λ be a Borel set of $I\!R^d$ with $0 \notin \bar{\Lambda}$, and f a Borel function defined on Λ. We have

$$\int_\Lambda f(x) N_t(\omega, dx) = \sum_{0<s\leq t} f(\Delta X_s(\omega)) 1\!\!1_\Lambda(\Delta X_s(\omega)) \,.$$

Proposition 2.4 (Compensation formula.) *If f is bounded and vanishes in a neighborhood of 0,*

$$E(\sum_{0<s\leq t} f(\Delta X_s)) = t \int_{I\!R^d} f(x)\nu(dx) \,.$$

More generally, for any bounded predictable process H

$$E\left[\sum_{s\leq t} H_s f(\Delta X_s)\right] = E\left[\int_0^t ds H_s \int f(x) d\nu(x)\right]$$

and if H is a predictable function (i.e. $H : \Omega \times I\!R^+ \times I\!R^d \to I\!R$ is $\mathcal{P} \times \mathcal{B}$ measurable)

$$E\left[\sum_{s\le t} H_s(\omega, \Delta X_s)\right] = E\left[\int_0^t ds \int d\nu(x) H_s(\omega, x)\right].$$

Both sides are well defined and finite if

$$E\left[\int_0^t ds \int d\nu(x) |H_s(\omega, x)|\right] < \infty$$

PROOF: Let Λ be a Borel set of $I\!R^d$ with $0 \notin \bar{\Lambda}$, f a Borel function defined on Λ. The process N^Λ being a Poisson process with intensity $\nu(\Lambda)$, we have

$$E\left(\int_\Lambda f(x) N_t(\cdot, dx)\right) = t \int_\Lambda f(x)\nu(dx),$$

and if $f \mathbb{1}_\Lambda \in L^2(d\nu)$

$$E\left(\int_\Lambda f(x) N_t(\cdot, dx) - t \int_\Lambda f(x)\nu(dx)\right)^2 = t\int_\Lambda f^2(x)\nu(dx)\,.$$

$\triangle$

Proposition 2.5 (Exponential formula.)
Let X be a Lévy process and ν its Lévy measure. For all t and all Borel function f defined on $\mathbb{R}^+ \times \mathbb{R}^d$ such that $\int_0^t ds \int |1-e^{f(s,x)}|\nu(dx) < \infty$, one has

$$E\left[\exp\left(\sum_{s\le t} f(s,\Delta X_s)\mathbb{1}_{\{\Delta X_s \neq 0\}}\right)\right] = \exp\left(-\int_0^t ds \int (1-e^{f(s,x)})\nu(dx)\right)\,.$$

Warning 2.6 The above property does not extend to predictable functions.

Proposition 2.7 (Lévy-Itô's decomposition.) *If X is a R^d-valued Lévy process, such that $\int_{\{|x|<1\}} |x|\nu(dx) < \infty$ it can be decomposed into $X = Y^{(0)} + Y^{(1)} + Y^{(2)} + Y^{(3)}$ where $Y^{(0)}$ is a constant drift, $Y^{(1)}$ is a linear transform of a Brownian motion, $Y^{(2)}$ is a compound Poisson process with jump size greater than or equal to 1 and $Y^{(3)}$ is a Lévy process.*

2.3 Lévy-Khintchine Representation

Proposition 2.8 *Let X be a Lévy process. There exists $m \in \mathbb{R}^d$, a non-negative semi-definite quadratic form A, a Lévy measure ν such that for $u \in \mathbb{R}^d$*

$$E(\exp(i(u \cdot X_1))) =$$

$$\exp\left(i(u \cdot m) - \frac{(u \cdot Au)}{2} + \int_{\mathbb{R}^d} (e^{i(u \cdot x)} - 1 - i(u \cdot x)\mathbb{1}_{|x| \leq 1})\nu(dx)\right) \quad (2.1)$$

where ν is the Lévy measure.

2.4 Representation Theorem

Proposition 2.9 *Let X be a $\mathbb{R}^d$-valued Lévy process and $\mathbf{F}^X$ its natural filtration. Let M be a locally square integrable martingale with $M_0 = m$. Then, there exists a family (φ, ψ) of predictable processes such that*

$$\int_0^t |\varphi_s^i|^2 ds < \infty, \text{ a.s.}$$

$$\int_0^t \int_{\mathbb{R}^d} |\psi_s(x)|^2 ds\, \nu(dx) < \infty, \text{ a.s.}$$

and

$$M_t = m + \sum_{i=1}^d \int_0^t \varphi_s^i dW_s^i + \int_0^t \int_{\mathbb{R}^d} \psi_s(x)(N(ds,dx) - ds\, \nu(dx)) \, .$$

3 Change of measure

3.1 Esscher transform

We define a probability Q, equivalent to P by the formula

$$Q|_{\mathcal{F}_t} = \frac{e^{(\lambda \cdot X_t)}}{E(e^{(\lambda \cdot X_t)})} P|_{\mathcal{F}_t} . \tag{3.1}$$

This particular choice of measure transformation, (called an Esscher transform) preserves the Lévy process property.

Proposition 3.1 *Let X be a P-Lévy process with parameters (m, A, ν) where $A = R^T R$. Let λ be such that $E(e^{(\lambda \cdot X_t)}) < \infty$ and suppose Q is defined by (3.1). Then X is a Lévy process under Q, and if the Lévy-Khintchine decomposition of X under P is (2.1), then the decomposition of X under Q is*

$$
\begin{aligned}
E_Q(\exp(i(u \cdot X_1))) &= \exp\Bigg(i(u \cdot m^{(\lambda)}) - \frac{(u \cdot Au)}{2} \\
&\quad + \int_{\mathbb{R}^d} (e^{i(u \cdot x)} - 1 - i(u \cdot x) 1\!\!1_{|x| \le 1}) \nu^{(\lambda)}(dx) \Bigg)
\end{aligned}
\tag{3.2}
$$

with

$$
\begin{aligned}
m^{(\lambda)} &= m + R\lambda + \int_{|x| \le 1} x(e^{\lambda x} - 1)\nu(dx) \\
\nu^{(\lambda)}(dx) &= e^{\lambda x} \nu(dx) \, .
\end{aligned}
$$

The characteristic exponent of X under Q is

$$\Phi^{(\lambda)}(u) = \Phi(u - i\lambda) - \Phi(-i\lambda)\,.$$

If $\Psi(\lambda) < \infty$, $\Psi^{(\lambda)}(u) = \Psi(u + \lambda) - \Psi(\lambda)$ *for* $u \geq \min(-\lambda, 0)$.

3.2 General case

More generally, any density $(L_t, t \geq 0)$ which is a positive martingale can be used.

$$dL_t = \sum_{i=1}^{d} \widetilde{\varphi}_t^i dW_t^i + \int_{x=-\infty}^{x=\infty} \widetilde{\psi}_t(x)[N(dt,dx) - dt\nu(dx)] \,.$$

From the strict positivity of L, there exists φ, ψ such that $\widetilde{\varphi}_t = L_{t-}\varphi_t$, $\widetilde{\psi}_t = L_{t-}(e^{\psi(t,x)} - 1)$, hence the process L satisfies

$$dL_t = L_{t-} \left(\sum_{i=1}^{d} \varphi_t^i dW_t^i + \int (e^{\psi(t,x)} - 1)[N(dt,dx) - dt\nu(dx)] \right) \quad (3.3)$$

Proposition 3.2 *Let $Q|_{\mathcal{F}_t} = L_t\, P|_{\mathcal{F}_t}$ where L is defined in (3.3). With respect to Q,*

(i) $W_t^\varphi \stackrel{def}{=} W_t - \int_0^t \varphi_s ds$ is a Brownian motion

(ii) The process N is compensated by $e^{\psi(s,x)} ds\nu(dx)$ meaning that for any Borel function h such that

$$\int_0^T \int_{I\!R} |h(s,x)| e^{\psi(s,x)} ds\nu(dx) < \infty \,,$$

the process

$$\int_0^t \int_{I\!R} h(s,x) \left(N(ds,dx) - e^{\psi(s,x)} ds\nu(dx) \right)$$

is a local martingale.

4 Fluctuation theory

Let $M_t = \sup_{s\leq t} X_s$ be the running maximum of the Lévy process X. The reflected process $M - X$ enjoys the strong Markov property.

Let θ be an exponential variable with parameter q, independent of X. Note that

$$E(e^{iuX_\theta}) = q\int E(e^{iuX_t})e^{-qt}dt = q\int e^{-t\Phi(u)}e^{-qt}dt\,.$$

Using excursion theory, the random variables M_θ and $X_\theta - M_\theta$ can be proved to be independent, hence

$$E(e^{iuM_\theta})E(e^{iu(X_\theta - M_\theta)}) = \frac{q}{q+\Phi(u)}\,. \tag{4.1}$$

The equality (4.1) is known as the Wiener-Hopf factorization.

Let $m_t = \min_{s\leq t}(X_s)$. Then

$$m_\theta \stackrel{law}{=} X_\theta - M_\theta\,.$$

If $E(e^{X_1}) < \infty$, using Wiener-Hopf factorization, Mordecki proves that the boundaries for perpetual American options are given by

$$b_p = KE(e^{m_\theta}), b_c = KE(e^{M_\theta})$$

where $m_t = \inf_{s \leq t} X_s$ and θ is an exponential r.v. independent of X with parameter r, hence $b_c b_p = \dfrac{rK^2}{1 - \ln E(e^{X_1})}$.

4.1 Pecherskii-Rogozin Identity

For $x > 0$, denote by T_x the first passage time above x defined as

$$T_x = \inf\{t > 0 \,:\, X_t > x\}$$

and by $K_x = X_{T_x} - x$ the so-called overshoot.

Proposition 4.1 (Pecherskii-Rogozin Identity.) *For every triple of positive numbers* (α, β, q),

$$\int_0^\infty e^{-qx} E(e^{-\alpha T_x - \beta K_x}) dx = \frac{\kappa(\alpha, q) - \kappa(\alpha, \beta)}{(q - \beta)\kappa(\alpha, q)} \tag{4.2}$$

5 Spectrally Negative Lévy Processes

A spectrally negative Lévy process is a Lévy process with no positive jumps, its Lévy measure is supported by $(-\infty, 0)$. Then, X admits exponential moments

$$E(\exp(\lambda X_t)) = \exp(t\Psi(\lambda)) < \infty, \ \forall \lambda > 0$$

where

$$\Psi(\lambda) = \lambda m + \frac{1}{2}\sigma^2\lambda^2 + \int_{-\infty}^{0} (e^{\lambda x} - 1 - \lambda x \mathbb{1}_{\{-1<x<0\}})\nu(dx)\,.$$

Let X be a Lévy process, $M_t = \sup_{s \leq t} X_s$ and $Z_t = M_t - X_t$. If X is spectrally negative, the process $M_t = e^{-\alpha Z_t} - 1 + \alpha Y_t - \Psi(\alpha)\int_0^t e^{-\alpha Z_s} ds$ is a martingale (Asmussen-Kella-Whitt martingale).

6 Exponential Lévy Processes as Stock Price Processes

6.1 Exponentials of Lévy Processes

Let $S_t = xe^{X_t}$ where X is a (m, σ^2, ν) real valued Lévy process.

Let us assume that $E(e^{-\alpha X_1}) <_i nfty$, for $\alpha \in [-\epsilon, , \epsilon]$. This implies that X has finite momments of all orders. In terms of Lévy measure,

$$\begin{aligned}
\int \mathbb{1}_{\{|x|\geq 1\}} e^{-\alpha x} \nu(dx) &< \infty\,, \\
\int \mathbb{1}_{\{|x|\geq 1\}} x^a e^{-\alpha x} \nu(dx) &< \infty \, \forall a > 0 \\
\int \mathbb{1}_{\{|x|\geq 1\}} \nu(dx) &< \infty
\end{aligned}$$

The solution of the SDE

$$dS_t = S_{t-}(b(t)dt + \sigma(t)dX_t)$$

is

$$S_t = S_0 \exp\left(\int_0^t \sigma(s)dX_s + \int_0^t (b(s) - \frac{\sigma^2}{2}\sigma(s)ds\right) \prod_{0<s\leq t} (1+\sigma(s)\Delta X_s) \exp(-\sigma(s)\Delta X_s$$

6.2 Option pricing with Esscher Transform

Let $S_t = S_0 e^{rt+X_t}$ where L is a Lévy process under a the historical probability P.

Proposition 6.1 *We assume that* $\Psi(\alpha) = E(e^{\alpha X_1}) < \infty$ *on some open interval* (a, b) *with* $b - a > 1$ *and that there exists a real number* θ *such that* $\Psi(\alpha) = \Psi(\alpha+1)$*. The process* $e^{-rt}S_t = S_0 e^{X_t}$ *is a martingale under the probability* Q *defined as* $Q = Z_t P$ *where* $Z_t = \dfrac{e^{\theta X_t}}{\Psi(\theta)}$

Hence, the value of a contingent claim $h(S_T)$ can be obtained, assuming that the emm chosen by the market is Q as

$$V_t = e^{-r(T-t)} E_Q(h(S_T)|\mathcal{F}_t) = e^{-r(T-t)} \frac{1}{\Psi(\theta)} E_P(h(ye^{r(T-t)+X_{T-t}} e^{\theta X_{T-t}})\Big|_{y=S_t}$$

6.3 A Differential Equation for Option Pricing

Assume that

$$V(t,x) = e^{-r(T-t)} E_Q(h(e^{x+X_{T-t}}))$$

belongs to $C^{1,2}$. Then

$$rV = \frac{1}{2}\sigma^2 \partial_{xx} V + \partial_t V + \int \left(V(t,x+y) - V(t,x)\right) \nu(dy)\,.$$

6.4 Put-call Symmetry

Let us study a financial market with a riskless asset with constant interest rate r, and a price process $S_t = S_0 e^{X_t}$ where X is a Lévy process such that $e^{-(r-\delta)t} S_t$ is a martingale. In terms of characteristic exponent, this condition means that $\psi(1) = r - \delta$, and the characteristic triple of X is such that

$$m = r - \delta - \sigma^2/2 - \int (e^y - 1 - y \mathbb{1}_{\{|y| \leq 1\}} \nu(dy) \,.$$

Then, the following symmetry between call and put prices holds:

$$C_E(S_0, K, r, \delta, T, \psi) = P_E(K, S_0, \delta, r, T, \widetilde{\psi}) \,.$$

7 Subordinators

A Lévy process which takes values in $[0,\infty[$ (i.e. with increasing paths) is a subordinator. In this case, the parameters in the Lévy-Khintchine decomposition are $m \geq 0, \sigma = 0$ and the Lévy measure ν is a measure on $]0,\infty[$ with $\displaystyle\int_{]0,\infty[} (1 \wedge x)\nu(dx) < \infty$. The Laplace exponent can be expressed as

$$\Phi(u) = \delta u + \int_{]0,\infty[} (1 - e^{-ux})\nu(dx)$$

where $\delta \geq 0$.

Definition 7.1 *Let Z be a subordinator and X an independent Lévy process. The process $\widetilde{X}_t = X_{Z_t}$ is a Lévy process, called subordinated Lévy process.*

Example 7.2 Compound Poisson process. A compound Poisson process with $Y_k \geq 0$ is a subordinator.

Example 7.3 Gamma process. The Gamma process $G(t;\gamma)$ is a subordinator which satisfies

$$G(t+h;\gamma) - G(t;\gamma) \overset{law}{=} \Gamma(h;\gamma)\,.$$

Here $\Gamma(h;\gamma)$ follows the Gamma law. The Gamma process is an increasing Lévy process, hence a subordinator, with one sided Lévy measure

$$\frac{1}{x}\exp(-\frac{x}{\gamma})\mathbb{1}_{x>0}\,.$$

Example 7.4 Let W be a BM, and

$$T_r = \inf\{t \geq 0 \,:\, W_t \geq r\}\,.$$

The process $(T_r, r \geq 0)$ is a stable $(1/2)$ subordinator, its Lévy measure is $\dfrac{1}{\sqrt{2\pi}\,x^{3/2}}\mathbb{1}_{x>0}dx$. Let B be a BM independent of W. The process B_{T_t} is a Cauchy process, its Lévy measure is $dx/(\pi x^2)$.

Proposition 7.5 (Changes of Lévy characteristics under subordination.) *Let X be a (a^X, A^X, ν^X) Lévy process and Z be a subordinator with drift β and Lévy measure ν^Z, independent of X. The process $\widetilde{X}_t = X_{Z_t}$ is a Lévy process with characteristic exponent*

$$\Phi(u) = i(\widetilde{a} \cdot u) + \frac{1}{2}\widetilde{A}(u) - \int (e^{i(u \cdot x)} - 1 - i(u \cdot x)\mathbb{1}_{|x| \leq 1})\widetilde{\nu}(dx)$$

with

$$\begin{aligned}
\widetilde{a} &= \beta a^X + \int \nu^Z(ds)\mathbb{1}_{|x|\leq 1} x P(X_s \in dx) \\
\widetilde{A} &= \beta A^X \\
\widetilde{\nu}(dx) &= \beta \nu^X dx + \int \nu^Z(ds) P(X_s \in dx)\,.
\end{aligned}$$

Example 7.6 Normal Inverse Gaussian. The NIG Lévy process is a subordinated process with Lévy measure $\dfrac{\delta\alpha}{\pi}\dfrac{e^{\beta x}}{|x|}K_1(\alpha|x|)dx$.

8 Variance-Gamma Model

The variance Gamma process is a Lévy process where X_t has a Variance Gamma law VG(σ, ν, θ). Its characteristic function is

$$E(\exp(iuX_t)) = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-t/\nu} .$$

The Variance Gamma process can be characterized as a time changed BM with drift as follows: let W be a BM, $\gamma(t)$ a $G(1/\nu, 1/\nu)$ process. Then

$$X_t = \theta\gamma(t) + \sigma W_{\gamma(t)}$$

is a VG(σ, ν, θ) process. The variance Gamma process is a finite variation process. Hence it is the difference of two increasing processes. Madan et al. showed that it is the difference of two independent Gamma processes

$$X_t = G(t; \mu_1, \gamma_1) - G(t; \mu_2, \gamma_2) .$$

Indeed, the characteristic function can be factorized

$$E(\exp(iuX_t)) = \left(1 - \frac{iu}{\nu_1}\right)^{-t/\gamma} \left(1 + \frac{iu}{\nu_2}\right)^{-t/\gamma}$$

with

$$\nu_1^{-1} = \frac{1}{2}\left(\theta\nu + \sqrt{\theta^2\nu^2 + 2\nu\sigma^2}\right)$$
$$\nu_2^{-1} = \frac{1}{2}\left(\theta\nu - \sqrt{\theta^2\nu^2 + 2\nu\sigma^2}\right)$$

The Lévy density of X is

$$\frac{1}{\gamma}\frac{1}{|x|}\exp(-\nu_1|x|) \qquad \text{for } x < 0$$
$$\frac{1}{\gamma}\frac{1}{x}\exp(-\nu_2 x) \qquad \text{for } x > 0\,.$$

The density of X_1 is

$$\frac{2e^{\frac{\theta x}{\sigma^2}}}{\gamma^{1/\gamma}\sqrt{2\pi}\sigma\Gamma(1/2)}\left(\frac{x^2}{\theta^2 + 2\sigma^2/\gamma}\right)^{\frac{1}{2\gamma}-\frac{1}{4}} K_{\frac{1}{\gamma}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x^2(\theta^2 + 2\sigma^2/\gamma)}\right)$$

where K_α is the modified Bessel function.

Stock prices driven by a Variance-Gamma process have dynamics

$$S_t = S_0 \exp\left(rt + X(t;\sigma,\nu,\theta) + \frac{t}{\nu}\ln(1 - \theta\nu - \frac{\sigma^2\nu}{2})\right)$$

From $E(e^{X_t}) = \exp\left(-\frac{t}{\nu}\ln(1 - \theta\nu - \frac{\sigma^2\nu}{2})\right)$, we get that $S_t e^{-rt}$ is a martingale. The parameters ν and θ give control on skewness and kurtosis.

The CGMY model, introduced by Carr et al. is an extension of the Variance-Gamma model. The Lévy density is

$$\begin{cases} \dfrac{C}{x^{Y+1}} e^{-Mx} & x > 0 \\ \dfrac{C}{|x|^{Y+1}} e^{Gx} & x < 0 \end{cases}$$

with $C > 0, M \geq 0, G \geq 0$ and $Y < 2, Y \notin \mathbb{Z}$.
If $Y < 0$, there is a finite number of jumps in any finite interval, if not,

the process has infinite activity. If $Y \in [1, 2[$, the process is with infinite variation. This process is also called KoBol.