Credit risk

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Lecture of M. Jeanblanc

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Contents

Notation

$B(t,T)$ Price at time t of a zero coupon with maturity T, 19

$D^{(R)}(t,T)$ Price at time t of a corporate bond which pays R at default time and 1 at maturity, if the default has not occurred before maturity, 22

$D^{(R,T)}$ Price of a defaultable bond, which pays 1 at maturity if no default and R at maturity T if the default has occurred before maturity, 20

F_t: conditional probability, 63

$\Gamma(t)$: hazard function, 21

Γ_t: hazard process, 63

$g_t^b(X)$: last time before t at which the process X is at the level b, 226

$\mathbf{T}$: trivial filtration , 172

Introduction

These notes are mainly based on the papers of Bielecki, Jeanblanc and Rutkowski:

Hedging of defaultable claims, Lecture Notes in Mathematics, 1847, pages 1–132, Paris-Princeton, Springer-Verlag, 2004, R.A. Carmona, E. Cinlar, I. Ekeland, E. Jouini, J.E. Scheinkman, N. Touzi, eds.

Stochastic Methods In Credit Risk Modelling, Valuation And Hedging, Lecture Notes in Mathematics, Frittelli, M. edt, CIME-EMS Summer School on Stochastic Methods in Finance, Bressanone, Springer, 2004.

Hedging of credit derivatives in models with totally unexpected default, in Stochastic processes and applications to mathematical finance, Akahori, J. Ogawa, S. and Watanabe S. Edt, p. 35-100 Proceedings of the 5th Ritsumeikan International conference, World Scientific

Pricing and hedging Credit default Swaps Work in progress.

and on the book of T.R. Bielecki and M. Rutkowski: *Credit risk : Modelling valuation and Hedging*, Springer Verlag, 2001.

The reader can find other interesting information on the web sites quoted at the end of the bibliography of this document.

The goal of this lecture is to present a survey of recent developments in the area of mathematical modeling of *credit risk* and *credit derivatives*. Credit risk embedded in a financial transaction is the risk that at least one of the parties involved in the transaction will suffer a financial loss due to decline in the creditworthiness of the counter-party to the transaction, or perhaps of some third party. For example:

- A holder of a corporate bond bears a risk that the (market) value of the bond will decline due to decline in credit rating of the issuer.
- A bank may suffer a loss if a bank's debtor defaults on payment of the interest due and (or) the principal amount of the loan.
- A party involved in a trade of a credit derivative, such as a credit default swap (CDS), may suffer a loss if a reference credit event occurs.
- The market value of individual tranches constituting a collateralized debt obligation (CDO) may decline as a result of changes in the correlation between the default times of the underlying defaultable securities (i.e., of the collateral).

The most extensively studied form of credit risk is the *default risk* – that is, the risk that a counterparty in a financial contract will not fulfil a contractual commitment to meet her/his obligations stated in the contract. For this reason, the main tool in the area of credit risk modeling is a judicious specification of the random time of default. A large part of the present text will be devoted to this issue.

Our main goal is to present the most important mathematical tools that are used for the arbitrage valuation of defaultable claims, which are also known under the name of credit derivatives. We also examine the important issue of hedging these claims.

In Chapter 1, we provide a concise summary of the main developments within the so-called *structural approach* to modeling and valuation of credit risk. We also study the random barrier case. Chapter 2 is devoted to the study of a toy model within the hazard process framework. Chapter 3 studies the case of Cox processes. Chapter 4 is devoted to the *reduced-form approach*. This approach is purely probabilistic in nature and, technically speaking, it has a lot in common with the reliability theory. Chapter 5 studies hedging strategies under assumption that a defaultable asset is traded. Chapter 6 studies different ways to give a price in incomplete market setting. Chapter 7 provides an introduction to the area of modeling dependent credit migrations and defaults. An appendix recalls some notion of stochastic calculus and probability theory.

Let us only mention that the proofs of most results can be found in Bielecki and Rutkowski [23], Bielecki et al. [15, 18, 20] and Jeanblanc and Rutkowski [119]. We quote some of the seminal papers; the reader can also refer to books of Bruyère [39], Bluhm et al. [28], Bielecki and Rutkowski [23], Cossin and Pirotte [52], Duffie and Singleton [74], Frey, McNeil and Embrechts [89], Lando [143], Schönbucher [170] for more information. At the end of the bibliography, we also mention some web addresses where articles can be downloaded.

Finally, it should be acknowledged that some results (especially within the reduced form approach) were obtained independently by various authors, who worked under different set of assumptions and within distinct setups, and thus we decided to omit detailed credentials in most cases. We hope our colleagues will accept our apologies for this deficiency, and we stress that this by no means signifies that these results that are not explicitly attributed are ours.

Begin at the beginning, and go on till you come to the end. Then, stop.

L. Carroll, Alice's Adventures in Wonderland

Chapter 1

Structural Approach

In this chapter, we present the *structural approach* to modeling credit risk (it is also known as the *value-of-the-firm approach*). This methodology directly refers to economic fundamentals, such as the capital structure of a company, in order to model credit events (a default event, in particular). As we shall see in what follows, the two major driving concepts in the structural modeling are: the total value of the firm's assets and the default triggering barrier. This was historically the first approach used in this area, and it goes back to the fundamental papers by Black and Scholes [26] and Merton [159].

1.1 Basic Assumptions

We fix a finite horizon date $T^* > 0$, and we suppose that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with some (reference) filtration $\mathbf{F} = (\mathcal{F}_t)_{0 \le t \le T^*}$, is sufficiently rich to support the following objects:

- The *short-term interest rate* process r, and thus also a default-free term structure model.
- The *firm's value process* V, which is interpreted as a model for the total value of the firm's assets.
- The *barrier process* v, which will be used in the specification of the default time τ.
- The *promised contingent claim* X representing the firm's liabilities to be redeemed at maturity date $T \le T^*$.
- The process C, which models the *promised dividends,* i.e., the liabilities stream that is redeemed continuously or discretely over time to the holder of a defaultable claim.
- The *recovery claim* $\widetilde{X}$ representing the recovery payoff received at time T, if default occurs prior to or at the claim's maturity date T.
- The *recovery process* Z, which specifies the recovery payoff at time of default, if it occurs prior to or at the maturity date T.

1.1.1 Defaultable Claims

Technical Assumptions. We postulate that the processes V, Z, C and v are progressively measurable with respect to the filtration $\mathbf{F}$, and that the random variables X and $\widetilde{X}$ are $\mathcal{F}_T$-measurable. In addition, C is assumed to be a process of finite variation, with $C_0 = 0$. We assume without mentioning that all random objects introduced above satisfy suitable integrability conditions.

Probabilities $\mathbb{P}$ and $\mathbb{Q}$. The probability $\mathbb{P}$ is assumed to represent the *real-world* (or *statistical*) probability, as opposed to the *martingale measure* (also known as the *risk-neutral probability*). The latter probability is denoted by $\mathbb{Q}$ in what follows.

Default Time. In the structural approach, the default time τ will be typically defined in terms of the firm's value process V and the barrier process v. We set

$$\tau = \inf\{\, t > 0 : \, t \in \mathcal{T} \text{ and } V_t \leq v_t\}$$

with the usual convention that the infimum over the empty set equals $+\infty$. In main cases, the set $\mathcal{T}$ is an interval $[0,T]$ (or $[0,\infty)$ in the case of perpetual claims). In first passage structural models, the default time τ is usually given by the formula:

$$\tau = \inf\{\, t > 0 : \, t \in [0,T] \text{ and } V_t \leq \bar{v}(t)\},$$

where $\bar{v} : [0,T] \to I\!R_+$ is some deterministic function, termed the *barrier*.

Predictability of Default Time. Since the underlying filtration $\mathbf{F}$ in most structural models is generated by a standard Brownian motion, τ will be an $\mathbf{F}$-predictable stopping time (as any stopping time with respect to a Brownian filtration): there exists a sequence of increasing stopping times announcing the default time.

Recovery Rules. If default does not occur before or at time T, the promised claim X is paid in full at time T. Otherwise, depending on the market convention, either (1) the amount $\widetilde{X}$ is paid at the maturity date T, or (2) the amount Z_τ is paid at time τ. In the case when default occurs at maturity, i.e., on the event $\{\tau = T\}$, we postulate that only the recovery payment $\widetilde{X}$ is paid. In a general setting, we consider simultaneously both kinds of recovery payoff, and thus a generic defaultable claim is formally defined as a quintuple $(X, C, \widetilde{X}, Z, \tau)$.

1.1.2 Risk-Neutral Valuation Formula

Suppose that our financial market model is arbitrage-free, in the sense that there exists a *martingale measure* (*risk-neutral probability*) $\mathbb{Q}$, meaning that price process of any tradeable security, which pays no coupons or dividends, becomes an $\mathbf{F}$-martingale under $\mathbb{Q}$, when discounted by the *savings account* B, given as

$$B_t = \exp\Big(\int_0^t r_u\, du\Big).$$

We introduce the jump process $H_t = 1\!\!1_{\{\tau \leq t\}}$, and we denote by D the process that models all cash flows received by the owner of a defaultable claim. Let us denote

$$X^d(T) = X 1\!\!1_{\{\tau > T\}} + \widetilde{X} 1\!\!1_{\{\tau \leq T\}}.$$

Definition 1.1.1 *The dividend process D of a defaultable contingent claim $(X, C, \widetilde{X}, Z, \tau)$, which settles at time T, equals*

$$D_t = X^d(T) 1\!\!1_{\{t \geq T\}} + \int_{]0,t]} (1 - H_u)\, dC_u + \int_{]0,t]} Z_u\, dH_u.$$

It is apparent that D is a process of finite variation, and

$$\int_{]0,t]} (1 - H_u)\, dC_u = \int_{]0,t]} 1\!\!1_{\{\tau > u\}}\, dC_u = C_{\tau-} 1\!\!1_{\{\tau \leq t\}} + C_t 1\!\!1_{\{\tau > t\}}.$$

Note that if default occurs at some date t, the promised dividend $C_t - C_{t-}$, which is due to be paid at this date, is not received by the holder of a defaultable claim. Furthermore, if we set $\tau \wedge t = \min\{\tau, t\}$ then

$$\int_{]0,t]} Z_u\, dH_u = Z_{\tau \wedge t} 1\!\!1_{\{\tau \leq t\}} = Z_\tau 1\!\!1_{\{\tau \leq t\}}.$$

Remark 1.1.1 In principle, the promised payoff X could be incorporated into the promised dividends process C. However, this would be inconvenient, since in practice the recovery rules concerning the promised dividends C and the promised claim X are different, in general. For instance, in the case of a defaultable coupon bond, it is frequently postulated that in case of default the future coupons are lost, but a strictly positive fraction of the face value is usually received by the bondholder.

We are in the position to define the ex-dividend price S_t of a defaultable claim. At any time t, the random variable S_t represents the current value of all future cash flows associated with a given defaultable claim.

Definition 1.1.2 *For any date $t \in [0,T[$, the ex-dividend price of the defaultable claim $(X, C, \widetilde{X}, Z, \tau)$ is given as*

$$S_t = B_t \,\mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{F}_t \Big). \tag{1.1}$$

In addition, we always set $S_T = X^d(T)$. The discounted ex-dividend price S_t^*, $t \in [0,T]$, satisfies

$$S_t^* = S_t B_t^{-1} - \int_{]0,t]} B_u^{-1}\, dD_u, \quad \forall\, t \in [0,T],$$

and thus it follows a supermartingale under $\mathbb{Q}$ if and only if the dividend process D is increasing. The process $S_t + B_t \int_{]0,t]} B_u^{-1}\, dD_u$ is also called the cum-dividend process.

1.1.3 Defaultable Zero-Coupon Bond

Assume that $C \equiv 0$, $Z \equiv 0$ and $X = L$ for some positive constant $L > 0$. Then the value process S represents the arbitrage price of a *defaultable zero-coupon bond* (also known as the *corporate discount bond*) with the face value L and recovery at maturity only. In general, the price $D(t,T)$ of such a bond equals

$$D(t,T) = B_t \,\mathbb{E}_{\mathbb{Q}}\big(B_T^{-1}(L\mathbb{1}_{\{\tau > T\}} + \widetilde{X}\mathbb{1}_{\{\tau \le T\}}) \,\big|\, \mathcal{F}_t\big).$$

It is convenient to rewrite the last formula as follows:

$$D(t,T) = LB_t \,\mathbb{E}_{\mathbb{Q}}\big(B_T^{-1}(\mathbb{1}_{\{\tau > T\}} + \delta(T)\mathbb{1}_{\{\tau \le T\}}) \,\big|\, \mathcal{F}_t\big),$$

where the random variable $\delta(T) = \widetilde{X}/L$ represents the so-called *recovery rate upon default.* It is natural to assume that $0 \le \widetilde{X} \le L$ so that $\delta(T)$ satisfies $0 \le \delta(T) \le 1$. Alternatively, we may re-express the bond price as follows:

$$D(t,T) = L\Big(B(t,T) - B_t \,\mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} w(T)\mathbb{1}_{\{\tau \le T\}} \,\big|\, \mathcal{F}_t\big)\Big),$$

where

$$B(t,T) = B_t \,\mathbb{E}_{\mathbb{Q}}(B_T^{-1} \,|\, \mathcal{F}_t)$$

is the price of a unit default-free zero-coupon bond, and $w(T) = 1 - \delta(T)$ is the *writedown rate upon default.* Generally speaking, the time-t value of a corporate bond depends on the joint probability distribution under $\mathbb{Q}$ of the three-dimensional random variable $(B_T, \delta(T), \tau)$ or, equivalently, $(B_T, w(T), \tau)$.

Example 1.1.1 Merton [159] postulates that the recovery payoff upon default (I.E., when $V_T < L$, equals $\widetilde{X} = V_T$, where the random variable V_T is the firm's value at maturity date T of a corporate bond. Consequently, the random recovery rate upon default equals $\delta(T) = V_T/L$, and the writedown rate upon default equals $w(T) = 1 - V_T/L$.

Expected Writedowns. For simplicity, we assume that the savings account B is non-random – that is, the short-term rate r is deterministic. Then the price of a default-free zero-coupon bond equals $B(t,T) = B_t B_T^{-1}$, and the price of a zero-coupon corporate bond satisfies

$$D(t,T) = L_t(1 - w^*(t,T)),$$

where $L_t = LB(t,T)$ is the present value of future liabilities, and $w^*(t,T)$ is the *conditional expected writedown rate* under $\mathbb{Q}$. It is given by the following equality:

$$w^*(t,T) = \mathbb{E}_{\mathbb{Q}}\big(w(T)\mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{F}_t\big).$$

The *conditional expected writedown rate upon default* equals, under $\mathbb{Q}$,

$$w_t^* = \frac{\mathbb{E}_{\mathbb{Q}}\big(w(T)\mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{F}_t\big)}{\mathbb{Q}\{\tau \le T \,|\, \mathcal{F}_t\}} = \frac{w^*(t,T)}{p_t^*},$$

where $p_t^* = \mathbb{Q}\{\tau \le T \,|\, \mathcal{F}_t\}$ is the *conditional risk-neutral probability of default.* Finally, let $\delta_t^* = 1 - w_t^*$ be the *conditional expected recovery rate upon default* under $\mathbb{Q}$. In terms of p_t^*, δ_t^* and p_t^*, we obtain

$$D(t,T) = L_t(1 - p_t^*) + L_t p_t^* \delta_t^* = L_t(1 - p_t^* w_t^*).$$

If the random variables $w(T)$ and τ are conditionally independent with respect to the σ-field $\mathcal{F}_t$ under $\mathbb{Q}$, then we have $w_t^* = \mathbb{E}_{\mathbb{Q}}(w(T) \,|\, \mathcal{F}_t)$.

Example 1.1.2 In practice, it is common to assume that the recovery rate is non-random. Let the recovery rate $\delta(T)$ be constant, specifically, $\delta(T) = \delta$ for some real number δ. In this case, the writedown rate $w(T) = w = 1 - \delta$ is non-random as well. Then $w^*(t,T) = wp_t^*$ and $w_t^* = w$ for every $0 \le t \le T$. Furthermore, the price of a defaultable bond has the following representation

$$D(t,T) = L_t(1 - p_t^*) + \delta L_t p_t^* = L_t(1 - wp_t^*).$$

We shall return to various recovery schemes later in the text.

1.2 Classic Structural Models

Classic structural models are based on the assumption that the risk-neutral dynamics of the value process of the assets of the firm V are given by the SDE:

$$dV_t = V_t\big((r - \kappa)\,dt + \sigma_V\,dW_t\big), \quad V_0 > 0,$$

where κ is the constant payout (dividend) ratio, and the process W is a standard Brownian motion under the martingale measure $\mathbb{Q}$.

1.2.1 Merton's Model

We present here the classic model due to Merton [159].

Basic assumptions. A firm has a single liability with promised terminal payoff L, interpreted as the zero-coupon bond with maturity T and face value $L > 0$. The ability of the firm to redeem its debt is determined by the total value V_T of firm's assets at time T. Default may occur at time T only, and the default event corresponds to the event $\{V_T < L\}$. Hence, the stopping time τ equals

$$\tau = T\mathbb{1}_{\{V_T < L\}} + \infty\mathbb{1}_{\{V_T \ge L\}}.$$

Moreover $C = 0$, $Z = 0$, and

$$X^d(T) = V_T\mathbb{1}_{\{V_T < L\}} + L\mathbb{1}_{\{V_T \ge L\}}$$

so that $\tilde{X} = V_T$. In other words, the payoff at maturity equals

$$D_T = \min(V_T, L) = L - \max(L - V_T, 0) = L - (L - V_T)^+.$$

The latter equality shows that the valuation of the corporate bond in Merton's setup is equivalent to the valuation of a European put option written on the firm's value with strike equal to the bond's face value. Let $D(t,T)$ be the price at time $t < T$ of the corporate bond. It is clear that the value $D(V_t)$ of the firm's debt equals

$$D(V_t) = D(t,T) = L\,B(t,T) - P_t,$$

where P_t is the price of a put option with strike L and expiration date T. It is apparent that the value $E(V_t)$ of the firm's equity at time t equals

$$E(V_t) = V_t - D(V_t) = V_t - LB(t,T) + P_t = C_t,$$

where C_t stands for the price at time t of a call option written on the firm's assets, with strike price L and exercise date T. To justify the last equality above, we may also observe that at time T we have

$$E(V_T) = V_T - D(V_T) = V_T - \min(V_T, L) = (V_T - L)^+.$$

We conclude that the firm's shareholders are in some sense the holders of a call option on the firm's assets.

Merton's Formula. Using the option-like features of a corporate bond, Merton [159] derived a closed-form expression for its arbitrage price. Let $\mathcal{N}$ denote the standard Gaussian cumulative distribution function:

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\,du, \quad \forall\, x \in I\!R.$$

Proposition 1.2.1 *For every $0 \le t < T$ the value $D(t,T)$ of a corporate bond equals*

$$D(t,T) = V_t e^{-\kappa(T-t)} \mathcal{N}\big(-d_+(V_t, T-t)\big) + L\,B(t,T)\mathcal{N}\big(d_-(V_t, T-t)\big)$$

where

$$d_\pm(V_t, T-t) = \frac{\ln(V_t/L) + \big(r - \kappa \pm \frac{1}{2}\sigma_V^2\big)(T-t)}{\sigma_V\sqrt{T-t}}.$$

The unique replicating strategy for a defaultable bond involves holding at any time $0 \le t < T$: $\phi_t^1 V_t$ units of cash invested in the firm's value and $\phi_t^2 B(t,T)$ units of cash invested in default-free bonds, where

$$\phi_t^1 = e^{-\kappa(T-t)} \mathcal{N}\big(-d_+(V_t, T-t)\big)$$

and

$$\phi_t^2 = \frac{D(t,T) - \phi_t^1 V_t}{B(t,T)} = L\mathcal{N}\big(d_-(V_t, T-t)\big).$$

Credit Spreads

For notational simplicity, we set $\kappa = 0$. Then Merton's formula becomes:

$$D(t,T) = LB(t,T)\big(\Gamma_t \mathcal{N}(-d) + \mathcal{N}(d - \sigma_V\sqrt{T-t})\big),$$

where we denote $\Gamma_t = V_t/LB(t,T)$ and

$$d = d(V_t, T-t) = \frac{\ln(V_t/L) + (r + \sigma_V^2/2)(T-t)}{\sigma_V\sqrt{T-t}}.$$

Since $LB(t,T)$ represents the current value of the face value of the firm's debt, the quantity Γ_t can be seen as a proxy of the asset-to-debt ratio $V_t/D(t,T)$. It can be easily verified that the inequality

$D(t,T) < LB(t,T)$ is valid. This property is equivalent to the positivity of the corresponding credit spread (see below).

Observe that in the present setup the continuously compounded yield $r(t,T)$ at time t on the T-maturity Treasury zero-coupon bond is constant, and equal to the short-term rate r. Indeed, we have

$$B(t,T) = e^{-r(t,T)(T-t)} = e^{-r(T-t)}.$$

Let us denote by $r^d(t,T)$ the continuously compounded yield on the corporate bond at time $t < T$, so that

$$D(t,T) = Le^{-r^d(t,T)(T-t)}.$$

From the last equality, it follows that

$$r^d(t,T) = -\frac{\ln D(t,T) - \ln L}{T-t}.$$

For $t < T$ the *credit spread* $S(t,T)$ is defined as the excess return on a defaultable bond:

$$S(t,T) = r^d(t,T) - r(t,T) = \frac{1}{T-t} \ln \frac{LB(t,T)}{D(t,T)}.$$

In Merton's model, we have

$$S(t,T) = -\frac{\ln\big(\mathcal{N}(d - \sigma_V\sqrt{T-t}) + \Gamma_t \mathcal{N}(-d)\big)}{T-t} > 0.$$

This agrees with the well-known fact that risky bonds have an expected return in excess of the risk-free interest rate. In other words, the yields on corporate bonds are higher than yields on Treasury bonds with matching notional amounts. Notice, however, when t tends to T, the credit spread in Merton's model tends either to infinity or to 0, depending on whether $V_T < L$ or $V_T > L$. Formally, if we define the *forward short spread at time* T as

$$FSS_T = \lim_{t \uparrow T} S(t,T)$$

then

$$FSS_T(\omega) = \begin{cases} 0, & \text{if } \omega \in \{V_T > L\}, \\ \infty, & \text{if } \omega \in \{V_T < L\}. \end{cases}$$

1.2.2 Black and Cox Model

By construction, Merton's model does not allow for a premature default, in the sense that the default may only occur at the maturity of the claim. Several authors put forward structural-type models in which this restrictive and unrealistic feature is relaxed. In most of these models, the time of default is given as the *first passage time* of the value process V to either a deterministic or a random barrier. In principle, the bond's default may thus occur at any time before or on the maturity date T. The challenge is to appropriately specify the lower threshold v, the recovery process Z, and to explicitly evaluate the conditional expectation that appears on the right-hand side of the risk-neutral valuation formula

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{F}_t \Big),$$

which is valid for $t \in [0,T[$. As one might easily guess, this is a non-trivial mathematical problem, in general. In addition, the practical problem of the lack of direct observations of the value process V largely limits the applicability of the first-passage-time models based on the value of the firm process V.

Corporate Zero-Coupon Bond Black and Cox [25] extend Merton's [159] research in several directions, by taking into account such specific features of real-life debt contracts as: safety covenants, debt subordination, and restrictions on the sale of assets. Following Merton [159], they assume that the firm's stockholders receive continuous dividend payments, which are proportional to the current value of firm's assets. Specifically, they postulate that

$$dV_t = V_t\big((r-\kappa)\,dt + \sigma_V\,dW_t\big), \quad V_0 > 0,$$

where W is a BM (under the risk-neutral probability $\mathbb{Q}$), the constant $\kappa \geq 0$ represents the payout ratio, and $\sigma_V > 0$ is the constant volatility. The short-term interest rate r is assumed to be constant.

Safety covenants. Safety covenants provide the firm's bondholders with the right to force the firm to bankruptcy or reorganization if the firm is doing poorly according to a set standard. The standard for a poor performance is set by Black and Cox in terms of a time-dependent deterministic barrier $\bar{v}(t) = Ke^{-\gamma(T-t)}$, $t \in [0,T[$, for some constant $K > 0$. As soon as the value of firm's assets crosses this lower threshold, the bondholders take over the firm. Otherwise, default takes place at debt's maturity or not depending on whether $V_T < L$ or not.

Default time. Let us set

$$v_t = \begin{cases} \bar{v}(t), & \text{for } t < T, \\ L, & \text{for } t = T. \end{cases}$$

The default event occurs at the first time $t \in [0,T]$ at which the firm's value V_t falls below the level v_t, or the default event does not occur at all. The default time equals ($\inf \emptyset = +\infty$)

$$\tau = \inf\{\, t \in [0,T] : V_t \leq v_t\}.$$

The recovery process Z and the recovery payoff $\widetilde{X}$ are proportional to the value process: $Z \equiv \beta_2 V$ and $\widetilde{X} = \beta_1 V_T$ for some constants $\beta_1, \beta_2 \in [0,1]$. The case examined by Black and Cox [25] corresponds to $\beta_1 = \beta_2 = 1$.

To summarize, we consider the following model:

$$X = L,\ C \equiv 0,\ Z \equiv \beta_2 V,\ \widetilde{X} = \beta_1 V_T,\ \tau = \bar{\tau} \wedge \widehat{\tau},$$

where the *early default time* $\bar{\tau}$ equals

$$\bar{\tau} = \inf\{\, t \in [0,T) \,: V_t \leq \bar{v}(t)\}$$

and $\widehat{\tau}$ stands for Merton's default time: $\widehat{\tau} = T\mathbb{1}_{\{V_T < L\}} + \infty \mathbb{1}_{\{V_T \geq L\}}$.

Bond Valuation Similarly as in Merton's model, it is assumed that the short term interest rate is deterministic and equal to a positive constant r. We postulate, in addition, that $\bar{v}(t) \leq LB(t,T)$ or, more explicitly,

$$Ke^{-\gamma(T-t)} \leq Le^{-r(T-t)}, \quad \forall\, t \in [0,T],$$

so that, in particular, $K \leq L$. This condition ensures that the payoff to the bondholder at the default time τ never exceeds the face value of debt, discounted at a risk-free rate.

PDE approach. Since the model for the value process V is given in terms of a Markovian diffusion, a suitable partial differential equation can be used to characterize the value process of the corporate bond. Let us write $D(t,T) = u(V_t,t)$. Then the pricing function $u = u(v,t)$ of a defaultable bond satisfies the following PDE:

$$u_t(v,t) + (r-\kappa)vu_v(v,t) + \frac{1}{2}\sigma_V^2 v^2 u_{vv}(v,t) - ru(v,t) = 0$$

on the domain

$$\{(v,t) \in I\!\!R_+ \times I\!\!R_+ :\, 0 < t < T,\, v > Ke^{-\gamma(T-t)}\},$$

with the boundary condition

$$u(Ke^{-\gamma(T-t)},t) = \beta_2 Ke^{-\gamma(T-t)}$$

and the terminal condition $u(v,T) = \min(\beta_1 v, L)$.

Probabilistic approach. For any $t < T$ the price $D(t,T) = u(V_t, t)$ of a defaultable bond has the following probabilistic representation, on the set $\{\tau > t\} = \{\bar{\tau} > t\}$

$$\begin{aligned} D(t,T) &= \mathbb{E}_{\mathbb{Q}}\Big(Le^{-r(T-t)}\mathbb{1}_{\{\bar{\tau}\geq T,\, V_T\geq L\}}\,\Big|\,\mathcal{F}_t\Big) \\ &\quad + \mathbb{E}_{\mathbb{Q}}\Big(\beta_1 V_T e^{-r(T-t)}\mathbb{1}_{\{\bar{\tau}\geq T,\, V_T<L\}}\,\Big|\,\mathcal{F}_t\Big) \\ &\quad + \mathbb{E}_{\mathbb{Q}}\Big(K\beta_2 e^{-\gamma(T-\bar{\tau})}e^{-r(\bar{\tau}-t)}\mathbb{1}_{\{t<\bar{\tau}<T\}}\,\Big|\,\mathcal{F}_t\Big). \end{aligned}$$

After default – that is, on the set $\{\tau \leq t\} = \{\bar{\tau} \leq t\}$, we clearly have

$$D(t,T) = \beta_2 \bar{v}(\tau)B^{-1}(\tau,T)B(t,T) = K\beta_2 e^{-\gamma(T-\tau)}e^{r(t-\tau)}.$$

To compute the expected values above, we observe that:

- the first two conditional expectations can be computed by using the formula for the conditional probability $\mathbb{Q}\{V_s \geq x,\, \tau \geq s \,|\, \mathcal{F}_t\}$,
- to evaluate the third conditional expectation, it suffices employ the conditional probability law of the first passage time of the process V to the barrier $\bar{v}(t)$.

Black and Cox Formula. Before we state the bond valuation result due to Black and Cox [25], we find it convenient to introduce some notation. We denote

$$\begin{aligned} \nu &= r - \kappa - \frac{1}{2}\sigma_V^2, \\ m &= \nu - \gamma = r - \kappa - \gamma - \frac{1}{2}\sigma_V^2 \\ b &= m\sigma^{-2}. \end{aligned}$$

For the sake of brevity, in the statement of Proposition 1.2.2 we shall write σ instead of σ_V. As already mentioned, the probabilistic proof of this result is based on the knowledge of the probability law of the first passage time of the geometric (exponential) Brownian motion to an exponential barrier (see Appendix equations (8.11) and (8.12)).

Proposition 1.2.2 *Assume that $m^2 + 2\sigma^2(r-\gamma) > 0$. Prior to bond's default, that is: on the set $\{\tau > t\}$, the price process $D(t,T) = u(V_t, t)$ of a defaultable bond equals*

$$\begin{aligned} D(t,T) = LB(t,T)\big(&\mathcal{N}\big(h_1(V_t,T-t)\big) - Z_t^{2b\sigma^{-2}}\mathcal{N}\big(h_2(V_t,T-t)\big)\big) \\ &+ \beta_1 V_t e^{-\kappa(T-t)}\big(\mathcal{N}\big(h_3(V_t,T-t)\big) - \mathcal{N}\big(h_4(V_t,T-t)\big)\big) \\ &+ \beta_1 V_t e^{-\kappa(T-t)} Z_t^{2b+2}\big(\mathcal{N}\big(h_5(V_t,T-t)\big) - \mathcal{N}\big(h_6(V_t,T-t)\big)\big) \\ &+ \beta_2 V_t\big(Z_t^{\theta+\zeta}\mathcal{N}\big(h_7(V_t,T-t)\big) + Z_t^{\theta-\zeta}\mathcal{N}\big(h_8(V_t,T-t)\big)\big), \end{aligned}$$

where $Z_t = \bar{v}(t)/V_t$, $\theta = b+1$, $\zeta = \sigma^{-2}\sqrt{m^2 + 2\sigma^2(r-\gamma)}$ and

$$\begin{aligned} h_1(V_t,T-t) &= \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\ h_2(V_t,T-t) &= \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\ h_3(V_t,T-t) &= \frac{\ln(L/V_t) - (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ h_4(V_t,T-t) &= \frac{\ln(K/V_t) - (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \end{aligned}$$

$$
\begin{aligned}
h_5(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(LV_t) + (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_6(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(KV_t) + (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_7(V_t, T-t) &= \frac{\ln(\bar{v}(t)/V_t) + \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\
h_8(V_t, T-t) &= \frac{\ln(\bar{v}(t)/V_t) - \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}
$$

Special Cases Assume that $\beta_1 = \beta_2 = 1$ and the barrier function $\bar{v}$ is such that $K = L$. Then necessarily $\gamma \geq r$. It can be checked that for $K = L$ we have $D(t,T) = D_1(t,T) + D_3(t,T)$ where:

$$D_1(t,T) = LB(t,T)\big(\mathcal{N}\big(h_1(V_t, T-t)\big) - Z_t^{2\hat{a}}\mathcal{N}\big(h_2(V_t, T-t)\big)\big)$$

$$D_3(t,T) = V_t\big(Z_t^{\theta+\zeta}\mathcal{N}\big(h_7(V_t, T-t)\big) + Z_t^{\theta-\zeta}\mathcal{N}\big(h_8(V_t, T-t)\big)\big).$$

• **Case $\gamma = r$.** If we also assume that $\gamma = r$ then $\zeta = -\sigma^{-2}\hat{\nu}$, and thus

$$V_t Z_t^{\theta+\zeta} = LB(t,T), \quad V_t Z_t^{\theta-\zeta} = V_t Z_t^{2\hat{a}+1} = LB(t,T) Z_t^{2\hat{a}}.$$

It is also easy to see that in this case

$$h_1(V_t, T-t) = \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}} = -h_7(V_t, T-t),$$

while

$$h_2(V_t, T-t) = \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}} = h_8(V_t, T-t).$$

We conclude that if $\bar{v}(t) = Le^{-r(T-t)} = LB(t,T)$ then $D(t,T) = LB(t,T)$. This result is quite intuitive. A corporate bond with a safety covenant represented by the barrier function, which equals the discounted value of the bond's face value, is equivalent to a default-free bond with the same face value and maturity.

• **Case $\gamma > r$.** For $K = L$ and $\gamma > r$, it is natural to expect that $D(t,T)$ would be smaller than $LB(t,T)$. It is also possible to show that when γ tends to infinity (all other parameters being fixed), then the Black and Cox price converges to Merton's price.

1.2.3 Further Developments

The Black and Cox first-passage-time approach was later developed by, among others: Brennan and Schwartz [35, 36] – an analysis of convertible bonds, Kim et al. [134] – a random barrier and random interest rates, Nielsen et al. [161] – a random barrier and random interest rates, Leland [147], Leland and Toft [148] – a study of an optimal capital structure, bankruptcy costs and tax benefits, Longstaff and Schwartz [152] – a constant barrier and random interest rates, Brigo [37].

One can study the problem

$$\tau = \inf\{t \,:\, V_t \leq L(t)\}$$

where $L(t)$ is a deterministic function and V a geometric Brownian motion. However, there exists few explicit results. See the appendix for some references.

• **Other stopping times** Moraux suggests to chose, as default time a Parisian stopping time For a continuous process V and a given $t > 0$, we introduce $g_t^b(V)$, the last time before t at which the process V was at level b, i.e.,

$$g_t^b(V) = \sup\{s \leq t : V_s = b\}.$$

The Parisian time is the first time at which the process V is under b for a period greater than D, i.e.,

$$G_D^{-,b}(V) = \inf\{t > 0 : (t - g_t^b(V))1\!\!1_{\{V_t<b\}} \geq D\}$$

This time is a stopping time. Let $\tau = G_D^{-,b}(V)$. See Appendix for results on the joint law of (τ, V_τ) in the case of a Black-Scholes dynamics.

Another default time is the first time where the process V has spend more than D time below a level, i.e., $\tau = \inf\{t : A_t^V > D\}$ where $A_t^V = \int_0^t 1\!\!1_{V_s>b}ds$. The law of this time is related with cumulative options.

Campi and Sbuelz [40] present the case where the default time is given by a first hitting time of a CEV process and study the difficult problem of pricing an equity default swap. [40] More precisely, hey assume that the dynamics of the firm is

$$dS_t = S_{t-}\left((r-\kappa)dt + \sigma S_t^\beta dW_t - dM_t\right)$$

where W is a BM and M the compensated martingale of a Poisson process (i.e., $M_t = N_t - \lambda t$), and they define

$$\tau = \inf\{t : S_t \leq 0\}$$

In other terms, they take $\tau = \tau^\beta \wedge \tau^N$ where τ^N is the first jump of the Poisson process and

$$\tau^\beta = \inf\{t : X_t \leq 0\}$$

where

$$dX_t = X_{t-}\left((r-\kappa+\lambda)dt + \sigma X_t^\beta dW_t\right) .$$

Using that a CEV process can be expressed in terms of a Bessel process time changed, and results on the hitting time of 0 for a Bessel process of dimension smaller than 2, they obtain closed from solutions.

• **Zhou's model** Zhou [180] studies the case where the dynamics of the firm is

$$dV_t = V_{t-}\left((\mu - \lambda\nu)dt + \sigma dW_t + dX_t\right)$$

where W is a Brownian motion, X a compound Poisson process $X_t = \sum_1^{N_t} e^{Y_i} - 1$ where $\ln Y_i \overset{law}{=} \mathcal{N}(a, b^2)$ with $\nu = \exp(a+b^2/2)-1$. This choice of parameters implies that $Ve^{\mu t}$ is a martingale. In a first part, Zhou studies Merton's problem in that setting. In a second part, he gives an approximation for the first passage problem when the default time is $\tau = \inf\{t : V_t \leq L\}$.

1.2.4 Optimal Capital Structure

We consider a firm that has an interest paying bonds outstanding. We assume that it is a consol bond, which pays continuously coupon rate c. Assume that $r > 0$ and the payout rate κ is equal to zero. This condition can be given a financial interpretation as the restriction on the sale of assets, as opposed to issuing of new equity. Equivalently, we may think about a situation in which the stockholders will make payments to the firm to cover the interest payments. However, they have the right to stop making payments at any time and either turn the firm over to the bondholders or pay them a lump payment of c/r per unit of the bond's notional amount.

Recall that we denote by $E(V_t)$ ($D(V_t)$, resp.) the value at time t of the firm equity (debt, resp.), hence the total value of the firm's assets satisfies $V_t = E(V_t) + D(V_t)$.

Black and Cox [25] argue that there is a critical level of the value of the firm, denoted as v^*, below which no more equity can be sold. The critical value v^* will be chosen by stockholders, whose aim is to minimize the value of the bonds (equivalently, to maximize the value of the equity). Let us observe that v^* is nothing else than a constant default barrier in the problem under consideration; the optimal default time τ^* thus equals $\tau^* = \inf\{\, t \geq 0 : V_t \leq v^*\}$.

To find the value of v^*, let us first fix the bankruptcy level $\bar{v}$. The ODE for the pricing function $u^\infty = u^\infty(V)$ of a consol bond takes the following form (recall that $\sigma = \sigma_V$)

$$\frac{1}{2} V^2 \sigma^2 u^\infty_{VV} + rV u^\infty_V + c - ru^\infty = 0,$$

subject to the lower boundary condition $u^\infty(\bar{v}) = \min(\bar{v}, c/r)$ and the upper boundary condition

$$\lim_{V \to \infty} u^\infty_V(V) = 0.$$

For the last condition, observe that when the firm's value grows to infinity, the possibility of default becomes meaningless, so that the value of the defaultable consol bond tends to the value c/r of the default-free consol bond. The general solution has the following form:

$$u^\infty(V) = \frac{c}{r} + K_1 V + K_2 V^{-\alpha},$$

where $\alpha = 2r/\sigma^2$ and K_1, K_2 are some constants, to be determined from boundary conditions. We find that $K_1 = 0$, and

$$K_2 = \begin{cases} \bar{v}^{\alpha+1} - (c/r)\bar{v}^\alpha, & \text{if } \bar{v} < c/r, \\ 0, & \text{if } \bar{v} \geq c/r. \end{cases}$$

Hence, if $\bar{v} < c/r$ then

$$u^\infty(V_t) = \frac{c}{r} + \left(\bar{v}^{\alpha+1} - \frac{c}{r}\bar{v}^\alpha\right) V_t^{-\alpha}$$

or, equivalently,

$$u^\infty(V_t) = \frac{c}{r}\left(1 - \left(\frac{\bar{v}}{V_t}\right)^\alpha\right) + \bar{v}\left(\frac{\bar{v}}{V_t}\right)^\alpha.$$

It is in the interest of the stockholders to select the bankruptcy level in such a way that the value of the debt, $D(V_t) = u^\infty(V_t)$, is minimized, and thus the value of firm's equity

$$E(V_t) = V_t - D(V_t) = V_t - \frac{c}{r}(1 - \bar{q}_t) - \bar{v}\bar{q}_t$$

is maximized. It is easy to check that the optimal level of the barrier does not depend on the current value of the firm, and it equals

$$v^* = \frac{c}{r}\frac{\alpha}{\alpha+1} = \frac{c}{r + \sigma^2/2}.$$

Given the optimal strategy of the stockholders, the price process of the firm's debt (i.e., of a consol bond) takes the form, on the set $\{\tau^* > t\}$,

$$D^*(V_t) = \frac{c}{r} - \frac{1}{\alpha V_t^\alpha}\left(\frac{c}{r + \sigma^2/2}\right)^{\alpha+1}$$

or, equivalently,

$$D^*(V_t) = \frac{c}{r}(1 - q_t^*) + v^* q_t^*,$$

where

$$q_t^* = \left(\frac{v^*}{V_t}\right)^\alpha = \frac{1}{V_t^\alpha}\left(\frac{c}{r + \sigma^2/2}\right)^\alpha.$$

Further Developments

We end this section by remarking that other important developments in the area of optimal capital structure were presented in the papers by Leland [147], Leland and Toft[148], Christensen et al. [46]. Chen and Kou [43], Dao [56], Hilberink and Rogers [100], LeCourtois and Quittard-Pinon [145] study the same problem modelling the firm value process as a diffusion with jumps. The reason for this extension was to eliminate an undesirable feature of previously examined models, in which short spreads tend to zero when a bond approaches maturity date.

1.3 Stochastic Interest Rates

In this section, we assume that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$, supports the short-term interest rate process r and the value process V. The dynamics under the martingale measure $\mathbb{Q}$ of the firm's value and of the price of a default-free zero-coupon bond $B(t,T)$ are

$$dV_t = V_t\big((r_t - \kappa(t))\,dt + \sigma(t)\,dW_t\big)$$

and

$$dB(t,T) = B(t,T)\big(r_t\,dt + b(t,T)\,dW_t\big)$$

respectively, where W is a d-dimensional standard $\mathbb{Q}$-Brownian motion. Furthermore, $\kappa : [0,T] \to I\!R$, $\sigma : [0,T] \to I\!R^d$ and $b(\cdot,T) : [0,T] \to I\!R^d$ are assumed to be bounded functions. The *forward value* $F_V(t,T) = V_t/B(t,T)$ of the firm satisfies under the *forward martingale measure* $\mathbb{P}_T$

$$dF_V(t,T) = -\kappa(t)F_V(t,T)\,dt + F_V(t,T)\big(\sigma(t) - b(t,T)\big)\,dW_t^T$$

where the process $W_t^T = W_t - \int_0^t b(u,T)\,du$, $t \in [0,T]$, is a d-dimensional SBM under $\mathbb{P}_T$. For any $t \in [0,T]$, we set

$$F_V^{\kappa}(t,T) = F_V(t,T)e^{-\int_t^T \kappa(u)\,du}.$$

Then

$$dF_V^{\kappa}(t,T) = F_V^{\kappa}(t,T)\big(\sigma(t) - b(t,T)\big)\,dW_t^T.$$

Furthermore, it is apparent that $F_V^{\kappa}(T,T) = F_V(T,T) = V_T$. We consider the following modification of the Black and Cox approach:

$$X = L,\ Z_t = \beta_2 V_t,\ \tilde{X} = \beta_1 V_T,\ \tau = \inf\{\, t \in [0,T] : V_t < v_t\},$$

where $\beta_2,\, \beta_1 \in [0,1]$ are constants, and the barrier v is given by the formula

$$v_t = \begin{cases} KB(t,T)e^{\int_t^T \kappa(u)\,du} & \text{for } t < T, \\ L & \text{for } t = T, \end{cases}$$

with the constant K satisfying $0 < K \leq L$.

Let us denote, for any $t \leq T$,

$$\kappa(t,T) = \int_t^T \kappa(u)\,du, \quad \sigma^2(t,T) = \int_t^T |\sigma(u) - b(u,T)|^2\,du$$

where $|\cdot|$ is the Euclidean norm in $I\!R^d$. For brevity, we write $F_t = F_V^{\kappa}(t,T)$, and we denote

$$\eta_+(t,T) = \kappa(t,T) + \frac{1}{2}\sigma^2(t,T), \quad \eta_-(t,T) = \kappa(t,T) - \frac{1}{2}\sigma^2(t,T).$$

The following result extends Black and Cox valuation formula for a corporate bond to the case of random interest rates.

Proposition 1.3.1 *For any $t < T$, the forward price of a defaultable bond $F_D(t,T) = D(t,T)/B(t,T)$ equals on the set $\{\tau > t\}$*

$$\begin{aligned} &L\big(\mathcal{N}\big(\widehat{h}_1(F_t,t,T)\big) - (F_t/K)e^{-\kappa(t,T)}\mathcal{N}\big(\widehat{h}_2(F_t,t,T)\big)\big) \\ &\quad + \beta_1 F_t e^{-\kappa(t,T)}\big(\mathcal{N}\big(\widehat{h}_3(F_t,t,T)\big) - \mathcal{N}\big(\widehat{h}_4(F_t,t,T)\big)\big) \\ &\quad + \beta_1 K\big(\mathcal{N}\big(\widehat{h}_5(F_t,t,T)\big) - \mathcal{N}\big(\widehat{h}_6(F_t,t,T)\big)\big) \\ &\quad + \beta_2 K J_+(F_t,t,T) + \beta_2 F_t e^{-\kappa(t,T)} J_-(F_t,t,T), \end{aligned}$$

where

$$\begin{aligned}
\widehat{h}_1(F_t,t,T) &= \frac{\ln(F_t/L)-\eta_+(t,T)}{\sigma(t,T)},\\
\widehat{h}_2(F_t,T,t) &= \frac{2\ln K-\ln(LF_t)+\eta_-(t,T)}{\sigma(t,T)},\\
\widehat{h}_3(F_t,t,T) &= \frac{\ln(L/F_t)+\eta_-(t,T)}{\sigma(t,T)},\\
\widehat{h}_4(F_t,t,T) &= \frac{\ln(K/F_t)+\eta_-(t,T)}{\sigma(t,T)},\\
\widehat{h}_5(F_t,t,T) &= \frac{2\ln K-\ln(LF_t)+\eta_+(t,T)}{\sigma(t,T)},\\
\widehat{h}_6(F_t,t,T) &= \frac{\ln(K/F_t)+\eta_+(t,T)}{\sigma(t,T)},
\end{aligned}$$

and for any fixed $0 \le t < T$ *and* $F_t > 0$ *we set*

$$J_\pm(F_t,t,T) = \int_t^T e^{\kappa(u,T)}\, d\mathcal{N}\left(\frac{\ln(K/F_t)+\kappa(t,T)\pm\frac{1}{2}\sigma^2(t,u)}{\sigma(t,u)}\right).$$

In the special case when $\kappa \equiv 0$, the formula of Proposition 1.3.1 covers as a special case the valuation result established by Briys and de Varenne [38]. In some other recent studies of first passage time models, in which the triggering barrier is assumed to be either a constant or an unspecified stochastic process, typically no closed-form solution for the value of a corporate debt is available, and thus a numerical approach is required (see, for instance, Kim et al. [134], Longstaff and Schwartz [152], Nielsen et al. [161], or Saá-Requejo and Santa-Clara [167]).

1.4 Random Barrier

In the case of full information and Brownian filtration, the first hitting time of a deterministic barrier is predictable. This is no longer the case when we deal with incomplete information (as in Duffie and Lando [71], see also Chapter 2, Section 2.2.7), or when an additional source of randomness is present. We present here a formula for credit spreads arising in a special case of a totally inaccessible time of default. For a more detailed study we refer to Babbs and Bielecki [7]. As we shall see, the method we use here is close to the general method presented in Chapter 4.

We suppose here that the default barrier is a random variable D defined on the underlying probability space $(\Omega, \mathbb{P})$. The default occurs at time τ where

$$\tau = \inf\{t \,:\, V_t \le D\},$$

where V is the value of the firm and, for simplicity, $V_0 = 1$. Note that

$$\{\tau > t\} = \{\inf_{u\le t} V_u > D\}.$$

We shall denote by m_t^V the running minimum of V, i.e. $m_t^V = \inf_{u\le t} V_u$. With this notation, $\{\tau > t\} = \{m_t^V > D\}$. Note that m^V is a decreasing process.

1.4.1 Independent barrier

In a first step we assume that, under the risk-neutral probability $\mathbb{Q}$, the barrier D is independent of the value of the firm. We denote by F_D the cumulative distribution function of the r.v. D, i.e. $F_D(z) = \mathbb{Q}(D \le z)$. We assume that F_D is differentiable and we denote f_D its derivative.

Lemma 1.4.1 *Let $F_t = \mathbb{Q}(\tau \le t|\mathcal{F}_t)$ and $\Gamma_t = -\ln(1-F_t)$. Then*

$$\Gamma_t = -\int_0^t \frac{f_D(m_u^V)}{F_D(m_u^V)} dm_u^V \,.$$

PROOF: If D is independent of $\mathcal{F}_\infty$,

$$F_t = \mathbb{Q}(\tau \le t|\mathcal{F}_t) = \mathbb{Q}(m_t^V \le D|\mathcal{F}_t) = 1 - F_D(m_t^V)\,.$$

The process m^V is decreasing. It follows that $\Gamma_t = -\ln F_D(m_t^V)$, hence $d\Gamma_t = -\frac{f_D(m_t^V)}{F_D(m_t^V)} dm_t^V$ and

$$\Gamma_t = -\int_0^t \frac{f_D(m_u^V)}{F_D(m_u^V)} dm_u^V \,.$$

$\square$

Example 1.4.1 Assume that D is uniformly distributed on the interval $[0,1]$. Then, $\Gamma_t = -\ln m_t^V$. The computation of quantities as $E(e^{\Gamma_T} f(V_T))$ requires the knowledge of the joined law of the pair (V_T, m_T^V).

We postulate now that the value process V is a geometric Brownian motion with a drift, that is, we set $V_t = e^{\Psi_t}$, where $\Psi_t = \mu t + \sigma W_t$. It is clear that $\tau = \inf\{t \ge 0 : \Psi_t^* \le \psi\}$, where Ψ^* is the *running minimum* of the process Ψ: $\Psi_t^* = \inf\{\Psi_s : 0 \le s \le t\}$.

We choose the Brownian filtration as the reference filtration, i.e., we set $\mathbf{F} = \mathbf{F}^W$. Let us denote by $G(z)$ the cumulative distribution function under $\mathbb{Q}$ of the barrier ψ. We assume that $G(z) > 0$ for $z < 0$ and that G admits the density g with respect to the Lebesgue measure (note that $g(z) = 0$ for $z > 0$). This means that we assume that the value process V (hence also the process Ψ) is perfectly observed. In addition, we suppose that the bond investor can observe the occurrence of the default time. Thus, he can observe the process $H_t = \mathbb{1}_{\{\tau \le t\}} = \mathbb{1}_{\{\Psi_t^* \le \psi\}}$. We denote by $\mathbf{H}$ the natural filtration of the process H. The information available to the investor is represented by the (enlarged) filtration $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$.

We assume that the default time τ and interest rates are independent under $\mathbb{Q}$. Then, it is possible to establish the following result (see Giesecke [92] or Babbs and Bielecki [7]). Note that the process Ψ^* is decreasing, so that the integral with respect to this process is a (pathwise) Stieltjes integral.

Proposition 1.4.1 *Under the assumptions stated above, and additionally assuming $L = 1$, $Z \equiv 0$ and $\tilde{X} = 0$, we have that for every $t < T$*

$$S(t,T) = -\mathbb{1}_{\{\tau > t\}} \frac{1}{T-t} \ln \mathbb{E}_{\mathbb{P}^*}\Big(e^{\int_t^T \frac{f_D(\Psi_u^*)}{F_D(\Psi_u^*)}\, d\Psi_u^*} \,\Big|\, \mathcal{F}_t \Big).$$

In the next chapter, we shall introduce the notion of a hazard process of a random time. For the default time τ defined above, the $\mathbf{F}$-hazard process Γ exists and is given by the formula

$$\Gamma_t = -\int_0^t \frac{f_D(\Psi_u^*)}{F_D(\Psi_u^*)}\, d\Psi_u^*.$$

This process is continuous, and thus the default time τ is a totally inaccessible stopping time with respect to the filtration $\mathbf{G}$.

`To be completed`

1.5 Comments on Structural Models

We end this chapter by commenting on merits and drawbacks of the structural approach to credit risk.

Advantages

- An approach based on the volatility of the total value of a firm. The credit risk is thus measured in a standard way. The random time of default is defined in an intuitive way. The default event is linked to the notion of the firm's insolvency.
- Valuation and hedging of defaultable claims relies on similar techniques as the valuation and hedging of exotic options in the standard default-free Black-Scholes setup.
- The concept of the distance to default, which measures the obligor's leverage relative to the volatility of its assets value, may serve to reflect credit ratings.
- Dependent defaults are easy to handle through correlation of processes corresponding to different names.

Disadvantages

- A stringent assumption that the total value of the firm's assets can be easily observed. In practice, continuous-time observations of the value process V are not available. This issue was recently addressed by Crouhy et al.[54], Duffie and Lando [71], Jeanblanc and Valchev [122], who showed that a structural model with incomplete accounting data can be dealt with using the intensity-based methodology. The paper of Guo [97] presents a case with delayed information. See also Section 4.4.2.
- An unrealistic postulate that the total value of the firm's assets is a tradeable security.
- This approach is known to generate low credit spreads for corporate bonds close to maturity. It requires a judicious specification of the default barrier in order to get a good fit to the observed spread curves.

Other issues

- A major problem with applying structural models is the difficulty with estimation of the volatility of assets value. For the classical Merton's model, there exists a simple formula that relates this volatility to the volatility of the firm's equity, which in principle can be easily estimated. However, no such simple expression exists in case of first-passage-time models. Certain market-oriented technologies, such as CreditGrades, attempt to produce such a formula.
- Structural models discussed above were at most one-factor models, with the only factor being the short-term interest rate. Two- and three-factor structural models have been also developed and closed-form valuation formulae were derived in some special cases.

Chapter 2

Hazard Function Approach

We provide in this chapter a detailed analysis of the relatively simple case of the *reduced form* methodology, when the flow of informations available to an agent reduces to the observations of the random time which models the default event. The focus is on the evaluation of conditional expectations with respect to the filtration generated by a default time with the use of the hazard function. We study hedging strategies based on CDS and/or with DZC. We also present a model with two default times. In the following chapters, we shall study the case when an additional information flow - formally represented by some filtration $\mathbf{F}$ - is present, with the use of the hazard process.

2.1 The Toy Model

We begin with the simple case where a riskless asset, with **deterministic** interest rate $(r(s); s \geq 0)$ is the only asset available in the default-free market.

The price of a risk-free zero-coupon bond with maturity T is $B(0,T) = \exp\left(-\int_0^T r(s)ds\right)$, whereas its time t price $B(t,T)$ is

$$B(t,T) \stackrel{def}{=} \exp\left(-\int_t^T r(s)ds\right) .$$

Default occurs at time τ (where τ is assumed to be a positive random variable with density f, constructed on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$). We denote by F the cumulative function of the r.v. τ defined as $F(t) = \mathbb{P}(\tau \leq t) = \int_0^t f(s)ds$ and we assume that $F(t) < 1$ for any $t < T$, where T is the maturity date (Otherwise there exists $t_0 < T$ such that $F(t_0) = 1$, and default occurs a.s. before t_0). We emphasize that the risk is not hedgeable. Indeed, a random payoff of the form $\mathbb{1}_{\{T<\tau\}}$ cannot be perfectly hedged with deterministic zero-coupon bonds which are the only tradeable assets in our model. To hedge the risk, we shall assume later on that some defaultable asset is traded, e.g., a defaultable zero-coupon bond or a CDS (Credit Default Swap).

Remark 2.1.1 It is not difficult to generalize the study presented in what follows to the case where τ does not admit a density by dealing with the right-continuous version of the cumulative function. The case where τ is bounded can also be studied along the same method. We leave the details to the reader.

2.1.1 Defaultable Zero-coupon with Payment at Maturity

A **defaultable zero-coupon** bond (DZC in short)- or a corporate bond- with maturity T and rebate R paid at maturity, consists of

- The payment of one monetary unit at time T if default has not occurred before time T, i.e., if $\tau > T$,
- A payment of R monetary units, made at maturity, if $\tau \le T$, where $0 < R < 1$.

Value of the defaultable zero-coupon bond

The "value" of the defaultable zero-coupon bond is defined as the expectation of discounted payoffs

$$\begin{aligned} D^{(R,T)}(0,T) &= \mathbb{E}(B(0,T)\,[\mathbb{1}_{\{T<\tau\}} + R\mathbb{1}_{\{\tau \le T\}}]) \\ &= B(0,T)\,\mathbb{E}(1-(1-R)\mathbb{1}_{\{\tau\le T\}}) \\ &= B(0,T)[1-(1-R)F(T)]\,, \end{aligned} \tag{2.1}$$

where the T in the superscript for $D^{(R,T)}$ means that the recovery R is paid at maturity T. In fact, this quantity is a net present value and is equal to the value of the default free ZC, minus the expected loss, computed under the historical probability. Obviously, this is not a hedging price.

The time-t value depends whether or not default has happened before this time. If default has occurred before time t, the payment of R will be made at time T, and the price of the DZC is $RB(t,T)$.
If the default has not yet occurred, the holder does not know when it will occur. The value $D^{(R,T)}(t,T)$ of the DZC is the conditional expectation of the discounted payoff $B(t,T)\,[\mathbb{1}_{\{T<\tau\}} + R\mathbb{1}_{\{\tau\le T\}}]$ given the information:

$$D^{(R,T)}(t,T) = \mathbb{1}_{\{\tau\le t\}}B(t,T)R + \mathbb{1}_{t<\tau}\widetilde{D}^{(R,T)}(t,T)$$

where the predefault value $\widetilde{D}^{(R,T)}$ is defined as

$$\begin{aligned} \widetilde{D}^{(R,T)}(t,T) &= \mathbb{E}\big(B(t,T)\,(\mathbb{1}_{\{T<\tau\}} + R\mathbb{1}_{\{\tau\le T\}})\,\big|t<\tau\big) \\ &= B(t,T)\big(1-(1-R)\mathbb{P}(\tau\le T\big|t<\tau)\big) \\ &= B(t,T)\left(1-(1-R)\frac{\mathbb{P}(t<\tau\le T)}{\mathbb{P}(t<\tau)}\right) \\ &= B(t,T)\left(1-(1-R)\frac{F(T)-F(t)}{1-F(t)}\right) \end{aligned} \tag{2.2}$$

Note that the value of the DZC is discontinuous at time τ, unless $F(T)=1$ (or $R=1$). In the case $F(T)=1$, the default appears with probability one before maturity and the DZC is equivalent to a payment of R at maturity. If $R=1$, the DZC is in fact a default-free zero coupon bond.

Formula (2.2) can be read as

$$D^{(R,T)}(t,T) = B(t,T) - \text{EDLGD } \times \text{DP}$$

where the **Expected Discounted Loss Given Default** (EDLGD) is defined as $B(t,T)(1-R)$ and the conditional **Default Probability** (DP) is

$$DP = \frac{\mathbb{P}(t<\tau\le T)}{\mathbb{P}(t<\tau)} = \mathbb{P}(\tau\le T|t<\tau)\,.$$

In case the payment is a function of the default time, say $R(\tau)$, the value of this defaultable zero-coupon is

$$\begin{aligned} D^{(R,t)}(0,T) &= \mathbb{E}\left(B(0,T)\,\mathbb{1}_{\{T<\tau\}} + B(0,T)R(\tau)\mathbb{1}_{\{\tau\le T\}}\right) \\ &= B(0,T)\left[\mathbb{P}(T<\tau) + \int_0^T R(s)f(s)ds\right]\,. \end{aligned}$$

If the default has not occurred before t, the predefault time-t value $\widetilde{D}^{(R,T)}(t,T)$ satisfies

$$\begin{aligned}\widetilde{D}^{(R,T)}(t,T) &= B(t,T)\mathbb{E}(\,1\!\!1_{\{T<\tau\}} + R(\tau)1\!\!1_{\{\tau\leq T\}}\big|t<\tau) \\ &= B(t,T)\left[\frac{\mathbb{P}(T<\tau)}{\mathbb{P}(t<\tau)} + \frac{1}{\mathbb{P}(t<\tau)}\int_t^T R(s)f(s)ds\right].\end{aligned}$$

To summarize,

$$D^{(R,T)}(t,T) = 1\!\!1_{1\!\!1_{\{t<\tau\}}}\widetilde{D}^{(R,T)}(t,T) + 1\!\!1_{1\!\!1_{\{\tau\leq t\}}}R(\tau)\,B(t,T)\,.$$

Hazard function

We introduce the **hazard function** Γ defined by

$$\Gamma(t) = -\ln(1-F(t))$$

and its derivative $\gamma(t) = \dfrac{f(t)}{1-F(t)}$ where $f(t) = F'(t)$, i.e.,

$$1-F(t) = e^{-\Gamma(t)} = \exp\left(-\int_0^t \gamma(s)ds\right) = \mathbb{P}(\tau > t)\,.$$

The quantity $\gamma(t)$ is the **hazard rate**. The interpretation of the hazard rate is the probability that the default occurs in a small interval dt given that the default did not occur before time t

$$\gamma(t) = \lim_{h\to 0}\frac{1}{h}P(\tau \leq t+h|\tau > t)\,.$$

Note that Γ is increasing.

Then, formula (2.2) reads

$$\begin{aligned}\widetilde{D}^{(R,T)}(t,T) &= B(t,T)\left(\frac{1-F(T)}{1-F(t)} + R\frac{F(T)-F(t)}{1-F(t)}\right) \\ &= \widetilde{D}(t,T) + R(B(t,T) - \widetilde{D}(t,T))\,,\end{aligned}$$

where

$$\widetilde{D}(t,T) = \exp\left(-\int_t^T (r+\gamma)(s)ds\right),$$

is the predefault value at time t of a DZC which pays one monetary unit at maturity, if the default did not occur before maturity. Hence, the spot rate has to be adjusted by means of a spread (equal to γ) in order to evaluate DZCs.

The dynamics of $\widetilde{D}^{(R,T)}$ can be easily written in terms of the function γ as

$$d_t\widetilde{D}^{(R,T)}(t,T) = (r(t)+\gamma(t))\,\widetilde{D}^{(R,T)}(t,T)\,dt - B(t,T)\gamma(t)R(t)dt$$

The dynamics of $D^{(R,T)}$ will be written in the next section.

If γ and R are constant, the credit spread is

$$\frac{1}{T-t}\ln\frac{B(t,T)}{\widetilde{D}^{(R,T)}(t,T)} = \gamma - \frac{1}{T-t}\ln\left(1+R(e^{\gamma(T-t)}-1)\right)$$

and goes to $\gamma(1-R)$ when t goes to T.

The quantity $\lambda(t,T) = \frac{f(t,T)}{1-F(t,T)}$ where

$$F(t,T) = \mathbb{P}(\tau \leq T|\tau > t)$$

and $f(t,T)\,dT = \mathbb{P}(\tau \in dT | \tau > t)$ is called the *conditional hazard rate.* One has

$$F(t,T) = 1 - \exp - \int_t^T \lambda(s,T) ds\,.$$

In our setting,

$$1 - F(t,T) = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)} = \exp - \int_t^T \gamma(s) ds$$

and $\lambda(s,T) = \gamma(s)$.

Remark 2.1.2 In case τ is the first jump of an inhomogeneous Poisson process with deterministic intensity $(\lambda(t), t \geq 0)$ (See Appendix if needed),

$$f(t) = \mathbb{P}(\tau \in dt)/dt = \lambda(t) \exp\left(-\int_0^t \lambda(s) ds\right) = \lambda(t) e^{-\Lambda(t)}$$

where $\Lambda(t) = \displaystyle\int_0^t \lambda(s) ds$ and $\mathbb{P}(\tau \leq t) = F(t) = 1 - e^{-\Lambda(t)}$, hence the hazard function is equal to the compensator of the Poisson process, i.e. $\Gamma(t) = \Lambda(t)$. Conversely, if τ is a random time with density f, setting $\Lambda(t) = -\ln(1 - F(t))$ allows us to interpret τ as the first jump time of an inhomogeneous Poisson process with intensity the derivative of Λ.

2.1.2 Defaultable Zero-coupon with Payment at Hit

Here, a defaultable zero-coupon bond with maturity T consists of

- The payment of one monetary unit at time T if default has not yet occurred,
- A payment of $R(\tau)$ monetary units, where R is a deterministic function, made at time τ if $\tau \leq T$.

Value of the defaultable zero-coupon

The value of this defaultable zero-coupon bond is

$$\begin{aligned} D^{(R)}(0,T) &= \mathbb{E}(B(0,T)\,\mathbb{1}_{\{T<\tau\}} + B(0,\tau)R(\tau)\mathbb{1}_{\{\tau \leq T\}}) \\ &= \mathbb{P}(T < \tau) B(0,T) + \int_0^T B(0,s) R(s) dF(s) \\ &= G(T) B(0,T) - \int_0^T B(0,s) R(s) dG(s)\,, \end{aligned} \tag{2.3}$$

where $G(t) = 1 - F(t) = \mathbb{P}(t < \tau)$ is the survival probability. Obviously, if the default has occurred before time t, the value of the DZC is null (this was not the case for payment of the rebate at maturity), and $D^{(R)}(t,T) = \mathbb{1}_{t<\tau} \widetilde{D}^{(R)}(t,T)$ where $\widetilde{D}^{(R)}(t,T)$ is a deterministic function (the predefault price). The predefault time-t value $\widetilde{D}^{(R)}(t,T)$ satisfies

$$\begin{aligned} B(0,t)\widetilde{D}^{(R)}(t,T) &= \mathbb{E}(B(0,T)\,\mathbb{1}_{\{T<\tau\}} + B(0,\tau)R(\tau)\mathbb{1}_{\{\tau \leq T\}} | t < \tau) \\ &= \frac{\mathbb{P}(T < \tau)}{\mathbb{P}(t < \tau)} B(0,T) + \frac{1}{\mathbb{P}(t < \tau)} \int_t^T B(0,s) R(s) dF(s)\,. \end{aligned}$$

Hence,

$$B(0,t) G(t) \widetilde{D}^{(R)}(t,T) = G(T) B(0,T) - \int_t^T B(0,s) R(s) dG(s)\,.$$

In terms of the hazard function,

$$\widetilde{D}^{(R)}(0,T) = e^{-\Gamma(T)}B(0,T) + \int_0^T B(0,s)e^{-\Gamma(s)}R(s)d\Gamma(s)\,. \tag{2.4}$$

The time-t value $\widetilde{D}^{(R)}(t,T)$ satisfies:

$$B(0,t)e^{-\Gamma(t)}\widetilde{D}^{(R)}(t,T) = e^{-\Gamma(T)}B(0,T) + \int_t^T B(0,s)e^{-\Gamma(s)}R(s)d\Gamma(s)\,.$$

The process $t \to D^{(R)}(t,T)$ admits a discontinuity at time τ.

A particular case

If F is differentiable, the function $\gamma = \Gamma'$ satisfies $f(t) = \gamma(t)e^{-\Gamma(t)}$. Then,

$$\begin{aligned}\widetilde{D}^{(R)}(0,T) &= e^{-\Gamma(T)}B(0,T) + \int_0^T B(0,s)\gamma(s)e^{-\Gamma(s)}R(s)ds\,, \qquad (2.5)\\ &= \widetilde{D}(0,t) + \int_0^T \widetilde{D}(0,s)\gamma(s)R(s)ds\,,\end{aligned}$$

and

$$\widetilde{D}(0,t)\widetilde{D}^{(R)}(t,T) = \widetilde{D}(0,T) + \int_t^T \widetilde{D}(0,s)\gamma(s)R(s)ds$$

with $\widetilde{D}(0,t) = \exp\left(-\int_0^t [r(s)+\gamma(s)]ds\right)$. The defaultable interest rate is $r+\gamma$ and is, as expected, greater than r (the value of a DZC with $R=0$ is smaller than the value of a default-free zero-coupon). The dynamics of $\widetilde{D}^{(R)}(t,T)$ are

$$d\widetilde{D}^{(R)}(t,T) = \{(r(t)+\gamma(t))\widetilde{D}^{(R)}(t,T) - R(t)\gamma(t))\}dt\,.$$

The dynamics of $D^{(R)}$ includes a jump at time τ and will be computed in a next section.

Fractional recovery of treasury value

This case corresponds to $R(t) = RB(t,T)$.

$$D^{(R)}(t,T) = \mathbb{1}_{t<\tau}\left(e^{-\int_t^T (r(s)+\gamma(s)ds} + RB(t,T)\int_t^T ds\gamma(s)e^{\int_t^s \gamma(u)du}\right)$$

Fractional recovery of market value

Let us assume here that the recovery is $R(t) = R\widetilde{D}^{(R)}(t,T)$ where R is a constant (i.e. the recovery is $RD^{(R)}(\tau-,T)$). The dynamics of $\widetilde{D}^{(R)}$ is

$$d\widetilde{D}^{(R)}(t,T) = \{r(t)+\gamma(t)(1-R(t))\}\widetilde{D}^{(R)}(t,T)dt\,,$$

hence

$$\widetilde{D}^{(R)}(t,T) = \exp\left(-\int_t^T r(s)ds - \int_t^T \gamma(u)(1-R(u))du\right).$$

2.1.3 Implied probabilities

If defaultable zero-coupon bonds with zero recovery are traded in the market at price $D^{(R,*)}(t,T)$, the implied survival probability is $\mathbb{Q}^*$ such that $\mathbb{Q}^*(\tau > T|\tau > t) = \frac{D^{(R,*)}(t,T)}{B(t,T)}$. Of course, this probability may differ from the historical probability. The implied hazard rate is the function $\lambda(t,T)$ such that

$$\lambda(t,T) = -\frac{\partial}{\partial T} \ln \frac{D^{(R,*)}(t,T)}{B(t,T)} = \gamma^*(T) .$$

In the toy model, the implies hazard rate is not very interesting. The aim is to obtain

$$\widetilde{D}^{(R,*)}(t,T) = B(t,T) \exp - \int_t^T \lambda(t,s) ds .$$

This approach will be useful when he predefault price is stochastic.

2.1.4 Spreads

A term structure of credit spreads associated with the zero-coupon bonds $S(t,T)$ is defined as

$$S(t,T) = -\frac{1}{T-t} \ln \frac{D^{(R,*)}(t,T)}{B(t,T)} .$$

In our setting, on the set $\{\tau > t\}$

$$S(t,T) = -\frac{1}{T-t} \ln \mathbb{Q}^*(\tau > T|\tau > t) ,$$

whereas $S(t,T) = \infty$ on the set $\{\tau \le t\}$.

2.2 Toy Model and Martingales

We now present the results of the previous section in a different form, following closely Dellacherie ([63], page 122). We keep the same notation for the cumulative function and the hazard function, assumed to be continuous. We denote by $(H_t, t \ge 0)$ the right-continuous increasing process $H_t = 1\!\!1_{\{t \ge \tau\}}$ and by $(\mathcal{H}_t)$ its natural filtration. The filtration $\mathbf{H}$ is the smallest filtration which makes τ a stopping time. The σ-algebra $\mathcal{H}_t$ is generated by the sets $\{\tau \le s\}$ for $s \le t$ (or by the r.v. $\tau \wedge t$) (note that the set $\{\tau > t\}$ is an atom). A key point is that any integrable $\mathcal{H}_t$-measurable r.v. H is of the form $H = h(\tau \wedge t) = h(\tau)1\!\!1_{\{\tau \le t\}} + h(t)1\!\!1_{\{t < \tau\}}$ where h is a Borel function.
We now give some elementary tools to compute the conditional expectation w.r.t. $\mathcal{H}_t$, as presented in Brémaud [32], Dellacherie [63], Elliott [80]. Note that if the cumulative distribution function F is continuous, then, τ is a $\mathbf{H}$-totally inaccessible stopping time. (See Dellacherie and Meyer [67] IV, 107.)

2.2.1 Key Lemma

Lemma 2.2.1 *If X is any integrable, $\mathcal{G}$-measurable r.v.*

$$\mathbb{E}(X|\mathcal{H}_s)1\!\!1_{\{s<\tau\}} = 1\!\!1_{\{s<\tau\}} \frac{\mathbb{E}(X 1\!\!1_{\{s<\tau\}})}{\mathbb{P}(s < \tau)} . \tag{2.6}$$

PROOF: The r.v. $\mathbb{E}(X|\mathcal{H}_s)$ is $\mathcal{H}_s$-measurable. Therefore, it can be written in the form $\mathbb{E}(X|\mathcal{H}_s) = h(\tau \wedge s) = h(\tau)1\!\!1_{\{s \ge \tau\}} + h(s)1\!\!1_{\{s<\tau\}}$ for some function h. By multiplying both members by $1\!\!1_{\{s<\tau\}}$, and taking the expectation, we obtain

$$\begin{aligned}\mathbb{E}[1\!\!1_{\{s<\tau\}}\mathbb{E}(X|\mathcal{H}_s)] &= \mathbb{E}[\mathbb{E}(1\!\!1_{\{s<\tau\}}X|\mathcal{H}_s)] = \mathbb{E}[1\!\!1_{\{s<\tau\}}X] \\ &= \mathbb{E}(h(s)1\!\!1_{\{s<\tau\}}) = h(s)\mathbb{P}(s<\tau)\,.\end{aligned}$$

Hence, $h(s) = \dfrac{\mathbb{E}(X1\!\!1_{\{s<\tau\}})}{\mathbb{P}(s<\tau)}$ gives the desired result. □

Corollary 2.2.1 *Assume that Y is $\mathcal{H}_\infty$-measurable, so that $Y = h(\tau)$ for some Borel measurable function $h : I\!\!R_+ \to I\!\!R$. If the hazard function Γ of τ is continuous then*

$$\mathbb{E}(Y|\mathcal{H}_t) = 1\!\!1_{\{\tau\le t\}}h(\tau) + 1\!\!1_{\{t<\tau\}}\int_t^\infty h(u)e^{\Gamma(t)-\Gamma(u)}\,d\Gamma(u). \tag{2.7}$$

If τ admits the intensity function γ then

$$\mathbb{E}(Y|\mathcal{H}_t) = 1\!\!1_{\{\tau\le t\}}h(\tau) + 1\!\!1_{\{t<\tau\}}\int_t^\infty h(u)\gamma(u)e^{-\int_t^u \gamma(v)\,dv}\,du.$$

In particular, for any $t \le s$ we have

$$\mathbb{P}(\tau > s|\mathcal{H}_t) = 1\!\!1_{\{t<\tau\}}e^{-\int_t^s \gamma(v)\,dv}$$

and

$$\mathbb{P}(t<\tau<s|\mathcal{H}_t) = 1\!\!1_{\{t<\tau\}}\Big(1-e^{-\int_t^s \gamma(v)\,dv}\Big).$$

2.2.2 Some Martingales

Proposition 2.2.1 *The process $(M_t, t \ge 0)$ defined as*

$$M_t = H_t - \int_0^{\tau\wedge t}\frac{dF(s)}{1-F(s)} = H_t - \int_0^t (1-H_{s-})\frac{dF(s)}{1-F(s)}$$

is a **H***-martingale.*

Proof: Let $s<t$. Then:

$$\mathbb{E}(H_t - H_s|\mathcal{H}_s) = 1\!\!1_{\{s<\tau\}}\mathbb{E}(1\!\!1_{\{s<\tau\le t\}}|\mathcal{H}_s) = 1\!\!1_{\{s<\tau\}}\frac{F(t)-F(s)}{1-F(s)}\,, \tag{2.8}$$

which follows from (2.6) with $X = 1\!\!1_{\{\tau\le t\}}$.
On the other hand, the quantity

$$C \stackrel{def}{=} \mathbb{E}\left[\int_s^t (1-H_{u-})\frac{dF(u)}{1-F(u)}\,|\,\mathcal{H}_s\right],$$

is equal to

$$\begin{aligned}C &= \int_s^t \frac{dF(u)}{1-F(u)}\mathbb{E}\left[1\!\!1_{\{\tau>u\}}|\mathcal{H}_s\right] \\ &= 1\!\!1_{\{\tau>s\}}\int_s^t \frac{dF(u)}{1-F(u)}\left(1-\frac{F(u)-F(s)}{1-F(s)}\right) \\ &= 1\!\!1_{\{\tau>s\}}\left(\frac{F(t)-F(s)}{1-F(s)}\right)\end{aligned}$$

which, from (2.8) proves the desired result. □

The function

$$\int_0^t \frac{dF(s)}{1-F(s)} = -\ln(1-F(t)) = \Gamma(t)$$

is the **hazard function**.
From Proposition 2.2.1, we obtain the Doob-Meyer decomposition of the submartingale H_t as $M_t + \Gamma(t \wedge \tau)$. The predictable process $A_t = \Gamma_{t\wedge\tau}$ is called the **compensator** of H.
In particular, if F is differentiable, the process

$$M_t = H_t - \int_0^{\tau\wedge t} \gamma(s)ds = H_t - \int_0^t \gamma(s)(1-H_s)ds$$

is a martingale, where $\gamma(s) = \dfrac{f(s)}{1-F(s)}$ is a deterministic non-negative function, called **the intensity of** τ.

Proposition 2.2.2 *Assume that F (and thus also Γ) is a continuous function. Then the process $M_t = H_t - \Gamma(t \wedge \tau)$ follows a $\mathbf{D}$-martingale.*

We can now write the dynamics of a defaultable zero-coupon bond with recovery R paid at hit, assuming that M is a martingale under the risk-neutral probability.

Proposition 2.2.3 *The risk-neutral dynamics of a DZC with recovery paid at hit is*

$$dD^{(R)}(t,T) = \Big(r(t)D^{(R)}(t,T) - R(t)\gamma(t)(1-H_t)\Big)\, dt - \widetilde{D}^{(R)}(t,T)dM_t \tag{2.9}$$

where M is the risk-neutral martingale $M_t = H_t - \int_0^t (1-H_s)\gamma_s ds$.

PROOF: From $D^{(R)}(t,T) = 1\!\!1_{t<\tau}\widetilde{D}^{(R)}(t,T) = (1-H_t)\widetilde{D}^{(R)}(t,T)$ and the dynamics of $\widetilde{D}^{(R)}(t,T)$, we obtain

$$\begin{aligned} dD^{(R)}(t,T) &= (1-H_t)d\widetilde{D}^{(R)}(t,T) - \widetilde{D}^{(R)}(t,T)dH_t \\ &= (1-H_t)\Big((r(t)+\gamma(t))\widetilde{D}^{(R)}(t,T) - R(t)\gamma(t)\Big)\, dt - \widetilde{D}^{(R)}(t,T))dH_t \\ &= \Big(r(t)D^{(R)}(t,T) - R(t)\gamma(t)(1-H_t)\Big)\, dt - \widetilde{D}^{(R)}(t,T)dM_t \end{aligned}$$

We emphazise that here, we are working under a risk-neutral probability. We shall see further on how to compute the risk-neutral hazard rate from the historical one, using the Radon-Nikodým density. □

Proposition 2.2.4 *The process $L_t \stackrel{def}{=} 1\!\!1_{\{\tau>t\}} \exp\left(\int_0^t \gamma(s)ds\right)$ is a $\mathbf{H}$-martingale and*

$$L_t = 1 - \int_{]0,t]} L_{u-}dM_u \tag{2.10}$$

In particular, for $t < T$,

$$\mathbb{E}(1\!\!1_{\{\tau>T\}}|\mathcal{H}_t) = 1\!\!1_{\{\tau>t\}} \exp\left(-\int_t^T \gamma(s)ds\right).$$

PROOF: We shall give 3 different arguments, each of which constitutes a proof.
a) Since the function γ is deterministic, for $t > s$

$$\mathbb{E}(L_t|\mathcal{H}_s) = \exp\Big(\int_0^t \gamma(u)du\Big)\, \mathbb{E}(1\!\!1_{\{t<\tau\}}|\mathcal{H}_s)\,.$$

From the equality (2.6)

$$\mathbb{E}(1\!\!1_{\{t<\tau\}}|\mathcal{H}_s) = 1\!\!1_{\{\tau>s\}}\frac{1-F(t)}{1-F(s)} = 1\!\!1_{\{\tau>s\}} \exp\left(-\Gamma(t)+\Gamma(s)\right).$$

Hence,

$$\mathbb{E}(L_t|\mathcal{H}_s) = 1\!\!1_{\{\tau>s\}} \exp\left(\int_0^s \gamma(u)du\right) = L_s.$$

b) Another method is to apply integration by parts formula (see Appendix 8.4.2 if needed) to the process $L_t = (1-H_t)\exp\left(\int_0^t \gamma(s)ds\right)$

$$\begin{aligned} dL_t &= -dH_t \exp\left(\int_0^t \gamma(s)ds\right) + \gamma(t)\exp\left(\int_0^t \gamma(s)ds\right)(1-H_t)dt \\ &= -\exp\left(\int_0^t \gamma(s)ds\right) dM_t\,. \end{aligned}$$

c) A third (sophisticated) method is to note that L is the exponential martingale of M (see Appendix), i.e., the solution of the SDE

$$dL_t = -L_{t-}dM_t\,,\; L_0 = 1.$$

□

Proposition 2.2.5 *Assume that* Γ *is a continuous function. Then for any* (*bounded*) *Borel measurable function* $h : \mathbb{R}_+ \to \mathbb{R}$, *the process*

$$M_t^h = 1\!\!1_{\{\tau\le t\}}h(\tau) - \int_0^{t\wedge\tau} h(u)\,d\Gamma(u) \tag{2.11}$$

is a $\mathbf{H}$*-martingale.*

Proof: Notice that the proof given below provides an alternative proof of the first part of Proposition 2.2.2. We wish to establish via direct calculations the martingale property of the process M^h given by formula (2.11). To this end, notice that formula (2.7) in Corollary 2.2.1 gives

$$E\big(h(\tau)1\!\!1_{\{t<\tau\le s\}}\,|\,\mathcal{H}_t\big) = 1\!\!1_{\{t<\tau\}}e^{\Gamma(t)}\int_t^s h(u)e^{-\Gamma(u)}\,d\Gamma(u).$$

On the other hand, using the same formula, we get

$$J := E\Big(\int_{t\wedge\tau}^{s\wedge\tau} h(u)\,d\Gamma(u)\Big) = E\big(\tilde h(\tau)1\!\!1_{\{t<\tau\le s\}} + \tilde h(s)1\!\!1_{\{\tau>s\}}\,|\,\mathcal{H}_t\big)$$

where we set $\tilde h(s) = \int_t^s h(u)\,d\Gamma(u)$. Consequently,

$$J = 1\!\!1_{\{t<\tau\}}e^{\Gamma(t)}\Big(\int_t^s \tilde h(u)e^{-\Gamma(u)}\,d\Gamma(u) + e^{-\Gamma(s)}\tilde h(s)\Big).$$

To conclude the proof, it is enough to observe that Fubini's theorem yields

$$\begin{aligned} &\int_t^s e^{-\Gamma(u)}\int_t^u h(v)\,d\Gamma(v)\,d\Gamma(u) + e^{-\Gamma(s)}\tilde h(s) \\ &= \int_t^s h(u)\int_u^s e^{-\Gamma(v)}\,d\Gamma(v)\,d\Gamma(u) + e^{-\Gamma(s)}\int_t^s h(u)\,d\Gamma(u) \\ &= \int_t^s h(u)e^{-\Gamma(u)}\,d\Gamma(u), \end{aligned}$$

as expected. □

Corollary 2.2.2 *Let $h : I\!R_+ \to I\!R$ be a (bounded) Borel measurable function. Then the process*

$$\widetilde{M}^h_t = \exp\big(1\!\!1_{\{\tau\le t\}} h(\tau)\big) - \int_0^{t\wedge\tau} (e^{h(u)} - 1)\, d\Gamma(u) \tag{2.12}$$

is a **D**-*martingale.*

Proof: In view of the preceding result applied to $e^h - 1$, it is enough to observe that

$$\exp\big(1\!\!1_{\{\tau\le t\}} h(\tau)\big) = 1\!\!1_{\{\tau\le t\}} e^{h(\tau)} + 1\!\!1_{\{t\ge\tau\}} = 1\!\!1_{\{\tau\le t\}} (e^{h(\tau)} - 1) + 1\,.$$

□

Proposition 2.2.6 *Assume that Γ is a continuous function. Let $h : I\!R_+ \to I\!R$ be a non-negative Borel measurable function such that the random variable $h(\tau)$ is integrable. Then the process*

$$\widehat{M}_t = (1 + 1\!\!1_{\tau\le t} h(\tau)) \exp\Big(- \int_0^{t\wedge\tau} h(u)\, d\Gamma(u)\Big) \tag{2.13}$$

is a **H**-*martingale.*

Proof: One notes that

$$\begin{aligned}\widehat{M}_t &= \exp\Big(- \int_0^t (1 - H_u) h(u)\, d\Gamma(u)\Big) + 1\!\!1_{\tau\le t} h(\tau) \exp\Big(- \int_0^\tau (1 - H_u) h(u)\, d\Gamma(u)\Big) \\ &= \exp\Big(- \int_0^t (1 - H_u) h(u)\, d\Gamma(u)\Big) + \int_0^t h(u) \exp\Big(- \int_0^u (1 - H_s) h(s)\, d\Gamma(s)\Big) dH_u\end{aligned}$$

From Itô's calculus,

$$\begin{aligned}d\widehat{M}_t &= \exp\Big(- \int_0^t (1 - H_u) h(u)\, d\Gamma(u)\Big)(-(1 - H_t) h(t)\, d\Gamma(t) + h(t) dH_t) \\ &= h(t) \exp\Big(- \int_0^t (1 - H_u) h(u)\, d\Gamma(u)\Big) dM_t\,.\end{aligned}$$

□

It is useful to compare with the Doleans-Dade exponential of hM (see Appendix, Section 8.4.4).

Example 2.2.1 In the case where N is an inhomogeneous Poisson process with deterministic intensity λ and τ is the first time when N jumps, let $H_t = N_{t\wedge\tau}$. It is well known that $N_t - \int_0^t \lambda(s) ds$ is a martingale (see Appendix). Therefore, the process stopped at time τ is also a martingale, i.e., $H_t - \int_0^{t\wedge\tau} \lambda(s) ds$ is a martingale. Furthermore, we have seen in Remark 2.1.2 that we can reduce our attention to this case, since any random time can be viewed as the first time where an inhomogeneous Poisson process jumps.

Exercise 2.2.1 In this exercise, F is only continuous on right, and $F(t-)$ is the left limit at point t. Prove that the process $(M_t, t \ge 0)$ defined as

$$M_t = H_t - \int_0^{\tau\wedge t} \frac{dF(s)}{1 - F(s-)} = H_t - \int_0^t (1 - H_{s-}) \frac{dF(s)}{1 - F(s-)}$$

is a **H**-martingale.

2.2.3 Representation Theorem

Proposition 2.2.7 *Let h be a (bounded) Borel function. Then, the martingale $M_t^h = \mathbb{E}(h(\tau)|\mathcal{H}_t)$ admits the representation*

$$\mathbb{E}(h(\tau)|\mathcal{H}_t) = \mathbb{E}(h(\tau)) - \int_0^{t\wedge\tau} (g(s) - h(s))\, dM_s \,,$$

where $M_t = H_t - \Gamma(t\wedge\tau)$ and

$$g(t) = -\frac{1}{G(t)}\int_t^\infty h(u)dG(u) = \frac{1}{G(t)}\mathbb{E}(h(\tau)\mathbb{1}_{\tau>t})\,. \tag{2.14}$$

Note that $g(t) = M_t^h$ on $\{t<\tau\}$. In particular, any square integrable $\mathbf{H}$-martingale $(X_t, t\geq 0)$ can be written as $X_t = X_0 + \int_0^t x_s dM_s$ where $(x_t, t\geq 0)$ is a predictable process.

PROOF: We give two different proofs.

• `First proof:`

From Lemma 2.2.1

$$\begin{aligned} M_t^h &= h(\tau)\mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{t<\tau\}}\frac{\mathbb{E}(h(\tau)\mathbb{1}_{\{t<\tau\}})}{\mathbb{P}(t<\tau)} \\ &= h(\tau)\mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{t<\tau\}}e^{\Gamma(t)}\mathbb{E}(h(\tau)\mathbb{1}_{\{t<\tau\}})\,. \end{aligned}$$

An integration by parts leads to

$$\begin{aligned} &e^{\Gamma_t}\mathbb{E}[h(\tau)\mathbb{1}_{\{t<\tau\}}] = e^{\Gamma_t}\int_t^\infty h(s)dF(s) = g(t) \\ &= \int_0^\infty h(s)dF(s) - \int_0^t e^{\Gamma(s)}h(s)dF(s) + \int_0^t \mathbb{E}(h(\tau)\mathbb{1}_{\{s<\tau\}})e^{\Gamma(s)}d\Gamma(s) \end{aligned}$$

Therefore, since $\mathbb{E}(h(\tau)) = \int_0^\infty h(s)dF(s)$ and $M_s^h = e^{\Gamma(s)}\mathbb{E}(h(\tau)\mathbb{1}_{\{s<\tau\}}) = g(s)$ on $\{s<\tau\}$, the following equality holds on the set $\{t<\tau\}$:

$$e^{\Gamma_t}\mathbb{E}[h(\tau)\mathbb{1}_{\{t<\tau\}}] = \mathbb{E}(h(\tau)) - \int_0^t e^{\Gamma(s)}h(s)dF(s) + \int_0^t g(s)d\Gamma(s)\,.$$

Hence,

$$\begin{aligned} \mathbb{1}_{\{t<\tau\}}\mathbb{E}(h(\tau)|\mathcal{H}_t) &= \mathbb{1}_{\{t<\tau\}}\left(\mathbb{E}(h(\tau)) + \int_0^{t\wedge\tau}(g(s)-h(s))\frac{dF(s)}{1-F(s)}\right) \\ &= \mathbb{1}_{\{t<\tau\}}\left(\mathbb{E}(h(\tau)) - \int_0^{t\wedge\tau}(g(s)-h(s))(dH_s - d\Gamma(s))\right)\,, \end{aligned}$$

where the last equality is due to $\mathbb{1}_{t<\tau}\displaystyle\int_0^{t\wedge\tau}(g(s)-h(s))dH_s = 0$.

On the complementary set $\{t\geq\tau\}$, we have seen that $\mathbb{E}(h(\tau)|\mathcal{H}_t) = h(\tau)$, whereas

$$\begin{aligned} &\int_0^{t\wedge\tau}(g(s)-h(s))(dH_s - d\Gamma(s)) = \int_{]0,\tau]}(g(s)-h(s))(dH_s - d\Gamma(s)) \\ &= \int_{]0,\tau[}(g(s)-h(s))(dH_s - d\Gamma(s)) + (g(\tau^-) - h(\tau))\,. \end{aligned}$$

Therefore,

$$\mathbb{E}(h(\tau)) - \int_0^{t\wedge\tau}(g(s)-h(s))(dH_s - d\Gamma(s)) = M_{\tau^-}^H - (M_{\tau^-}^H - h(\tau)) = h(\tau)\,.$$

The predictable representation theorem follows immediately.

• `Second proof` Another proof consists in computing the conditional expectation

$$\begin{aligned} M_t^h &= \mathbb{E}(h(\tau)|\mathcal{H}_t) = h(\tau)1\!\!1_{\{\tau<t\}} + 1\!\!1_{\{\tau>t\}} e^{-\Gamma(t)} \int_t^\infty h(u)dF(u) \\ &= \int_0^t h(s)dH_s + (1-H_t)e^{-\Gamma(t)} \int_t^\infty h(u)dF(u) = \int_0^t h(s)dH_s + (1-H_t)g(t) \end{aligned}$$

and to use Itô's formula and that $dM_t = dH_t - \gamma(t)(1-H_t)dt$. We obtain, using that $dF(t) = e^{\Gamma(t)}d\Gamma(t) = e^{\Gamma(t)}\gamma(t)dt = -dG(t)$

$$\begin{aligned} dM_t^h &= h(t)dH_t + (1-H_t)h(t)\gamma(t)dt - g(t)\,dH_t - (1-H_t)g(t)\gamma(t)dt \\ &= (h(t)-g(t))dH_t + (1-H_t)(h(t)-g(t))\gamma(t)dt = (h(t)-g(t))dM_t \end{aligned}$$

□

Exercise 2.2.2 If Γ is not continuous, prove that

$$\mathbb{E}(h(\tau)|\mathcal{H}_t) = \mathbb{E}(h(\tau)) - \int_0^{t\wedge\tau} e^{\Delta\Gamma(s)}(g(s)-h(s))\,dM_s\,.$$

2.2.4 Change of a Probability Measure

Let $\mathbb{P}^*$ be an arbitrary probability measure on $(\Omega, \mathcal{H}_\infty)$, which is absolutely continuous with respect to $\mathbb{P}$. We denote by η the $\mathcal{H}_\infty$-measurable density of $\mathbb{P}^*$ with respect to $\mathbb{P}$

$$\eta := \frac{d\mathbb{P}^*}{d\mathbb{P}} = h(\tau) \geq 0, \quad \mathbb{P}\text{-a.s.}, \tag{2.15}$$

where $h : I\!\!R \to I\!\!R_+$ is a Borel measurable function satisfying

$$\mathbb{E}_{\mathbb{P}}(h(\tau)) = \int_0^\infty h(u)\,dF(u) = 1.$$

We can use Girsanov's theorem. Nevertheless, we prefer here to establish this theorem in our particular setting. Of course, the probability measure $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ if and only if the inequality in (2.15) is strict $\mathbb{P}$-a.s. Furthermore, we shall assume that $\mathbb{P}^*(\tau = 0) = 0$ and $\mathbb{P}^*(\tau > t) > 0$ for any $t \in I\!\!R_+$. Actually the first condition is satisfied for any $\mathbb{P}^*$ absolutely continuous with respect to $\mathbb{P}$. For the second condition to hold, it is sufficient and necessary to assume that for every t

$$\mathbb{P}^*(\tau > t) = 1 - F^*(t) = \int_{]t,\infty[} h(u)\,dF(u) > 0,$$

where the c.d.f. F^* of τ under $\mathbb{P}^*$

$$F^*(t) := \mathbb{P}^*(\tau \leq t) = \int_{[0,t]} h(u)\,dF(u). \tag{2.16}$$

Put another way, we assume that

$$g(t) \stackrel{def}{=} e^{\Gamma(t)}\mathbb{E}\big(1\!\!1_{\tau>t}h(\tau)\big) = e^{\Gamma(t)} \int_{]t,\infty[} h(u)\,dF(u) = e^{\Gamma(t)}\,\mathbb{P}^*(\tau > t) > 0.$$

We assume throughout that this is the case, so that the hazard function Γ^* of τ with respect to $\mathbb{P}^*$ is well defined. Our goal is to examine relationships between hazard functions Γ^* and Γ. It is easily seen that in general we have

$$\frac{\Gamma^*(t)}{\Gamma(t)} = \frac{\ln\Big(\int_{]t,\infty[} h(u)\,dF(u)\Big)}{\ln(1-F(t))}, \tag{2.17}$$

since by definition $\Gamma^*(t) = -\ln(1 - F^*(t))$.

Assume first that F is an absolutely continuous function, so that the intensity function γ of τ under $\mathbb{P}$ is well defined. Recall that γ is given by the formula

$$\gamma(t) = \frac{f(t)}{1 - F(t)}.$$

On the other hand, the c.d.f. F^* of τ under $\mathbb{P}^*$ now equals

$$F^*(t) := \mathbb{P}^*(\tau \le t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\tau \le t} h(\tau)) = \int_0^t h(u) f(u)\, du.$$

so that F^* follows an absolutely continuous function. Therefore, the intensity function γ^* of the random time τ under $\mathbb{P}^*$ exists, and it is given by the formula

$$\gamma^*(t) = \frac{h(t)f(t)}{1 - F^*(t)} = \frac{h(t)f(t)}{1 - \int_0^t h(u)f(u)\, du}.$$

To derive a more straightforward relationship between the intensities γ and γ^*, let us introduce an auxiliary function $h^* : I\!R_+ \to I\!R$, given by the formula $h^*(t) = h(t)/g(t)$.

Notice that

$$\gamma^*(t) = \frac{h(t)f(t)}{1 - \int_0^t h(u)f(u)\, du} = \frac{h(t)f(t)}{\int_t^\infty h(u)f(u)\, du} = \frac{h(t)f(t)}{e^{-\Gamma(t)} g(t)} = h^*(t)\,\frac{f(t)}{1 - F(t)} = h^*(t)\gamma(t).$$

This means also that $d\Gamma^*(t) = h^*(t)\, d\Gamma(t)$. It appears that the last equality holds true if F is merely a continuous function. Indeed, if F (and thus F^*) is continuous, we get

$$d\Gamma^*(t) = \frac{dF^*(t)}{1 - F^*(t)} = \frac{d(1 - e^{-\Gamma(t)} g(t))}{e^{-\Gamma(t)} g(t)} = \frac{g(t) d\Gamma(t) - dg(t)}{g(t)} = h^*(t)\, d\Gamma(t).$$

To summarize, if the hazard function Γ is continuous then Γ^* is also continuous and $d\Gamma^*(t) = h^*(t)\, d\Gamma(t)$.

To understand better the origin of the function h^*, let us introduce the following non-negative $\mathbb{P}$-martingale (which is strictly positive when the probability measures $\mathbb{P}^*$ and $\mathbb{P}$ are equivalent)

$$\eta_t := \frac{d\mathbb{P}^*}{d\mathbb{P}}_{|\mathcal{H}_t} = \mathbb{E}_{\mathbb{P}}(\eta | \mathcal{H}_t) = \mathbb{E}_{\mathbb{P}}(h(\tau) | \mathcal{H}_t), \tag{2.18}$$

so that $\eta_t = M_t^h$. The general formula for η_t reads (cf. (2.2.1))

$$\eta_t = \mathbb{1}_{\tau \le t} h(\tau) + \mathbb{1}_{\tau > t}\, e^{\Gamma(t)} \int_{]t,\infty[} h(u)\, dF(u) = \mathbb{1}_{\tau \le t} h(\tau) + \mathbb{1}_{\tau > t} g(t).$$

Assume now that F is a continuous function. Then

$$\eta_t = \mathbb{1}_{\tau \le t} h(\tau) + \mathbb{1}_{\tau > t} \int_t^\infty h(u) e^{\Gamma(t) - \Gamma(u)}\, d\Gamma(u).$$

On the other hand, using the representation theorem, we get

$$M_t^h = M_0^h + \int_{]0,t]} M_{u-}^h (h^*(u) - 1)\, dM_u$$

where $h^*(u) = h(u)/g(u)$. We conclude that

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-} (h^*(u) - 1)\, dM_u. \tag{2.19}$$

It is thus easily seen that

$$\eta_t = \big(1 + 1\!\!1_{\tau \le t} v(\tau)\big) \exp\Big(- \int_0^{t\wedge\tau} v(u)\, d\Gamma(u)\Big), \tag{2.20}$$

where we write $v(t) = h^*(t) - 1$. Therefore, the martingale property of the process η, which is obvious from (2.18), is also a consequence of Proposition 2.2.6.

Remark 2.2.1 In view of (2.19), we have

$$\eta_t = \mathcal{E}_t\Big(\int_0^{\cdot} (h^*(u) - 1)\, dM_u\Big),$$

where $\mathcal{E}$ stands for the Doléans exponential. Representation (2.20) for the random variable η_t can thus be obtained from the general formula for the Doléans exponential. (See Appendix 8.4.4.)

We are in the position to formulate the following result (all statements were already established above).

Proposition 2.2.8 *Let $\mathbb{P}^*$ be any probability measure on $(\Omega, \mathcal{H}_\infty)$ absolutely continuous with respect to $\mathbb{P}$, so that (2.15) holds for some function h. Assume that $\mathbb{P}^*(\tau > t) > 0$ for every $t \in I\!R_+$. Then*

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}_{\,|\mathcal{H}_t} = \mathcal{E}_t\Big(\int_0^{\cdot} (h^*(u) - 1)\, dM_u\Big), \tag{2.21}$$

where

$$h^*(t) = h(t)/g(t), \quad g(t) = e^{\Gamma(t)} \int_t^\infty h(u)\, dF(u),$$

and $\Gamma^*(t) = g^*(t)\Gamma(t)$ with

$$g^*(t) = \frac{\ln\Big(\int_{]t,\infty[} h(u)\, dF(u)\Big)}{\ln(1 - F(t))}. \tag{2.22}$$

If, in addition, the random time τ admits the intensity function γ under $\mathbb{P}$, then the intensity function γ^* of τ under $\mathbb{P}^*$ satisfies $\gamma^*(t) = h^*(t)\gamma(t)$ a.e. on $I\!R_+$. More generally, if the hazard function Γ of τ under $\mathbb{P}$ is continuous, then the hazard function Γ^* of τ under $\mathbb{P}^*$ is also continuous, and it satisfies $d\Gamma^*(t) = h^*(t)\, d\Gamma(t)$.

Corollary 2.2.3 *If F is continuous then $M^*_t = H_t - \Gamma^*(t \wedge \tau)$ is a $\mathbf{H}$-martingale under $\mathbb{P}^*$.*

Proof: In view Proposition 2.2.2, the corollary is an immediate consequence of the continuity of Γ^*. Alternatively, we may check directly that the product $U_t = \eta_t M^*_t = \eta_t(H_t - \Gamma^*(t\wedge\tau))$ follows a $\mathbf{H}$-martingale under $\mathbb{P}$. To this end, observe that the integration by parts formula for functions of finite variation yields

$$\begin{aligned}
U_t &= \int_{]0,t]} \eta_{t-}\, dM^*_t + \int_{]0,t]} M^*_t\, d\eta_t \\
&= \int_{]0,t]} \eta_{t-}\, dM^*_t + \int_{]0,t]} M^*_{t-}\, d\eta_t + \sum_{u\le t} \Delta M^*_u \Delta\eta_u \\
&= \int_{]0,t]} \eta_{t-}\, dM^*_t + \int_{]0,t]} M^*_{t-}\, d\eta_t + 1\!\!1_{\tau\le t}(\eta_\tau - \eta_{\tau-}).
\end{aligned}$$

Using (2.19), we obtain

$$\begin{aligned}
U_t &= \int_{]0,t]} \eta_{t-}\, dM^*_t + \int_{]0,t]} M^*_{t-}\, d\eta_t + \eta_{\tau-} 1\!\!1_{\tau\le t}(h^*(\tau) - 1) \\
&= \int_{]0,t]} \eta_{t-}\, d\big(\Gamma(t\wedge\tau) - \Gamma^*(t\wedge\tau) + 1\!\!1_{\tau\le t}(h^*(\tau) - 1)\big) + N_t,
\end{aligned}$$

where the process N, which equals

$$N_t = \int_{]0,t]} \eta_{t-}\, dM_t + \int_{]0,t]} M^*_{t-}\, d\eta_t$$

is manifestly a **H**-martingale with respect to $\mathbb{P}$. It remains to show that the process

$$N^*_t := \Gamma(t \wedge \tau) - \Gamma^*(t \wedge \tau) + 1\!\!1_{\tau \le t}(h^*(\tau) - 1)$$

follows a **H**-martingale with respect to $\mathbb{P}$. By virtue of Proposition 2.2.5, the process

$$1\!\!1_{\tau \le t}(h^*(\tau) - 1) + \Gamma(t \wedge \tau) - \int_0^{t\wedge\tau} h^*(u)\, d\Gamma(u)$$

is a **H**-martingale. Therefore, to conclude the proof it is enough to notice that

$$\int_0^{t\wedge\tau} h^*(u)\, d\Gamma(u) - \Gamma^*(t \wedge \tau) = \int_0^{t\wedge\tau} (h^*(u)\, d\Gamma(u) - d\Gamma^*(u)) = 0,$$

where the last equality is a consequence of the relationship $d\Gamma^*(t) = h^*(t)\, d\Gamma(t)$ established in Proposition 2.2.8. □

By virtue of Proposition 2.2.2 if Γ^* is a continuous function then the process $M^* = H_t - \Gamma^*(t \wedge \tau)$ follows a **H**-martingale under $\mathbb{P}^*$. The next result suggests that this martingale property uniquely characterizes the (continuous) hazard function of a random time.

Lemma 2.2.2 *Suppose that an equivalent probability measure $\mathbb{P}^*$ is given by formula* (2.15) *for some function h. Let $\Lambda^* : I\!R_+ \to I\!R_+$ be an arbitrary continuous increasing function, with $\Lambda^*(0) = 0$. If the process $M^*_t := H_t - \Lambda^*(t \wedge \tau)$ follows a* **H**-*martingale under $\mathbb{P}^*$, then $\Lambda^*(t) = -\ln(1 - F^*(t))$ with F^* given by formula* (2.21).

PROOF: The Bayes rule implies

$$\mathbb{E}_{\mathbb{P}^*}(M^*_t|\mathcal{H}_s) = \frac{\mathbb{E}_{\mathbb{P}}(M^*_t\eta|\mathcal{H}_s)}{\mathbb{E}_{\mathbb{P}}(\eta|\mathcal{H}_s)} = \eta_s^{-1}\, \mathbb{E}_{\mathbb{P}}(M^*_t\eta_t|\mathcal{H}_s)$$

and thus

$$\mathbb{E}_{\mathbb{P}^*}(M^*_t|\mathcal{H}_s) = \frac{\mathbb{E}_{\mathbb{P}}\big((H_t - \Lambda^*(t\wedge\tau))(H_t h(\tau) + (1 - H_t)g(t))\,\big|\,\mathcal{H}_s\big)}{H_s h(\tau) + (1 - H_s)g(s)}\,,$$

or equivalently

$$\mathbb{E}_{\mathbb{P}^*}(M^*_t|\mathcal{H}_s) = \frac{\mathbb{E}_{\mathbb{P}}\big(H_t h(\tau) - H_t\Lambda^*(t\wedge\tau)h(\tau) - (1 - H_t)\Lambda^*(t\wedge\tau)g(t)\,\big|\,\mathcal{H}_s\big)}{H_s h(\tau) + (1 - H_s)g(s)}\,.$$

This means that

$$\mathbb{E}_{\mathbb{P}^*}(M^*_t|\mathcal{H}_s) = \frac{J}{H_s h(\tau) + (1 - H_s)g(s)}\,,$$

where we write

$$J = \mathbb{E}_{\mathbb{P}}\big(H_t h(\tau) - H_t\Lambda^*(t\wedge\tau)h(\tau) - (1 - H_t)\Lambda^*(t\wedge\tau)g(t)\,\big|\,\mathcal{H}_s\big).$$

We obtain

$$J = H_s h(\tau) - H_s\Lambda^*(\tau)h(\tau) - (1 - H_s)(1 - F(s))^{-1}\,\mathbb{E}_{\mathbb{P}}\big(1\!\!1_{\{s<\tau\le t\}}(\Lambda^*(\tau) - 1)h(\tau) + 1\!\!1_{\{\tau>t\}}\Lambda^*(t)g(t)\big)$$

and thus the martingale condition $\mathbb{E}_{\mathbb{P}^*}(M^*_t|\mathcal{H}_s) = M^*_s$, is equivalent to the following equality

$$(1 - H_s)(1 - F(s))^{-1}\,\mathbb{E}_{\mathbb{P}}\big(1\!\!1_{\{s<\tau\le t\}}(\Lambda^*(\tau) - 1)h(\tau) + 1\!\!1_{\{\tau>t\}}\Lambda^*(t)g(t)\big) = \Lambda^*(s)(1 - H_s)g(s).$$

Therefore, for every $s \le t$ we have

$$\mathbb{E}_{\mathbb{P}}\big(1\!\!1_{\{s<\tau\le t\}}(\Lambda^*(\tau)-1)h(\tau)+1\!\!1_{\{\tau>t\}}\Lambda^*(t)g(t)\big)=\Lambda^*(s)(1-F(s))g(s)$$

so that

$$\int_s^t(\Lambda^*(u)-1)h(u)\,dF(u)+\Lambda^*(t)g(t)(1-F(t))=\Lambda^*(s)\int_s^\infty h(u)\,dF(u),$$

and finally,

$$\int_s^t(\Lambda^*(u)-1)\,dF^*(u)+\Lambda^*(t)(1-F^*(t))=\Lambda^*(s)(1-F^*(s)).$$

After simple manipulations involving the integration by parts, we get for $s \le t$

$$\int_s^t(1-F^*(u))\,d\Lambda^*(u)=F^*(t)-F^*(s),$$

and since $\Lambda^*(0)=F^*(0)=0$, we find that $\Lambda^*=-\ln(1-F^*(t))$. $\square$

Representation Theorem

We now recall a suitable version of the predictable representation theorem and we shall present a different proof. For any RCLL function $\widehat{h}:\mathbb{R}_+\to\mathbb{R}$ such that the random variable $\widehat{h}(\tau)$ is integrable, we set $\widehat{M}_t=\mathbb{E}_{\mathbb{Q}}(\widehat{h}(\tau)\,|\,\mathcal{H}_t)$ for every $t\in\mathbb{R}_+$. It is clear that $\widehat{M}$ is an $\mathbf{H}$-martingale under $\mathbb{Q}$. The following version of the martingale representation theorem is well known (see, for instance, Blanchet-Scalliet and Jeanblanc [27], Jeanblanc and Rutkowski [121] or Proposition 4.3.2 in Bielecki and Rutkowski [23]).

Proposition 2.2.9 *Assume that G is continuous and $\widehat{h}$ is an RCLL function such that the random variable $\widehat{h}(\tau)$ is $\mathbb{Q}$-integrable. Then the $\mathbf{H}$-martingale $\widehat{M}$ admits the following integral representation*

$$\widehat{M}_t=\widehat{M}_0+\int_{]0,t]}(\widehat{h}(u)-\widehat{g}(u))\,dM_u, \tag{2.23}$$

where the continuous function $\widehat{g}:\mathbb{R}_+\to\mathbb{R}$ is given by the formula

$$\widehat{g}(t)=\frac{1}{G(t)}\,\mathbb{E}_{\mathbb{Q}}\big(1\!\!1_{\{\tau>t\}}\widehat{h}(\tau)\big)=-\frac{1}{G(t)}\int_t^\infty\widehat{h}(u)\,dG(u). \tag{2.24}$$

Remark 2.2.2 It is easily seen that on the set $\{t\le\tau\}$ we have $\widehat{g}(t)=\widehat{M}_{t-}$. Therefore, formula (2.23) can also be rewritten as follows

$$\widehat{M}_t=\widehat{M}_0+\int_{]0,t]}\big(\widehat{h}(u)-\widehat{M}_{u-}\big)\,dM_u=\widehat{M}_0+\int_{]0,t]}\big(\widehat{h}(u)-\widetilde{M}(u-)\big)\,dM_u, \tag{2.25}$$

where $\widetilde{M}=\widehat{g}$ is the unique function such that $\widehat{M}_t1\!\!1_{\{\tau>t\}}=\widetilde{M}(t)1\!\!1_{\{\tau>t\}}$ for every $t\in\mathbb{R}_+$.

Lemma 2.2.3 *Let M^1 and M^2 be arbitrary two $\mathbf{H}$-martingales under $\mathbb{Q}$. If for every $t\in[0,T]$ we have $1\!\!1_{\{t<\tau\}}M^1_t=1\!\!1_{\{t<\tau\}}M^2_t$ then $M^1_t=M^2_t$ for every $t\in[0,T]$.*

Proof: We have $M^i_t=\mathbb{E}_{\mathbb{Q}}(h_i(\tau)\,|\,\mathcal{H}_t)$ for some functions $h_i:\mathbb{R}_+\to\mathbb{R}$ such that $h_i(\tau)$ is $\mathbb{Q}$-integrable. Using the well known formula for the conditional expectation

$$\mathbb{E}_{\mathbb{Q}}(h_i(\tau)\,|\,\mathcal{H}_t)=1\!\!1_{\{t\ge\tau\}}h_i(\tau)-1\!\!1_{\{t<\tau\}}\frac{1}{G(t)}\int_t^\infty h_i(u)\,dG(u)=1\!\!1_{\{t\ge\tau\}}h_i(\tau)+1\!\!1_{\{t<\tau\}}\widehat{g}_i(t),$$

and the assumption that $1\!\!1_{\{t<\tau\}} M_t^1 = 1\!\!1_{\{t<\tau\}} M_t^2$, we obtain the equality $\widehat{g}_1(t) = \widehat{g}_2(t)$ for every $t \in [0,T]$ (recall that $\mathbb{Q}(\tau > t) > 0$ for every $t \in [0,T]$). Therefore, we have

$$\int_t^\infty h_1(u)\, dG(u) = \int_t^\infty h_2(u)\, dG(u), \quad \forall\, t \in [0,T].$$

This immediately implies that $h_1(t) = h_2(t)$ on $[0,T]$, almost everywhere with respect to the distribution of τ, and thus we have $h_1(\tau \wedge T) = h_2(\tau \wedge T)$, $\mathbb{Q}$-a.s. Consequently, $M_t^1 = M_t^2$ for every $t \in [0,T]$. □

2.2.5 Incompleteness of the Toy model

In order to study the completeness of the financial market, we first need to define the tradeable assets.

If the market consists only of the risk-free zero-coupon bond, there exists infinitely many e.m.m's. The discounted asset prices are constant, hence the set $\mathcal{Q}$ of equivalent martingale measures is the set of probabilities equivalent to the historical one. For any $\mathbb{Q} \in \mathcal{Q}$, we denote by F_Q the cumulative function of τ under $\mathbb{Q}$, i.e.,

$$F_Q(t) = \mathbb{Q}(\tau \le t)\,.$$

The range of prices is defined as the set of prices which do not induce arbitrage opportunities. For a DZC with a constant rebate R paid at maturity, the range of prices is equal to the set

$$\{\mathbb{E}_{\mathbb{Q}}(B(0,T)(1\!\!1_{\{T<\tau\}} + R1\!\!1_{\{\tau<T\}})), \mathbb{Q} \in \mathcal{Q}\}\,.$$

This set is exactly the interval $]RB(0,T), B(0,T)[$. Indeed, it is obvious that the range of prices is included in the interval $]RB(0,T), B(0,T)[$. Now, in the set $\mathcal{Q}$, one can select a sequence of probabilities $\mathbb{Q}_n$ which converge weakly to the Dirac measure at point 0 (resp. at point T) (the bounds are obtained as limit cases: the default appears at time 0^+, or never). Obviously, this range is too large to be efficient. (See Hugonnier for a generalization of this result)

2.2.6 Risk Neutral Probability Measures

It is usual to interpret the absence of arbitrage opportunities as the existence of an e.m.m. . If DZCs are traded, their prices are *given by the market*, and the equivalent martingale measure $\mathbb{Q}$, *chosen by the market,* is such that, on the set $\{t < \tau\}$,

$$D^{(R)}(t,T) = B(t,T)\mathbb{E}_{\mathbb{Q}}\big([1\!\!1_{T<\tau} + R1\!\!1_{t<\tau\le T}] \,\big|t < \tau\big)\,.$$

Therefore, we can characterize the cumulative function of τ under $\mathbb{Q}$ from the market prices of the DZC as follows.

Zero Recovery

If a DZC with zero recovery of maturity T is traded at a price $D^{(R)}(t,T)$ which belongs to the interval $]0, B(t,T)[$, then, under any risk-neutral probability $\mathbb{Q}$, the process $B(0,t)D^{(R)}(t,T)$ is a martingale (for the moment, we do not know if the market is complete, so we can not claim that the e.m.m. is unique), the following equality holds

$$D^{(R)}(t,T)B(0,t) = \mathbb{E}_{\mathbb{Q}}(B(0,T)1\!\!1_{\{T<\tau\}}|\mathcal{H}_t) = B(0,T)1\!\!1_{\{t<\tau\}} \exp\left(-\int_t^T \lambda^Q(s)ds\right)$$

where $\lambda^Q(s) = \dfrac{dF_Q(s)/ds}{1 - F_Q(s)}$. It is obvious that if $D^{(R)}(t,T)$ belongs to the range of viable prices $]0, B(0,T)[$, the process λ^Q is stricly positive (and the converse holds true). The process λ^Q is the

$\mathbb{Q}$-intensity of τ. Therefore, the value of $\int_t^T \lambda^Q(s)ds$ is known for any t as soon as there are DZC bonds for each maturity, and the unique risk-neutral intensity can be obtained from the prices of DZCs as $r(t) + \lambda^Q(t) = -\partial_T \ln D^{(R)}(t,T)|_{T=t}$.

Remark 2.2.3 It is important to note that there is no relation between the risk-neutral intensity and the historical one. The risk-neutral intensity can be greater (resp. smaller) than the historical one. The historical intensity can be deduced from observation of default time, the risk-neutral one is obtained from the prices of traded defaultable claims.

Fixed Payment at maturity

If the prices of DZCs with different maturities are known, then from (2.1)

$$\frac{B(0,T) - D^{(R,T)}(0,T)}{B(0,T)(1-R)} = F_Q(T)$$

where $F_Q(t) = \mathbb{Q}(\tau \le t)$, so that the law of τ is known under the e.m.m.. However, as noticed in Hull and White [107], *extracting default probabilities from bond prices [is] in practice, usually more complicated. First, the recovery rate is usually non-zero. Second, most corporate bonds are not zero-coupon bonds.*

Payment at hit

In this case the cumulative function can be obtained using the derivative of the defaultable zero-coupon price with respect to the maturity. Indeed, denoting by $\partial_T D^{(R)}$ the derivative of the value of the DZC at time 0 with respect to the maturity, and assuming that $G = 1 - F$ is differentiable, we obtain from (2.3)

$$\partial_T D^{(R)}(0,T) = g(T)B(0,T) - G(T)B(0,T)r(T) - R(T)g(T)B(0,T)\,,$$

where $g(t) = G'(t)$. Therefore, solving this equation leads to

$$\mathbb{Q}(\tau > t) = G(t) = \Delta(t)\left[1 + \int_0^t \partial_T D^{(R)}(0,s)\frac{1}{B(0,s)(1-R(s))}(\Delta(s))^{-1}ds\right]\,,$$

where $\Delta(t) = \exp\left(\int_0^t \frac{r(u)}{1-R(u)}du\right)$.

2.2.7 Partial information: Duffie and Lando's model

Duffie and Lando [71] study the case where $\tau = \inf\{t : V_t \le m\}$ where V satisfies

$$dV_t = \mu(t,V_t)dt + \sigma(t,V_t)dW_t\,.$$

Here the process W is a Brownian motion. If the information is the Brownian filtration, the time τ is a stopping time w.r.t. a Brownian filtration, therefore is predictable and admits no intensity. We will discuss this point latter on. If the agents do not know the behavior of V, but only the minimal information $\mathcal{H}_t$, i.e. he knows when the default appears, the price of a zero-coupon is, in the case where the default is not yet occurred, $\exp\left(-\int_t^T \lambda(s)ds\right)$ where $\lambda(s) = \frac{f(s)}{G(s)}$ and $G(s) = \mathbb{P}(\tau > s)$, $f = -G'$, as soon as the cumulative function of τ is differentiable. Duffie and Lando have obtained that the intensity is $\lambda(t) = \frac{1}{2}\sigma^2(t,0)\frac{\partial f}{\partial x}(t,0)$ where $f(t,x)$ is the conditional density of V_t when $T_0 > t$, i.e. the differential w.r.t. x of $\dfrac{\mathbb{P}(V_t \le x, T_0 > t)}{\mathbb{P}(T_0 > t)}$, where $T_0 = \inf\{t\,;\, V_t = 0\}$. In the case where V is an homogenous diffusion, i.e. $dV_t = \mu(V_t)dt + \sigma(V_t)dW_t$, the equality between Duffie-Lando and our result is not so obvious. See Elliott et al. [81] for comments.

2.3 Pricing and Trading Defaultable Claims

This section gives an overview of basic results concerning the valuation and trading of defaultable claims. Here, we assume that the interest r is constant.

2.3.1 Recovery at maturity

Let S be the price of an asset which delivers only a recovery $R(\tau)$ at time T where R is a deterministic function. We know that the process $dM_t = dH_t - (1-H_t)\gamma_t dt$ is a martingale where $\gamma(t) = f(t)/G(t)$, f is the density of τ and $G(t) = Q(\tau > t)$. Therefore,

$$\begin{aligned}
e^{-rt}S_t &= E_Q\left((R(\tau)e^{-rT}|\mathcal{G}_t\right) = e^{-rT}\mathbb{1}_{\tau<t}R(\tau) + e^{-rT}\mathbb{1}_{\tau>t}\frac{E_Q\left(R(\tau)\mathbb{1}_{t<\tau<T}\right)}{G(t)} \\
&= e^{-rT}\int_0^t R(u)dH_u + e^{-rT}\mathbb{1}_{\tau>t}\widetilde{S}_t
\end{aligned}$$

where $\widetilde{S}_t$ is the predefault price given by the deterministic function

$$\widetilde{S}_t = \frac{E_Q(R(\tau)\mathbb{1}_{t<\tau<T})}{G(t)} = \frac{\int_t^T R(u)f(u)du}{G(t)}$$

Hence,

$$d\widetilde{S}_t = f(t)\frac{\int_t^T R(u)f(u)du}{G^2(t)}dt - \frac{R(t)f(t)}{G(t)}dt = \widetilde{S}_t\frac{f(t)}{G(t)}dt - \frac{R(t)f(t)}{G(t)}dt\,.$$

It follows that

$$\begin{aligned}
d(e^{-rt}S_t) &= e^{-rT}\left(R(t)dH_t + (1-H_t)\frac{f(t)}{G(t)}\left(\widetilde{S}_t - R(t)\right)dt - \widetilde{S}_{t-}dH_t\right) \\
&= (e^{-rT}R(t) - e^{-rt}S_{t-})(dH_t - (1-H_t)\gamma(t)dt) \\
&= e^{-rt}(e^{-r(T-t)}R(t) - S_{t-})\,dM_t\,.
\end{aligned}$$

In that case, the discounted price is a martingale under the risk-neutral probability, and the price S does not vanishes (as soon as R does not)

2.3.2 Recovery at default time

In the case where the recovery is paid at default time, the price of the derivative is obviously equal to 0 after the default time, and

$$\begin{aligned}
e^{-rt}S_t &= E_Q(R(\tau)e^{-r\tau}\mathbb{1}_{t<\tau\le T}|\mathcal{G}_t) = \mathbb{1}_{\tau>t}\frac{E_Q(e^{-r\tau}R(\tau)\mathbb{1}_{t<\tau<T})}{G(t)} \\
&= \mathbb{1}_{\tau>t}\widetilde{S}_t
\end{aligned}$$

where the predefault price is the deterministic function

$$\widetilde{S}_t = \frac{1}{G(t)}\int_t^T R(u)e^{-ru}f(u)du.$$

Then

$$
\begin{aligned}
d\widetilde{S}_t &= -R(t)e^{-rt}f(t)/G(t) + f(t)(\int_t^T R(u)e^{-ru}f(u)du/Q(\tau > t)^2 \\
&= -R(t)e^{-rt}f(t)/G(t) + f(t)\widetilde{S}_t/G(t) \\
&= \frac{f(t)}{G(t)}(-R(t)e^{-rt} + \widetilde{S}_t)dt
\end{aligned}
$$

and

$$
\begin{aligned}
d(e^{-rt}S_t) &= (1-H_t)\frac{f(t)}{G(t)}(-R(t)e^{-rt} + \widetilde{S}_t)dt - \widetilde{S}_t dH_t \\
&= -\widetilde{S}_t(dH_t - (1-H_t)\gamma(t)dt) = (R(t)e^{-rt} - \widetilde{S}_t)dM_t - R(t)e^{-rt}(1-H_t)\gamma(t)dt \\
&= e^{-rt}(R(t) - S_{t-})dM_t - R(t)e^{-rt}(1-H_t)\gamma(t)dt\,.
\end{aligned}
$$

In that case, the discounted process is not a martingale under the risk-neutral probability. The process $S_t e^{-rt} + \int_0^t R(s)e^{-rs}(1-H_s)\gamma(s)ds$ is a martingale. The recovery has to be understood as a dividend process, paid up time τ, at rate γ.

2.4 Pricing and Trading a CDS

We are now in the position to apply the general theory to the case of a particular class contracts, specifically, credit default swaps. We work throughout under a spot martingale measure $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$. We shall work under additional assumption that the interest rate r is null. Subsequently, these restrictions will be relaxed.

2.4.1 Valuation of a Credit Default Swap

A stylized credit default swap is formally introduced through the following definition.

Definition 2.4.1 *A credit default swap with a constant rate κ and recovery at default is a defaultable claim $(0, A, Z, \tau)$, where $Z_t \equiv R(t)$ and $A_t = -\kappa t$ for every $t \in [0,T]$. An RCLL function $R : [0,T] \to I\!R$ represents the default protection, and a constant $\kappa \in I\!R$ represents the CDS rate (also termed the spread, premium or annuity of a CDS).*

We shall first analyze the valuation and trading credit default swaps. We denote by F the cumulative distribution function of the default time τ under $\mathbb{Q}$, and we assume that F is a continuous function, with $F(0) = 0$ and $F(T) < 1$ for some fixed date $T > 0$. Also, we write $G = 1 - F$ to denote the *survival probability function* of τ, so that $G(t) > 0$ for every $t \in [0,T]$. For simplicity of exposition, we assume in this section that the interest rate $r = 0$, so that the price of a savings account $B_t = 1$ for every t. Note also that we have only one tradeable asset in our model (a savings account), and we wish to value a defaultable claim within this model. It is clear that any probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{H}_T)$, equivalent to $\mathbb{Q}$, can be chosen as a spot martingale measure for our model. The choice of $\mathbb{Q}$ is reflected in the cumulative distribution function F (in particular, in the default intensity if F is absolutely continuous).

Ex-dividend Price of a CDS

Consider a CDS with the rate κ, which was initiated at time 0 (or indeed at any date prior to the current date t). Its market value at time t does not depend on the past otherwise than through the level of the rate κ. Unless explicitly stated otherwise, we assume that κ is an arbitrary constant.

Unless explicitly stated otherwise, we assume that the default protection payment is received at the time of default, and it is equal $R(t)$ if default occurs at time t, prior to or at maturity date T.

In view of (4.34), the ex-dividend price of a CDS maturing at T with rate κ is given by the formula

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau\leq T\}}R(\tau)\,\Big|\,\mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau\}}\kappa\big((\tau\wedge T)-t\big)\,\Big|\,\mathcal{H}_t\Big), \tag{2.26}$$

where the first conditional expectation represents the current value of the *default protection stream* (or the *protection leg*), and the second is the value of the *survival annuity stream* (or the *fee leg*).

Note that in Lemma 2.4.1, we do not need to specify the inception date s of a CDS. We only assume that the maturity date T, the rate κ, and the protection payment R are given.

Lemma 2.4.1 *The ex-dividend price at time $t \in [s,T]$ of a credit default swap started at s, with rate κ and protection payment $R(\tau)$ at default, equals*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\frac{1}{G(t)}\left(-\int_t^T R(u)\,dG(u) - \kappa\int_t^T G(u)\,du\right). \tag{2.27}$$

PROOF: We have, on the set $\{t<\tau\}$,

$$\begin{aligned} S_t(\kappa) &= -\frac{\int_t^T R(u)\,dG(u)}{G(t)} - \kappa\left(\frac{-\int_t^T u\,dG(u) + TG(T)}{G(t)} - t\right) \\ &= \frac{1}{G(t)}\left(-\int_t^T R(u)\,dG(u) - \kappa\Big(TG(T) - tG(t) - \int_t^T u\,dG(u)\Big)\right). \end{aligned}$$

Since

$$\int_t^T G(u)\,du = TG(T) - tG(t) - \int_t^T u\,dG(u), \tag{2.28}$$

we conclude that (2.27) holds. □

The ex-dividend price of a CDS can also be represented as follows (see (4.35))

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\widetilde{S}_t(\kappa), \quad \forall\, t\in[0,T], \tag{2.29}$$

where $\widetilde{S}_t(\kappa)$ stands for the *ex-dividend pre-default price* of a CDS. It is useful to note that formula (2.27) yields an explicit expression for $\widetilde{S}_t(\kappa)$, and that $\widetilde{S}(\kappa)$ follows a continuous function, provided that G is continuous.

2.4.2 Market CDS Rate

Assume now that a CDS was initiated at some date $s \leq t$ and its initial price was equal to zero. Since a CDS with this property plays an important role, we introduce a formal definition. In Definition 2.4.2, it is implicitly assumed that a recovery function R is given.

Definition 2.4.2 *A market CDS started at s is a CDS initiated at time s whose initial value is equal to zero. A T-maturity market CDS rate (also known as the fair CDS spread) at time s is the level of the rate $\kappa=\kappa(s,T)$ that makes a T-maturity CDS started at s valueless at its inception. A market CDS rate at time s is thus determined by the equation $S_s(\kappa(s,T))=0$, where S is defined by (2.26). By assumption, $\kappa(s,T)$ is an $\mathcal{F}_s$-measurable random variable (hence, a constant if the reference filtration is trivial).*

Under the present assumptions, by virtue of Lemma 2.4.1, the T-maturity market CDS rate $\kappa(s,T)$ solves the following equation

$$\int_s^T R(u)\,dG(u) + \kappa(s,T)\int_s^T G(u)\,du = 0,$$

and thus we have, for every $s \in [0,T]$,

$$\kappa(s,T) = -\frac{\int_s^T R(u)\, dG(u)}{\int_s^T G(u)\, du}. \tag{2.30}$$

Remarks 2.4.1 Let us comment briefly on a model calibration. Suppose that at time 0 the market gives the premium of a CDS for any maturity T. In this way, the market chooses the risk-neutral probability measure $\mathbb{Q}$. Specifically, if $\kappa(0,T)$ is the T-maturity market CDS rate for a given recovery function R then we have

$$\kappa(0,T) = -\frac{\int_0^T R(u)\, dG(u)}{\int_0^T G(u)\, du}.$$

Hence, if credit default swaps with the same recovery function R and various maturities are traded at time 0, it is possible to find the implied risk-neutral c.d.f. F (and thus the default intensity γ under $\mathbb{Q}$) from the *term structure of CDS rates* $\kappa(0,T)$ by solving an ordinary differential equation.

Standing assumptions. We fix a maturity date T, and we write briefly $\kappa(s)$ instead of $\kappa(s,T)$. In addition, we assume that all credit default swaps have a common recovery function R.

Note that the ex-dividend pre-default value at time $t \in [0,T]$ of a CDS with any fixed rate κ can be easily related to the market rate $\kappa(t)$. We have the following result, in which the quantity $\nu(t,s) = \kappa(t) - \kappa(s)$ represents the *calendar CDS market spread* (for a given maturity T).

Proposition 2.4.1 *The ex-dividend price of a market CDS started at s with recovery R at default and maturity T equals, for every $t \in [s,T]$,*

$$S_t(\kappa(s)) = \mathbb{1}_{\{t<\tau\}} \left(\kappa(t) - \kappa(s)\right) \frac{\int_t^T G(u)\, du}{G(t)} = \mathbb{1}_{\{t<\tau\}}\, \nu(t,s) \frac{\int_t^T G(u)\, du}{G(t)}, \tag{2.31}$$

or more explicitly,

$$S_t(\kappa(s)) = \mathbb{1}_{\{t<\tau\}} \frac{\int_t^T G(u)\, du}{G(t)} \left(\frac{\int_s^T R(u)\, dG(u)}{\int_s^T G(u)\, du} - \frac{\int_t^T R(u)\, dG(u)}{\int_t^T G(u)\, du} \right). \tag{2.32}$$

PROOF: To establish equality (2.32), it suffices to observe that $S_t(\kappa(s)) = S_t(\kappa(s)) - S_t(\kappa(t))$, and to use (2.27) and (2.30). □

Remark 2.4.1 Note that the price of a CDS can take negative values.

Forward Start CDS

A representation of the value of a swap in terms of the market swap rate, similar to (2.31), is well known to hold for default-free interest rate swaps. It is particularly useful if the calendar spread is modeled as a stochastic process. In particular, it leads to the Black swaption formula within the framework of Jamshidian's [112] model of co-terminal forward swap rates.

In the present context, it is convenient to consider a *forward start CDS* initiated at time $s \in [0,U]$ and giving default protection over the future time interval $[U,T]$. If the reference entity defaults prior to the start date U the contract is terminated and no payments are made. The price of this contract at any date $t \in [s,U]$ equals

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{U<\tau\le T\}} R(\tau) \,\Big|\, \mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{U<\tau\}} \kappa\big((\tau \wedge T) - U\big) \,\Big|\, \mathcal{H}_t\Big). \tag{2.33}$$

Since a forward start CDS does not pays any dividends prior to the start date U, the price $S_t(\kappa)$, $t \in [s,U]$, can be considered here as either the cum-dividend price or the ex-dividend price. Note that since G is continuous, the probability of default occurring at time U equals zero, and thus for $t = U$ the last formula coincides with (2.26). This is by no means surprising, since at time T a forward start CDS becomes a standard (i.e., spot) CDS.

If G is continuous, representation (2.33) can be made more explicit, namely,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \left(- \int_U^T R(u)\, dG(u) - \kappa \int_U^T G(u)\, du \right).$$

A *forward start market CDS* at time $t \in [0, U]$ is a forward CDS in which κ is chosen at time t in such a way that the contract is valueless at time t. The corresponding (pre-default) *forward CDS rate* $\kappa(t, U, T)$ is thus determined by the the following equation

$$S_t(\kappa(t,U,T)) = \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{U<\tau\le T\}} R(\tau) \,\Big|\, \mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{U<\tau\}} \kappa(t,U,T)\big((\tau \wedge T) - U\big) \,\Big|\, \mathcal{H}_t\Big) = 0,$$

which yields, for every $t \in [0, U]$,

$$\kappa(t, U, T) = -\frac{\int_U^T R(u)\, dG(u)}{\int_U^T G(u)\, du}.$$

The price of an arbitrary forward CDS can be easily expressed in terms of κ and $\kappa(t, U, T)$. We have, for every $t \in [0, U]$,

$$S_t(\kappa) = S_t(\kappa) - S_t(\kappa(t,U,T)) = (\kappa(t,U,T) - \kappa)\, \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{U<\tau\}}\big((\tau \wedge T) - U\big) \,\Big|\, \mathcal{H}_t\Big),$$

or more explicitly,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} (\kappa(t,U,T) - \kappa)\, \frac{\int_U^T G(u)\, du}{G(t)}.$$

Under the assumption of a deterministic default intensity, the formulae above are of rather limited interest. Let us stress, however, that similar representations are also valid in the case of a stochastic default intensity, where they prove useful in pricing of options on a forward start CDS (equivalently, options on a forward CDS rate).

Case of a Constant Default Intensity

Assume that $R(t) = R$ is independent of t, and $F(t) = 1 - e^{-\gamma t}$ for a constant default intensity $\gamma > 0$ under $\mathbb{Q}$. In this case, the valuation formulae for a CDS can be further simplified. In view of Lemma 2.4.1, the ex-dividend price of a (spot) CDS with rate κ equals, for every $t \in [0, T]$,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} (R\gamma - \kappa)\gamma^{-1}\Big(1 - e^{-\gamma(T-t)}\Big).$$

The last formula (or the general formula (2.30)) yields $\kappa(s) = R\gamma$ for every $s < T$, so that the market rate $\kappa(s)$ is independent of s. As a consequence, the ex-dividend price of a market CDS started at s equals zero not only at the inception date s, but indeed at any time $t \in [s, T]$, both prior to and after default). Hence, this process follows a trivial martingale under $\mathbb{Q}$. As we shall see in what follows, this martingale property the ex-dividend price of a market CDS is an exception, rather than a rule, so that it no longer holds if default intensity is not constant.

2.4.3 Price Dynamics of a CDS

Unless explicitly stated otherwise, we consider a spot CDS and we assume that

$$G(t) = \mathbb{Q}(\tau > t) = \exp\left(- \int_0^t \gamma(u)\, du \right),$$

where the default intensity $\gamma(t)$ under $\mathbb{Q}$ is a strictly positive deterministic function. We first focus on the dynamics of the ex-dividend price of a CDS with rate κ started at some date $s < T$.

Lemma 2.4.2 *The dynamics of the ex-dividend price $S_t(\kappa)$ on $[s,T]$ are*

$$dS_t(\kappa) = -S_{t-}(\kappa)\,dM_t + (1-H_t)(\kappa - R(t)\gamma(t))\,dt, \tag{2.34}$$

where the **H**-*martingale M under $\mathbb{Q}$ is given by the formula*

$$M_t = H_t - \int_0^t (1-H_u)\gamma(u)\,du, \quad \forall\, t \in I\!R_+. \tag{2.35}$$

Hence, the process $\bar{S}_t(\kappa)$, $t\in[s,T]$, given by the expression

$$\bar{S}_t(\kappa) = S_t(\kappa) + \int_s^t R(u)\,dH_u - \kappa\int_s^t (1-H_u)\,du \tag{2.36}$$

is a $\mathbb{Q}$-martingale for $t\in[s,T]$. Specifically,

$$d\bar{S}_t(\kappa) = \big(R(t) - S_{t-}(\kappa)\big)\,dM_t = \big(R(t) - (1-H_{t-})\widetilde{S}_{t-}(\kappa)\big)\,dM_t\,. \tag{2.37}$$

Proof: It suffices to recall that

$$S_t(\kappa) = 1\!\!1_{\{t<\tau\}}\widetilde{S}_t(\kappa) = (1-H_t)\widetilde{S}_t(\kappa)$$

so that

$$dS_t(\kappa) = (1-H_t)\,d\widetilde{S}_t(\kappa) - \widetilde{S}_{t-}(\kappa)\,dH_t.$$

Using formula (2.27), we find easily that we have

$$d\widetilde{S}_t(\kappa) = \gamma(t)\widetilde{S}_t(\kappa)\,dt + (\kappa - R(t)\gamma(t))\,dt. \tag{2.38}$$

In view of (2.35) and the fact that $S_{\tau-}(\kappa) = \widetilde{S}_{\tau-}(\kappa)$, the proof of (2.34) is complete. To prove the second statement, it suffices to observe that the process N given by

$$N_t = S_t(\kappa) - \int_s^t (1-H_u)(\kappa - R(u)\gamma(u))\,du = -\int_s^t S_{u-}(\kappa)\,dM_u$$

is an **H**-martingale under $\mathbb{Q}$. But for every $t\in[s,T]$

$$\bar{S}_t(\kappa) = N_t + \int_s^t R(u)\,dM_u,$$

so that $\bar{S}(\kappa)$ also follows an **H**-martingale under $\mathbb{Q}$. Note that the process $\bar{S}(\kappa)$ given by (2.36) represents the cum-dividend price of a CDS, so that the martingale property $\bar{S}(\kappa)$ is expected. □

Equality (2.34) emphasizes the fact that a single cash flow of $R(\tau)$ occurring at time τ can be formally treated as a dividend stream at the rate $R(t)\gamma(t)$ paid continuously prior to default. It is clear that we also have

$$dS_t(\kappa) = -\widetilde{S}_{t-}(\kappa)\,dM_t + (1-H_t)(\kappa - R(t)\gamma(t))\,dt. \tag{2.39}$$

In some instances, it can be useful to reformulate the dynamics of a market CDS in terms of market observables, such as CDS spreads.

Corollary 2.4.1 *The dynamics of the ex-dividend price $S_t(\kappa(s))$ on $[s,T]$ are also given as*

$$dS_t(\kappa(s)) = -S_{t-}(\kappa(s))\,dM_t + (1-H_t)\left(\frac{\int_t^T G(u)\,du}{G(t)}\,d_t\nu(t,s) - \nu(t,s)\,dt\right). \tag{2.40}$$

Proof: Under the present assumptions, for any fixed s, the calendar spread $\nu(t,s)$, $t\in[s,T]$ is a continuous function of bounded variation. In view of (2.34), it suffices to check that

$$\frac{\int_t^T G(u)\,du}{G(t)}\,d_t\nu(t,s) - \nu(t,s)\,dt = (\kappa(s) - R(t)\gamma(t))\,dt, \tag{2.41}$$

where $d_t\nu(t,s) = d_t(\kappa(t)-\kappa(s)) = d\kappa(t)$. Equality (2.41) follows by elementary computations. □

Trading a Credit Default Swap

We shall show that, in the present set-up, in order to replicate an arbitrary contingent claim Y settling at time T and satisfying the usual integrability condition, it suffices to deal with two traded assets: a CDS with maturity $U \geq T$ and a constant savings account $B = 1$. Since one can always work with discounted values, the last assumption is not restrictive.

According to Section 4.3.4, a strategy $\phi_t = (\phi_t^0, \phi_t^1)$, $t \in [0, T]$, is self-financing if the wealth process $V(\phi)$, defined as

$$V_t(\phi) = \phi_t^0 + \phi_t^1 S_t(\kappa), \tag{2.42}$$

satisfies

$$dV_t(\phi) = \phi_t^1 \big(dS_t(\kappa) + dD_t\big) = \phi_t^1 d\bar{S}_t(\kappa), \tag{2.43}$$

where $S(\kappa)$ is the ex-dividend price of a CDS with the dividend stream D , and so, $\bar{S}(\kappa) = S(\kappa) + D$ is the corresponding cum-dividend price process. As usual, we say that a strategy ϕ replicates a contingent claim Y if $V_T(\phi) = Y$. On the set $\{\tau \leq t \leq T\}$ the ex-dividend price $S(\kappa)$ equals zero, and thus the total wealth is necessarily invested in B, so that it is constant. This means that ϕ replicates Y if and only if $V_{\tau\wedge T}(\phi) = Y$.

Lemma 2.4.3 *For any self-financing strategy ϕ we have, on the set $\{\tau \leq T\}$,*

$$\Delta_\tau V(\phi) := V_\tau(\phi) - V_{\tau-}(\phi) = \phi_\tau^1 (R(\tau) - \widetilde{S}_\tau(\kappa)). \tag{2.44}$$

PROOF: In general, the process ϕ^1 is **G**-predictable. In our model, ϕ^1 is assumed to be an RCLL function. The jump of the wealth process $V(\phi)$ at time τ equals, on the set $\{\tau \leq T\}$,

$$\Delta_\tau V(\phi) = \phi_\tau^1 \Delta_\tau S + \phi_\tau^1 \Delta_\tau D = \phi_\tau^1 \Delta_\tau \bar{S},$$

where $\Delta_\tau S(\kappa) = S_\tau(\kappa) - S_{\tau-}(\kappa) = -\widetilde{S}_\tau(\kappa)$ (recall that the ex-dividend price $S(\kappa)$ drops to zero at default time) and manifestly $\Delta_\tau D = R(\tau)$. □

2.4.4 Hedging of Defaultable Claims

An $\mathcal{H}_T$-measurable random variable Y is known to admit the following representation

$$Y = 1\!\!1_{\{T \geq \tau\}} Z(\tau) + 1\!\!1_{\{T < \tau\}} X, \tag{2.45}$$

where $Z : [0, T] \to I\!\!R$ is a Borel measurable function, and X is a constant. For definiteness, we shall deal with claims Y such that h is an RCLL function, but this formal restriction is not essential.

Using results of Section 2.3.1

$$\begin{aligned} E(V_T|\mathcal{H}_t) = V_t &= Z_\tau 1\!\!1_{\{\tau \leq t\}} + 1\!\!1_{\{t<\tau\}} \frac{1}{G_t} \left(X G_T + \int_t^T Z_s dG_s \right) \\ &= \int_0^t Z_t dH_s + (1 - H_s) \frac{1}{G_t} \left(X G_T + \int_t^T Z_s dG_s \right) \end{aligned}$$

hence $dV_t = (Z_t - \widetilde{g}) dM_t$ with $\widetilde{g}(t) = \frac{1}{G_t}(\int_t^T Z_s dG_s + X G_T)$.

Our aim is to provide an hedging strategy for Y, using a CDS. In view of Lemma 2.4.2, the dynamics of the price $S(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - R(t)\gamma(t))\, dt.$$

From Corollary 4.3.1, we know that the wealth $V(\phi)$ of any admissible self-financing strategy ϕ^0, ϕ_1 built on savings account and CDS, is an **H**-martingale under $\mathbb{Q}$. Furthermore, using the dynamics obtained in Section 2.3.1, we have

$$dV_t(\phi) = -\phi_t^1\,(S_{t-} - R_t)dM_t \tag{2.46}$$

Then, by identification

Proposition 2.4.2 *Assume that the inequality $\widetilde{S}_t(\kappa) \neq R(t)$ holds for every $t \in [0,T]$. Let ϕ^1 be an RCLL function given by the formula*

$$\phi_t^1 = \frac{h(t) - \widetilde{g}(t)}{R(t) - \widetilde{S}_t(\kappa)}, \tag{2.47}$$

and let $\phi_t^0 = V_t(\phi) - \phi_t^1 S_t(\kappa)$, where the process $V(\phi)$ is given by (2.43) *with the initial condition $V_0(\phi) = \mathbb{E}_{\mathbb{Q}}(Y)$, where Y is given by* (2.45). *Then the self-financing trading strategy $\phi = (\phi^0, \phi^1)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$, where $X = c(T)$ and $Z_t = h(t)$.*

Replication of a Defaultable Claim

We now give a different proof, based on the representation theorem.

To deal with a claim Y given (2.45), we shall apply Proposition 2.2.9 to the function $\widehat{h}$, where $\widehat{h}(t) = h(t)$ for $t \leq T$ and $\widehat{h}(t) = c(T)$ for $t > T$ (recall that $\mathbb{Q}(\tau = T) = 0$). In this case, we obtain

$$\widehat{g}(t) = \frac{1}{G(t)} \left(-\int_t^T h(u)\, dG(u) + c(T)G(T) \right), \tag{2.48}$$

and thus for the process $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}}(Y \,|\, \mathcal{H}_t)$, $t \in [0,T]$, we have

$$\widehat{M}_t = \mathbb{E}_{\mathbb{Q}}(Y) + \int_{]0,t]} (h(u) - \widehat{g}(u))\, dM_u \tag{2.49}$$

with $\widehat{g}$ given by (2.48). Recall that $\widetilde{S}(\kappa)$ is the pre-default ex-dividend price process of a CDS with rate κ and maturity T. We know that $\widetilde{S}(\kappa)$ is a continuous function of t if G is continuous.

Proposition 2.4.3 *Assume that the inequality $\widetilde{S}_t(\kappa) \neq R(t)$ holds for every $t \in [0,T]$. Let ϕ^1 be an RCLL function given by the formula*

$$\phi_t^1 = \frac{h(t) - \widehat{g}(t)}{R(t) - \widetilde{S}_t(\kappa)}, \tag{2.50}$$

and let $\phi_t^0 = V_t(\phi) - \phi_t^1 S_t(\kappa)$, where the process $V(\phi)$ is given by (2.43) *with the initial condition $V_0(\phi) = \mathbb{E}_{\mathbb{Q}}(Y)$, where Y is given by* (2.45). *Then the self-financing trading strategy $\phi = (\phi^0, \phi^1)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$, where $X = c(T)$ and $Z_t = h(t)$.*

Proof: The idea of the proof is based on the observation that it is enough to concentrate on the formula for trading strategy prior to default. In view of Lemma 2.4.2, the dynamics of the price $S(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - R(t)\gamma(t))\, dt.$$

and thus we have, on the set $\{\tau > t\}$,

$$dS_t(\kappa) = d\widetilde{S}_t(\kappa) = \big(\gamma(t)\widetilde{S}_t(\kappa) + \kappa - R(t)\gamma(t)\big)dt. \tag{2.51}$$

From Corollary 4.3.1, we know that the wealth $V(\phi)$ of any admissible self-financing strategy is an $\mathbf{H}$-martingale under $\mathbb{Q}$. Since under the present assumptions $dB_t = 0$, for the wealth process $V(\phi)$ we obtain, on the set $\{\tau > t\}$,

$$dV_t(\phi) = \phi_t^1\,(d\widetilde{S}_t(\kappa) - \kappa\, dt) = -\phi_t^1 \gamma(t)\big(R(t) - \widetilde{S}_t(\kappa)\big)\, dt. \tag{2.52}$$

For the martingale $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}}(Y \mid \mathcal{H}_t)$ associated with Y, in view of (2.49) we obtain, on the set $\{\tau > t\}$,

$$d\widehat{M}_t = -\gamma(t)(h(t) - \widehat{g}(t))\,dt. \tag{2.53}$$

We wish to find ϕ^1 such that $V_t(\phi) = \widehat{M}_t$ for every $t \in [0,T]$. To this end, we first focus on the equality $\mathbb{1}_{\{t<\tau\}} V_t(\phi) = \mathbb{1}_{\{t<\tau\}} \widehat{M}_t$ for pre-default values. Since $\gamma(t)$ is assumed to be strictly positive, a comparison of (2.52) with (2.53) yields

$$\phi^1_t = \frac{h(t) - \widehat{g}(t)}{R(t) - \widetilde{S}_t(\kappa)}, \quad \forall\, t \in [0,T]. \tag{2.54}$$

We thus see that if $V_0(\phi) = \widehat{M}_0$ then also $\mathbb{1}_{\{t<\tau\}} V_t(\phi) = \mathbb{1}_{\{t<\tau\}} \widehat{M}_t$ for every $t \in [0,T]$. As usual, the second component of a self-financing strategy ϕ is given by (2.42), that is, $\phi^0_t = V_t(\phi) - \phi^1_t S_t(\kappa)$, where $V(\phi)$ is given by (2.43) with the initial condition $V_0(\phi) = \mathbb{E}_{\mathbb{Q}}(Y)$. In particular, we have that $\phi^0_0 = \mathbb{E}_{\mathbb{Q}}(Y) - \phi^1_0 S_0(\kappa)$.

To complete the proof, that is, to show that $V_t(\phi) = \widehat{M}_t$ for every $t \in [0,T]$, it suffices to compare the jumps of both processes at time τ (both martingales are stopped at τ). It is clear from (2.49) that the jump of $\widehat{M}$ equals $\Delta_\tau \widehat{M} = h(\tau) - \widehat{g}(\tau)$. Using (2.44), we get for the jump of the wealth process

$$\Delta_\tau V(\phi) = \phi^1_\tau (R(\tau) - \widetilde{S}_\tau(\kappa)) = h(\tau) - \widehat{g}(\tau),$$

and thus we conclude that $V_t(\phi) = \widehat{M}_t$ for every $t \in [0,T]$. In particular, ϕ is admissible and $V_T(\phi) = V_{\tau\wedge T}(\phi) = h(\tau \wedge T) = Y$, so that ϕ replicates a claim Y. Note that if $\kappa = \kappa(0)$ then $S_0(\kappa(0)) = 0$, so that $\phi^0_0 = V_0(\phi) = \mathbb{E}_{\mathbb{Q}}(Y)$. □

Let us now analyze the condition $\widetilde{S}_t(\kappa) \neq R(t)$ for every $t \in [0,T]$. It ensures, in particular, that the wealth process $V(\phi)$ has a non-zero jump at default time for any the self-financing trading strategy such that $\phi^1_t \neq 0$ for every $t \in [0,T]$. It appears that this condition is not restrictive, since it is satisfied under mild assumptions.

Indeed, if $\kappa > 0$ and R is a non-increasing function then the inequality $\widetilde{S}_t(\kappa) < R(t)$ is valid for every $t \in [0,T]$ (this follows easily from (2.26)). For instance, if $\gamma(t) > 0$ and the protection payment $R > 0$ is constant then it is clear from (2.30) that the market rate $\kappa(0)$ is strictly positive. Consequently, formula (2.26) implies that $\widetilde{S}_t(\kappa(0)) < R$ for every $t \in [0,T]$, as was required. To summarize, when a tradeable asset is a market CDS with a constant $R > 0$ and the default intensity is strictly positive then the inequality holds. Let us finally observe that if the default intensity vanishes on some set then we do not need to impose the inequality $\widetilde{S}_t(\kappa) \neq R(t)$ on this set in order to equate (2.52) with (2.53), since the desired equality holds anyway.

It is useful to note that the proof of Proposition 2.4.3 was implicitly based on the following observation. In our case, Lemma 2.2.3 can be applied to the following **H**-martingales under $\mathbb{Q}$: $M^1 = V(\phi)$, that is, the wealth process of an admissible self-financing strategy ϕ and $M^2 = \widehat{M}$, that is, the conjectured price of a claim Y, as given by the risk-neutral valuation formula.

The method presented above can be extended to replicate a defaultable claim (X, A, Z, τ), where $X = c(T)$, $A_t = \int_0^t a(u)\,du$ and $Z_t = h(t)$ for some RCLL functions a and h. In this case, it is natural to expect that the cum-dividend price process π_t associated with a defaultable claim (X, A, Z, τ), is given by the formula, for every $t \in [0,T]$,

$$\pi_t = \mathbb{1}_{\{t<\tau\}} \widehat{M}_t + \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \int_t^T a(u)G(u)\,du + \int_0^t h(u)\,dH_u + \int_0^t a(u)(1 - H_u)\,du, \tag{2.55}$$

where $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}}(Y \mid \mathcal{H}_t)$ with Y is given by (2.45). Let us denote by Π_t the corresponding ex-dividend price, that is: $\Pi_t = \mathbb{1}_{\{t<\tau\}} \widehat{M}_t + \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \int_t^T a(u)G(u)\,du$. It is rather straightforward to verify that π_t satisfies

$$\pi_t = \mathbb{E}_{\mathbb{Q}}(Y) + \int_0^T a(t)G(t)\,dt + \int_{(0,t]} (h(u) - \Pi_{u-})\,dM_u, \quad t \in [0,T],$$

so that it is a martingale. Consequently, the dynamics of π_t are

$$d\pi_t = (h(t) - \Pi_{t-})\, dM_t, \quad t \in [0,T].$$

From this, or directly from (2.55), we see that the pre-default dynamics of process π_t are

$$d\pi_t = d\widehat{M}_t + \gamma(t)\widehat{a}(t)\, dt = -\gamma(t)\big(h(t) - \widehat{g}(t) - \widehat{a}(t)\big)\, dt = -\gamma(t)\big(h(t) - \widetilde{\Pi}(t)\big)\, dt, \quad t \in [0,\tau),$$

where we set $\widehat{a}(t) = (G(t))^{-1}\int_t^T a(u)G(u)\, du$ and $\widetilde{\Pi}(t)$ is the pre-default value of Π_t. Note that $\widehat{a}(t)$ represents the pre-default value of the future promised dividends associated with A. Therefore, arguing as in the proof of Proposition 2.4.3, we find the following expression for the component ϕ^1 of a replicating strategy for a defaultable claim (X, A, Z, τ)

$$\phi^1_t = \frac{h(t) - \widehat{g}(t) - \widehat{a}(t)}{R(t) - \widetilde{S}_t(\kappa)}, \quad \forall\, t \in [0,T]. \tag{2.56}$$

It is easy to see that the jump condition at time τ, mentioned in the second part of the proof of Proposition 2.4.3, is also satisfied in this case. In fact, it is enough to observe that $\Delta\pi_\tau = h(\tau) - \Pi_{\tau-} = h(\tau) - \widehat{g}(\tau) - \widehat{a}(\tau)$.

Remark 2.4.2 Of course, if we take as (X, A, Z, τ) a CDS with rate κ and recovery function R , then we have $h(t) = R(t)$ and $\widehat{g}(t) + \widehat{a}(t) = \widetilde{S}_t(\kappa)$, so that clearly $\phi^1_t = 1$ for every $t \in [0,T]$.

The following immediate corollary to Proposition 2.4.3 is worth stating.

Corollary 2.4.2 *Assume that $\widetilde{S}_t(\kappa) \neq R(t)$ for every $t \in [0,T]$. Then the market is complete, in the sense, that any defaultable claim (X, A, Z, τ), where $X = c(T)$, $A_t = \int_0^t a(u)\, du$ and $Z_t = h(t)$ for some constant $c(T)$ and RCLL functions a and h, is attainable through continuous trading in a CDS and a bond. The cum-dividend arbitrage price π_t of such defaultable claim satisfies, for every $t \in [0,T]$,*

$$\pi_t = V_t(\phi) = \pi_0 + \int_{]0,t]} (h(u) - \Pi_{u-})\, dM_u,$$

where

$$\pi_0 = \mathbb{E}_{\mathbb{Q}}(Y) + \int_0^T a(t)G(t)\, dt,$$

with Y given by (2.45). Its pre-default price is $\widetilde{\pi}(t) = \widehat{g}(t) + \widehat{a}(t) + A_t$, so that we have, for every $t \in [0,T]$

$$\pi_t = \mathbb{1}_{\{t<\tau\}}(\widehat{g}(t) + \widehat{a}(t) + A_t) + \mathbb{1}_{\{t\geq\tau\}}(h(\tau) + A_\tau) = \mathbb{1}_{\{t<\tau\}}\widetilde{\pi}(t) + \mathbb{1}_{\{t\geq\tau\}}\pi_\tau.$$

Case of a Constant Default Intensity

As a partial check of the calculations above, we shall consider once again the case of constant default intensity and constant protection payment. In this case, $\kappa(0) = R\gamma$ and $S_t(\kappa(0)) = 0$ for every $t \in [0,T]$, so that

$$dV_t(\phi) = -\phi^1_t R\gamma\, dt = -\phi^1_t \kappa(0)\, dt. \tag{2.57}$$

Furthermore, for any RCLL function h, formula (2.54) yields

$$\phi^1_t = R^{-1}\Big(h(t) + e^{\gamma t}\int_t^T h(u)\, d\big(e^{-\gamma u}\big) - c(T)e^{-\gamma T}\Big). \tag{2.58}$$

Assume, for instance, that $h(t) = R$ for $t \in [0,T[$ and $c(T) = 0$. Then (2.58) gives $\phi_t^1 = e^{-\gamma(T-t)}$. Since $S_0(\kappa(0)) = 0$, we have $\phi_0^0 = \pi_0(Y) = V_0(\phi) = R(1 - e^{-\gamma T})$. In view of (2.57), the gains/losses from positions in market CDSs over the time interval $[0,t]$ equal, on the set $\{\tau > t\}$,

$$V_t(\phi) - V_0(\phi) = -R\gamma \int_0^t \phi_u^1 \, du = -R\gamma \int_0^t e^{-\gamma(T-u)} \, du = -Re^{-\gamma T}\big(e^{\gamma t} - 1\big) < 0.$$

Suppose that default occurs at some date $t \in [0,T]$. Then the protection payments is collected, and the wealth at time t becomes

$$V_t(\phi) = V_{t-}(\phi) + \phi_t^1 R = R(1 - e^{-\gamma T}) - Re^{-\gamma T}\big(e^{\gamma t} - 1\big) + Re^{-\gamma(T-t)} = R.$$

The last equality shows that the strategy is indeed replicating on the set $\{\tau \le T\}$. On the set $\{\tau > T\}$, the wealth at time T equals

$$V_T(\phi) = R(1 - e^{-\gamma T}) - Re^{-\gamma T}\big(e^{\gamma T} - 1\big) = 0.$$

Since $S_t(\kappa(0)) = 0$ for every $t \in [0,T]$, we have that $\phi_t^0 = V_t(\phi)$ for every $t \in [0,T]$.

Short Sale of a CDS

As usual, we assume that the maturity T of a CDS is fixed and we consider the situation where the default has not yet occurred.

1. Long position. We say that an agent has a long position at time t in a CDS if he owns at time t a CDS contract that had been created (initiated) at time s_0 by some two parties and was sold to the agent (by means of assignment for example) at time s. If $s_0 = s$ then the agent is an original counter-party to the contract, that is the agent owns the contract from initiation. If an agent owns a CDS contract, the agent is entitled to receive the protection payment for which the agent pays the premium. The long position in a contract may be liquidated at any time $s < t < T$ by means of assignment or offsetting.

2. Short position. We stress that the short position, namely, selling a CDS contract to a dealer, can only be created for a newly initiated contract. It is not possible to sell to a dealer at time t a CDS contract initiated at time $s_0 < t$.

3. Offsetting a long position. If an agent has purchased at time $s_0 \le s < T$ a CDS contract initiated at s_0, he can offset his long position by creating a short position at time t. A new contract is initiated at time t, with the initial price $S_t(\kappa(s_0))$, possibly with a new dealer. This short position offsets the long position outstanding, so that the agent effectively has a zero position in the contract at time t and thereafter.

4. Market constraints. The above taxonomy of positions may have some bearing on portfolios involving short positions in CDS contracts. It should be stressed that not all trades involving a CDS are feasible in practice. Let us consider the CDS contract initiated at time t_0 and maturing at time T. Recall that the ex-dividend price of this contract for any $t \in [t_0, \tau \wedge T[$ is $S_t(\kappa(t_0))$. This is the theoretical price at which the contract should trade so to avoid arbitrage. This price also provides substance for the P&L analysis as it really marks-to-market positions in the CDS contract.

Let us denote the time-t position in the CDS contract of an agent as ϕ_t^1, where $t \in [t_0, \tau \wedge T]$. The strategy is subject to the following constraints: $\phi_t^1 \ge 0$ if $\phi_{t_0}^1 \ge 0$ and $\phi_t^1 \ge \phi_{t_0}^1$ if $\phi_{t_0}^1 \le 0$. It is clear that both restrictions are related to short sale of a CDS. The next result shows that under some assumptions a replicating strategy for a claim Y does not require a short sale of a CDS.

Corollary 2.4.3 *Assume that $\widetilde{S}_t(\kappa) < R(t)$ for every $t \in [0,T]$. Let h be a non-increasing function and let $c(T) \le h(T)$. Then $\phi_t^1 \ge 0$ for every $t \in [0,T]$.*

PROOF: It is enough to observe that if h be a non-increasing function and $c(T) \le h(T)$ then it follows easily from the first equality in (2.24) that for the function $\widehat{g}$ given by (2.48) we have that $h(t) \ge \widehat{g}(t)$ for every $t \in [0,T]$. In view of (2.50), this shows that $\phi_t^1 \ge 0$ for every $t \in [0,T]$. $\square$

2.5 Successive default times

The previous results can easily be generalized to the case of successive default times. We assume in this section that $r = 0$.

2.5.1 Two times

Let us first study the case with two random times τ_1, τ_2. We denote by $\tau_{(1)} = \inf(\tau_1, \tau_2)$ and $\tau_{(2)} = \sup(\tau_1, \tau_2)$, and we assume, for simplicity, that $\mathbb{P}(\tau_1 = \tau_2) = 0$. We denote by $(H^i_t, t \geq 0)$ the default process associated with $\tau_i, (i = 1, 2)$, and by $H_t = H^1_t + H^2_t$ the process associated with two defaults. As before, $\mathbf{H}^i$ is the filtration generated by the process H^i and $\mathbf{H}$ is the filtration generated by the process H. The σ-algebra $\mathcal{G}_t = \mathcal{H}^1_t \vee \mathcal{H}^2_t$ is equal to $\sigma(\tau_1 \wedge t) \vee \sigma(\tau_2 \wedge t)$. It is useful to note that $\mathcal{G}_t$ is strictly greater than $\mathcal{H}_t$.

Exemple: assume that τ_1 and τ_2 are independent and identically distributed. Then, obviously, for $u < t$

$$P(\tau_1 < \tau_2 | \tau_{(1)} = u, \tau_{(2)} = t) = 1/2\,,$$

hence $\sigma(\tau_1, \tau_2) \neq \sigma(\tau_{(1)}, \tau_{(2)})$.

Computation of joint laws

A $\mathcal{H}^1_t \vee \mathcal{H}^2_t$-measurable random variable is equal to

- a constant on the set $t < \tau_{(1)}$,

- a $\sigma(\tau_{(1)})$-measurable random variable on the set $\tau_{(1)} \leq t < \tau_{(2)}$, i.e., a $\sigma(\tau_1)$-measurable random variable on the set $\tau_1 \leq t < \tau_2$, and a $\sigma(\tau_2)$-measurable random variable on the set $\tau_2 \leq t < \tau_1$

- a $\sigma(\tau_1, \tau_2)$-measurable random variable on the set $\tau_2 \leq t$.

We note G the survival probability of the pair (τ_1, τ_2), i.e.,

$$G(t, s) = \mathbb{P}(\tau_1 > t, \tau_2 > s)\,.$$

We shall also use the notation

$$g(s) = \frac{d}{ds} G(s, s) = \partial_1 G(s, s) + \partial_2 G(s, s)$$

where $\partial_1 G$ is the partial derivative of G with respect to the first variable.

• We present in a first step some computations of conditional laws.

$$\begin{aligned} \mathbb{P}(\tau_{(1)} > s) &= \mathbb{P}(\tau_1 > s, \tau_2 > s) = G(s, s) \\ \mathbb{P}(\tau_{(2)} > t | \tau_{(1)} = s) &= \frac{1}{g(s)} \left(\partial_1 G(s, t) + \partial_2 G(t, s)\right), \text{ for } t > s \end{aligned}$$

• We also compute conditional expectation in the filtration $\mathbf{G} = \mathbf{H}^1 \vee \mathbf{H}^2$: For $t < T$

$$\begin{aligned} \mathbb{P}(T < \tau_{(1)} | \mathcal{H}^1_t \vee \mathcal{H}^2_t) &= 1\!\!1_{t < \tau_{(1)}} \frac{\mathbb{P}(T < \tau_{(1)})}{\mathbb{P}(t < \tau_{(1)})} = 1\!\!1_{t < \tau_{(1)}} \frac{G(T, T)}{G(t, t)} \\ \mathbb{P}(T < \tau_1 | \mathcal{H}^1_t \vee \mathcal{H}^2_t) &= 1\!\!1_{t < \tau_1} \frac{\mathbb{P}(T < \tau_1 | \mathcal{H}^2_t)}{\mathbb{P}(t < \tau_1 | \mathcal{H}^2_t)} + 1\!\!1_{\tau_1 < t} \\ &= 1\!\!1_{t < \tau_1} \left(1\!\!1_{t < \tau_2} \frac{\mathbb{P}(T < \tau_1, t < \tau_2)}{\mathbb{P}(t < \tau_1, t < \tau_2)} + 1\!\!1_{\tau_2 \leq t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) + 1\!\!1_{\tau_1 < t} \\ &= 1\!\!1_{t < \tau_1} \left(1\!\!1_{t < \tau_2} \frac{G(T, t)}{G(t, t)} + 1\!\!1_{\tau_2 < t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) + 1\!\!1_{\tau_1 < t} \end{aligned}$$

$$\begin{aligned}\mathbb{P}(\tau_{(2)} \le T|\mathcal{H}_t^1 \vee \mathcal{H}_t^2) &= 1\!\!1_{t<\tau_{(1)}} \frac{\mathbb{P}(t \le \tau_{(1)} < \tau_{(2)} < T)}{\mathbb{P}(t < \tau_{(1)})} + 1\!\!1_{\tau_1 \le t < \tau_2} \frac{\mathbb{P}(t < \tau_2 < T|\tau_1)}{\mathbb{P}(t < \tau_2|\tau_1)} \\ &\quad + 1\!\!1_{\tau_2 \le t < \tau_1} \frac{\mathbb{P}(t < \tau_1 < T|\tau_2)}{\mathbb{P}(t < \tau_1|\tau_2)} + 1\!\!1_{\tau_{(2)}<t} \,.\end{aligned}$$

• The computation of $\mathbb{P}(T < \tau_1|\tau_2)$ can be done as follows: the function h such that $\mathbb{P}(T < \tau_1|\tau_2) = h(\tau_2)$ satisfies

$$\mathbb{E}(h(\tau_2)\varphi(\tau_2) 1\!\!1_{\tau_2<t}) = \mathbb{E}(\varphi(\tau_2) 1\!\!1_{\tau_2<t} 1\!\!1_{T<\tau_1})$$

for any function φ. This implies that (assuming that the pair (τ_1, τ_2) has a density f)

$$\int_0^t dv h(v)\varphi(v) \int_0^\infty du f(u,v) = \int_0^t dv \varphi(v) \int_T^\infty du f(u,v)$$

or

$$\int_0^t dv h(v)\varphi(v) \partial_2 G(0,v) = \int_0^t dv \varphi(v) \partial_2 G(T,v)$$

hence, $h(v) = \frac{\partial_2 G(T,v)}{\partial_2 G(0,v)}$.
We can also write

$$\mathbb{P}(T < \tau_1|\tau_2 = v) = \frac{\mathbb{P}(T < \tau_1, \tau_2 \in dv)}{\mathbb{P}(\tau_2 \in dv)} = -\frac{1}{\mathbb{P}(\tau_2 \in dv)} \frac{d}{dv} \mathbb{P}(\tau_1 > T, \tau_2 > v) = \frac{\partial_2 G(T,v)}{\partial_2 G(0,v)}$$

hence, on the set $\tau_2 < T$,

$$\mathbb{P}(T < \tau_1|\tau_2) = h(\tau_2) = \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(0,\tau_2)}$$

• In the same way, for $T > t$

$$\mathbb{P}(\tau_1 \le T < \tau_2|\mathcal{H}_t^1 \vee \mathcal{H}_t^2) 1\!\!1_{\{\tau_1 \le t < \tau_2\}} = 1\!\!1_{\{\tau_1 \le t < \tau_2\}} \Psi(\tau_1)$$

where Ψ satisfies

$$\mathbb{E}(\varphi(\tau_1) 1\!\!1_{\tau_1 \le t < T < \tau_2}) = \mathbb{E}(\varphi(\tau_1)\Psi(\tau_1) 1\!\!1_{\{\tau_1 \le t < \tau_2\}})$$

for any function φ. In other terms

$$\int_0^t du \varphi(u) \int_T^\infty dv f(u,v) = \int_0^t du \varphi(u) \Psi(u) \int_t^\infty dv f(u,v)$$

or

$$\int_0^t du \varphi(u) \partial_1 G(u,T) = \int_0^t du \varphi(u) \Psi(u) \partial_1 G(u,t) \,.$$

This implies that

$$\Psi(u) = \frac{\partial_1 G(u,T)}{\partial_1 G(u,t)}$$

$$\mathbb{P}(\tau_1 \le T < \tau_2|\mathcal{H}_t^1 \vee \mathcal{H}_t^2) 1\!\!1_{\{\tau_1 \le t < \tau_2\}} = 1\!\!1_{\{\tau_1 \le t < \tau_2\}} \frac{\partial_1 G(\tau_1,T)}{\partial_1 G(\tau_1,t)} \,.$$

Value of credit derivatives

We introduce different credit derivatives

A defaultable zero-coupon related to the default times D^i delivers 1 monetary unit if τ_i is greater that T: $D^i(t,T) = \mathbb{E}(1\!\!1_{\{T<\tau_i\}}|\mathcal{H}_t^1 \vee \mathcal{H}_t^2)$

A contract which pays R^1 is one default occurs before T and R_2 if the two default occur before T: $CD_t = \mathbb{E}(R_1 1\!\!1_{\{0<\tau_{(1)} \le T\}} + R_2 1\!\!1_{\{0<\tau_{(2)} \le T\}}|\mathcal{H}_t^1 \vee \mathcal{H}_t^2)$

We obtain

$$D^1(t,T) = 1\!\!1_{\{\tau_1>t\}}\left(1\!\!1_{\{\tau_2\le t\}}\frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + 1\!\!1_{\{\tau_2>t\}}\frac{G(T,t)}{G(t,t)}\right) \tag{2.59}$$

$$D^2(t,T) = 1\!\!1_{\{\tau_2>t\}}\left(1\!\!1_{\{\tau_1\le t\}}\frac{\partial_1 G(\tau_1,T)}{\partial_2 G(\tau_1,t)} + 1\!\!1_{\{\tau_1>t\}}\frac{G(t,T)}{G(t,t)}\right) \tag{2.60}$$

$$CD_t = R_1 1\!\!1_{\{\tau_{(1)}>t\}}\left(\frac{G(t,t)-G(T,T)}{G(t,t)}\right) + R_2 1\!\!1_{\{\tau_{(2)}\le t\}} + R_1 1\!\!1_{\{\tau_{(1)}\le t\}} \tag{2.61}$$

$$+R_2 1\!\!1_{\{\tau_{(2)}>t\}}\left\{I_t(0,1)\left(1-\frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)}\right) + I_t(1,0)\left(1-\frac{\partial_1 G(\tau_1,T)}{\partial_1 G(\tau_1,t)}\right)\right. \tag{2.62}$$

$$\left. +I_t(0,0)\left(1-\frac{G(t,T)+G(T,t)-G(T,T)}{G(t,t)}\right)\right\} \tag{2.63}$$

where by

$$I_t(1,1) = 1\!\!1_{\{\tau_1\le t,\tau_2\le t\}}\,, \qquad I_t(0,0) = 1\!\!1_{\{\tau_1>t,\tau_2>t\}}$$
$$I_t(1,0) = 1\!\!1_{\{\tau_1\le t,\tau_2>t\}}\,, \qquad I_t(0,1) = 1\!\!1_{\{\tau_1>t,\tau_2\le t\}}$$

More generally, some easy computation leads to

$$\mathbb{E}(h(\tau_1,\tau_2)|\mathcal{H}_t) = I_t(1,1)h(\tau_1,\tau_2) + I_t(1,0)\Psi_{1,0}(\tau_1) + I_t(0,1)\Psi_{0,1}(\tau_2) + I_t(0,0)\Psi_{0,0}$$

where

$$\Psi_{1,0}(u) = -\frac{1}{\partial_1 G(u,t)}\int_t^\infty h(u,v)\partial_1 G(u,dv)$$
$$\Psi_{0,1}(v) = -\frac{1}{\partial_2 G(t,v)}\int_t^\infty h(u,v)\partial_2 G(du,v)$$
$$\Psi_{0,0} = \frac{1}{G(t,t)}\int_t^\infty\int_t^\infty h(u,v)G(du,dv)$$

The next result deals with the valuation of a first-to-default claim in a bivariate set-up. Let us stress that the concept of the (tentative) price will be later supported by strict replication arguments. In this section, by a *pre-default price* associated with a $\mathbf{G}$-adapted price process π, we mean here the function $\widetilde{\pi}$ such that $\pi_t 1\!\!1_{\{\tau_{(1)}>t\}} = \widetilde{\pi}(t)1\!\!1_{\{\tau_{(1)}>t\}}$ for every $t\in[0,T]$. In other words, the pre-default price $\widetilde{\pi}$ and the price π coincide prior to the first default only.

Definition 2.5.1 *Let Z_i be two functions, and X a constant. A FtD claim pays $Z_1(\tau_1)$ at time τ_1 if $\tau_1<T, \tau_1<\tau_2$, pays $Z_2(\tau_2)$ at time τ_2 if $\tau_2<T,\tau_2<\tau_1$, and X at maturity if $\tau_1\wedge\tau_2>T$*

Proposition 2.5.1 *The pre-default price of a FtD claim $(X,0,Z,\tau_{(1)})$, where $Z=(Z_1,Z_2)$ and $X=c(T)$, equals*

$$\frac{1}{G(t,t)}\left(-\int_t^T Z_1(u)\,G(du,u) - \int_t^T Z_2(v)G(v,dv) + XG(T,T)\right).$$

PROOF: The price can be expressed as

$$\mathbb{E}_{\mathbb{Q}}(Z_1(\tau_1)1\!\!1_{\{\tau_1\le T,\tau_2>\tau_1\}}|\mathcal{H}_t) + \mathbb{E}_{\mathbb{Q}}(Z_2(\tau_2)1\!\!1_{\{\tau_2\le T,\tau_1>\tau_2\}}|\mathcal{H}_t) + \mathbb{E}_{\mathbb{Q}}(c(T)1\!\!1_{\{\tau_{(1)}>T\}}|\mathcal{H}_t).$$

The pricing formula now follows by evaluating the conditional expectation, using the joint distribution of default times under the martingale measure $\mathbb{Q}$. □

Comments 2.5.1 Same computations appear in Kurtz and Riboulet [139]

2.5.2 Poisson Jumps

Suppose that the default times are modeled via a Poisson process with intensity h. (See the appendix for definitions and main properties of Poisson processes) The terminal payoff is $\prod_{T_i \leq T}(1 - R(T_i))$, where R is a deterministic function valued in $[0, 1]$. The value of this payoff is $\mathbb{E}(\prod_{T_i \leq T}(1 - R(T_i)))$. In the case of constant $R(s) = R$, we get

$$\mathbb{E}\left(\prod_{T_i \leq T}(1 - R(T_i))\right) = \mathbb{E}\left((1 - R)^{N_T}\right) = \exp\left(R\int_0^T h(s)ds\right) .$$

In the general case,

$$\mathbb{E}\left(\prod_{T_i \leq T}(1 - R(T_i))\right) = \mathbb{E}\left(\exp\left(\sum_{s \leq T}\ln(1 - R(s))\Delta N_s\right)\right) = \mathbb{E}\left(\exp\left(\int_0^T \ln(1 - R_s)dN_s\right)\right) .$$

Hence (See the appendix)

$$\mathbb{E}\left(\prod_{T_i \leq T}(1 - R(T_i))\right) = \exp\left(\int_0^T R(s)h(s)ds\right) .$$

Chapter 3

Cox Processes and Extensions

We now present a case where the default time is defined through a given process, as the first hitting time of a random barrier. Hence, some information is given by the default free market. In this framework, we present the valuation of defaultable claims.

3.1 Construction of Cox Processes with a given stochastic intensity

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbf{F}$. *A nonnegative* $\mathbf{F}$*-adapted process* λ *is given.* We assume that there exists, on the space $(\Omega, \mathcal{G}, \mathbb{P})$, a random variable Θ, independent of $\mathcal{F}_\infty$, with an exponential law: $\mathbb{P}(\Theta \geq t) = e^{-t}$. We define the default time τ as the first time when the increasing process $\Lambda_t = \int_0^t \lambda_s \, ds$ is above the random level Θ, i.e.,

$$\tau = \inf \{t \geq 0 \, : \, \Lambda_t \geq \Theta\}.$$

In particular, using the increasing property of Λ, on gets $\{\tau > s\} = \{\Lambda_s < \Theta\}$. We assume that $\Lambda_t < \infty, \forall t$, $\Lambda_\infty = \infty$, hence τ is a real-valued r.v.. One can also define τ as

$$\tau = \inf \{t \geq 0 \, : \, \Lambda_t \geq -\ln U\}$$

where U has a uniform law. Indeed, the r.v. $-\ln U$ has an exponential law of parameter 1 since $\{-\ln U > a\} = \{U < e^{-a}\}$.

Comments 3.1.1 (i) In order to construct the r.v. Θ, one needs to enlarge the probability space. Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ be an auxiliary probability space with a r.v. Θ with exponential law. We introduce the product probability space $(\widetilde{\Omega}, \widetilde{\mathcal{G}}, \widetilde{\mathbb{Q}}) = (\Omega \times \hat{\Omega}, \mathcal{F}_\infty \otimes \hat{\mathcal{F}}, \mathbb{Q} \otimes \hat{\mathbb{P}})$.
(ii) Another construction is to choose $\tau = \inf \{t \geq 0 \, : \, \tilde{N}_{\Lambda_t} = 1\}$, where $\Lambda_t = \int_0^t \lambda_s \, ds$ and $\tilde{N}$ is a Poisson process with intensity 1, independent of the filtration $\mathbf{F}$. The second method is in fact equivalent to the first. Cox processes are used in a great number of studies (see, e.g., [141]).

3.2 Conditional Expectations

Lemma 3.2.1 *The conditional distribution function of* τ *given the* σ*-field* $\mathcal{F}_t$ *is for* $t \geq s$

$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \exp\Big(- \Lambda_s \Big).$$

PROOF: The proof follows from the equality $\{\tau > s\} = \{\Lambda_s < \Theta\}$. From the independence assumption and the $\mathcal{F}_t$-measurability of Λ_s for $s \le t$, we obtain

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \mathbb{P}\Big(\Lambda_s < \Theta \,\Big|\, \mathcal{F}_t\Big) = \exp\Big(-\Lambda_s\Big).$$

In particular, we have

$$\mathbb{P}(\tau \le t|\mathcal{F}_t) = \mathbb{P}(\tau \le t|\mathcal{F}_\infty), \tag{3.1}$$

and, for $t \ge s$, $\mathbb{P}(\tau > s|\mathcal{F}_t) = \mathbb{P}(\tau > s|\mathcal{F}_s)$. Let us notice that the process $F_t = \mathbb{P}(\tau \le t|\mathcal{F}_t)$ is here an increasing process, as the right-hand side of 3.1 is. $\square$

Remark 3.2.1 If the process λ is not non-negative, we get,

$$\{\tau > s\} = \{\sup_{u \le s} \Lambda_u < \Theta\},$$

hence for $s < t$

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp(-\sup_{u \le s} \Lambda_u).$$

More generally, some authors define the default time as

$$\tau = \inf\{t \ge 0 \,:\, X_t \ge \Theta\}$$

where X is a given $\mathbf{F}$-semi-martingale. Then, for $s \le t$

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp(-\sup_{u \le s} X_u).$$

3.3 Choice of filtration

We write as in the previous chapter $H_t = 1\!\!1_{\{\tau \le t\}}$ and $\mathcal{H}_t = \sigma(H_s \,:\, s \le t)$. We introduce the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, that is, the enlarged filtration generated by the underlying filtration $\mathbf{F}$ and the process H. (We denote by $\mathbf{F}$ the original **F**iltration and by $\mathbf{G}$ the enlar**G**ed one.) We shall frequently write $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$.

It is easy to describe the events which belong to the σ-field $\mathcal{G}_t$ on the set $\{\tau > t\}$. Indeed, if $G_t \in \mathcal{G}_t$, then $G_t \cap \{\tau > t\} = B_t \cap \{\tau > t\}$ for some event $B_t \in \mathcal{F}_t$.

Therefore any $\mathcal{G}_t$-measurable random variable Y_t satisfies $1\!\!1_{\{\tau > t\}} Y_t = 1\!\!1_{\{\tau > t\}} y_t$, where y_t is an $\mathcal{F}_t$-measurable random variable.

3.4 Key lemma

Proposition 3.4.1 *Let Y be an integrable r.v. Then,*

$$1\!\!1_{\{\tau > t\}} \mathbb{E}(Y|\mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} \frac{\mathbb{E}(Y 1\!\!1_{\{\tau > t\}}|\mathcal{F}_t)}{\mathbb{E}(1\!\!1_{\{\tau > t\}}|\mathcal{F}_t)} = 1\!\!1_{\{\tau > t\}} e^{\Lambda_t} \mathbb{E}(Y 1\!\!1_{\{\tau > t\}}|\mathcal{F}_t).$$

PROOF: From the remarks on the $\mathcal{G}_t$-measurability, if $Y_t = \mathbb{E}(Y|\mathcal{G}_t)$, then there exists an $\mathcal{F}_t$-measurable r.v. y_t such that

$$1\!\!1_{\{\tau > t\}} \mathbb{E}(Y|\mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} y_t$$

and taking conditional expectation w.r.t. $\mathcal{F}_t$ of both members, we deduce $y_t = \dfrac{\mathbb{E}(Y 1\!\!1_{\{\tau > t\}}|\mathcal{F}_t)}{\mathbb{E}(1\!\!1_{\{\tau > t\}}|\mathcal{F}_t)}$. $\square$

Corollary 3.4.1 *If X is an integrable $\mathcal{F}_T$-measurable random variable*

$$\mathbb{E}(X 1\!\!1_{\{T < \tau\}}|\mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} e^{\Lambda_t} \mathbb{E}(X e^{-\Lambda_T}|\mathcal{F}_t). \tag{3.2}$$

PROOF: Let X be an $\mathcal{F}_T$-measurable r.v. From Proposition 3.4.1, the r.v. $\mathbb{E}(X1\!\!1_{\{\tau>T\}}|\mathcal{G}_t)$ is equal to 0 on the $\mathcal{G}_t$-measurable set $\tau < t$, whereas

$$\mathbb{E}(X1\!\!1_{\{\tau>T\}}|\mathcal{F}_t) = \mathbb{E}(X1\!\!1_{\{\tau>T\}}|\mathcal{F}_T|\mathcal{F}_t) = \mathbb{E}(Xe^{\Lambda_T}|\mathcal{F}_t).$$

□

Comments 3.4.1 This corollary admits an interesting interpretation. If $X1\!\!1_{\{T<\tau\}}$ is some defaultable payoff, its value is the value of the default free payoff X when the interest rate is higher that the spot rate and the difference, i.e., λ can be interpreted as a spread. However, we emphasize that we are not dealing with a risk neutral probability. In the case where the market is assumed to be complete, that means in particular that a defaultable zero-coupon is traded (or duplicable). Then, for pricing purpose, the intensity has to be evaluated under the risk-neutral probability given by the market.

Definition 3.4.1 *The process λ is called the intensity of τ.*

We now compute the expectation of a value at time τ of a predictable process.

Lemma 3.4.1 (i) *If h is an $\mathbf{F}$-predictable (bounded) process then*

$$\mathbb{E}(h_\tau|\mathcal{F}_t) = \mathbb{E}\Big(\int_0^\infty h_u\lambda_u \exp\big(-\Lambda_u\big)\,du\,\Big|\,\mathcal{F}_t\Big)$$

and

$$\mathbb{E}(h_\tau|\mathcal{G}_t) = \mathbb{E}\Big(\int_t^\infty h_u\lambda_u \exp\big(\Lambda_t-\Lambda_u\big)\,du\,\Big|\,\mathcal{F}_t\Big)1\!\!1_{\{\tau>t\}} + h_\tau 1\!\!1_{\{\tau\le t\}}. \tag{3.3}$$

In particular

$$\mathbb{E}(h_\tau) = \mathbb{E}\Big(\int_0^\infty h_u\lambda_u\,\exp\big(-\Lambda_u\big)\,du\Big) = \mathbb{E}\Big(\int_0^\infty h_u dF_u\Big)$$

(ii) *The process $(H_t - \int_0^{t\wedge\tau}\lambda_s ds, t\ge 0)$ is a $\mathbf{G}$-martingale.*

PROOF: Let $h_t = 1\!\!1_{]v,w]}(t)B_v$ where $B_v \in \mathcal{F}_v$ be an elementary predictable process. Then, from Corollary 3.4.1

$$\begin{aligned}
\mathbb{E}(h_\tau|\mathcal{F}_t) &= \mathbb{E}\Big(1\!\!1_{]v,w]}(\tau)B_v|\mathcal{F}_t\Big) = \mathbb{E}\Big(\mathbb{E}(1\!\!1_{]v,w]}(\tau)B_v|\mathcal{F}_\infty)\,\Big|\,\mathcal{F}_t\Big)\\
&= \mathbb{E}\Big(B_v\mathbb{P}(v<\tau\le w|\mathcal{F}_\infty)\,\Big|\,\mathcal{F}_t\Big) = \mathbb{E}\Big(B_v\big(e^{-\Lambda_v}-e^{-\Lambda_w}\big)|\mathcal{F}_t\Big)
\end{aligned}$$

It follows that

$$\mathbb{E}(h_\tau|\mathcal{F}_t) = \mathbb{E}\Big(B_v\int_v^w \lambda_u e^{-\Lambda_u}\,du\,\Big|\,\mathcal{F}_t\Big) = \mathbb{E}\Big(\int_0^\infty h_u\lambda_u e^{-\Lambda_u}du\,\Big|\,\mathcal{F}_t\Big)$$

and the result is derived from the monotone class theorem.

The martingale property (ii) follows from integration by parts formula. Indeed, let $t<s$. Then, on the one hand from Corollary 3.4.1

$$\begin{aligned}
\mathbb{E}(H_s - H_t|\mathcal{G}_t) &= \mathbb{P}(t<\tau\le s|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}\frac{\mathbb{P}(t<\tau\le s|\mathcal{F}_t)}{\mathbb{P}(t<\tau|\mathcal{F}_t)}\\
&= 1\!\!1_{\{t<\tau\}}\mathbb{E}(1-\exp(\Lambda_s-\Lambda_t)|\mathcal{F}_t)
\end{aligned}$$

On the other hand, from part (i)

$$\mathbb{E}\Big(\int_{t\wedge\tau}^{s\wedge\tau}\lambda_u du\,\Big|\,\mathcal{G}_t\Big) = \mathbb{E}\big(\Lambda_{s\wedge\tau}-\Lambda_{t\wedge\tau}|\mathcal{G}_t\big) = 1\!\!1_{\{t<\tau\}}\mathbb{E}\Big(\int_t^\infty h_u\lambda_u e^{-(\Lambda_u-\Lambda_t)}du\,\Big|\,\mathcal{F}_t\Big)$$

where $h_u = \Lambda(s \wedge u) - \Lambda(t \wedge u)$. Consequently,

$$\begin{aligned}\int_t^\infty h_u \lambda_u e^{-(\Lambda_u - \Lambda_t)} du &= \int_t^s (\Lambda_u - \Lambda_s)\lambda_u e^{-(\Lambda_u - \Lambda_t)} du + (\Lambda_t - \Lambda_s) \int_s^\infty \lambda_u e^{-(\Lambda_u - \Lambda_t)} du \\ &= -(\Lambda_s - \Lambda_t) e^{-(\Lambda_s - \Lambda_t)} + \int_t^s \lambda(u) e^{-(\Lambda_u - \Lambda_t)} du + (\Lambda_s - \Lambda_t) e^{-(\Lambda_s - \Lambda_t)} \\ &= 1 - e^{-(\Lambda_s - \Lambda_t)}.\end{aligned}$$

This ends the proof. □

Proposition 3.4.2 *The process L defined as $L_t = \mathbb{1}_{\{t<\tau\}} e^{\Lambda_t} = (1 - H_t)e^{\Lambda_t}$ is a $\mathbf{G}$-martingale, and $dL_t = -L_{t-} dM_t$.*

PROOF: The proof is left to the reader. □

Comments 3.4.2 The proof extends easily to $\mathbf{G}$ predictable processes, using that any $\mathbf{G}$-predictable process coincides on the set $[0, \tau]$ to an $\mathbf{F}$-predictable process.
Using elementary processes of the form $\mathbb{1}_{\{[v,w[\}} B_v$, the proof extends to càdlàg $\mathbf{F}$-adapted processes. However, the result do not extend to càdlàg $\mathbf{G}$-adapted processes as one can chech in the case $h_t = H_t$.

3.5 Conditional Expectation of $\mathcal{F}_\infty$-Measurable Random Variables

Lemma 3.5.1 *Let X be an $\mathcal{F}_\infty$-measurable r.v.. Then*

$$\mathbb{E}(X|\mathcal{G}_t) = \mathbb{E}(X|\mathcal{F}_t). \tag{3.4}$$

PROOF: Let X be an $\mathcal{F}_\infty$-measurable r.v. To prove that $\mathbb{E}(X|\mathcal{G}_t) = \mathbb{E}(X|\mathcal{F}_t)$, it suffices to check that

$$\mathbb{E}(B_t h(\tau \wedge t) X) = \mathbb{E}(B_t h(\tau \wedge t) \mathbb{E}(X|\mathcal{F}_t))$$

for any $B_t \in \mathcal{F}_t$ and any $h = \mathbb{1}_{[0,a]}$. For $t \le a$, the equality is obvious. For $t > a$, we have from (3.1)

$$\begin{aligned}\mathbb{E}(B_t \mathbb{1}_{\{\tau \le a\}} \mathbb{E}(X|\mathcal{F}_t)) &= \mathbb{E}(B_t \mathbb{E}(X|\mathcal{F}_t) \mathbb{E}(\mathbb{1}_{\{\tau \le a\}} | \mathcal{F}_\infty)) = \mathbb{E}(\mathbb{E}(B_t X|\mathcal{F}_t) \mathbb{E}(\mathbb{1}_{\{\tau \le a\}} | \mathcal{F}_t)) \\ &= \mathbb{E}(X B_t \mathbb{E}(\mathbb{1}_{\{\tau \le a\}} | \mathcal{F}_t)) = \mathbb{E}(B_t X \mathbb{1}_{\{\tau \le a\}})\end{aligned}$$

as expected. □

Remark 3.5.1 Let us remark that (3.4) implies that every $\mathbf{F}$-square integrable martingale is a $\mathbf{G}$-martingale. However, equality (3.4) does not apply to any $\mathcal{G}$-measurable random variable; in particular $\mathbb{P}(\tau \le t|\mathcal{G}_t) = \mathbb{1}_{\{\tau \le t\}}$ is not equal to $F_t = \mathbb{P}(\tau \le t|\mathcal{F}_t)$.

3.6 Extension

In Wong [178], the time of default is given as

$$\tau = \inf\{t \,:\, \Lambda_t \ge \Sigma\}$$

where Σ a non-negative r.v. independent of $\mathcal{F}_\infty$. This model reduces to the previous one: if Φ is the cumulative function of Σ, the r.v. $\Phi(\Sigma)$ has a uniform distribution and

$$\tau = \inf\{t \,:\, \Phi(\Lambda_t) \ge \Phi(\Sigma)\} = \inf\{t \,:\, \Psi^{-1}[\Phi(\Lambda_t)] \ge \Theta\}$$

where Ψ is the cumulative function of the exponential law. Then,

$$F_t = \mathbb{P}(\tau \le t|\mathcal{F}_t) = \mathbb{P}(\Lambda_t \ge \Sigma|\mathcal{F}_t) = 1 - \exp\left(-\Psi^{-1}(\Phi(\Lambda_t))\right).$$

3.7 Dynamics of prices

We assume here that **F**-martingales are continuous. Let R be an **F**-adapted process.

3.7.1 Defaultable Zero-Coupon Bond

From Corollary 3.4.1, for $t < T$

$$\mathbb{E}(1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\mathbb{E}\Big(\exp\big(-\int_t^T \lambda_s\,ds\big)\,\Big|\,\mathcal{F}_t\Big).$$

Let $\mathbb{Q}$ be a risk-neutral probability and $B(t,T)$ be the price at time t of a default-free bond paying 1 at maturity T satisfies

$$B(t,T) = \mathbb{E}_{\mathbb{Q}}\Big(\exp\big(-\int_t^T r_s\,ds\big)\,\Big|\,\mathcal{F}_t\Big).$$

The market price $D(t,T)$ of a defaultable zero-coupon bond with maturity T is

$$\begin{aligned} D(t,T) &= \mathbb{E}_{\mathbb{Q}}\Big(1\!\!1_{\{T<\tau\}}\exp\big(-\int_t^T r_s\,ds\big)\,\Big|\,\mathcal{G}_t\Big) \\ &= 1\!\!1_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\Big(\exp\big(-\int_t^T [r_s+\lambda_s^Q]\,ds\big)\,\Big|\,\mathcal{F}_t\Big). \end{aligned}$$

Then, in the case $r = 0$,

$$D(t,T) = L_t\mathbb{Q}(\tau > T|\mathcal{F}_t) = L_t m_t$$

with $m_t = \mathbb{Q}(\tau > T|\mathcal{F}_t) = \mathbb{E}(e^{-\Lambda_T}|\mathcal{F}_t)$.
In the particular case where λ is deterministic, $m_t = e^{-\Lambda_T}$ and $dm_t = 0$. Hence $D(t,T) = L_t e^{-\Lambda_T}$ and

$$dD(t,T) = m_t dL_t = -m_t L_{t-}dM_t = -e^{-\Lambda_T}L_{t-}dM_t\,.$$

3.7.2 Recovery with Payment at maturity

We consider a contract which pays R_τ at date T, if $\tau \le T$ and no payment in the case $\tau > T$. We assume here that $r = 0$.

The price at time t of this contract is

$$\begin{aligned} S_t &= E(R_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) = R_\tau 1\!\!1_{\tau<t} + 1\!\!1_{t<\tau}E(R_\tau 1\!\!1_{t<\tau<T}|\mathcal{G}_t) \\ &= R_\tau 1\!\!1_{\tau<t} + 1\!\!1_{t<\tau}e^{\Lambda_t}E(\int_t^T R_u dF_u|\mathcal{F}_t) \end{aligned}$$

where $F_u = P(\tau \le u|\mathcal{F}_u) = 1 - e^{-\Lambda_u}$, or

$$S_t = R_\tau 1\!\!1_{\tau<t} + 1\!\!1_{t<\tau}e^{\Lambda_t}E(\int_t^T R_u e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t)$$

or

$$S_t = \int_0^t R_u dH_u + L_t\left(-\int_0^t R_u e^{-\Lambda_u}\lambda_u du + m_t^R\right)$$

where $m_t^R = E(\int_0^T R_u e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t)$. From $dL_t = -L_{t-}dM_t$ and

$$d(Lm^R) = Ldm^R + m^R dL + d[m^R, L] = Ldm^R + m^R dL$$

we deduce that

$$dS_t = R_t(dH_t - \lambda_t(1-H_t)dt) - S_{t-}dM_t + L_t dm_t^R = (R_t - S_{t-})dM_t + L_t dm_t^R$$

(Note that, since m^R is continuous, its covariation process with L is null an that one can write $L_t dm_t^R$ instead of $L_{t-}dm_t^R$. Note also that, from the definition the process S is a martingale.

3.7.3 Recovery with Payment at Default time

If the payment R is done at time τ, in the case $r = 0$,

$$S_t = \mathbb{1}_{t<\tau} E(R_\tau \mathbb{1}_{t<\tau<T}|\mathcal{G}_t) = \mathbb{1}_{t<\tau} e^{\Lambda_t} E(\int_t^T R_u dF_u|\mathcal{F}_t)\,.$$

The dynamics of S is

$$dS_t = -S_{t-}dM_t + L_t(dm_t^R - R_t e^{-\Lambda_t}\lambda_t)dt$$

and the process

$$S_t + R_\tau \mathbb{1}_{\{\tau<t\}} = S_t + \int_0^t R_s dH_s$$

is a martingale.

3.7.4 Price and Hedging a Defaultable Call

We assume that

- the savings account $Y_t^0 = 1$
- a risky asset with risk-neutral dynamics

$$dY_t = Y_t \sigma dW_t$$

  where W is a Brownian motion
- a DZC of maturity T with price $D(t,T)$

are traded. The reference filtration is that of the BM W. The price of a defaultable call with payoff $\mathbb{1}_{T<\tau}(Y_T - K)^+$ is

$$\begin{aligned} C_t &= \mathbb{E}(\mathbb{1}_{T<\tau}(Y_T - K)^+|\mathcal{G}_t) = \mathbb{1}_{t<\tau} e^{\Lambda_t}\mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+|\mathcal{F}_t) \\ &= L_t m_t^Y \end{aligned}$$

with $m_t^Y = \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+|\mathcal{F}_t)$. hence

$$dC_t = L_t dm_t^Y - m_t^Y L_{t-} dM_t$$

- In the particular case where λ is deterministic,

$$m_t^Y = e^{-\Lambda_T} E((Y_T - K)^+|\mathcal{F}_t) = e^{-\Lambda_T} C_t^Y$$

where C^Y is the price of a call in the Black Scholes model. This quantity is $C_t^Y = C^Y(t, Y_t)$ and satisfies $dC_t^Y = \Delta_t dY_t$ where $\Delta_t$is the Delta-hedge ($\Delta_t = \partial_y C^Y(t, Y_t)$).

$$C_t = \mathbb{1}_{t<\tau} e^{\Lambda_t} e^{-\Lambda_T} C^Y(t, Y_t) = L_t e^{-\Lambda_T} C^Y(t, Y_t) = D(t,T) C^Y(t, Y_t)$$

From

$$C_t = D(t,T) C^Y(t, Y_t)$$

we deduce

$$\begin{aligned} dC_t &= e^{-\Lambda_T}(L_t dC^Y + C^Y dL_t) = e^{-\Lambda_T}(L_t \Delta_t dY_t - C^Y L_t dM_t) \\ &= e^{-\Lambda_T}(L_t \Delta_t dY_t - C^Y L_t dM_t) \end{aligned}$$

Therefore, using that $dD(t,T) = m_t dM_t = -e^{-\Lambda_T} L_t dM_t$ we get

$$dC_t = e^{-\Lambda_T} L_t \Delta_t dY_t - C^Y dD(t,T) = e^{-\Lambda_T} L_t \Delta_t dY_t + \frac{C_t}{D(t,T)} dD(t,T)$$

hence, an hedging strategy consists of holding $\frac{C_t}{D(t,T)}$ DZCs.

• In the general case, one obtains

$$dC_t = \frac{C_{t-}}{D(t,T)} dD(t,T) + L \frac{m_t^Y}{m_t} dm_t + L dm_t^Y$$

An hedging strategy consists of holding $\frac{C_{t-}}{D(t,T)}$ DZCs.

3.7.5 Corporate bond

The time-t value of a corporate bond, which pays R_τ at time T in case of default and 1 otherwise, is given by

$$\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s\, ds} (R_\tau 1\!\!1_{\{\tau \leq T\}} + 1\!\!1_{\{\tau > T\}}) \Big| \mathcal{G}_t\Big) = \mathbb{E}_{\mathbb{Q}}\Big(\exp(-\int_t^T r_s ds)\,(1 - (1 - R_\tau) 1\!\!1_{\{\tau \leq T\}}) \Big| \mathcal{G}_t\Big).$$

Then, setting $\Lambda_t^Q = \int_0^t \lambda_u^Q du$,

$$D^{(R,T)}(t,T) = B(t,T) - \mathbb{E}\left(\exp(-\int_t^T (r_s + \lambda_s^Q) ds) \int_t^T ds(1 - R_s) \lambda_s^Q e^{-\Lambda_s^Q} ds | \mathcal{F}_t\right)$$

Therefore, given the price of a DZC, we can deduce the risk neutral intensity.

In the case where the compensation is paid at default time,

$$\begin{aligned} D^{(R)}(t,T) &= \mathbb{E}_{\mathbb{Q}}\left(\exp - \int_t^T (r_s + \lambda_s^Q) ds | \mathcal{F}_t\right) \\ &+ \mathbb{E}_{\mathbb{Q}}\left(\int_t^T ds R_s \lambda_s^Q \exp(-\int_t^s (r_u + \lambda_u^Q) du) | \mathcal{F}_t\right) \end{aligned}$$

3.8 Term Structure Models

Some authors choose to model the intensity. A substantial literature proposes to model both the default free term structure and the term structure representing the relative prices of different maturities of default-risky debt, using an extension of the method developed by Heath-Jarrow-Morton.

Major papers in this area include Jarrow and Turnbull [116], Schönbucher [171, 172], Hubner [105, 104], and Bielecki and Rutkowski [23].

Other authors choice to model directly credit spreads (see Duffie and Singleton [74], Douady and Jeanblanc [68]).

3.8.1 Jarrow and Turnbull's model

Jarrow and Turnbull consider a situation where the interest rate follows a Vasicek's dynamics and where the intensity γ is a linear function of the interest rate and a factor Z, modeled as a Brownian motion.

$$dr_t = \kappa(r_\infty - r_t)dt + \sigma dW_t$$

and $\gamma_t = a_0(t) + a_1(t)r_t + a_2(t)Z_t$. The problem is that γ is not a non-negative value. nevertheless, the corporate bond follows

$$D(t,T) = \delta B(t,T) + (1-\delta)\exp(-\mu + \frac{1}{2}v)$$

where μ and v are the mean and variance of $R_T + \Gamma_T$.
TO BE COMPLETED

3.8.2 Vacicek Model

In Schonbucher, the dynamics of interest rate and of the intensity are

$$\begin{aligned} dr_t &= (k(t) - ar_t)dt + \sigma(t)dW_t \\ d\lambda_t &= (\widehat{k}(t) - \widehat{a}\lambda_t)dt + \widehat{\sigma}(t)dB_t \end{aligned}$$

where W and B are two Brownian motion with correlation ρ.

Proposition 3.8.1 *(i)The price of a default free zero-coupon with maturity T is*

$$B(t,T) = \exp(A(t,T;a,k,\sigma) - \kappa(t,T;a)r_t)$$

where

$$A(t,T;a,k,\sigma) = \frac{1}{2}\int_t^T \sigma^2(u)\kappa^2(t,u;a)du - \int_t^T \kappa(t,u;a)k(u)du$$

and $\kappa(t,u:a) = \frac{1}{a}(1 - e^{-a(T-t)})$.
(ii) The price of a defaultable zero-coupon with maturity T with zero recovery is

$$D(t,T) = B(t,T)B(t,T) = \exp(A(t,T;\widehat{h},\widehat{\sigma}) - \kappa(t,T;\widehat{a})\lambda_t))$$

with

$$\widehat{h}(t) = \widehat{k}(t) - \rho\widehat{\sigma}(t)\sigma(t)\kappa(t,T;a)$$

PROOF: See Appendix for (i). For (ii) write

$$D(t,T) = B(t,T)\mathbb{E}_{\mathbb{Q}_T}(\exp - \int_t^T \lambda_s ds)$$

where $\mathbb{Q}_T$ is the T-forward probability measure. The dynamics of λ under $\mathbb{Q}_T$ are

$$d\lambda_t = (\widehat{h}(t) - \widehat{a}\lambda_t)dt + \widehat{\sigma}(t)d\widehat{B}_t$$

□

3.8.3 The CIR model

`To be written`

Duffee's model

Duffee [69] assumes that the value of a default free bond is

$$\mathbb{E}_{\mathbb{Q}}\left(\exp\left(-\int_t^T r_s ds\right)|\mathcal{F}_t\right)$$

and that the defaultable bond is priced as

$$E_Q\left(\exp\left(-\int_t^T (r_s+\gamma_s)ds\right)|\mathcal{F}_t\right)$$

where

$$r_t = s_{1,t} + s_{2,t}, \quad h_t = \beta + \beta_1 s_{1,t} + \beta_2 s_{2,t} + s_{3,t}$$

and

$$ds_{i,t} = \kappa_i(\theta_i - s_{i,t})dt + \sigma_i\sqrt{s_{i,t}}dW_i(t)$$

are CIR processes. To be completed

3.9 Analysis of Several Random Times

Assume that τ_i are default times defined as

$$\tau_i = \inf\{t \geq 0 \,:\, \int_0^t \lambda_u^i du \geq -\ln U_i\},\ i = 1, \dots n,$$

where the λ^i are given $\mathbf{F}$-processes and U_i uniform random variables, independent of $\mathcal{F}_\infty$. We denote $H_t^i = 1\!\!1_{\tau_i \leq t}$ and $\mathcal{H}_t^i = \sigma(H_s^i, s \leq t)$. We define the filtrations $\mathcal{G}_t^i = \mathcal{F}_t \vee \mathcal{H}_t^i,$ and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^n$. It is easy to check that, for any integrable random variable $Y \in \mathcal{F}_\infty$, then

$$\mathbb{E}(Y|\mathcal{G}_t) = \mathbb{E}(Y|\mathcal{F}_t)$$

which implies that, for $t_1, \cdots, t_n < t$

$$\mathbb{P}(\tau_1 > t_1, \cdots, \tau_n > t_n|\mathcal{F}_\infty) = \mathbb{P}(\tau_1 > t_1, \cdots, \tau_n > t_n|\mathcal{F}_t)\,.$$

Let $\widehat{C}$ the survival copule of $(U_i, i \leq n)$, i.e.

$$\widehat{C}(u_1, \cdots, u_n) = \mathbb{P}(U_1 > u_1, \cdots, U_n > u_n)$$

Then, for $t_1, \cdots, t_n < t$

$$\begin{aligned}
\mathbb{P}(\tau_1 > t_1, \cdots, \tau_n > t_n|\mathcal{F}_t) &= \mathbb{P}(\Lambda_{t_1} < -\ln U_1, \cdots, \Lambda_{t_n} < -\ln U_n|\mathcal{F}_t) \\
&= \mathbb{P}(\exp -\Lambda_{t_1} > U_1, \cdots, \exp -\Lambda_{t_n} > U_n|\mathcal{F}_t) \\
&= \widehat{C}(\exp -\Lambda_{t_1}, \cdots, \exp -\Lambda_{t_n})
\end{aligned}$$

Some authors make the following assumption

$$\mathbb{P}(\tau_1 > t_1, \cdots, \tau_n > t_n|\mathcal{F}_t) = \mathbb{P}(\tau_1 > t_1|\mathcal{F}_t) \cdots \mathbb{P}(\tau_n > t_n|\mathcal{F}_t)\,.$$

This is in particular the case when the r.v. U_i are independent. In that case, the processes

$$H_t^i - \int_0^t \lambda_s^i (1 - H_s^i) ds$$

are $\mathbf{G}$-martingales as well as $\mathbf{G}^i$ martingales.

Chapter 4

Hazard process Approach

In this chapter, we present a general model of default time, based on a specific choice of filtration. We show how this setup is related with Cox process model, and we discuss the intensity based approach.

4.1 General case

4.1.1 The model

In reduced form approach, we shall deal with two kinds of information: the information from the asset's prices, denoted as $(\mathcal{F}_t, t \geq 0)$ and the information from the default time, i.e. the knowledge of the time were the default occured in the past, it the default has appeared. More precisely, this information is modeled by the filtration $\mathbf{H}$ generated by the default process H (completed with negligeable sets).

At the intuitive level, $\mathbf{F}$ is generated by prices of some assets, or by other economic factors (e.g., interest rates). This filtration can also be a subfiltration of the prices. The case where $\mathbf{F}$ is the trivial filtration is exactly what we have studied in the toy example. Though in typical examples $\mathbf{F}$ is chosen to be the Brownian filtration, most theoretical results do not rely on such a specification of the filtration $\mathbf{F}$. We denote by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$.

Special attention is paid here to the hypothesis $(\mathcal{H})$, which postulates the invariance of the martingale property with respect to the enlargement of $\mathbf{F}$ by the observations of a default time. We establish a representation theorem, in order to understand the meaning of complete market in a defaultable world and we deduce the hedging strategies for credit derivatives. The main part of this section can be found in the surveys of Jeanblanc and Rutkowski [119, 120].

4.1.2 Key lemma

It is straightforward to establish that any $\mathcal{G}_t$-random variable is equal, on the set $\{\tau > t\}$, to an $\mathcal{F}_t$-measurable random variable. We denote by $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$ the conditional law of τ given the information $\mathcal{F}_t$.

Lemma 4.1.1 *Let X be an $\mathcal{F}_T$-measurable integrable r.v. Then,*

$$\mathbb{E}(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \frac{\mathbb{E}(X 1\!\!1_{\{\tau>T\}}|\mathcal{F}_t)}{\mathbb{E}(1\!\!1_{\{\tau>t\}}|\mathcal{F}_t)} = 1\!\!1_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}(X e^{-\Gamma_T}|\mathcal{F}_t). \tag{4.1}$$

where $\Gamma_t = -\ln(1 - F_t)$

Proof: The proof is exactly the same as Corollary 3.4.1. Indeed,

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}(X|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}x_t$$

where x_t is $\mathcal{F}_t$-measurable, and taking conditional expectation w.r.t. $\mathcal{F}_t$ of both members, we deduce

$$x_t = \frac{\mathbb{E}(X\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}(Xe^{-\Gamma_T}|\mathcal{F}_t)\,.$$

□

The main point is that here, the process Γ is not necessarily increasing.

Lemma 4.1.2 *Let h be an* **F***-predictable process. Then,*

$$\mathbb{E}(h_\tau\mathbb{1}_{\tau<T}|\mathcal{G}_t) = h_\tau\mathbb{1}_{\{\tau<t\}} + \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}(\int_t^T h_u dF_u|\mathcal{F}_t) \tag{4.2}$$

We are not interested with **G**-predictable processes, mainly because any **G**-predictable process is equal, on $\{t \le \tau\}$ to an **F**-predictable process. As we shall see, this elementary result will allow us to compute the value of credit derivatives, as soon as some elementary defaultable asset is priced by the market.

Comments 4.1.1 It can be useful to understand the meaning of the lemma in the case where, as in the structural model, the default time is an $(\mathcal{F}_t)$ stopping time.

Remark 4.1.1 We emphasize that, in the Cox process approach, the enlarged filtration $\mathbf{G} = \mathbf{F}\vee\mathbf{H}$ is here the filtration which should be taken into account; the filtration generated by $\mathcal{F}_t$ and $\sigma(\Theta)$ is too large. In the latter filtration, in the case where **F** is a Brownian filtration, τ would be a predictable stopping time.

4.1.3 Martingales

Proposition 4.1.1 *The process $(F_t, t \ge 0)$ is a* **F***-submartingale.*

Proof: From definition, and form the increasing property of the process H,

$$\begin{aligned}\mathbb{E}(F_t|\mathcal{F}_s) &= \mathbb{E}\left(\mathbb{E}(H_t|\mathcal{F}_t)\,|\mathcal{F}_s\right) = \mathbb{E}\left(H_t|\mathcal{F}_s\right)\\ &\ge \mathbb{E}\left(H_s|\mathcal{F}_s\right) = F_s\end{aligned}$$

□

Proposition 4.1.2 *(i) The process $L_t = (1-H_t)e^{\Gamma_t}$ is a* **G** *-martingale.*

(ii) If X is a **F***-martingale, XL is a* **G** *-martingale.*

(iii) If the process Γ is increasing and continuous, the process $M_t = H_t - \Gamma(t\wedge\tau)$ is a **G** *-martingale.*

Proof: (i) From the key lemma, for $t > s$

$$\mathbb{E}(L_t|\mathcal{G}_s) = \mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{G}_s) = \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}\mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{F}_s) = \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s} = L_s$$

since $\mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{F}_s) = \mathbb{E}(\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)e^{\Gamma_t}|\mathcal{F}_s) = 1$.

(ii) From the key lemma,

$$\begin{aligned}\mathbb{E}(L_tX_t|\mathcal{G}_s) &= \mathbb{E}(\mathbb{1}_{\{\tau>t\}}L_tX_t|\mathcal{G}_s)\\ &= \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}\mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{-\Gamma_t}X_t|\mathcal{F}_s)\\ &= \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}\mathbb{E}(\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)e^{-\Gamma_t}X_t|\mathcal{F}_s)\\ &= L_sX_s.\end{aligned}$$

(*iii*) From integration by parts formula (H is a finite variation process, and Γ an increasing continuous process):

$$dL_t = (1 - H_t)e^{\Gamma_t} d\Gamma_t - e^{\Gamma_t} dH_t$$

and the process $M_t = H_t - \Gamma(t \wedge \tau)$ can be written

$$M_t \equiv \int_{]0,t]} dH_u - \int_{]0,t]} (1 - H_u) d\Gamma_u = - \int_{]0,t]} e^{-\Gamma_u} dL_u$$

and is a **G**-local martingale since L is **G**-martingale. (It can be noted that, if Γ is not increasing, the differential of e^Γ is more complicated.) □

Comments 4.1.2 Assertion (ii) seems to be related with a change of probability. It is important to note that here, one changes the filtration, not the probability measure. Moreover, setting $Q^* = LP$ does not define a probability Q **equivalent** to P, since the positive martingale L vanishes. The probability Q^* would be absolutely continuous wrt P. See Collin-Dufresne and Hugonnier [49].

Lemma 4.1.3 *Let $\widetilde{V}$ and R be* **F***-predictable processes. The process*

$$V_t = \widetilde{V}_t \mathbb{1}_{\{t<\tau\}} + R_\tau \mathbb{1}_{\{\tau \le t\}}$$

is a **G***-martingale if and only if the process*

$$\widetilde{V}_t e^{-\Gamma_t} + \int_0^t R_u e^{-\Gamma_u} d\Gamma_u$$

is an **F***-martingale*

Proof: The direct part comes from the fact that if V is a **G**-martingale, then $\mathbb{E}_{\mathbb{Q}}(V_t|\mathcal{F}_t)$ is an **F**-martingale. The converse is an immediate application of Lemma 2.4.1 and Lemma 4.1.2. □

Lemma 4.1.4 *Let P be the price process of a claim which delivers R_τ at default time and pays a cumulative coupon C till the default time, i.e. the discounted cum-dividend process*

$$B_t^{-1} P_t + \mathbb{1}_{\{\tau \le t\}} B_\tau^{-1} R_\tau + \int_0^{t \wedge \tau} B_u^{-1} dC_u$$

is a **G***-martingale. Let $\widetilde{P}t$ be the predefault price of the process P, i.e., $\tilde{P}$ is* **F***-predictable and $P_t = \mathbb{1}_{\{t<\tau\}} \widetilde{P}_t$. Then the process*

$$P_t^* = \alpha_t \widetilde{P}_t + \int_0^t \alpha_s dC_s + \int_0^t R_u \alpha_u \, d\Gamma_u$$

is an **F***-martingale, where $\alpha_t = B_t^{-1} e^{-\Gamma_t}$.*

Conversely, if $\widetilde{V}$ is an **F***-predictable process such that the process $\alpha_t \widetilde{V}_t + \int_0^t \alpha_s dC_s + \int_0^t R_u \alpha_u \, d\Gamma_u$ is an* **F***-martingale, then (the discounted cum-dividend) process*

$$B_t^{-1} \widetilde{V}_t \mathbb{1}_{\{t<\tau\}} + \mathbb{1}_{\{\tau \le t\}} B_\tau^{-1} R_\tau + \int_0^{t \wedge \tau} B_u^{-1} dC_u$$

is a **G***-martingale.*

Proof: This is an application of the Lemma 4.1.3. □

The following lemma are of great interest while dealing with convertible bonds.

Lemma 4.1.5 *For any* **F***-stopping time* θ*, we have:*

$$\mathbb{Q}(\tau > \theta \,|\, \mathcal{F}_\theta) = e^{-\Gamma_\theta}. \tag{4.3}$$

Let us be given $t \in I\!R_+$ *and* θ *an* **F** *stopping time, valued in* $(t, T]$.
(i) *For any bounded from below,* $\mathcal{F}_\theta$*-measurable random variable* χ*, we have:*

$$\mathbb{E}_\mathbb{Q}(1\!\!1_{\{t<\tau\le\theta\}}\chi \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau_{(2)}>t\}}\,\mathbb{E}_\mathbb{Q}\big((1-e^{\Gamma_t-\Gamma_\theta})\chi \,\big|\, \mathcal{F}_t\big)\ ,\ \mathbb{E}_\mathbb{Q}(1\!\!1_{\{\tau>\theta\}}\chi \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\,\mathbb{E}_\mathbb{Q}\big(e^{-\Gamma_\theta}\chi \,\big|\, \mathcal{F}_t\big)\ .$$

(ii) *For any bounded from below,* **F***-predictable process* Z*, we have:*

$$\mathbb{E}_\mathbb{Q}(Z_\tau 1\!\!1_{\{t<\tau\le\theta\}}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}e^{\Gamma_t}\,\mathbb{E}_\mathbb{Q}\Big(\int_t^\theta Z_u e^{-\Gamma_u}\,d\Gamma_u \,\Big|\, \mathcal{F}_t\Big). \tag{4.4}$$

(iii) *For any* **F***-predictable process process* A *with finite variation over* $[0, T]$*, we have:*

$$\mathbb{E}_\mathbb{Q}\Big(\int_{t\wedge\tau}^{\theta\wedge\tau} dA_u \,\Big|\, \mathcal{G}_t\Big) = 1\!\!1_{\{t<\tau\}}e^{\Gamma_t}\,\mathbb{E}_\mathbb{Q}\Big(\int_t^\theta e^{-\Gamma_u}dA_u \,\Big|\, \mathcal{F}_t\Big)\ . \tag{4.5}$$

4.1.4 Interpretation of the intensity

The submartingale property of F implies, from the Doob-Meyer decomposition that $F_t = Z_t + A_t$ where Z is a **F**-martingale and A a **F**-predictable increasing process. In terms of A,

$$\mathbb{E}(h_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) = h_\tau 1\!\!1_{\{\tau<t\}} + 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}(\int_t^T h_u dA_u|\mathcal{F}_t)\,.$$

In this general setting, the process Γ is not with finite variation. Hence, (iii) in Lemma 4.1.2 does not give the Doob-Meyer decomposition of H.

Proposition 4.1.3 *We assume for simplicity that* F *is continuous. The process*

$$M_t = H_t - \int_0^{t\wedge\tau}\frac{dA_u}{1-F_u}$$

is a **G***-martingale.*

PROOF: Let $s < t$. We give the proof in two steps, using the Doob-Meyer decomposition of F as $F_t = Z_t + A_t$.
First step: we prove

$$\mathbb{E}(H_t|\mathcal{G}_s) = H_s + 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

Indeed,

$$\begin{aligned}
\mathbb{E}(H_t|\mathcal{G}_s) &= 1 - \mathbb{P}(t<\tau|\mathcal{G}_s) = 1 - 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}(1-F_t|\mathcal{F}_s) \\
&= 1 - 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}(1-Z_t-A_t|\mathcal{F}_s) \\
&= 1 - 1\!\!1_{s<\tau}\frac{1}{1-F_s}(1-Z_s-A_s-\mathbb{E}(A_t-A_s|\mathcal{F}_s) \\
&= 1 - 1\!\!1_{s<\tau}\frac{1}{1-F_s}(1-F_s-\mathbb{E}(A_t-A_s|\mathcal{F}_s) \\
&= 1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}(A_t-A_s|\mathcal{F}_s)
\end{aligned}$$

In a second step, we prove that, setting $\Lambda_t = \int_0^t (1-H_s)\frac{dA_s}{1-F_s}$,

$$\mathbb{E}(\Lambda_{t\wedge\tau}|\mathcal{G}_s) = \Lambda_{s\wedge\tau} + 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

From the key formula,

$$\begin{aligned}
\mathbb{E}(\Lambda_{t\wedge\tau}|\mathcal{G}_s) &= \Lambda_{s\wedge\tau}1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}\left(\int_s^\infty \Lambda_{t\wedge u}dF_u|\mathcal{F}_s\right)\\
&= \Lambda_{s\wedge\tau}1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}\left(\int_s^t \Lambda_u dF_u + \int_t^\infty \Lambda_t dF_u|\mathcal{F}_s\right)\\
&= \Lambda_{s\wedge\tau}1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}\left(\int_s^t \Lambda_u dF_u + \Lambda_t(1-F_t)|\mathcal{F}_s\right)
\end{aligned}$$

We now use IP formula, using that Λ is bounded variation and continuous

$$d(\lambda_t(1-F_t)) = -\Lambda_t dF_t + (1-F_t)d\Lambda_t = -\Lambda_t dF_t + dA_t$$

hence

$$\int_s^t \Lambda_u dF_u + \Lambda_t(1-F_t) = -\Lambda_t(1-F_t) + \Lambda_s(1-F_s) + A_t - A_s + \Lambda_t(1-F_t) = \Lambda_s(1-F_s) + A_t - A_s$$

It follows that

$$\begin{aligned}
\mathbb{E}(\Lambda_{t\wedge\tau}|\mathcal{G}_s) &= \Lambda_{s\wedge\tau}1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}\left(\Lambda_s(1-F_s) + A_t - A_s|\mathcal{F}_s\right)\\
&= \Lambda_{s\wedge\tau} + 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}\left(A_t - A_s|\mathcal{F}_s\right)
\end{aligned}$$

Assuming that A is absolutely continuous wrt the Lebesgue measure and denote by a its derivative, we have proved the existence of a **F**-adapted process γ, called the intensity such that the process

$$H_t - \int_0^{t\wedge\tau}\gamma_u du = H_t - \int_0^t (1-H_u)\gamma_u du$$

is a **G**-martingale. More precisely, $\gamma_s = \frac{a_s}{1-F_s}$.

Lemma 4.1.6 *The process γ satisfies*

$$\gamma_t = \lim_{h\to 0}\frac{1}{h}\frac{\mathbb{P}(t<\tau<t+h|\mathcal{F}_t)}{\mathbb{P}(t<\tau|\mathcal{F}_t)}$$

PROOF: The martingale property of M implies that

$$\mathbb{E}(1\!\!1_{t<\tau<t+h}|\mathcal{G}_t) - \int_t^{t+h}\mathbb{E}((1-H_s)\lambda_s|\mathcal{G}_t)ds = 0$$

It follows that, by projection on $\mathcal{F}_t$

$$\mathbb{P}(t<\tau<t+h|\mathcal{F}_t) = \int_t^{t+h}\lambda_s\mathbb{P}(s<\tau|\mathcal{F}_t)ds$$

□

4.1.5 Restricting the information

Suppose from now on that $\widetilde{\mathcal{F}}_t \subset \mathcal{F}_t$ and define the σ-algebra $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$ and the associated hazard process $\widetilde{\Gamma}_t = -\ln(\widetilde{G}_t)$ with

$$\widetilde{G}_t = \mathbb{P}(t < \tau|\widetilde{\mathcal{F}}_t) = \mathbb{E}(G_t|\widetilde{\mathcal{F}}_t)\,.$$

Then, the key lemma implies that

$$\mathbb{E}(\mathbb{1}_{\{\tau>t\}}Y|\widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{\tau>t\}}e^{\widetilde{\Gamma}_t}\mathbb{E}\left(\mathbb{1}_{\{\tau>t\}}Y|\widetilde{\mathcal{F}}_t\right)$$

and if Y is a $\widetilde{\mathcal{F}}_T$-measurable variable,

$$\mathbb{E}(\mathbb{1}_{\{\tau>T\}}Y|\widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{\tau>t\}}e^{\widetilde{\Gamma}_t}\mathbb{E}\left(\widetilde{G}_T Y|\widetilde{\mathcal{F}}_t\right).$$

From $\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>T\}}Y|\widetilde{\mathcal{G}}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>T\}}Y|\mathcal{G}_t|\widetilde{\mathcal{G}}_t)$, we deduce

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{\{\tau>T\}}Y|\widetilde{\mathcal{G}}_t) &= \mathbb{E}\left(e^{\Gamma_t}\ \mathbb{1}_{\{\tau>t\}}\mathbb{E}(G_T Y|\mathcal{F}_t)\big|\,\widetilde{\mathcal{G}}_t\right)\\
&= \mathbb{1}_{\{\tau>t\}}e^{\widetilde{\Gamma}_t}\mathbb{E}\left(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}(G_T Y|\mathcal{F}_t)\big|\,\widetilde{\mathcal{F}}_t\right).
\end{aligned}$$

It can be noted that, from the uniqueness of the predefault $\mathbf{F}$-adapted value, for any t,

$$\mathbb{E}(\widetilde{G}_T Y|\widetilde{\mathcal{F}}_t) = \mathbb{E}\left(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}(G_T Y|\mathcal{F}_t)\big|\,\widetilde{\mathcal{F}}_t\right).$$

As a check, a simple computation shows

$$\begin{aligned}
\mathbb{E}\left(\mathbb{1}_{\{\tau>t\}}\mathbb{E}(G_T Y|\mathcal{F}_t)e^{\Gamma_t}\big|\,\widetilde{\mathcal{F}}_t\right) &= \mathbb{E}\left(\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)e^{\Gamma_t}\mathbb{E}(G_T Y|\mathcal{F}_t)\big|\,\widetilde{\mathcal{F}}_t\right)\\
&= \mathbb{E}\left(\mathbb{E}(G_T Y|\mathcal{F}_t)|\,\widetilde{\mathcal{F}}_t\right) = \mathbb{E}\left(G_T Y|\widetilde{\mathcal{F}}_t\right)\\
&= \mathbb{E}\left(\mathbb{E}\left(G_T|\widetilde{\mathcal{F}}_T\right)Y|\widetilde{\mathcal{F}}_t\right) = \mathbb{E}\left(\widetilde{G}_T Y|\widetilde{\mathcal{F}}_t\right).
\end{aligned}$$

since Y is $\widetilde{\mathcal{F}}_T$-measurable.

Let $F_t = Z_t + A_t$ be the $\mathbf{F}$-Doob-Meyer decomposition of the $\mathbf{F}$-submartingale F and assume that A is differentiable with respect to t: $A_t = \int_0^t a_s ds$. The process $\widetilde{A}_t = \mathbb{E}(A_t|\widetilde{\mathcal{F}}_t)$ is a $\widetilde{\mathbf{F}}$-submartingale and its $\widetilde{\mathbf{F}}$-Doob-Meyer decomposition is

$$\widetilde{A}_t = \widetilde{z}_t + \widetilde{\alpha}_t\,.$$

Hence, setting $\widetilde{Z}_t = \mathbb{E}(Z_t|\widetilde{\mathcal{F}}_t)$, the sub-martingale

$$\widetilde{F}_t = \mathbb{P}(t \geq \tau|\widetilde{\mathcal{F}}_t) = \mathbb{E}(F_t|\widetilde{\mathcal{F}}_t)$$

admits a $\widetilde{\mathbf{F}}$-Doob-Meyer decomposition as

$$\widetilde{F}_t = \widetilde{Z}_t + \widetilde{z}_t + \widetilde{\alpha}_t$$

where $\widetilde{Z}_t + \widetilde{z}_t$ is the martingale part. The computation of $\widetilde{\alpha}$in terms of a is given in the next lemma:

Lemma 4.1.7 *The compensator of $\widetilde{F}$ is $\widetilde{\alpha}_t = \int_0^t \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds$.*

Proof: Let us prove that the process $M_t^F = \mathbb{E}(F_t|\widetilde{\mathcal{F}}_t) - \int_0^t \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds$ is a $\widetilde{\mathbf{F}}$-martingale. It is integrable and $\widetilde{\mathbb{F}}$-adapted. From definition

$$\begin{aligned}
\mathbb{E}(M_T^F|\widetilde{\mathcal{F}}_t) &= \mathbb{E}\left(\mathbb{E}(F_T|\widetilde{\mathcal{F}}_T) - \int_0^T \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds\,\middle|\,\widetilde{\mathcal{F}}_t\right)\\
&= \mathbb{E}(F_T|\widetilde{\mathcal{F}}_t) - \int_0^t \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds - \mathbb{E}\left(\int_t^T \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds\,\middle|\,\widetilde{\mathcal{F}}_t\right)\\
&= \mathbb{E}(Z_T|\widetilde{\mathcal{F}}_t) + \mathbb{E}(A_T|\widetilde{\mathcal{F}}_t) - \int_0^t \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds - \int_t^T \mathbb{E}(a_s|\widetilde{\mathcal{F}}_t)ds
\end{aligned}$$

Since Z is a $\mathbf{F}$ martingale,$\mathbb{E}(Z_T|\widetilde{\mathcal{F}}_t) = \mathbb{E}(Z_t|\widetilde{\mathcal{F}}_t)$, hence

$$\begin{aligned}
\mathbb{E}(M_T^F|\widetilde{\mathcal{F}}_t) &= \mathbb{E}(Z_t|\widetilde{\mathcal{F}}_t) + \mathbb{E}\left(\int_0^t a_s ds \middle| \widetilde{\mathcal{F}}_t\right) - \int_0^t \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds \\
&= \mathbb{E}(Z_t + \int_0^t a_s ds|\widetilde{\mathcal{F}}_t) - \int_0^t \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds \\
&= M_t^F
\end{aligned}$$

Hence $(\widetilde{F}_t - \int_0^t \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds, t \geq 0)$ is a $\widetilde{\mathbf{F}}$-martingale. Obviously, the process $\int_0^\cdot \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds$ is predictable. The uniqueness in Doob Meyer theorem implies $\widetilde{\alpha}_t = \int_0^t \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds$. □

It follows that

$$H_t - \int_0^{t\wedge\tau} \frac{\widetilde{f}_s}{1-\widetilde{F}_s} ds$$

is a $\widetilde{\mathbf{G}}$-martingale and that the $\widetilde{\mathbf{F}}$-intensity of τ is equal to $\mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)/\widetilde{G}_s$, and not "as we could think" to $\mathbb{E}(a_s/G_s|\widetilde{\mathcal{F}}_s)$. Note that even if $(\mathcal{H})$ hypothesis holds between $\widetilde{\mathbf{F}}$ and $\mathbf{F}$, this proof can not be simplified since $\widetilde{F}_t$ is increasing but not $\widetilde{\mathbf{F}}$-predictable (there is no raison for $\widetilde{F}_t$ to have an intensity).

This result can be directly proved thanks to Bremaud's following result: $H_t - \int_0^{t\wedge\tau} \lambda_s ds$ is a $\mathbf{G}$-martingale hence $H_t - \int_0^{t\wedge\tau} \mathbb{E}(\lambda_s|\widetilde{\mathcal{G}}_s)ds$ is a $\widetilde{\mathbf{G}}$-martingale.

Note that

$$\int_0^{t\wedge\tau} \mathbb{E}(\lambda_s|\widetilde{\mathcal{G}}_s)ds = \int_0^t \mathbb{1}_{\{s\leq\tau\}}\mathbb{E}(\lambda_s|\widetilde{\mathcal{G}}_s)ds = \int_0^t \mathbb{E}(\mathbb{1}_{\{s\leq\tau\}}\lambda_s|\widetilde{\mathcal{G}}_s)ds$$

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{\{s\leq\tau\}}\lambda_s|\widetilde{\mathcal{G}}_s) &= \frac{\mathbb{1}_{\{s\leq\tau\}}}{\widetilde{G}_s}\mathbb{E}(\mathbb{1}_{\{s\leq\tau\}}\lambda_s|\widetilde{\mathcal{F}}_s) \\
&= \frac{\mathbb{1}_{\{s\leq\tau\}}}{\widetilde{G}_s}\mathbb{E}(G_s\lambda_s|\widetilde{\mathcal{F}}_s) = \frac{\mathbb{1}_{\{s\leq\tau\}}}{\widetilde{G}_s}\mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)
\end{aligned}$$

hence $H_t - \int_0^{t\wedge\tau} \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)/\widetilde{G}_s ds$ is a $\widetilde{\mathbf{G}}$-martingale, and we are done.

4.1.6 Enlargement of filtration

Working in the $\mathbf{G}$ filtration is possible, because the decomposition of any $\mathbf{F}$-martingale in this filtration is known up to time τ. For example, if B is an $\mathbf{F}$-Brownian motion, its decomposition in the $\mathbf{G}$ filtration up to time τ is

$$B_{t\wedge\tau} = \widehat{B}_{t\wedge\tau} + \int_0^{t\wedge\tau} \frac{d<B,G>_s}{G_{s-}},$$

where $(\widehat{B}_{t\wedge\tau}\ ,\ t \geq 0)$ is a continuous $\mathbf{G}$-martingale with increasing process $(t\wedge\tau)$. If the dynamics of an asset S are given by $dS_t = S_t(r_t dt + \sigma_t dB_t)$ in a default free framework, where B is a Brownian motion under the EMM, its dynamics will be

$$dS_t = S_t(r_t dt + \sigma_t \frac{d<B,G>_t}{G_{t-}} + \sigma_t d\widehat{B}_t)$$

in the default filtration, if we restrict our attention to time before default. Therefore, the default will act as a change of drift term on the prices.

See the Appendix for more details.

4.2 $(\mathcal{H})$ Hypothesis

In a general setting, **F** martingales do not remains **G**-martingales. We study here a specific case.

4.2.1 Complete model case

Proposition 4.2.1 *Let S be a semi-martingale on $(\Omega, \mathcal{G}, \mathbb{P})$ such that there exists a unique probability $\mathbb{Q}$, equivalent to $\mathbb{P}$ on $\mathcal{F}_T$, where $\mathcal{F}_t = \mathcal{F}_t^S = \sigma(S_s, s \le t)$ such that $(\widetilde{S}_t = S_t R_t, 0 \le t \le T)$ is an $\mathbf{F}^S$-martingale under the probability Q. We assume that there exists a probability $\tilde{\mathbb{Q}}$, equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ such that $(\widetilde{S}_t, 0 \le t \le T)$ is a $\mathbf{G}$-martingale under the probability $\tilde{\mathbb{Q}}$. Then, square integrable $(\mathbf{F}, \mathbb{Q})$-martingales are $(\mathbf{G}, \widetilde{\mathbb{Q}})$-martingales and the restriction of $\tilde{\mathbb{Q}}$ to $\mathcal{F}_T$ is equal Q.*

PROOF: We give a "financial proof". Under the hypothesis, any square integrable $\mathbf{F} - Q$ martingale can be thought as the discounted value of a contingent claim $\xi \in \mathcal{F}_T$. Since the same claim exists in the larger market, which is assumed to be arbitrage free, the claim process is also a $\mathbf{G} - \tilde{\mathbb{Q}}$ martingale. From the uniqueness of price for hedgeable claims, for any contingent claim $X \in \mathcal{F}_T$ and any **G**-e.m.m. $\tilde{\mathbb{Q}}$,

$$\mathbb{E}_{\mathbb{Q}}(X R_T | \mathcal{F}_t) = \mathbb{E}_{\tilde{Q}}(X R_T | \mathcal{G}_t) .$$

In particular, $\mathbb{E}_{\mathbb{Q}}(Z) = \mathbb{E}_{\tilde{Q}}(Z)$ for any $Z \in \mathcal{F}_T$ (take $t = 0$ and $X = Z R_T^{-1}$), hence the restriction of any e.m.m. $\tilde{Q}$ to the σ-algebra $\mathcal{F}_T$ equals $\mathbb{Q}$. Moreover, since any square integrable $\mathbf{F}$-$\mathbb{Q}$-martingale can be written as $\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t) = \mathbb{E}_{\tilde{Q}}(X|\mathcal{G}_t)$, we get that any square integrable $\mathbf{F}$-$\tilde{Q}$-martingale is a $\mathbf{G}$-$\tilde{Q}$-martingale. □

4.2.2 Definition and Properties of $(\mathcal{H})$ Hypothesis

We shall now examine the hypothesis $(\mathcal{H})$ which reads:

$(\mathcal{H})$ Every **F** square-integrable martingale is a **G** square-integrable martingale.

This hypothesis implies that the **F**-Brownian motion remains a Brownian motion in the enlarged filtration. It was studied by Brémaud and Yor [34] and Mazziotto and Szpirglas [158], and for financial purpose by Kusuoka [140]. This can be written in any of the equivalent forms (see, e.g. Dellacherie and Meyer [65]) :

Lemma 4.2.1 *Assume that $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$, where $\mathbf{F}$ is an arbitrary filtration and $\mathbf{H}$ is generated by the process $H_t = \mathbb{1}_{\{\tau \le t\}}$. Then the following conditions are equivalent to the hypothesis $(\mathcal{H})$.*
(i) For any $t, h \in \mathbb{R}_+$, we have

$$\mathbb{P}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_{t+h}). \tag{4.6}$$

(i′) For any $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_\infty). \tag{4.7}$$

(ii) For any $t \in \mathbb{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$ under $\mathbb{P}$, that is,

$$\mathbb{E}_{\mathbb{P}}(\xi\,\eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)\, \mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t)$$

for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ and bounded, $\mathcal{G}_t$-measurable random variable η.
(iii) For any $t \in \mathbb{R}_+$, and any $u \ge t$ the σ-fields $\mathcal{F}_u$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$.
(iv) For any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{F}_\infty$-measurable random variable ξ: $\mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)$.
(v) For any $t \in \mathbb{R}_+$, and any bounded, $\mathcal{G}_t$-measurable random variable η: $\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_\infty)$.

PROOF:

If $(\mathcal{H})$ holds, then (4.7) holds too. If (4.7) holds, the fact that $\mathcal{H}_t$ is generated by the sets $\{\tau \leq s\}, s \leq t$ proves that $\mathcal{F}_\infty$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$. The property follows. This result can be also found in [66]. The equivalence between (4.7) and (4.6) is left to the reader.

Using monotone class theorem it can be shown that conditions (i) and (i′) are equivalent. The proof of equivalence of conditions (i′)–(v) can be found, for instance, in Section 6.1.1 of Bielecki and Rutkowski [23] (for related results, see Elliott *et al.* [81]). Hence, we shall only show that condition (iv) and the hypothesis $(\mathcal{H})$ are equivalent.

Assume first that the hypothesis $(\mathcal{H})$ holds. Consider any bounded, $\mathcal{F}_\infty$-measurable random variable ξ_∞. Let $\xi_t = \mathbb{E}_{\mathbb{P}}(\xi_\infty \,|\, \mathcal{F}_t)$ be the martingale associated with ξ_∞. Then, $(\mathcal{H})$ implies that ξ is also a local martingale with respect to $\mathbf{G}$, and thus a $\mathbf{G}$-martingale, since ξ is bounded (recall that any bounded local martingale is a martingale). We conclude that $\xi_t = \mathbb{E}_{\mathbb{P}}(\xi_\infty \,|\, \mathcal{G}_t)$ and thus (iv) holds.

Suppose now that (iv) holds. First, we note that the standard truncation argument shows that the boundedness of ξ_∞ in (iv) can be replaced by the assumption that ξ_∞ is $\mathbb{P}$-integrable. Hence, any $\mathbf{F}$-martingale ξ is an $\mathbf{G}$-martingale, since ξ is clearly $\mathbf{G}$-adapted and we have, for every $t \leq s$,

$$\xi_t = \mathbb{E}_{\mathbb{P}}(\xi_s \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi_s \,|\, \mathcal{G}_t).$$

Now, suppose that L is an $\mathbf{F}$-local martingale so that there exists an increasing sequence of $\mathbf{F}$-stopping times τ_n such that $\lim_{n\to\infty} \tau_n = \infty$, for any n the stopped process ξ^{τ_n} follows a uniformly integrable $\mathbf{F}$-martingale. Hence, ξ^{τ_n} is also a uniformly integrable $\mathbf{G}$-martingale, and this means that ξ follows a $\mathbf{G}$-local martingale. □

Remarks 4.2.1 (i) Equality (4.7) appears in several papers on default risk, typically without any reference to the $(\mathcal{H})$ hypothesis. For example, in the Madan-Unal paper [153], the main theorem follows from the fact that (4.7) holds (See the proof of B9 in the appendix of their paper). This is also the case for Wong's model [178].
(ii) If τ is $\mathcal{F}_\infty$-measurable, and if (4.7) holds, then τ is an $\mathbf{F}$-stopping time. If τ is a $\mathbf{F}$-stopping time, equality (4.6) holds. If $\mathbf{F}$ is the Brownian filtration, τ is predictable and $\Lambda = H$.
(iii) Though condition $(\mathcal{H})$ does not necessarily hold true, in general, it is satisfied when τ is constructed through a standard approach (See Cox processes). This hypothesis is quite natural under the historical probability, and is stable under some change of measure. However, Kusuoka provides an example where $(\mathcal{H})$ holds under the historical probability and does not hold after a change of probability. This counter example is linked with dependency between default of different firms.
(iv) Hypothesis $(\mathcal{H})$ holds in particular if τ is independent from $\mathcal{F}_\infty$. See Greenfield thesis. [94].
(v) If $(\mathcal{H})$ hypothesis holds, from

$$\forall t,\, P(\tau \leq t|\mathcal{F}_t) = P(\tau \leq t|\mathcal{F}_\infty)\,,$$

we obtain that F is an increasing process.

Comments 4.2.1 See Elliott et al. [81] for more comments. The increasing property of F is equivalent to the fact that any $\mathbf{F}$-martingale, stopped at time τ is a $\mathbf{G}$ martingale. Nikeghbali and Yor [162] proved that this is equivalent to $E(m_\tau) = m_0$ for any bounded $\mathbf{F}$ martingale. The $(\mathcal{H})$ hypothesis is studied also in Florens and Fougere [86], under the name noncausality.

Proposition 4.2.2 *Assume that $\mathcal{H}$-hypothesis holds. If X is a $\mathbf{F}$-martingale, XL and $[L,X]$ are $\mathbf{G}$-local martingales.*

PROOF: We have seen in Proposition 4.1.2 that XL is a $\mathbf{G}$-martingale. Since $[L,X] = LX - \int L_- dX - \int X_- dL$, and that X is a $\mathbf{F}$, hence a $\mathbf{G}$-martingale, the process $[L,X]$ is the sum of three $\mathbf{G}$-martingales. □

4.2.3 Change of a probability measure

Kusuoka [140] shows, by means of a counter-example, that the hypothesis $(\mathcal{H})$ is not invariant with respect to an equivalent change of the underlying probability measure, in general. It is worth noting that his counter-example is based on two filtrations, $\mathbf{H}^1$ and $\mathbf{H}^2$, generated by the two random times τ^1 and τ^2, and he chooses $\mathbf{H}^1$ to play the role of the reference filtration $\mathbf{F}$. We shall argue that in the case where $\mathbf{F}$ is generated by a Brownian motion (or, more generally, by some martingale orthogonal to M under $\mathbb{P}$), the above-mentioned invariance property is valid under mild technical assumptions.

Girsanov's theorem

From Proposition 4.1.3 we know that the process $M_t = H_t - \Gamma_{t\wedge\tau}$ is a $\mathbf{G}$-martingale. We fix $T > 0$. For a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ we introduce the $\mathbf{G}$-martingale η_t, $t \le T$, by setting

$$\eta_t := \frac{d\mathbb{Q}}{d\mathbb{P}}_{\,|\mathcal{G}_t} = \mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t), \quad \mathbb{P}\text{-a.s.}, \tag{4.8}$$

where X is a $\mathcal{G}_T$-measurable integrable random variable, such that $\mathbb{P}(X > 0) = 1$. In view of Corollary **??** the Radon-Nikodým density process η admits the following representation

$$\eta_t = 1 + \int_0^t \xi_u \, dW_u + \int_{]0,t]} \zeta_u \, dM_u$$

where ξ and ζ are $\mathbf{G}$-predictable stochastic processes. Since η is a strictly positive process, we get

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-}\big(\beta_u \, dW_u + \kappa_u \, dM_u\big) \tag{4.9}$$

where β and κ are $\mathbf{G}$-predictable processes, with $\kappa > -1$.

Proposition 4.2.3 *Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{G}_T)$ equivalent to $\mathbb{P}$. If the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by* (4.8) *with η satisfying* (4.9)*, then the process*

$$W_t^* = W_t - \int_0^t \beta_u \, du, \quad \forall\, t \in [0, T], \tag{4.10}$$

follows a Brownian motion with respect to $\mathbf{G}$ under $\mathbb{Q}$, and the process

$$M_t^* := M_t - \int_{]0,t\wedge\tau]} \kappa_u \, d\Gamma_u = H_t - \int_{]0,t\wedge\tau]} (1+\kappa_u) \, d\Gamma_u, \quad \forall\, t \in [0, T], \tag{4.11}$$

is a $\mathbf{G}$-martingale orthogonal to W^*.

Proof: Notice first that for $t \le T$ we have

$$\begin{aligned} d(\eta_t W_t^*) &= W_t^* \, d\eta_t + \eta_{t-} \, dW_t^* + d[W^*, \eta]_t \\ &= W_t^* \, d\eta_t + \eta_{t-} \, dW_t - \eta_{t-}\beta_t \, dt + \eta_{t-}\beta_t \, d[W, W]_t \\ &= W_t^* \, d\eta_t + \eta_{t-} \, dW_t. \end{aligned}$$

This shows that W^* is a $\mathbf{G}$-martingale under $\mathbb{Q}$. Since the quadratic variation of W^* under $\mathbb{Q}$ equals $[W^*, W^*]_t = t$ and W^* is continuous, by virtue of Lévy's theorem it is clear that W^* follows a Brownian motion under $\mathbb{Q}$. Similarly, for $t \le T$

$$\begin{aligned} d(\eta_t M_t^*) &= M_t^* \, d\eta_t + \eta_{t-} \, dM_t^* + d[M^*, \eta]_t \\ &= M_t^* \, d\eta_t + \eta_{t-} \, dM_t - \eta_{t-}\kappa_t \, d\Gamma_{t\wedge\tau} + \eta_{t-}\kappa_t \, dH_t \\ &= M_t^* \, d\eta_t + \eta_{t-}(1+\kappa_t) \, dM_t. \end{aligned}$$

We conclude that M^* is a $\mathbf{G}$-martingale under $\mathbb{Q}$. To conclude it is enough to observe that W^* is a continuous process and M^* follows a process of finite variation. □

Corollary 4.2.1 *Let Y be a $\mathbf{G}$-martingale with respect to $\mathbb{Q}$. Then Y admits the following decomposition*

$$Y_t = Y_0 + \int_0^t \xi^*_u \, dW^*_u + \int_{]0,t]} \zeta^*_u \, dM^*_u, \tag{4.12}$$

where ξ^* and ζ^* are $\mathbf{G}$-predictable stochastic processes.

Proof: Consider the process $\widetilde{Y}$ given by the formula

$$\widetilde{Y}_t = \int_{]0,t]} \eta^{-1}_{u-} \, d(\eta_u Y_u) - \int_{]0,t]} \eta^{-1}_{u-} Y_{u-} \, d\eta_u.$$

It is clear that $\widetilde{Y}$ is a $\mathbf{G}$-martingale under $\mathbb{P}$. Notice also that Itô's formula yields

$$\eta^{-1}_{u-} \, d(\eta_u Y_u) = dY_u + \eta^{-1}_{u-} Y_{u-} \, d\eta_u + \eta^{-1}_{u-} \, d[Y,\eta]_u,$$

and thus

$$Y_t = Y_0 + \widetilde{Y}_t - \int_{]0,t]} \eta^{-1}_{u-} \, d[Y,\eta]_u. \tag{4.13}$$

From Corollary **??** we know that

$$\widetilde{Y}_t = Y_0 + \int_0^t \widetilde{\xi}_u \, dW_u + \int_{]0,t]} \widetilde{\zeta}_u \, dM_u \tag{4.14}$$

for some $\mathbf{G}$-predictable processes $\widetilde{\xi}$ and $\widetilde{\zeta}$. Therefore

$$\begin{aligned} dY_t &= \widetilde{\xi}_t \, dW_t + \widetilde{\zeta}_t \, dM_t - \eta^{-1}_{t-} \, d[Y,\eta]_t \\ &= \widetilde{\xi}_t \, dW^*_t + \widetilde{\zeta}_t (1+\kappa_t)^{-1} \, dM^*_t \end{aligned}$$

since (4.9) combined with (4.13)-(4.14) yield

$$\eta^{-1}_{t-} \, d[Y,\eta]_t = \widetilde{\xi}_t \beta_t \, dt + \widetilde{\zeta}_t \kappa_t (1+\kappa_t)^{-1} \, dH_t.$$

To derive the last equality we observe, in particular, that in view of (4.13) we have (we take into account continuity of Γ)

$$\Delta[Y,\eta]_t = \eta_{t-} \widetilde{\zeta}_t \kappa_t \, dH_t - \kappa_t \Delta[Y,\eta]_t.$$

We conclude that Y satisfies (4.12) with $\xi^* = \widetilde{\xi}$ and $\zeta^* = \widetilde{\zeta}(1+\kappa)^{-1}$. □

Preliminary lemma

Let us first examine a general set-up in which $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$, where $\mathbf{F}$ is an arbitrary filtration and $\mathbf{H}$ is generated by the default process H. We say that $\mathbb{Q}$ is locally equivalent to $\mathbb{P}$ if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in I\!R_+$. Then there exists the Radon-Nikodým density process η such that

$$d\mathbb{Q} \,|\, _{\mathcal{G}_t} = \eta_t \, d\mathbb{P} \,|\, _{\mathcal{G}_t}, \quad \forall \, t \in I\!R_+. \tag{4.15}$$

Part (i) in the next lemma is well known (see Jamshidian [114]). We assume that the hypothesis ($\mathcal{H}$) holds under $\mathbb{P}$.

Lemma 4.2.2 *(i) Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in I\!R_+$, with the associated Radon-Nikodým density process η. If the density process η is $\mathbf{F}$-adapted then we have $\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t)$ for every $t \in I\!R_+$. Hence, the hypothesis ($\mathcal{H}$) is also valid under $\mathbb{Q}$ and the $\mathbf{F}$-intensities of τ under $\mathbb{Q}$ and under $\mathbb{P}$ coincide.*
(ii) Assume that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$ and $d\mathbb{Q} = \eta_\infty \, d\mathbb{P}$, so that $\eta_t = \mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{G}_t)$. Then the hypothesis ($\mathcal{H}$) is valid under $\mathbb{Q}$ whenever we have, for every $t \in I\!R_+$,

$$\frac{\mathbb{E}_{\mathbb{P}}(\eta_\infty H_t \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{F}_\infty)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t H_t \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_\infty)}. \tag{4.16}$$

Proof: To prove (i), assume that the density process η is $\mathbf{F}$-adapted. We have for each $t \leq s \in \mathbb{R}_+$

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_s) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_s),$$

where the last equality follows by another application of the Bayes formula. The assertion now follows from part (i) in Lemma 4.2.1.

To prove part (ii), it suffices to establish the equality

$$\widehat{F}_t := \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty), \quad \forall\, t \in \mathbb{R}_+. \tag{4.17}$$

Note that since the random variables $\eta_t \mathbb{1}_{\{\tau \leq t\}}$ and η_t are $\mathbb{P}$-integrable and $\mathcal{G}_t$-measurable, using the Bayes formula, part (v) in Lemma 4.2.1, and assumed equality (4.16), we obtain the following chain of equalities

$$\begin{aligned}\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_\infty)} \\ &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_\infty \mathbb{1}_{\{\tau \leq t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{F}_\infty)} = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty).\end{aligned}$$

We conclude that the hypothesis $(\mathcal{H})$ holds under $\mathbb{Q}$ if and only if (4.16) is valid. $\square$

Unfortunately, straightforward verification of condition (4.16) is rather cumbersome. For this reason, we shall provide alternative sufficient conditions for the preservation of the hypothesis $(\mathcal{H})$ under a locally equivalent probability measure.

Case of the Brownian filtration

Let W be a Brownian motion under $\mathbb{P}$ and $\mathbf{F}$ its natural filtration. Since we work under the hypothesis $(\mathcal{H})$, the process W is also a $\mathbf{G}$-martingale, where $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$. Hence, W is a Brownian motion with respect to $\mathbf{G}$ under $\mathbb{P}$. Our goal is to show that the hypothesis $(\mathcal{H})$ is still valid under $\mathbb{Q} \in \mathcal{Q}$ for a large class $\mathcal{Q}$ of (locally) equivalent probability measures on $(\Omega, \mathcal{G})$.

Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$. Kusuoka [140] (see also Section 5.2.2 in Bielecki and Rutkowski [23]) proved that, under the hypothesis $(\mathcal{H})$, any $\mathbf{G}$-martingale under $\mathbb{P}$ can be represented as the sum of stochastic integrals with respect to the Brownian motion W and the jump martingale M. In our set-up, Kusuoka's representation theorem implies that there exist $\mathbf{G}$-predictable processes θ and $\zeta > -1$, such that the Radon-Nikodým density η of $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfies the following SDE

$$d\eta_t = \eta_{t-}\big(\theta_t \, dW_t + \zeta_t \, dM_t\big) \tag{4.18}$$

with the initial value $\eta_0 = 1$. More explicitly, the process η equals

$$\eta_t = \mathcal{E}_t\left(\int_0^{\cdot} \theta_u \, dW_u\right) \mathcal{E}_t\left(\int_0^{\cdot} \zeta_u \, dM_u\right) = \eta_t^{(1)} \eta_t^{(2)}, \tag{4.19}$$

where we write

$$\eta_t^{(1)} = \mathcal{E}_t\left(\int_0^{\cdot} \theta_u \, dW_u\right) = \exp\left(\int_0^t \theta_u \, dW_u - \frac{1}{2}\int_0^t \theta_u^2 \, du\right), \tag{4.20}$$

and

$$\eta_t^{(2)} = \mathcal{E}_t\left(\int_0^{\cdot} \zeta_u \, dM_u\right) = \exp\left(\int_0^t \ln(1+\zeta_u) \, dH_u - \int_0^{t\wedge\tau} \zeta_u \gamma_u \, du\right). \tag{4.21}$$

Moreover, by virtue of a suitable version of Girsanov's theorem, the following processes $\widehat{W}$ and $\widehat{M}$ are $\mathbf{G}$-martingales under $\mathbb{Q}$

$$\widehat{W}_t = W_t - \int_0^t \theta_u \, du, \quad \widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}} \gamma_u \zeta_u \, du. \tag{4.22}$$

Proposition 4.2.4 *Assume that the hypothesis* ($\mathcal{H}$) *holds under* $\mathbb{P}$. *Let* $\mathbb{Q}$ *be a probability measure locally equivalent to* $\mathbb{P}$ *with the associated Radon-Nikodým density process* η *given by formula (4.19) . If the process* θ *is* $\mathbf{F}$*-adapted then the hypothesis* ($\mathcal{H}$) *is valid under* $\mathbb{Q}$ *and the* $\mathbf{F}$*-intensity of* τ *under* $\mathbb{Q}$ *equals* $\widehat{\gamma}_t = (1+\widetilde{\zeta}_t)\gamma_t$, *where* $\widetilde{\zeta}$ *is the unique* $\mathbf{F}$*-predictable process such that the equality* $\widetilde{\zeta}_t \mathbb{1}_{\{t\leq\tau\}} = \zeta_t \mathbb{1}_{\{t\leq\tau\}}$ *holds for every* $t \in I\!R_+$.

PROOF: Let $\widetilde{\mathbb{P}}$ be the probability measure locally equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{G})$, given by

$$d\widetilde{\mathbb{P}}\,|_{\mathcal{G}_t} = \mathcal{E}_t\left(\int_0^{\cdot} \zeta_u\, dM_u\right) d\mathbb{P}\,|_{\mathcal{G}_t} = \eta_t^{(2)}\, d\mathbb{P}\,|_{\mathcal{G}_t}. \tag{4.23}$$

We claim that the hypothesis ($\mathcal{H}$) holds under $\widetilde{\mathbb{P}}$. From Girsanov's theorem, the process W follows a Brownian motion under $\widetilde{\mathbb{P}}$ with respect to both $\mathbf{F}$ and $\mathbf{G}$. Moreover, from the predictable representation property of W under $\widetilde{\mathbb{P}}$, we deduce that any $\mathbf{F}$-local martingale L under $\widetilde{\mathbb{P}}$ can be written as a stochastic integral with respect to W. Specifically, there exists an $\mathbf{F}$-predictable process ξ such that

$$L_t = L_0 + \int_0^t \xi_u\, dW_u.$$

This shows that L is also a $\mathbf{G}$-local martingale, and thus the hypothesis ($\mathcal{H}$) holds under $\widetilde{\mathbb{P}}$. Since

$$d\mathbb{Q}\,|_{\mathcal{G}_t} = \mathcal{E}_t\left(\int_0^{\cdot} \theta_u\, dW_u\right) d\widetilde{\mathbb{P}}\,|_{\mathcal{G}_t},$$

by virtue of part (i) in Lemma 4.2.2, the hypothesis ($\mathcal{H}$) is valid under $\mathbb{Q}$ as well. The last claim in the statement of the lemma can be deduced from the fact that the hypothesis ($\mathcal{H}$) holds under $\mathbb{Q}$ and, by Girsanov's theorem, the process

$$\widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}}\gamma_u\zeta_u\, du = H_t - \int_0^t \mathbb{1}_{\{u<\tau\}}(1+\widetilde{\zeta}_u)\gamma_u\, du$$

is a $\mathbb{Q}$-martingale. $\square$

We claim that the equality $\widetilde{\mathbb{P}} = \mathbb{P}$ holds on the filtration $\mathbf{F}$. Indeed, we have $d\widetilde{\mathbb{P}}\,|_{\mathcal{F}_t} = \widetilde{\eta}_t\, d\mathbb{P}\,|_{\mathcal{F}_t}$, where we write $\widetilde{\eta}_t = \mathbb{E}_{\mathbb{P}}(\eta_t^{(2)}\,|\,\mathcal{F}_t)$, and

$$\mathbb{E}_{\mathbb{P}}(\eta_t^{(2)}\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^{\cdot} \zeta_u\, dM_u\right)\Big|\,\mathcal{F}_\infty\right) = 1, \quad \forall\, t \in I\!R_+, \tag{4.24}$$

where the first equality follows from part (v) in Lemma 4.2.1.

To establish the second equality in (4.24), we first note that since the process M is stopped at τ, we may assume, without loss of generality, that $\zeta = \widetilde{\zeta}$ where the process $\widetilde{\zeta}$ is $\mathbf{F}$-predictable. Moreover,the conditional cumulative distribution function of τ given $\mathcal{F}_\infty$ has the form $1-\exp(-\Gamma_t(\omega))$. Hence, for arbitrarily selected sample paths of processes ζ and Γ, the claimed equality can be seen as a consequence of the martingale property of the Doléans exponential.

Formally, it can be proved by following elementary calculations, where the first equality is a consequence of (4.21)),

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^{\cdot} \widetilde{\zeta}_u\, dM_u\right)\Big|\,\mathcal{F}_\infty\right) &= \mathbb{E}_{\mathbb{P}}\left(\big(1+\mathbb{1}_{\{t\geq\tau\}}\widetilde{\zeta}_\tau\big)\exp\Big(-\int_0^{t\wedge\tau} \widetilde{\zeta}_u\gamma_u\, du\Big)\Big|\,\mathcal{F}_\infty\right)\\
&= \mathbb{E}_{\mathbb{P}}\left(\int_0^\infty \big(1+\mathbb{1}_{\{t\geq u\}}\widetilde{\zeta}_u\big)\exp\Big(-\int_0^{t\wedge u} \widetilde{\zeta}_v\gamma_v\, dv\Big)\gamma_u e^{-\int_0^u \gamma_v\, dv} du\,\Big|\,\mathcal{F}_\infty\right)\\
&= \mathbb{E}_{\mathbb{P}}\left(\int_0^t (1+\widetilde{\zeta}_u)\gamma_u \exp\Big(-\int_0^u (1+\widetilde{\zeta}_v)\gamma_v\, dv\Big)du\,\Big|\,\mathcal{F}_\infty\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \exp\Big(-\int_0^t \widetilde{\zeta}_v \gamma_v\, dv\Big)\, \mathbb{E}_{\mathbb{P}}\left(\int_t^\infty \gamma_u e^{-\int_0^u \gamma_v\, dv} du \,\Big|\, \mathcal{F}_\infty\right)\\
&= \int_0^t (1+\widetilde{\zeta}_u)\gamma_u \exp\Big(-\int_0^u (1+\widetilde{\zeta}_v)\gamma_v\, dv\Big) du\\
&+ \exp\Big(-\int_0^t \widetilde{\zeta}_v \gamma_v\, dv\Big) \int_t^\infty \gamma_u e^{-\int_0^u \gamma_v\, dv} du\\
&= 1 - \exp\Big(-\int_0^t (1+\widetilde{\zeta}_v)\gamma_v\, dv\Big) + \exp\Big(-\int_0^t \widetilde{\zeta}_v \gamma_v\, dv\Big) \exp\Big(-\int_0^t \gamma_v\, dv\Big) = 1,
\end{aligned}
$$

where the second last equality follows by an application of the chain rule.

Extension to orthogonal martingales

Equality (4.24) suggests that Proposition 4.2.4 can be extended to the case of arbitrary orthogonal local martingales. Such a generalization is convenient, if we wish to cover the situation considered in Kusuoka's counterexample.

Let N be a local martingale under $\mathbb{P}$ with respect to the filtration $\mathbf{F}$. It is also a $\mathbf{G}$-local martingale, since we maintain the assumption that the hypothesis $(\mathcal{H})$ holds under $\mathbb{P}$. Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$. We assume that the Radon-Nikodým density process η of $\mathbb{Q}$ with respect to $\mathbb{P}$ equals

$$
d\eta_t = \eta_{t-}\big(\theta_t\, dN_t + \zeta_t\, dM_t\big) \tag{4.25}
$$

for some $\mathbf{G}$-predictable processes θ and $\zeta > -1$ (the properties of the process θ depend, of course, on the choice of the local martingale N). The next result covers the case where N and M are orthogonal $\mathbf{G}$-local martingales under $\mathbb{P}$, so that the product MN follows a $\mathbf{G}$-local martingale.

Proposition 4.2.5 *Assume that the following conditions hold:*
(a) N and M are orthogonal $\mathbf{G}$-local martingales under $\mathbb{P}$,
(b) N has the predictable representation property under $\mathbb{P}$ with respect to $\mathbf{F}$, in the sense that any $\mathbf{F}$-local martingale L under $\mathbb{P}$ can be written as

$$
L_t = L_0 + \int_0^t \xi_u\, dN_u, \quad \forall\, t \in I\!\!R_+,
$$

for some $\mathbf{F}$-predictable process ξ,
(c) $\widetilde{\mathbb{P}}$ is a probability measure on $(\Omega, \mathcal{G})$ such that (4.23) holds.
Then we have:
(i) the hypothesis $(\mathcal{H})$ is valid under $\widetilde{\mathbb{P}}$,
(ii) if the process θ is $\mathbf{F}$-adapted then the hypothesis $(\mathcal{H})$ is valid under $\mathbb{Q}$.

The proof of the proposition hinges on the following simple lemma.

Lemma 4.2.3 *Under the assumptions of Proposition 4.2.5, we have:*
(i) N is a $\mathbf{G}$-local martingale under $\widetilde{\mathbb{P}}$,
(ii) N has the predictable representation property for $\mathbf{F}$-local martingales under $\widetilde{\mathbb{P}}$.

Proof: In view of (c), we have $d\widetilde{\mathbb{P}}\,|\,_{\mathcal{G}_t} = \eta^{(2)}_t\, d\mathbb{P}\,|\,_{\mathcal{G}_t}$, where the density process $\eta^{(2)}$ is given by (4.21), so that $d\eta^{(2)}_t = \eta^{(2)}_{t-}\zeta_t\, dM_t$. From the assumed orthogonality of N and M, it follows that N and $\eta^{(2)}$ are orthogonal $\mathbf{G}$-local martingales under $\mathbb{P}$, and thus $N\eta^{(2)}$ is a $\mathbf{G}$-local martingale under $\mathbb{P}$ as well. This means that N is a $\mathbf{G}$-local martingale under $\widetilde{\mathbb{P}}$, so that (i) holds.

To establish part (ii) in the lemma, we first define the auxiliary process $\widetilde{\eta}$ by setting $\widetilde{\eta}_t = \mathbb{E}_{\mathbb{P}}(\eta^{(2)}_t\,|\,\mathcal{F}_t)$. Then manifestly $d\widetilde{\mathbb{P}}\,|\,_{\mathcal{F}_t} = \widetilde{\eta}_t\, d\mathbb{P}\,|\,_{\mathcal{F}_t}$, and thus in order to show that any $\mathbf{F}$-local martingale under $\widetilde{\mathbb{P}}$ follows an **F**-local martingale under $\mathbb{P}$, it suffices to check that $\widetilde{\eta}_t = 1$ for every $t \in I\!R_+$, so that $\widetilde{\mathbb{P}} = \mathbb{P}$ on **F**. To this end, we note that

$$\mathbb{E}_{\mathbb{P}}(\eta_t^{(2)} \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^\cdot \zeta_u \, dM_u\right) \,\Big|\, \mathcal{F}_\infty\right) = 1, \quad \forall\, t \in I\!R_+,$$

where the first equality follows from part (v) in Lemma 4.2.1, and the second one can established similarly as the second equality in (4.24).

We are in a position to prove (ii). Let L be an **F**-local martingale under $\widetilde{\mathbb{P}}$. Then it follows also an **F**-local martingale under $\mathbb{P}$ and thus, by virtue of (b), it admits an integral representation with respect to N under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$. This shows that N has the predictable representation property with respect to **F** under $\widetilde{\mathbb{P}}$. □

We now proceed to the proof of Proposition 4.2.5.

Proof of Proposition 4.2.5. We shall argue along the similar lines as in the proof of Proposition 4.2.4. To prove (i), note that by part (ii) in Lemma 4.2.3 we know that any **F**-local martingale under $\widetilde{\mathbb{P}}$ admits the integral representation with respect to N. But, by part (i) in Lemma 4.2.3, N is a **G**-local martingale under $\widetilde{\mathbb{P}}$. We conclude that L is a **G**-local martingale under $\widetilde{\mathbb{P}}$, and thus the hypothesis ($\mathcal{H}$) is valid under $\widetilde{\mathbb{P}}$. Assertion (ii) now follows from part (i) in Lemma 4.2.2. □

Remark 4.2.1 It should be stressed that Proposition 4.2.5 is not directly employed in what follows. We decided to present it here, since it sheds some light on specific technical problems arising in the context of modeling dependent default times through an equivalent change of a probability measure (see Kusuoka [140]).

Example 4.2.1 Kusuoka [140] presents a counter-example based on the two independent random times τ_1 and τ_2 given on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We write $M_t^i = H_t^i - \int_0^{t\wedge\tau_i} \gamma_i(u)\, du$, where $H_t^i = 1\!\!1_{\{t \geq \tau_i\}}$ and γ_i is the deterministic intensity function of τ_i under $\mathbb{P}$. Let us set $d\mathbb{Q}\,|\,_{\mathcal{G}_t} = \eta_t \, d\mathbb{P}\,|\,_{\mathcal{G}_t}$, where $\eta_t = \eta_t^{(1)}\eta_t^{(2)}$ and, for $i = 1, 2$ and every $t \in I\!R_+$,

$$\eta_t^{(i)} = 1 + \int_0^t \eta_{u-}^{(i)} \zeta_u^{(i)} \, dM_u^i = \mathcal{E}_t\left(\int_0^\cdot \zeta_u^{(i)} \, dM_u^i\right)$$

for some **G**-predictable processes $\zeta^{(i)}$, $i = 1, 2$, where $\mathbf{G} = \mathbf{H}^1 \vee \mathbf{H}^2$. We set $\mathbf{F} = \mathbf{H}^1$ and $\mathbf{H} = \mathbf{H}^2$. Manifestly, the hypothesis ($\mathcal{H}$) holds under $\mathbb{P}$. Moreover, in view of Proposition 4.2.5, it is still valid under the equivalent probability measure $\widetilde{\mathbb{P}}$ given by

$$d\widetilde{\mathbb{P}}\,|\,_{\mathcal{G}_t} = \mathcal{E}_t\left(\int_0^\cdot \zeta_u^{(2)} \, dM_u^2\right) d\mathbb{P}\,|\,_{\mathcal{G}_t}.$$

It is clear that $\widetilde{\mathbb{P}} = \mathbb{P}$ on **F**, since

$$\mathbb{E}_{\mathbb{P}}(\eta_t^{(2)} \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^\cdot \zeta_u^{(2)} \, dM_u^2\right) \,\Big|\, \mathcal{H}_t^1\right) = 1, \quad \forall\, t \in I\!R_+.$$

However, the hypothesis ($\mathcal{H}$) is not necessarily valid under $\mathbb{Q}$ if the process $\zeta^{(1)}$ fails to be **F**-adapted. In Kusuoka's counter-example, the process $\zeta^{(1)}$ was chosen to be explicitly dependent on both random times, and it was shown that the hypothesis ($\mathcal{H}$) does not hold under $\mathbb{Q}$. For an alternative approach to Kusuoka's example, through an absolutely continuous change of a probability measure, the interested reader may consult Collin-Dufresne *et al.* [48].

4.2.4 Stochastic Barrier

Suppose that

$$P(\tau \leq t | \mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is an arbitrary continuous strictly increasing $\mathbf{F}$-adapted process. Our goal is to show that there exists a random variable Θ, independent of $\mathcal{F}_\infty$, with exponential law of parameter 1, such that $\tau \stackrel{law}{=} \inf\{t \geq 0 \,:\, \Gamma_t > \Theta\}$. Let us set $\Theta \stackrel{def}{=} \Gamma_\tau$. Then

$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of Γ, so that $\Gamma_{C_t} = t$. Therefore

$$P(\Theta > u | \mathcal{F}_\infty) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established the required properties, namely, the probability law of Θ and its independence of the σ-field $\mathcal{F}_\infty$. Furthermore, $\tau = \inf\{t : \Gamma_t > \Gamma_\tau\} = \inf\{t : \Gamma_t > \Theta\}$.

4.3 Representation theorem

Kusuoka [140] establishes the following representation theorem.

Théorème 4.1 *Under $(\mathcal{H})$, any $\mathbf{G}$-square integrable martingale admits a representation as the sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale M.*

We assume for simplicity that F is continuous and $F_t < 1, \forall t \in \mathbb{R}^+$. Since $(\mathcal{H})$ hypothesis holds, F is an increasing process.

Proposition 4.3.1 *Suppose that hypothesis $(\mathcal{H})$ holds under $\mathbb{P}$ and that any $\mathbf{F}$-martingale is continuous. Then, the martingale $M_t^h = \mathbb{E}_\mathbb{P}(h_\tau | \mathcal{G}_t)$, where h is an $\mathbf{F}$-predictable process such that $\mathbb{E}(h_\tau) < \infty$, admits the following decomposition as the sum of a continuous martingale and a discontinuous martingale*

$$M_t^h = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u} dm_u^h + \int_{]0,t\wedge\tau]} (h_u - J_u)\, dM_u, \tag{4.26}$$

where m^h is the continuous $\mathbf{F}$-martingale $m_t^h = \mathbb{E}_\mathbb{P}\Big(\int_0^\infty h_u dF_u \,|\, \mathcal{F}_t\Big)$,, J is the process $J_t = e^{\Gamma_t}(m_t^h - \int_0^t h_u dF_u)$ and M is the discontinuous $\mathbf{G}$-martingale $M_t = H_t - \Gamma_{t\wedge\tau}$ where $d\Gamma_u = \dfrac{dF_u}{1-F_u}$.

Proof: From (3.3) we know that

$$\begin{aligned} M_t^h &= \mathbb{E}(h_\tau | \mathcal{G}_t) = \mathbb{1}_{\{\tau \leq t\}} h_\tau + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}\Big(\int_t^\infty h_u dF_u \,\Big|\, \mathcal{F}_t\Big) \qquad (4.27) \\ &= \mathbb{1}_{\{\tau \leq t\}} h_\tau + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \left(m_t^h - \int_0^t h_u dF_u \right). \end{aligned}$$

Note that

$$dF_t = e^{-\Gamma_t} d\Gamma_t$$

and

$$d(e^{\Gamma_t}) = e^{\Gamma_t} d\Gamma_t = e^{\Gamma_t} \frac{dF_t}{1-F_t}. \tag{4.28}$$

We give now different proofs of the Proposition:

Proof 1. From the facts that Γ is an increasing process and m^h a continuous martingale, and using the integration by parts formula, we deduce that

$$\begin{aligned} dJ_t &= e^{\Gamma_t} dm_t^h + (m_t^h - \int_0^t h_u dF_u)\gamma_t e^{\Gamma_t} dt - e^{\Gamma_t} h_t dF_t \\ &= e^{\Gamma_t} dm_t^h + J_t \gamma_t e^{\Gamma_t} dt - e^{\Gamma_t} h_t dF_t \end{aligned}$$

Therefore, from (4.28)

$$dJ_t = e^{\Gamma_t} dm_t^h + (J_t - h_t)\frac{dF_t}{1-F_t}$$

or, in an integrated form,

$$J_t = m_0 + \int_0^t e^{\Gamma_u} dm_u^h + \int_0^t (J_u - h_u) d\Gamma_u .$$

Note that $J_u = M_u^h$ for $u < \tau$. Therefore, on $\{t < \tau\}$

$$M_t^h = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u} dm_u^h + \int_0^{t\wedge\tau} (J_u - h_u) d\Gamma_u$$

From (4.27), the jump of M^h at time τ is $h_\tau - J_\tau = h_\tau - M_{\tau-}^h$. Then, (4.26) follows.

Proof 2: The equality 4.27 can be written

$$M_t^h = \int_0^t h_s dH_s + 1\!\!1_{\{\tau > t\}} e^{\Gamma_t} \left(m_t^h - \int_0^t h_u dF_u \right)$$

and the result is obtained form IP formula.

□

Remark 4.3.1 Since hypothesis $(\mathcal{H})$ holds and Γ is $\mathbf{F}$-adapted, the processes $(m_t, t \geq 0)$ and $(\int_0^{t\wedge\tau} e^{\Gamma_u} dm_u,\, t \geq 0)$ are also $\mathbf{G}$-martingales.

4.3.1 Generic Defaultable Claims

A strictly positive random variable τ, defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, is termed a *random time*. In view of its interpretation, it will be later referred to as a *default time*. We introduce the jump process $H_t = 1\!\!1_{\{\tau \leq t\}}$ associated with τ, and we denote by $\mathbf{H}$ the filtration generated by this process. We assume that we are given, in addition, some auxiliary filtration $\mathbf{F}$, and we write $\mathbf{G} = \mathbf{H} \vee \mathbf{F}$, meaning that we have $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in I\!R_+$.

Definition 4.3.1 *By a defaultable claim maturing at T we mean the quadruple (X, A, Z, τ), where X is an $\mathcal{F}_T$-measurable random variable, A is an $\mathbf{F}$-adapted process of finite variation, Z is an $\mathbf{F}$-predictable process, and τ is a random time.*

The financial interpretation of the components of a defaultable claim becomes clear from the following definition of the dividend process D, which describes all cash flows associated with a defaultable claim over the lifespan $]0, T]$, that is, after the contract was initiated at time 0. Of course, the choice of 0 as the date of inception is arbitrary.

Definition 4.3.2 *The dividend process D of a defaultable claim maturing at T equals, for every $t \in [0, T]$,*

$$D_t = X 1\!\!1_{\{\tau > T\}} 1\!\!1_{[T,\infty[}(t) + \int_{]0,t]} (1 - H_u)\, dA_u + \int_{]0,t]} Z_u\, dH_u .$$

The financial interpretation of the definition above justifies the following terminology: X is the *promised payoff*, A represents the process of *promised dividends*, and the process Z, termed the *recovery process*, specifies the recovery payoff at default. It is worth stressing that, according to our convention, the cash payment (premium) at time 0 is not included in the dividend process D associated with a defaultable claim.

When dealing with a credit default swap, it is natural to assume that the premium paid at time 0 equals zero, and the process A represents the fee (annuity) paid in instalments up to maturity date or default, whichever comes first. For instance, if $A_t = -\kappa t$ for some constant $\kappa > 0$, then the 'price' of a stylized credit default swap is formally represented by this constant, referred to as the continuously paid *credit default rate* or *premium* (see Section 2.4.1 for details).

If the other covenants of the contract are known (i.e., the payoffs X and Z are given), the valuation of a swap is equivalent to finding the level of the rate κ that makes the swap valueless at inception. Typically, in a credit default swap we have $X = 0$, and Z is determined in reference to recovery rate of a reference credit-risky entity. In a more realistic approach, the process A is discontinuous, with jumps occurring at the premium payment dates. In this note, we shall only deal with a stylized CDS with a continuously paid premium.

Let us return to the general set-up. It is clear that the dividend process D follows a process of finite variation on $[0, T]$. Since

$$\int_{]0,t]} (1 - H_u)\, dA_u = \int_{]0,t]} 1\!\!1_{\{\tau > u\}}\, dA_u = A_{\tau-} 1\!\!1_{\{\tau \le t\}} + A_t 1\!\!1_{\{\tau > t\}},$$

it is also apparent that if default occurs at some date t, the 'promised dividend' $A_t - A_{t-}$ that is due to be received or paid at this date is disregarded. If we denote $\tau \wedge t = \min(\tau, t)$ then we have

$$\int_{]0,t]} Z_u\, dH_u = Z_{\tau \wedge t} 1\!\!1_{\{\tau \le t\}} = Z_\tau 1\!\!1_{\{\tau \le t\}}.$$

Let us stress that the process $D_u - D_t$, $u \in [t, T]$, represents all cash flows from a defaultable claim received by an investor who purchases it at time t. Of course, the process $D_u - D_t$ may depend on the past behavior of the claim (e.g., through some intrinsic parameters, such as credit spreads) as well as on the history of the market prior to t. The past dividends are not valued by the market, however, so that the current market value at time t of a claim (i.e., the price at which it trades at time t) depends only on future dividends to be paid or received over the time interval $]t, T]$.

Suppose that our underlying financial market model is arbitrage-free, in the sense that there exists a *spot martingale measure* $\mathbb{Q}$ (also referred to as a *risk-neutral probability*), meaning that $\mathbb{Q}$ is equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$, and the price process of any tradeable security, paying no coupons or dividends, follows a $\mathbf{G}$-martingale under $\mathbb{Q}$, when discounted by the *savings account* B, given by

$$B_t = \exp\left(\int_0^t r_u\, du\right), \quad \forall\, t \in I\!\!R_+. \tag{4.29}$$

4.3.2 Buy-and-hold Strategy

We write S^i, $i = 1, \ldots, k$ to denote the price processes of k primary securities in an arbitrage-free financial model. We make the standard assumption that the processes S^i, $i = 1, \ldots, k-1$ follow semimartingales. In addition, we set $S^k_t = B_t$ so that S^k represents the value process of the savings account. The last assumption is not necessary, however. We can assume, for instance, that S^k is the price of a T-maturity risk-free zero-coupon bond, or choose any other strictly positive price process as as numéraire.

For the sake of convenience, we assume that S^i, $i = 1, \ldots, k-1$ are non-dividend-paying assets, and we introduce the discounted price processes S^{i*} by setting $S^{i*}_t = S^i_t / B_t$. All processes are assumed to be given on a filtered probability space $(\Omega, \mathbf{G}, \mathbb{Q})$, where $\mathbb{Q}$ is interpreted as the real-life (i.e., statistical) probability measure.

Let us now assume that we have an additional traded security that pays dividends during its lifespan, assumed to be the time interval $[0,T]$, according to a process of finite variation D, with $D_0 = 0$. Let S denote a (yet unspecified) price process of this security. In particular, we do not postulate a priori that S follows a semimartingale. It is not necessary to interpret S as a price process of a defaultable claim, though we have here this particular interpretation in mind.

Let a **G**-predictable, $I\!R^{k+1}$-valued process $\phi = (\phi^0, \phi^1, \dots, \phi^k)$ represent a generic trading strategy, where ϕ^j_t represents the number of shares of the j^{th} asset held at time t. We identify here S^0 with S, so that S is the 0^{th} asset. In order to derive a pricing formula for this asset, it suffices to examine a simple trading strategy involving S, namely, the buy-and-hold strategy.

Suppose that one unit of the 0^{th} asset was purchased at time 0, at the initial price S_0, and it was hold until time T. We assume all the proceeds from dividends are re-invested in the savings account B. More specifically, we consider a *buy-and-hold* strategy $\psi = (1, 0, \dots, 0, \psi^k)$, where ψ^k is a **G**-predictable process. The associated *wealth process* $V(\psi)$ equals

$$V_t(\psi) = S_t + \psi^k_t B_t, \quad \forall\, t \in [0,T], \tag{4.30}$$

so that its initial value equals $V_0(\psi) = S_0 + \psi^k_0$.

Definition 4.3.3 *We say that a strategy* $\psi = (1, 0, \dots, 0, \psi^k)$ *is self-financing if*

$$dV_t(\psi) = dS_t + dD_t + \psi^k_t \, dB_t,$$

or more explicitly, for every $t \in [0,T]$,

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + D_t + \int_{]0,t]} \psi^k_u \, dB_u. \tag{4.31}$$

We assume from now on that the process ψ^k is chosen in such a way (with respect to S, D and B) that a buy-and-hold strategy ψ is self-financing. Also, we make a standing assumption that the random variable $Y = \int_{]0,T]} B_u^{-1} dD_u$ is $\mathbb{Q}$-integrable.

Lemma 4.3.1 *The discounted wealth* $V^*_t(\psi) = B_t^{-1} V_t(\psi)$ *of any self-financing buy-and-hold trading strategy* ψ *satisfies, for every* $t \in [0,T]$,

$$V^*_t(\psi) = V^*_0(\psi) + S^*_t - S^*_0 + \int_{]0,t]} B_u^{-1} dD_u. \tag{4.32}$$

Hence we have, for every $t \in [0,T]$,

$$V^*_T(\psi) - V^*_t(\psi) = S^*_T - S^*_t + \int_{]t,T]} B_u^{-1} \, dD_u. \tag{4.33}$$

Proof: We define an auxiliary process $\widehat{V}(\psi)$ by setting $\widehat{V}_t(\psi) = V_t(\psi) - S_t = \psi^k_t B_t$ for $t \in [0,T]$. In view of (4.31), we have

$$\widehat{V}_t(\psi) = \widehat{V}_0(\psi) + D_t + \int_{]0,t]} \psi^k_u \, dB_u,$$

and so the process $\widehat{V}(\psi)$ follows a semimartingale. An application of Itô's product rule yields

$$\begin{aligned} d\big(B_t^{-1}\widehat{V}_t(\psi)\big) &= B_t^{-1} d\widehat{V}_t(\psi) + \widehat{V}_t(\psi)\, dB_t^{-1} \\ &= B_t^{-1} dD_t + \psi^k_t B_t^{-1} dB_t + \psi^k_t B_t \, dB_t^{-1} \\ &= B_t^{-1} dD_t, \end{aligned}$$

where we have used the obvious identity: $B_t^{-1} dB_t + B_t \, dB_t^{-1} = 0$. Integrating the last equality, we obtain

$$B_t^{-1}\big(V_t(\psi) - S_t\big) = B_0^{-1}\big(V_0(\psi) - S_0\big) + \int_{]0,t]} B_u^{-1} dD_u,$$

and this immediately yields (4.32). □

It is worth noting that Lemma 4.3.1 remains valid if the assumption that S^k represents the savings account B is relaxed. It suffices to assume that the price process S^k is a *numéraire*, that is, a strictly positive continuous semimartingale. For the sake of brevity, let us write $S^k = \beta$. We say that $\psi = (1, 0, \ldots, 0, \psi^k)$ is self-financing it the wealth process

$$V_t(\psi) = S_t + \psi^k_t \beta_t, \quad \forall\, t \in [0, T],$$

satisfies, for every $t \in [0, T]$,

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + D_t + \int_{]0,t]} \psi^k_u \, d\beta_u.$$

Lemma 4.3.2 *The relative wealth $V^*_t(\psi) = \beta^{-1}_t V_t(\psi)$ of a self-financing trading strategy ψ satisfies, for every $t \in [0, T]$,*

$$V^*_t(\psi) = V^*_0(\psi) + S^*_t - S^*_0 + \int_{]0,t]} \beta^{-1}_u dD_u,$$

where $S^* = \beta^{-1}_t S_t$.

Proof: The proof proceeds along the same lines as before, noting that $\beta^1 d\beta + \beta d\beta^1 + d\langle \beta, \beta^1 \rangle = 0$. □

4.3.3 Spot Martingale Measure

Our next goal is to derive the risk-neutral valuation formula for the ex-dividend price S_t. To this end, we assume that our market model is arbitrage-free, meaning that it admits a (not necessarily unique) martingale measure $\mathbb{Q}$, equivalent to $\mathbb{Q}$, which is associated with the choice of B as a numéraire.

Definition 4.3.4 *We say that $\mathbb{Q}$ is a spot martingale measure if the discounted price S^{i*} of any non-dividend paying traded security follows a $\mathbb{Q}$-martingale with respect to $\mathbf{G}$.*

It is well known that the discounted wealth process $V^*(\phi)$ of any self-financing trading strategy $\phi = (0, \phi^1, \phi^2, \ldots, \phi^k)$ is a local martingale under $\mathbb{Q}$. In what follows, we shall only consider *admissible* trading strategies, that is, strategies for which the discounted wealth process $V^*(\phi)$ is a martingale under $\mathbb{Q}$. A market model in which only admissible trading strategies are allowed is *arbitrage-free*, that is, there are no arbitrage opportunities in this model.

Following this line of arguments, we postulate that the trading strategy ψ introduced in Section 4.3.2 is also *admissible*, so that its discounted wealth process $V^*(\psi)$ follows a martingale under $\mathbb{Q}$ with respect to $\mathbf{G}$. This assumption is quite natural if we wish to prevent arbitrage opportunities to appear in the extended model of the financial market. Indeed, since we postulate that S is traded, the wealth process $V(\psi)$ can be formally seen as an additional non-dividend paying tradeable security.

To derive a pricing formula for a defaultable claim, we make a natural assumption that the market value at time t of the 0^{th} security comes exclusively from the future dividends stream, that is, from the cash flows occurring in the open interval $]t, T[$. Since the lifespan of S is $[0, T]$, this amounts to postulate that $S_T = S^*_T = 0$. To emphasize this property, we shall refer to S as the *ex-dividend price* of the 0^{th} asset.

Definition 4.3.5 *A process S with $S_T = 0$ is the ex-dividend price of the 0^{th} asset if the discounted wealth process $V^*(\psi)$ of any self-financing buy-and-hold strategy ψ follows a $\mathbf{G}$-martingale under $\mathbb{Q}$.*

As a special case, we obtain the ex-dividend price a defaultable claim with maturity T.

Proposition 4.3.2 *The ex-dividend price process S associated with the dividend process D satisfies, for every $t \in [0,T]$,*

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big). \tag{4.34}$$

Proof: The postulated martingale property of the discounted wealth process $V^*(\psi)$ yields, for every $t \in [0,T]$,

$$\mathbb{E}_{\mathbb{Q}}\big(V_T^*(\psi) - V_t^*(\psi) \,\big|\, \mathcal{G}_t \big) = 0.$$

Taking into account (4.33), we thus obtain

$$S_t^* = \mathbb{E}_{\mathbb{Q}}\Big(S_T^* + \int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big).$$

Since, by virtue of the definition of the ex-dividend price we have $S_T = S_T^* = 0$, the last formula yields (4.34). □

It is not difficult to show that the ex-dividend price S satisfies, for every $t \in [0,T]$,

$$S_t = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t, \tag{4.35}$$

where the process $\widetilde{S}$ represents the *ex-dividend pre-default price* of a defaultable claim.

The *cum-dividend price* process $\bar{S}$ associated with the dividend process D is given by the formula, for every $t \in [0,T]$,

$$\bar{S}_t = B_t \mathbb{E}_{\mathbb{Q}}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big). \tag{4.36}$$

The corresponding discounted cum-dividend price process, $\widehat{S} := B^{-1}\bar{S}$, is a $\mathbf{G}$-martingale under $\mathbb{Q}$.

The savings account B can be replaced by an arbitrary numéraire β. The corresponding valuation formula becomes, for every $t \in [0,T]$,

$$S_t = \beta_t \, \mathbb{E}_{\mathbb{Q}^\beta}\Big(\int_{]t,T]} \beta_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big), \tag{4.37}$$

where $\mathbb{Q}^\beta$ is a martingale measure on $(\Omega, \mathcal{G}_T)$ associated with a numéraire β, that is, a probability measure on $(\Omega, \mathcal{G}_T)$ given by the formula

$$\frac{d\mathbb{Q}^\beta}{d\mathbb{Q}} = \frac{\beta_T}{\beta_0 B_T}, \quad \mathbb{Q}\text{-a.s.}$$

4.3.4 Self-Financing Trading Strategies

Let us now examine a general trading strategy $\phi = (\phi^0, \phi^1, \ldots, \phi^k)$ with $\mathbf{G}$-predictable components. The associated *wealth process* $V(\phi)$ equals $V_t(\phi) = \sum_{i=0}^k \phi_t^i S_t^i$, where, as before $S^0 = S$. A strategy ϕ is said to be *self-financing* if $V_t(\phi) = V_0(\phi) + G_t(\phi)$ for every $t \in [0,T]$, where the *gains process* $G(\phi)$ is defined as follows:

$$G_t(\phi) = \int_{]0,t]} \phi_u^0 \, dD_u + \sum_{i=0}^k \int_{]0,t]} \phi_u^i \, dS_u^i.$$

Corollary 4.3.1 *Let $S^k = B$. Then for any self-financing trading strategy ϕ, the discounted wealth process $V^*(\phi) = B_t^{-1} V_t(\phi)$ follows a martingale under $\mathbb{Q}$.*

Proof: Since B is a continuous process of finite variation, Itô's product rule gives

$$dS_t^{i*} = S_t^i \, dB_t^{-1} + B_t^{-1} dS_t^i$$

for $i = 0, 1, \ldots, k$, and so

$$\begin{aligned} dV_t^*(\phi) &= V_t(\phi) \, dB_t^{-1} + B_t^{-1} dV_t(\phi) \\ &= V_t(\phi) \, dB_t^{-1} + B_t^{-1} \Big(\sum_{i=0}^{k} \phi_t^i \, dS_t^i + \phi_t^0 \, dD_t \Big) \\ &= \sum_{i=0}^{k} \phi_t^i \big(S_t^i \, dB_t^{-1} + B_t^{-1} dS_t^i \big) + \phi_t^0 B_t^{-1} dD_t \\ &= \sum_{i=1}^{k-1} \phi_t^i \, dS_t^{i*} + \phi_t^0 \big(dS_t^* + B_t^{-1} dD_t \big) = \sum_{i=1}^{k-1} \phi_t^i \, dS_t^{i*} + \phi_t^0 \, d\widehat{S}_t, \end{aligned}$$

where the auxiliary process $\widehat{S}$ is given by the following expression:

$$\widehat{S}_t = S_t^* + \int_{]0,t]} B_u^{-1} \, dD_u.$$

To conclude, it suffices to observe that in view of (4.34) the process $\widehat{S}$ satisfies

$$\widehat{S}_t = \mathbb{E}_{\mathbb{Q}} \Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big), \tag{4.38}$$

and thus it follows a martingale under $\mathbb{Q}$. □

It is worth noting that $\widehat{S}_t$, given by formula (4.38), represents the discounted *cum-dividend price* at time t of the 0^{th} asset, that is, the arbitrage price at time t of all past and future dividends associated with the 0^{th} asset over its lifespan. To check this, let us consider a buy-and-hold strategy such that $\psi_0^k = 0$. Then, in view of (4.33), the terminal wealth at time T of this strategy equals

$$V_T(\psi) = B_T \int_{]0,T]} B_u^{-1} \, dD_u. \tag{4.39}$$

It is clear that $V_T(\psi)$ represents all dividends from S in the form of a single payoff at time T. The arbitrage price $\pi_t(\widehat{Y})$ at time $t < T$ of a claim $\widehat{Y} = V_T(\psi)$ equals (under the assumption that this claim is attainable)

$$\pi_t(\widehat{Y}) = B_t \, \mathbb{E}_{\mathbb{Q}} \Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big)$$

and thus $\widehat{S}_t = B_t^{-1} \pi_t(\widehat{Y})$. It is clear that discounted cum-dividend price follows a martingale under $\mathbb{Q}$ (under the standard integrability assumption).

Remarks 4.3.1 (i) Under the assumption of uniqueness of a spot martingale measure $\mathbb{Q}$, any $\mathbb{Q}$-integrable contingent claim is attainable, and the valuation formula established above can be justified by means of replication.
(ii) Otherwise – that is, when a martingale probability measure $\mathbb{Q}$ is not uniquely determined by the model $(S^1, S^2, \ldots, S^k)$ – the right-hand side of (4.34) may depend on the choice of a particular martingale probability, in general. In this case, a process defined by (4.34) for an arbitrarily chosen spot martingale measure $\mathbb{Q}$ can be taken as the no-arbitrage price process of a defaultable claim. In some cases, a market model can be completed by postulating that S is also a traded asset.

4.3.5 Martingale Properties of Prices of a Defaultable Claim

In the next result, we summarize the martingale properties of prices of a generic defaultable claim.

Corollary 4.3.2 *The discounted cum-dividend price $\widehat{S}_t$, $t \in [0,T]$, of a defaultable claim is a $\mathbb{Q}$-martingale with respect to* $\mathbf{G}$. *The discounted ex-dividend price S^*_t, $t \in [0,T]$, satisfies*

$$S^*_t = \widehat{S}_t - \int_{]0,t]} B_u^{-1}\, dD_u, \quad \forall\, t \in [0,T],$$

and thus it follows a supermartingale under $\mathbb{Q}$ if and only if the dividend process D is increasing.

In an application considered in Section 2.4, the finite variation process A is interpreted as the positive premium paid in instalments by the claimholder to the counterparty in exchange for a positive recovery (received by the claimholder either at maturity or at default). It is thus natural to assume that A is a decreasing process, and all other components of the dividend process are increasing processes (that is, we postulate that $X \geq 0$, and $Z \geq 0$). It is rather clear that, under these assumptions, the discounted ex-dividend price S^* is neither a super- or submartingale under $\mathbb{Q}$, in general.

Assume now that $A \equiv 0$, so that the premium for a defaultable claim is paid upfront at time 0, and it is not accounted for in the dividend process D. We postulate, as before, that $X \geq 0$, and $Z \geq 0$. In this case, the dividend process D is manifestly increasing, and thus the discounted ex-dividend price S^* is a supermartingale under $\mathbb{Q}$. This feature is quite natural since the discounted expected value of future dividends decreases when time elapses.

The final conclusion is that the martingale properties of the price of a defaultable claim depend on the specification of a claim and conventions regarding the prices (ex-dividend price or cum-dividend price). This point will be illustrated below by means of a detailed analysis of prices of credit default swaps.

4.4 Partial information

As pointed out by Jamshidian [113], *"one may wish to apply the general theory perhaps as an intermediate step, to a subfiltration that is not equal to the default-free filtration. In that case,* $\mathbf{F}$ *rarely satisfies hypothesis* $(\mathcal{H})$*"*. We present here simple cases of such a situation.

4.4.1 Information at discrete times

Assume that

$$dV_t = V_t(\mu dt + \sigma dW_t),\ V_0 = v$$

i.e., $V_t = ve^{\sigma(W_t+\nu t)} = ve^{\sigma X_t}$, with $\nu = (\mu - \sigma^2/2)/\sigma$ and $X_t = W_t + \nu t$. The default time is assumed to be the first hitting time of α with $\alpha < v$, i.e.,

$$\tau = \inf\{t \,:\, V_t \leq \alpha\} = \inf\{t \,:\, X_t \leq a\}$$

where $a = \sigma^{-1}\ln(\alpha/v)$. Here, $\mathbf{F}$ is the filtration of the observations of V at discrete times $t_1, \cdots t_n$ where $t_n \leq t < t_{n+1}$, i.e.,

$$\mathcal{F}_t = \sigma(V_{t_1}, \cdots, V_{t_n},\, t_i \leq t)$$

and we compute $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$. Let us recall that (See Section 8.1.2)

$$\mathbb{P}(\inf_{s\leq t} X_s > z) = \Phi(\nu, t, z)\,, \tag{4.40}$$

where

$$\begin{aligned}\Phi(\nu,t,z) &= \mathcal{N}\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z}\mathcal{N}\left(\frac{z+\nu t}{\sqrt{t}}\right), \qquad \text{for } z<0,\ t>0,\\ &= 0, \qquad \text{for } z\geq 0,\ t\geq 0,\\ \Phi(\nu,0,z) &= 1, \qquad \text{for } z<0.\end{aligned}$$

On $t < t_1$

In that case, F_t is the cumulative function of τ. Since $a<0$, we obtain

$$\begin{aligned}F_t &= \mathbb{P}(\tau\leq t) = \mathbb{P}(\inf_{s\leq t} X_s \leq a)\\ &= 1-\Phi(\nu,t,a) = \mathcal{N}\left(\frac{a-\nu t}{\sqrt{t}}\right) + e^{2\nu a}\mathcal{N}\left(\frac{a+\nu t}{\sqrt{t}}\right).\end{aligned}$$

On $t_1 < t < t_2$

We denote by $\mathcal{F}^W_t = \sigma(W_s, s\leq t)$ the natural filtration of the Brownian motion (this is also the natural filtration of X)

$$\begin{aligned}F_t &= \mathbb{P}(\tau\leq t|X_{t_1}) = 1-\mathbb{P}(\tau>t|X_{t_1})\\ &= \mathbb{E}(\mathbb{1}_{\{\inf_{s<t_1} X_s > a\}}\,\mathbb{P}(\inf_{t_1\leq s<t} X_s > a|\mathcal{F}^W_{t_1})|X_{t_1})\end{aligned}$$

The independence and stationarity of the increments of X yield to

$$\mathbb{P}(\inf_{t_1\leq s<t} X_s > a|\mathcal{F}^W_{t_1}) = \Phi(\nu, t-t_1, a-X_{t_1}).$$

Hence

$$F_t = 1-\Phi(\nu,t-t_1,a-X_{t_1})\mathbb{P}(\inf_{s<t_1} X_s > a|X_{t_1}).$$

From results on Brownian bridges, for $X_{t_1}>a$, we obtain (we skip the parameter ν in the definition of Φ)

$$F_t = 1-\Phi(t-t_1,a-X_{t_1})\left[1-\exp\left(-\frac{2\,a}{t_1}(a-X_{t_1})\right)\right]. \tag{4.41}$$

The case $X_{t_1}\leq a$ corresponds to default and, therefore, for $X_{t_1}\leq a$, $F_t=1$.

The process F is continuous and increasing in $[t_1,t_2[$. When t approaches t_1 from above, for $X_{t_1}>a$, $F_{t_1^+} = \exp\left[-\frac{2a}{t_1}(a-X_{t_1})\right]$, because $\lim_{t\to t_1^+}\Phi(t-t_1,a-X_{t_1})=1$.
For $X_{t_1}>a$, the jump of F at t_1 is

$$\Delta F^2_{t_1} = \exp\left[-\frac{2a}{t_1}(a-X_{t_1})\right] - 1 + \Phi(t_1,a).$$

For $X_{t_1}\leq a$, $\Phi(t-t_1,a-X_{t_1})=0$ by the definition of $\Phi(\cdot)$ and

$$\Delta F_{t_1} = \Phi(t_1,a).$$

General observation times $t_i < t < t_{i+1} < T$**,** $i\geq 2$

For $t_i<t<t_{i+1}$,

$$\begin{aligned}\mathbb{P}(\tau>t|X_{t_1},\dots,X_{t_i}) &= \mathbb{P}\left(\inf_{s\leq t_i} X_s > a\,\mathbb{P}(\inf_{t_i\leq s<t} X_s>a|\mathcal{F}_{t_i})|X_{t_1},\dots,X_{t_i}\right)\\ &= \Phi(t-t_i,a-X_{t_i})\mathbb{P}\left(\inf_{s\leq t_i} X_s>a|X_{t_1},\dots,X_{t_i}\right).\end{aligned}$$

Write K_i for the second term on the right-hand-side

$$\begin{aligned} K_i &= \mathbb{P}\left(\inf_{s\le t_i} X_s > a|X_{t_1},\dots,X_{t_i}\right) \\ &= \mathbb{P}\left(\inf_{s\le t_{i-1}} X_s > a\,\mathbb{P}(\inf_{t_{i-1}\le s<t_i} X_s > a|\mathcal{F}_{t_{i-1}}\vee X_{t_i})|X_{t_1},\dots,X_{t_i}\right). \end{aligned}$$

Obviously,

$$\begin{aligned} \mathbb{P}(\inf_{t_{i-1}\le s<t_i} X_s > a|\mathcal{F}_{t_{i-1}}\vee X_{t_i}) &= \mathbb{P}(\inf_{t_{i-1}\le s<t_i} X_s > a|X_{t_{i-1}},X_{t_i})) \\ &= \exp\left(-\frac{2}{t_i-t_{i-1}}(a-X_{t_{i-1}})(a-X_{t_i})\right). \end{aligned}$$

Therefore,

$$K_i = K_{i-1}\exp\left(-\frac{2}{t_i-t_{i-1}}(a-X_{t_{i-1}})(a-X_{t_i})\right). \tag{4.42}$$

Hence,

$$\begin{aligned} \mathbb{P}(\tau\le t|\mathcal{F}_t) &= 1 \qquad \text{if } X_{t_j}<a \text{ for at least one } t_j,\ t_j<t \\ &= 1-\Phi(t-t_i,a-X_{t_i})K_i, \end{aligned}$$

where

$$K_i = k(t_1,X_{t_1},0)k(t_2-t_1,X_{t_1},X_{t_2})\cdots k(t_i-t_{i-1},X_{t_{i-1}},X_{t_i})$$

and $k(s,x,y) = 1-\exp\left(-\frac{2}{s}(a-x)(a-y)\right)$.

Lemma 4.4.1 *The process ζ defined by*

$$\zeta_t = \sum_{i,t_i\le t}\Delta F_{t_i}.$$

is an $\mathbf{F}$*-martingale.*

PROOF: Consider first the times $t_i\le s<t\le t_{i+1}$. In this case, it is obvious that $\mathbb{E}(\zeta_t|\mathcal{H}_s)=\zeta_s$ since $\zeta_t=\zeta_s=\zeta_{t_i}$, which is $\mathcal{H}_s$-measurable.

It suffices to show that $\mathbb{E}(\zeta_t|\mathcal{F}_s)=\zeta_s$ for $t_i\le s<t_{i+1}\le t<t_{i+2}$. In this case, $\zeta_s=\zeta_{t_i}$ and $\zeta_t=\zeta_{t_i}+\Delta F_{t_{i+1}}$. Therefore,

$$\begin{aligned} \mathbb{E}(\zeta_t|\mathcal{F}_s) &= \mathbb{E}(\zeta_{t_i}+\Delta F_{t_{i+1}}|\mathcal{F}_s) \\ &= \zeta_{t_i}+\mathbb{E}(\Delta F_{t_{i+1}}|\mathcal{F}_s), \end{aligned}$$

which shows that it is necessary to prove that $\mathbb{E}(\Delta F_{t_{i+1}}|\mathcal{F}_s)=0$.

Let $s<u<t_{i+1}<v<t$. Then,

$$\mathbb{E}(F_v-F_u|\mathcal{F}_s)=\mathbb{E}(\mathbb{1}_{u<\tau\le v}|\mathcal{F}_s).$$

When

$$\begin{aligned} &v\to t_{i+1}, \quad v>t_{i+1} \quad \text{and} \\ &u\to t_{i+1}, \quad u<t_{i+1}, \qquad F_v-F_u\to\Delta F_{t_{i+1}}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}(\Delta F_{t_{i+1}}|\mathcal{F}_s) &= \lim_{u\to t_{i+1},v\to t_{i+1}}\mathbb{E}(\mathbb{1}_{u<\tau\le v}|\mathcal{F}_s) \\ &= \mathbb{E}(\mathbb{1}_{\tau=t_{i+1}}|\mathcal{F}_s)=0. \end{aligned}$$

□

The Doob-Meyer decomposition of F is

$$F_t = \zeta_t + (F_t - \zeta_t),$$

where ζ is an $\mathbf{F}$-martingale and $F_t - \zeta_t$ is a predictable increasing process.

The intensity of the default time would be the process λ defined as

$$\lambda_t dt = \frac{d(F_t - \zeta_t)}{1 - F_{t-}}.$$

Comments 4.4.1 It is also possible, as in Duffie and Lando [71], to assume that the observation at time $[t]$ is only $V_{[t]} + \epsilon$ where ϵ is a noise, modelled as a random variable independent of V. Another example, related with Parisian stopping times is presented in Çetin et al. [42]

4.4.2 Delayed information

In Guo et al. [97] the authors study a structural model with delayed information. More precisely, they start from a structural model where τ is a $\mathbf{F}$-stopping time, and they set $\widetilde{\mathcal{F}}_t = \mathcal{F}_{t-\delta}$ where $\delta > 0$ and $\mathcal{F}_t$ is the trivial filtration for negative s. We set $\mathcal{G}_t = \mathcal{F}_t$ and $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$. We prove here that the process $\widetilde{F}$ is not increasing.

Let $T_b = \inf\{t : W_t = b\}$. Then, for $t > \delta$,

$$\begin{aligned}
\widetilde{F}_t &= \mathbb{P}(T_b \le t|\widetilde{\mathcal{F}}_t) = \mathbb{P}(\inf_{s\le t} W_s \ge b|\widetilde{\mathcal{F}}_t) \\
&= 1\!\!1_{\inf_{s\le t-\delta} W_s < b}\mathbb{P}(\inf_{t-\delta<s\le t} W_s \ge b|\widetilde{\mathcal{F}}_t) \\
&= 1\!\!1_{\inf_{s\le t-\delta} W_s < b}\mathbb{P}(\inf_{t-\delta<s\le t} W_s - W_{t-\delta} \ge b - W_{t-\delta}|\widetilde{\mathcal{F}}_t) = 1\!\!1_{\inf_{s\le t-\delta} W_s < b}\Phi(\delta, b - W_{t-\delta}
\end{aligned}$$

where $\Phi(u,x) = \mathbb{P}(\inf_{s\le u} B_s \ge x) = \mathbb{P}(\sup_{s\le u} W_s \le -x) = \mathbb{P}(|W_u| \le -x) = \mathcal{N}(-x) - \mathcal{N}(x)$.
For $t < \delta$, $\widetilde{F}_t = \mathbb{P}(T_b \le t)$
TO BE COMPLETED.

4.5 Intensity approach

In the so-called intensity approach, the starting point is the knowledge of default time τ and some filtration $\mathbf{G}$ such that τ is a $\mathbf{G}$-stopping time. The intensity is defined as any non-negative process λ, such that

$$M_t \overset{def}{=} H_t - \int_0^{t\wedge\tau} \lambda_s ds$$

is a $\mathbf{G}$-martingale. The existence of the intensity relies on the fact that H is an increasing process, therefore a sub-martingale and can be written as a martingale M plus a predictable increasing process A. The increasing process A is such that $A_t 1\!\!1_{t\ge\tau} = A_\tau 1\!\!1_{t\ge\tau}$. In the case where τ is a predictable stopping time, obviously $A = H$. The intensity exists only if τ is a totally inaccessible stopping time.
Under some additional properties, Duffie et al. [72] establish formulae similar to (3.2). We emphasize that, in that setting the intensity is not well defined after time τ, i.e., if λ is an intensity, for any non-negative predictable process g the process $\tilde{\lambda}_t = \lambda_t 1\!\!1_{t\le\tau} + g_t 1\!\!1_{\{t>\tau\}}$ is also an intensity.

Lemma 4.5.1 *The process* $L_t = 1\!\!1_{\{t<\tau\}} \exp\left(\int_0^t \lambda_s ds\right)$ *is a martingale.*

PROOF: From integration by parts formula (see Section 8.4.2)

$$dL_t = \exp\left(\int_0^t \lambda_s ds\right)(-dH_t + (1-H_{t-})\lambda_t dt) = -\exp\left(\int_0^t \lambda_s ds\right) dM_t\,.$$

□

Proposition 4.5.1 *If the process* $Y_t = \mathbb{E}\left(X\exp\left(-\int_0^T \lambda_u du\right)|\mathcal{G}_t\right)$ *is continuous at time* τ*, then*

$$\mathbb{E}(X 1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}\mathbb{E}\left(X\exp\left(-\int_t^T \lambda_u du\right)|\mathcal{G}_t\right) \tag{4.43}$$

PROOF: The process $U_t = 1\!\!1_{t<\tau}\exp\left(\int_0^t \lambda_s ds\right)\mathbb{E}(X\exp - \int_0^T \lambda_u du|\mathcal{G}_t) = L_t Y_t$ is a martingale. Indeed, $dU_t = L_{t-}dY_t + Y_t dL_t$ and

$$\mathbb{E}(U_T|\mathcal{G}_t) = \mathbb{E}(X 1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = U_t\,.$$

The result follows. □

It can be mentioned that the continuity of the process depends on the choice of λ after time τ.

Proposition 4.5.2 *If the process* Y *is not continuous, then*

$$\mathbb{E}(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{t<\tau}\exp\left(\int_0^t \lambda_s ds\right)\mathbb{E}(X\exp - \Lambda_T|\mathcal{G}_t) - \mathbb{E}(\Delta Y_\tau e^{\Lambda_\tau} 1\!\!1_{t<\tau\le t}|\mathcal{G}_t)\,.$$

PROOF: We apply integration by parts formula

$$dU_t = L_{t-}dY_t + Y_{t-}dL_t + d[L,Y]_t = L_{t-}dY_t + Y_{t-}dL_t + \Delta L_t \Delta Y_t$$

and

$$\mathbb{E}(U_T|\mathcal{G}_t) = \mathbb{E}(X 1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = U_t - e^{\Lambda_\tau}\mathbb{E}(\Delta Y_\tau e^{\Lambda_\tau}|\mathcal{G}_t)\,.$$

Then, for any $X \in \mathcal{G}_T$:

$$\mathbb{E}(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\tau>t}\left(e^{\Lambda_t}\mathbb{E}(e^{-\Lambda_T}X|\mathcal{G}_t) - \mathbb{E}(e^{\Lambda_\tau}\,\Delta Y_\tau|\mathcal{G}_t)\right)$$

where $Y_t = \mathbb{E}\left(X\exp\left(-\Lambda_T\right)|\mathcal{G}_t\right)$ and $\Lambda_t = \int_0^t \lambda_u du$. □ Nevertheless, in practise, it is difficult to compute the size of the jump.

Comments 4.5.1 In Cox process model, for $X \in \mathcal{F}_T$,

$$\mathbb{E}\left(X\exp\left(-\Lambda_T\right)|\mathcal{G}_t\right) = \mathbb{E}\left(X\exp\left(-\Lambda_T\right)|\mathcal{F}_t\right)$$

and can not jump at time t.

4.5.1 Aven's Lemma

We recall Aven's lemma [6]

Lemma 4.5.2 *Let* $(\Omega, \mathcal{G}_t, \mathbb{P})$ *be a filtered probability space and* N *be a counting process. Assume that* $E(N_t) < \infty$ *for any* t*. Let* $(h_n, n \ge 1)$ *be a sequence of real numbers converging to 0, and*

$$Y_t^{(n)} = \frac{1}{h_n}E(N_{t+h_n} - N_t|\mathcal{G}_t)$$

Assume that there exists λ_t *and* y_t *non-negative* **G***-adapted processes such that* *(i) For any t, $\lim Y_t^{(n)} = \lambda_t$*

(ii) For any t, there exists for almost all ω an $n_0 = n_0(t,\omega)$ such that

$$|Y_s^{(n)} - \lambda_s(\omega)| \leq y_s(\omega)\,,\ s \leq t,\, n \geq n_0(t,\omega)$$

(iii) $\int_0^t y_s ds < \infty, \forall t$

Then, $N_t - \int_0^t \lambda_s ds$ is a $\mathbf{G}$-martingale.

We emphazise that, using this theorem when $N_t = H_t$ gives a value of the intensity which is equal to 0 after the default time. This is not convenient for using Duffie's no jump criteria, since, with this choice of intensity, the process Y in Proposition 4.5.1 has a jump at time τ. See Jeanblanc and LeCam [118] for a comparison between intensity and hazard process approaches.

In this section, we shall deal with several random times. Suppose we are given random times $\tau_1, \ldots, \tau_n$, defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a filtration $\mathbf{F}$. We are interested in the study of the hazard functions and processes associated with these random times. For $i = 1, \ldots, n$ we set

$$H_t^i = \mathbb{1}_{\{\tau_i \leq t\}}, \quad \forall\, t \in I\!\!R_+,$$

and we introduce the associated filtration $\mathbf{H}^i$ generated by the process H^i. We introduce the enlarged filtration $\mathbf{G} := \mathbf{H}^1 \vee \ldots \vee \mathbf{H}^n \vee \mathbf{F}$. It is thus evident that $\tau_1, \ldots, \tau_n$ are stopping times with respect to the filtration $\mathbf{G}$.

Definition 4.5.1 *For two filtration $\mathbf{F} \subset \mathbf{G}$, a $\mathbf{F}$-predictable right-continuous increasing process Λ^i is a $(\mathbf{F}, \mathbf{G})$-martingale hazard process of a random time τ_i if the process*

$$M_t^i = H_t^i - \Lambda_{t \wedge \tau_i}^i$$

is a $\mathbf{G}$-martingale.
If τ is a $\mathbf{G}$-stopping time, the $\mathbf{F}$-hazard process of τ is $\Gamma_t = -\ln(1 - F_t)$ where $F_t = P(\tau \leq t|\mathcal{F}_t)$

One of our goals will be to examine the relationship between the $(\mathbf{F}, \mathbf{G})$-martingale hazard processes of stopping times τ_i and the $(\mathbf{F}, \mathbf{G})$-martingale hazard process of their minimum $\tau = \min\,(\tau_1, \ldots, \tau_n)$.

4.5.2 Ordered Random Times

Consider two $\mathbf{F}$-adapted increasing continuous processes, Ψ^1 and Ψ^2, which satisfy $\Psi_0^2 = \Psi_0^1 = 0$ and $\Psi_t^2 > \Psi_t^1$ for every $t \in I\!\!R_+$. Let ξ be a random variable which is uniformly distributed on $[0,1]$, and is independent of the process Ψ. For $i = 1, 2$ we set

$$\tau_i = \inf\,\{\, t \in I\!\!R_+ \,:\, e^{-\Psi_t^i} \leq \xi \,\} = \inf\,\{\, t \in I\!\!R_+ \,:\, \Psi_t^i \geq -\ln \xi \,\}. \tag{4.44}$$

so that obviously $\tau_1 < \tau_2$ with probability 1.

We shall write $\mathbf{G}^i = \mathbf{H}^i \vee \mathbf{F}$, for $i = 1, 2$, and $\mathbf{G} = \mathbf{H}^1 \vee \mathbf{H}^2 \vee \mathbf{F}$. An analysis of each random time τ_i with respect to its 'natural' enlarged filtration $\mathbf{G}^i$ can be done along the same lines as in the previous section. It is clear that for each i the process Ψ^i represents: (i) the $(\mathbf{F})$-hazard process Γ^i of a random time τ_i, (ii) the $(\mathbf{F}, \mathbf{G}^i)$-martingale hazard process Λ^i of a random time τ_i, and (iii) the $\mathbf{G}^i$-martingale hazard process of τ_i when τ_i is considered as a $\mathbf{G}^i$-stopping time.

We shall focus on the study of hazard processes with respect to the enlarged filtrations. We find it convenient to introduce the following auxiliary notation: (Though in the present setup $\mathbf{F}^i = \mathbf{G}^i$, this double notation will appear useful in what follows.) $\mathbf{F}^i = \mathbf{H}^i \vee \mathbf{F}$, so that $\mathbf{G} = \mathbf{H}^1 \vee \mathbf{F}^2$ and $\mathbf{G} = \mathbf{H}^2 \vee \mathbf{F}^1$. Let us start by an analysis of τ_1. We are looking for the $(\mathbf{F}^2)$-hazard process $\widetilde{\Gamma}^1$ of

τ_1, as well as for the $(\mathbf{F}^2, \mathbf{G})$-martingale hazard process $\widetilde{\Lambda}^1$ of τ_1. We shall first check that $\widetilde{\Gamma}^1 \neq \Gamma^1$. Indeed, by virtue of the definition of a hazard process we have, for $t \in I\!R_+$,

$$\exp(-\Gamma^1_t) = P(\tau_1 > t|\mathcal{F}_t) = \exp(-\Psi^1_t)$$

and

$$e^{-\widetilde{\Gamma}^1_t} = \mathbb{P}(\tau_1 > t|\mathcal{F}^2_t) = \mathbb{P}(\tau_1 > t|\mathcal{F}_t \vee \mathcal{H}^2_t).$$

Equality $\widetilde{\Gamma}^1 = \Gamma^1$ would thus imply the following relationship, for every $t \in I\!R_+$,

$$\mathbb{P}(\tau_1 > t|\mathcal{F}_t \vee \mathcal{H}^2_t) = \mathbb{P}(\tau_1 > t|\mathcal{F}_t), \tag{4.45}$$

The relationship above is manifestly not valid, however. In effect, the inequality $\tau_2 \leq t$ implies $\tau_1 \leq t$, therefore on the set $\{\tau_2 \leq t\}$, which clearly belongs to the σ-field $\mathcal{H}^2_t$, we have $\mathbb{P}(\tau_1 > t|\mathcal{F}_t \vee \mathcal{H}^2_t) = 0$, and this contradicts (4.45). This shows also that the $(\mathbf{F}^2)$-hazard process $\widetilde{\Gamma}^1$ is well defined only strictly before τ_2.

Lemma 4.5.3 *The $\mathbf{G}^1$-martingale $M_t = M_{t\wedge\tau_1} = H^1_t - \Psi^1_{t\wedge\tau_1}$, is a $\mathbf{G}$-martingale*

PROOF: This seems rather obvious. Here is a detailed proof.From the key lemma

$$\begin{aligned} E(M_{t\wedge\tau_1}|\mathcal{G}_s) &= 1\!\!1_{\{s\leq\tau_2\}} N_t + 1\!\!1_{\{\tau_2<s\}} M_{\tau_1} \\ &= 1\!\!1_{\{s\leq\tau_2\}} 1\!\!1_{s<\tau_1} N_s + 1\!\!1_{\tau_1\leq s\leq\tau_2} N_s + 1\!\!1_{\{\tau_2<s\}} M_{\tau_1} \end{aligned}$$

where

$$N_s = \frac{E(M_{t\wedge\tau_1} 1\!\!1_{\{s<\tau_2\}}|\mathcal{F}_s \vee \mathcal{H}^1_s)}{P(s<\tau_2|\mathcal{F}_s \vee \mathcal{H}^1_s)}$$

Since $\tau_2 > \tau_1$, on the set $s < \tau_1$ one has $P(s < \tau_2|\mathcal{F}_s \vee \mathcal{H}^1_s) = 1$ and, from the $\mathbf{F} \vee \mathbf{H}^1$-martingale property of M,

$$E(M_{t\wedge\tau_1} 1\!\!1_{s<\tau_2}|\mathcal{F}_s \vee \mathcal{H}^1_s) = E(M_{t\wedge\tau_1}|\mathcal{F}_s \vee \mathcal{H}^1_s) = M_s$$

Now,

$$1\!\!1_{\tau_1<s<\tau_2} \frac{E(M_{t\wedge\tau_1} 1\!\!1_{s<\tau_2}|\mathcal{F}_s \vee \mathcal{H}^1_s)}{P(s<\tau_2|\mathcal{F}_s \vee \mathcal{H}^1_s)} = 1\!\!1_{\tau_1<s<\tau_2} M_{\tau_1} \frac{E(1\!\!1_{s<\tau_2}|\mathcal{F}_s \vee \mathcal{H}^1_s)}{P(s<\tau_2|\mathcal{F}_s \vee \mathcal{H}^1_s)} = 1\!\!1_{\tau_1<s<\tau_2} M_{\tau_1}$$

It follows that $E(M_{t\wedge\tau_1}|\mathcal{G}_s) = M_{s\wedge\tau_1}$. □

On the other hand, since $\mathbf{G} = \mathbf{G}^1 \vee \mathbf{H}^2$, it is clear that the process $H^1_t - \Psi^1_{t\wedge\tau_1}$, which is of course stopped at τ_1, is not only a $\mathbf{G}^1$-martingale, but also a $\mathbf{G}$-martingale. We conclude that Ψ^1 coincides with the $(\mathbf{F}^2, \mathbf{G})$-martingale hazard process $\widetilde{\Lambda}^1$ of τ_1. A similar reasoning shows that Ψ^1 represents also the $\mathbf{G}$-martingale hazard process $\widehat{\Lambda}^1$ of τ_1.

As one might easily guess, the properties of τ_2 with respect to the filtration $\mathbf{F}^1$ are slightly different. First, we have

$$e^{-\widetilde{\Gamma}^2_t} = \mathbb{P}(\tau_2 > t|\mathcal{F}^1_t) = \mathbb{P}(\tau_2 > t|\mathcal{F}_t \vee \mathcal{H}^1_t).$$

We claim that $\widetilde{\Gamma}^2 \neq \Gamma^2$, that is, the equality

$$\mathbb{P}(\tau_2 > t|\mathcal{F}_t \vee \mathcal{H}^1_t) = \mathbb{P}(\tau_2 > t|\mathcal{F}_t) \tag{4.46}$$

is not valid, in general. Indeed, the inequality $\tau_1 > t$ implies $\tau_2 > t$, and thus on set $\{\tau_1 > t\}$, which belongs to $\mathcal{H}^1_t$, we have $\mathbb{P}(\tau_2 > t|\mathcal{F}_t \vee \mathcal{H}^1_t) = 1$, in contradiction with (4.46). Notice that the process $\widetilde{\Gamma}^2$ is not well defined after time τ_1.

Furthermore, the process $H^2_t - \Psi^2_{t\wedge\tau_2}$ is a $\mathbf{G}^2$-martingale; it does not follow a $\mathbf{G}$-martingale, however (otherwise, the equality $\widetilde{\Gamma}^2 = \Gamma^2 = \Psi^2$ would hold on $[0, \tau_2]$, but this is clearly not true). The exact evaluation of the $(\mathbf{F}^1, \mathbf{G})$-martingale hazard process $\widetilde{\Lambda}^2$ of τ_2 seems to be rather difficult. Let us only mention that it is reasonable to expect that $\widetilde{\Lambda}^2$ it is discontinuous at τ_1.

Let us finally notice that τ_1 is a totally inaccessible stopping time not only with respect to $\mathbf{G}^1$, but also with respect to the filtration $\mathbf{G}$. On the other hand, τ_2 is a totally inaccessible stopping time with respect to $\mathbf{G}^1$, but it is a predictable stopping time with respect to $\mathbf{G}$. Indeed, we may easily find an announcing sequence τ_2^n of $\mathbf{G}$-stopping times, for instance,

$$\tau_2^n = \inf \{ t \geq \tau_1 \,:\, \Psi_t^2 \geq -\ln \xi - \tfrac{1}{n} \}.$$

Therefore the $\mathbf{G}$-martingale hazard process $\widehat{\Lambda}^2$ of τ_2 coincides with the $\mathbf{G}$-predictable process $H_t^2 = \mathbb{1}_{\{\tau_2 \leq t\}}$. Let us set $\tau = \tau_1 \wedge \tau_2$. In the present setup, it is evident that $\tau = \tau_1$, and thus the $\mathbf{G}$-martingale hazard process $\widehat{\Lambda}$ of τ is equal to Ψ^1. It is also equal to the sum of $\mathbf{G}$-martingale hazard processes $\widehat{\Lambda}^i$ of τ_i, $i = 1, 2,$ stopped at τ. Indeed, we have

$$\widehat{\Lambda}_{t\wedge\tau} = \Psi^1_{t\wedge\tau} = \Psi^1_{t\wedge\tau} + H^2_{t\wedge\tau} = \widehat{\Lambda}^1_{t\wedge\tau} + \widehat{\Lambda}^2_{t\wedge\tau}.$$

We shall see in the next section that this property is universal (though not always very useful).

4.5.3 Properties of the Minimum of Several Random Times

We shall examine the following problem: given a finite family of random times τ_i, $i = 1, \ldots, n,$ and the associated hazard processes, find the hazard process of the random time $\tau = \tau_1 \wedge \ldots \wedge \tau_n$. The problem above cannot be solved in such a generality, that is, without the knowledge of the joint law of $(\tau_1, \ldots, \tau_n)$. Indeed, as we shall see in what follows the solution depends heavily on specific assumptions on random times and the choice of filtrations (we follow Duffie [70] and Kusuoka [140]).

Hazard Function of the Minimum of Several Random Times

Let us first consider a simple result, in which we focus on the calculation of the hazard function of the minimum of several independent random times.

Lemma 4.5.4 *Let τ_i, $i = 1, \ldots, n,$ be n random times defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Assume that τ_i admits the hazard function Γ^i. If τ_i, $i = 1, \ldots, n,$ are mutually independent random variables, then the hazard function Γ of τ is equal to the sum of hazard functions Γ^i, $i = 1, \ldots, n$.*

Proof: For any $t \in \mathbb{R}_+$ we have

$$\begin{aligned} e^{-\Gamma(t)} &= 1 - F(t) = \mathbb{P}(\tau > t) = \mathbb{P}(\min(\tau_1, \ldots, \tau_n) > t) = \prod_{i=1}^n \mathbb{P}(\tau_i > t) \\ &= \prod_{i=1}^n (1 - F_i(t)) = \prod_{i=1}^n e^{-\Gamma^i(t)} = e^{-\sum_{i=1}^n \Gamma^i(t)}. \qquad \square \end{aligned}$$

Let us now focus on the case of continuous distribution functions F_i, $i = 1, \ldots, n$. In this case, we get also $\Lambda(t) = \sum_{i=1}^n \Lambda^i(t)$. In particular, if τ_i admits the intensity $\gamma^i(t) = \lambda^i(t) = f_i(t)(1 - F_i(t))^{-1}$, for each $i,$ then the process

$$H_t - \sum_{i=1}^n \int_0^{t\wedge\tau} \gamma^i(u)\, du = \mathbb{1}_{\{\tau \leq t\}} - \sum_{i=1}^n \int_0^{t\wedge\tau} \lambda^i(u)\, du$$

follows a $\mathbf{H}$-martingale, where $\mathbf{H} = \mathbf{H}^1 \vee \ldots \vee \mathbf{H}^n$.

Conversely, if the hazard function of τ satisfies $\Lambda(t) = \Gamma(t) = \sum_{i=1}^n \Gamma^i(t) = \sum_{i=1}^n \Lambda^i(t)$ for every t then we obtain

$$\mathbb{P}(\tau_1 > t, \ldots, \tau_n > t) = \prod_{i=1}^n \mathbb{P}(\tau_i > t), \quad \forall\, t \in \mathbb{R}_+.$$

Martingale Hazard Process of the Minimum of Several Random Times

We borrow from Duffie [70] the following simple result (see Lemma 1 therein).

Lemma 4.5.5 *Let τ_i, $i = 1, \dots, n$, be random times such that $\mathbb{P}(\tau_i = \tau_j) = 0$ for $i \neq j$. Then the $(\mathbf{F}, \mathbf{G})$-martingale hazard process Λ of $\tau = \tau_1 \wedge \ldots \wedge \tau_n$ is equal to the sum of $(\mathbf{F}, \mathbf{G})$-martingale hazard processes Λ^i stopped at τ, that is,*

$$\Lambda_t = \sum_{i=1}^n \Lambda^i_{t\wedge\tau}, \quad \forall\, t \in I\!\!R_+. \tag{4.47}$$

If Λ is a continuous process then the process $\widetilde{L}$ given by the formula $\widetilde{L}_t = (1 - H_t)e^{\Lambda_t}$ is a $\mathbf{G}$-martingale.

PROOF: By assumption, for any $i = 1, \dots, n,$ the process $\widetilde{M}^i_t := H^i_t - \Lambda^i_{t\wedge\tau_i}$ is a $\mathbf{G}$-martingale. Therefore, by the well-known properties of martingales the stopped process

$$(\widetilde{M}^i_t)^\tau = H^i_{t\wedge\tau} - \Lambda^i_{t\wedge\tau_i\wedge\tau} = H^i_{t\wedge\tau} - \Lambda^i_{t\wedge\tau}$$

also follows a $\mathbf{G}$-martingale for any fixed i. (Of course, if τ_i, $i = 1 \dots, n$ are stopping times with respect to the filtration $\mathbf{G}$, then τ is also a $\mathbf{G}$-stopping time.) On the other hand, since $\mathbb{P}(\tau_i = \tau_j) = 0$ for $i \neq j,$ we have

$$\sum_{i=1}^n H^i_{t\wedge\tau} = H_t = 1\!\!1_{\{\tau\le t\}}.$$

Therefore, the process

$$\widetilde{M}_t := H_t - \sum_{i=1}^n \Lambda^i_{t\wedge\tau} = \sum_{i=1}^n (\widetilde{M}^i_t)^\tau$$

obviously follows a $\mathbf{G}$-martingale, as a sum of $\mathbf{G}$-martingales. We conclude that the $(\mathbf{F}, \mathbf{G})$-martingale hazard process Λ of τ satisfies (4.47). The second statement is an easy consequence of Itô's formula, which gives

$$\widetilde{L}_t = 1 - \int_{]0,t]} \widetilde{L}_{u-}\, d\widetilde{M}_u. \tag{4.48}$$

This ends the proof. □

The striking feature of Lemma 4.5.5 is that the $(\mathbf{F}, \mathbf{G})$-martingale hazard process of τ can be easily found without the knowledge the joint probability law of random times $\tau_1, \dots, \tau_n$. It should thus be observed that in order to make use of the notion of a $(\mathbf{F}, \mathbf{G})$-martingale hazard process Λ we need to show in addition that Λ actually possesses desired properties. For instance, it would be useful to know whether the equality

$$\mathbb{P}(\tau > s|\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\mathbb{E}\big(e^{\Lambda_t-\Lambda_s}|\mathcal{F}_t\big) \tag{4.49}$$

holds for every $t \le s,$ or more generally whether we have

$$\mathbb{E}(1\!\!1_{\{\tau>s\}}Y|\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\mathbb{E}\big(Ye^{\Lambda_t-\Lambda_s}|\mathcal{F}_t\big) \tag{4.50}$$

for any bounded $\mathcal{G}_s$-measurable (or $\mathcal{F}_s$-measurable) random variable Y.

From now on, we shall assume that the following hypothesis (H) is satisfied.

H hypothesis

Assume that $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$, where $\mathbf{F}$ is an arbitrary filtration. Any bounded $\mathbf{F}$-martingale is a $\mathbf{G}$-martingale.

This hypothesis can be written in different forms:

Lemma 4.5.6 *Assume that $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$, where $\mathbf{F}$ is an arbitrary filtration. Then the following conditions are equivalent:*
(i) For any $t \in I\!\!R_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$ under $\mathbb{P}$, that is,

$$\mathbb{E}_{\mathbb{P}}(\xi\,\eta\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi\,|\,\mathcal{F}_t)\,\mathbb{E}_{\mathbb{P}}(\eta\,|\,\mathcal{F}_t)$$

for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ and bounded, $\mathcal{G}_t$-measurable random variable η.
(ii) For any $t \in I\!\!R_+$, and any $u \geq t$ the σ-fields $\mathcal{F}_u$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$.
(iii) For any $t \in I\!\!R_+$ and any bounded, $\mathcal{F}_\infty$-measurable random variable ξ: $\mathbb{E}_{\mathbb{P}}(\xi\,|\,\mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\xi\,|\,\mathcal{F}_t)$.
(iv) For any $t \in I\!\!R_+$, and any bounded, $\mathcal{G}_t$-measurable random variable η: $\mathbb{E}_{\mathbb{P}}(\eta\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\eta\,|\,\mathcal{F}_\infty)$.

Combining Lemma 4.5.5 with Corollary **??**, we get immediately the following result, which gives only a partial answer to the last question, however (see also Proposition 4.5.4 for related results).

Proposition 4.5.3 *Let τ_i, $i = 1, \ldots, n$, be random times such that $\mathbb{P}(\tau_i = \tau_j) = 0$ for $i \neq j$. Assume that hypothesis (H) is satisfied and that each random time τ_i admits a continuous $(\mathbf{F}, \mathbf{G})$-martingale hazard process Λ^i. Let us set $\Lambda = \sum_{i=1}^n \Lambda^i$, and let Y be a bounded $\mathcal{G}_s$-measurable random variable. Assume that the process V given by the formula*

$$V_t = \mathbb{E}\big(Ye^{\Lambda_t - \Lambda_s}|\mathcal{F}_t\big), \quad \forall\, t \in [0, s], \tag{4.51}$$

is continuous at τ. Then for any $t < s$ we have

$$\mathbb{E}(1\!\!1_{\{\tau > s\}}Y|\mathcal{G}_t) = 1\!\!1_{\{\tau > t\}}\mathbb{E}\big(Ye^{\Lambda_t - \Lambda_s}|\mathcal{F}_t\big). \tag{4.52}$$

In the case of absolutely continuous processes Λ^i we have

$$V_t = \mathbb{E}\big(Ye^{-\sum_{i=1}^n \int_t^s \lambda_u^i\,du}|\mathcal{F}_t\big), \tag{4.53}$$

and

$$\mathbb{E}(1\!\!1_{\{\tau > s\}}Y|\mathcal{G}_t) = 1\!\!1_{\{\tau > t\}}\mathbb{E}\big(Ye^{-\sum_{i=1}^n \int_t^s \lambda_u^i\,du}|\mathcal{F}_t\big). \tag{4.54}$$

At the first glance Proposition 4.5.3 seems to be a very useful and powerful result, since apparently it covers the case of independent and dependent random times. Notice, however, that the assumptions in Proposition 4.5.3 are rather restrictive: (i) any $\mathbf{F}$-martingale is a $\mathbf{G}$-martingale, (ii) each $(\mathbf{F}, \mathbf{G})$-martingale hazard process Λ^i is continuous. In addition, we deal here with a rather delicate issue of checking the continuity of V at τ. Therefore, the number of circumstances when Proposition 4.5.3 can be easily applied is in fact rather limited. One of them is examined in the foregoing example, in which random times τ_i are assumed conditionally independent given the filtration $\mathbf{F}$.

Example 4.5.1 Let ψ^1 and ψ^2 be two $\mathbf{F}$-progressively measurable non-negative stochastic processes defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{P})$, endowed with the filtration $\mathbf{F}$. We assume that

$$\int_0^\infty \psi_u^1\,du = \int_0^\infty \psi_u^2\,du = \infty$$

and we set

$$\tau_i = \inf\,\{\,t \in I\!\!R_+ \,:\, \int_0^t \psi_u^i\,du \geq -\ln \xi^i\,\}, \tag{4.55}$$

where ξ^1, ξ^2 are mutually independent random variables, defined on $(\Omega, \mathcal{G}, \mathbb{P})$, which are also independent of processes ψ^i, $i = 1, 2$, and are uniformly distributed on the unit interval $[0, 1]$. For each i, the enlarged filtration $\mathbf{G}^i := \mathbf{H}^i \vee \mathbf{F}$ thus satisfies $\mathcal{G}_t^i = \mathcal{F}_t \vee \mathcal{H}_t^i \subset \mathcal{F}_t \vee \sigma(\xi^i)$ for every t.

From Section **??** we know that the process $\Psi^i = \int_0^t \psi_u^i \, du$ represents the **F**-hazard process of τ_i. In particular, for any $\mathcal{F}_s$-measurable random variable Y we have for every $t \leq s$ (cf. (**??**))

$$\mathbb{E}(1\!\!1_{\{\tau_i > s\}} Y | \mathcal{G}_t^i) = 1\!\!1_{\{\tau_i > t\}} \mathbb{E}\big(Y e^{-\int_t^s \psi_u^i \, du} | \mathcal{F}_t\big). \tag{4.56}$$

In addition, the process Ψ^i is also the $(\mathbf{F}, \mathbf{G}^i)$-martingale hazard process of a random time τ_i. Finally, τ_i is a totally inaccessible stopping time with respect to $\mathbf{G}^i$, and the continuity condition of Corollary **??** is satisfied. Indeed, for any fixed $s > 0$, the process

$$V_t^i = \mathbb{E}\big(e^{\Psi_t^i - \Psi_s^i} | \mathcal{F}_t\big) = \mathbb{E}\big(e^{\Psi_t^i - \Psi_s^i} | \mathcal{G}_t^i\big), \quad \forall\, t \in [0, s],$$

is obviously continuous at τ_i. We conclude that for any $t \leq s$

$$\mathbb{P}(\tau_i > s | \mathcal{G}_t^i) = 1\!\!1_{\{\tau_i > t\}} \mathbb{E}\big(e^{-\int_t^s \psi_u^i \, du} | \mathcal{F}_t\big) = 1\!\!1_{\{\tau_i > t\}} \mathbb{E}\big(e^{-\int_t^s \psi_u^i \, du} | \mathcal{G}_t^i\big). \tag{4.57}$$

We introduce the filtration $\mathbf{G}$ by setting $\mathbf{G} = \mathbf{F} \vee \mathbf{H}^1 \vee \mathbf{H}^2$. Then τ_1, τ_2, as well as $\tau = \min(\tau_1, \tau_2)$ are $\mathbf{G}$-stopping times. It is not obvious, however, that the process Ψ^i is the $(\mathbf{F}, \mathbf{G})$-martingale hazard process of τ_i. We know that Ψ^i is a $\mathbf{G}$-adapted continuous process such that $\widetilde{M}_t^i = H_t^i - \Psi_{t \wedge \tau_i}^i$ is a $\mathbf{G}^i$-martingale. To conclude, we need to show that $\widetilde{M}^i$ is also a $\mathbf{G}$-martingale.

Let us consider, for instance, $i = 1$. The random variable $\widetilde{M}_t^1$ is manifestly $\mathcal{G}_t$-measurable. It is thus enough to check that for any $t \leq s$

$$\mathbb{E}(H_s^1 - \Psi_{s \wedge \tau_1}^1 | \mathcal{G}_t) = \mathbb{E}(H_s^1 - \Psi_{s \wedge \tau_1}^1 | \mathcal{G}_t^1).$$

Notice that the σ-fields $\mathcal{G}_s^1$ and $\mathcal{H}_t^2$ are conditionally independent given $\mathcal{G}_t^1$. Consequently,

$$\mathbb{E}(H_s^1 - \Psi_{s \wedge \tau_1}^1 | \mathcal{G}_t) = \mathbb{E}(H_s^1 - \Psi_{s \wedge \tau_1}^1 | \mathcal{G}_t^1 \vee \mathcal{H}_t^2) = \mathbb{E}(H_s^1 - \Psi_{s \wedge \tau_1}^1 | \mathcal{G}_t^1).$$

Since we have shown that Ψ^1 is the $(\mathbf{F}, \mathbf{G})$-martingale hazard process of τ_1, we have (under mild assumption on $\mathcal{G}_s$-measurable random variable Y)

$$\mathbb{E}(1\!\!1_{\{\tau_1 > s\}} Y | \mathcal{G}_t) = 1\!\!1_{\{\tau_1 > t\}} \mathbb{E}\big(Y e^{\Psi_t^1 - \Psi_s^1} | \mathcal{G}_t\big).$$

In particular, we have for any $t \leq s$ (cf. (4.57))

$$\mathbb{P}(\tau_1 > s | \mathcal{G}_t) = 1\!\!1_{\{\tau_1 > t\}} \mathbb{E}\big(e^{\Psi_t^1 - \Psi_s^1} | \mathcal{G}_t\big) = 1\!\!1_{\{\tau_i > t\}} \mathbb{E}\big(e^{-\int_t^s \psi_u^i \, du} | \mathcal{F}_t\big) \tag{4.58}$$

since the process

$$\widetilde{V}_t^1 := \mathbb{E}\big(e^{\Psi_t^1 - \Psi_s^1} | \mathcal{G}_t\big) = \mathbb{E}\big(e^{\Psi_t^1 - \Psi_s^1} | \mathcal{F}_t\big), \quad \forall\, t \in [0, s],$$

is continuous at τ_1.

In view of Lemma 4.5.5, the $(\mathbf{F}, \mathbf{G})$-martingale hazard process Ψ of τ, when stopped at τ, is the sum of $(\mathbf{F}, \mathbf{G})$-martingale hazard processes Ψ^i, $i = 1, 2$, associated with random times τ_i, $i = 1, 2$, also stopped at τ. We have for $t \leq s$

$$\mathbb{P}(\tau > s | \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} \mathbb{E}\big(e^{\Psi_t - \Psi_s} | \mathcal{G}_t\big) = 1\!\!1_{\{\tau > t\}} \mathbb{E}\big(e^{-\int_t^s (\psi_u^1 + \psi_u^2) \, du} | \mathcal{F}_t\big). \tag{4.59}$$

In should be stressed that the last formula is a consequence of the assumption that the underlying random variables ξ^1 and ξ^2 are independent. The case of dependent random variables ξ^1 and ξ^2 is much more involved; let us only observe that we cannot expect formula (4.59) to hold in this case. Indeed, it seems plausible that the $\mathbf{G}$-martingale hazard process of τ_1 will have a jump at τ_2 on the set $\{\tau_2 < \tau_1\}$, and conversely, the $\mathbf{G}$-martingale hazard process of τ_2 will be discontinuous at τ_1 on the set $\{\tau_1 < \tau_2\}$. Consequently, one may conjecture that the sum of these processes will have a discontinuity at τ, and thus it will not be possible to use the **G**-martingale hazard process of τ to directly represent the survival probability $\mathbb{P}(\tau > s | \mathcal{G}_t)$ through a counterpart of formula (4.59).

At the intuitive level, if the underlying random variables ξ^1 and ξ^2 are not independent, the observed occurence of τ_2 (τ_1, resp.) has a sudden impact on our assessments of the likelihood of the occurence of τ_1 (τ_2, resp.) in a given time interval in the future. A very special case of such a situation, when $\xi^1 = \xi^2$, was examined in Section 4.5.2. The general case remains, to our knowledge, an open problem.

Remark 4.5.1 Alternatively, we may check that Ψ^1 is also the $(\mathbf{G}^2, \mathbf{G})$-martingale hazard process of τ_1. Since Ψ^1 is a continuous $\mathbf{G}^2$-adapted process and $\mathbf{G} = \mathbf{G}^2 \vee \mathbf{H}^1$, it is enough to verify that Ψ^1 coincides with the $\mathbf{G}^2$-hazard process of τ_1, or equivalently, that

$$\mathbb{P}(\tau_1 > t | \mathcal{G}_t^2) = e^{-\Psi_t^1}, \quad \forall\, t \in I\!R_+.$$

The last equality is clear, however, since the σ-fields $\mathcal{G}_t^1$ and $\mathcal{H}_t^2$ are conditionally independent given $\mathcal{F}_t$, and thus (the event $\{\tau_1 > t\}$ belongs, of course, to $\mathcal{G}_t^1$)

$$\mathbb{P}(\tau_1 > t | \mathcal{G}_t^2) = \mathbb{P}(\tau_1 > t | \mathcal{F}_t \vee \mathcal{H}_t^2) = \mathbb{P}(\tau_1 > t | \mathcal{F}_t) = e^{-\Psi_t^1}.$$

Case of a Brownian Filtration

In this section, we consider once again the case of the Brownian filtration, that is, we assume that $\mathbf{F} = \mathbf{F}^W$ for some Brownian motion W. We postulate that W remains a martingale (and thus a Brownian motion) with respect to the enlarged filtration $\mathbf{G} = \mathbf{H}^1 \vee \ldots \vee \mathbf{H}^n \vee \mathbf{F}$. In view of the martingale representation property of the Brownian filtration this means, of course, that any **F**-local martingale follows also a local martingale with respect to **G** (or indeed with respect to any filtration $\mathbf{F} \subseteq \widetilde{\mathbf{F}} \subseteq \mathbf{G}$), so that (H) holds. It is worthwhile to stress that the case when **F** is a trivial filtration is also covered by the results of this section, however.

The first result is a generalization of the martingale representation property established in Corollary **??** (see also Proposition **??**). Recall that in Corollary **??** we have assumed that the **F**-hazard process Γ of a random time τ is an increasing continuous process. Also, by virtue of results of Section **??** (see Proposition **??**) under the assumptions of Corollary **??** we have $\Gamma = \Lambda$, that is, the **F**-hazard process Γ and the $(\mathbf{F}, \mathbf{G})$-martingale hazard process Λ coincide.

In the present setup, we find it convenient to make assumptions directly about the $(\mathbf{F}, \mathbf{G})$-martingale hazard processes Λ^i of random times τ_i, $i = 1, \ldots, n$. As before, we assume that $\mathbb{P}(\tau_i = \tau_j) = 0$ for $i \neq j$. Recall that by virtue of the definition of the $(\mathbf{F}, \mathbf{G})$-martingale hazard process Λ^i of a random time τ_i the process

$$\widetilde{M}_t^i = H_t^i - \Lambda_{t \wedge \tau_i}^i \tag{4.60}$$

follows a **G**-martingale. It is thus easily seen that the process

$$\widetilde{L}_t^i = (1 - H_t^i) e^{\Lambda_t^i}$$

also follows a **G**-martingale, since clearly (cf. (**??**) or (4.48))

$$\widetilde{L}_t^i = 1 - \int_{]0,t]} \widetilde{L}_{u-}^i \, d\widetilde{M}_u^i. \tag{4.61}$$

It is easily seen that $\widetilde{L}^i$ and $\widetilde{L}^j$ are mutually orthogonal **G**-martingales for any $i \neq j$ (a similar remark applies to $\widetilde{M}^i$ and $\widetilde{M}^j$).

For a fixed k with $0 \leq k \leq n$, we introduce the filtration $\widetilde{\mathbf{G}} = \mathbf{H}^1 \vee \ldots \vee \mathbf{H}^k \vee \mathbf{F}$. Then obviously $\widetilde{\mathbf{G}} = \mathbf{G}$ if $k = n$, and by convention $\widetilde{\mathbf{G}} = \mathbf{F}$ for $k = 0$. It is clear that for any fixed k and arbitrary

$i \leq k$ processes $\widetilde{L}^i$ and $\widetilde{M}^i$ are $\widetilde{\mathbf{G}}$-adapted. More specifically, $\widetilde{L}^i$ and $\widetilde{L}^j$ are mutually orthogonal $\widetilde{\mathbf{G}}$-martingales for $i, j \leq k$ provided that $i \neq j$.

A trivial modification of Lemma 4.5.5 shows that the $(\mathbf{F}, \widetilde{\mathbf{G}})$-martingale hazard process of the random time $\widetilde{\tau} := \tau_1 \wedge \ldots \wedge \tau_k$ equals $\widetilde{\Lambda} = \sum_{i=1}^k \Lambda^i$. In other words, the process $\widetilde{H}_t - \sum_{i=1}^k \Lambda^i_{t\wedge\widetilde{\tau}}$ is a $\widetilde{\mathbf{G}}$-martingale, where we set $\widetilde{H}_t = 1\!\!1_{\{\widetilde{\tau} \leq t\}}$.

Proposition 4.5.4 *Assume that the $\mathbf{F}$-Brownian motion W remains a Brownian motion with respect to the enlarged filtration $\mathbf{G}$. Let Y be a bounded $\mathcal{F}_T$-measurable random variable, and let $\widetilde{\tau} = \tau_1 \wedge \ldots \wedge \tau_k$. Then for any $t \leq s \leq T$ we have*

$$\mathbb{E}(1\!\!1_{\{\widetilde{\tau}>s\}} Y | \mathcal{G}_t) = \mathbb{E}(1\!\!1_{\{\widetilde{\tau}>s\}} Y | \widetilde{\mathcal{G}}_t) = 1\!\!1_{\{\widetilde{\tau}>t\}}\, \mathbb{E}\big(Y e^{\widetilde{\Lambda}_t - \widetilde{\Lambda}_s} | \mathcal{F}_t\big), \tag{4.62}$$

so that

$$\mathbb{P}(\widetilde{\tau} > s | \mathcal{G}_t) = \mathbb{P}(\widetilde{\tau} > s | \widetilde{\mathcal{G}}_t) = 1\!\!1_{\{\widetilde{\tau}>t\}} \mathbb{E}(e^{\widetilde{\Lambda}_t - \widetilde{\Lambda}_s} | \mathcal{F}_t). \tag{4.63}$$

In particular, for $\tau = \tau_1 \wedge \ldots \wedge \tau_n$ we have

$$\mathbb{P}(\tau > s | \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \mathbb{E}(e^{\Lambda_t - \Lambda_s} | \mathcal{F}_t), \tag{4.64}$$

where $\Lambda = \sum_{i=1}^n \Lambda^i$.

Proof: For a fixed $s \leq T$, we set

$$\widetilde{Y}_t = \mathbb{E}(Y e^{-\widetilde{\Lambda}_s} | \mathcal{F}_t), \quad \forall\, t \in [0, T].$$

Let the process U be given by the formula

$$U_t = (1 - \widetilde{H}_{t\wedge s}) e^{\widetilde{\Lambda}_{t\wedge s}} = \prod_{i=1}^k \widetilde{L}^i_{t\wedge s}, \quad \forall\, t \in [0, T]. \tag{4.65}$$

Under the present assumptions the process $\widetilde{Y}$ is a continuous $\mathbf{G}$-martingale and thus also a $\widetilde{\mathbf{G}}$-martingale. The process U, which is manifestly of finite variation, is also a $\widetilde{\mathbf{G}}$-martingale as a product of mutually orthogonal $\widetilde{\mathbf{G}}$-martingales $\widetilde{L}^1, \ldots, \widetilde{L}^k$ (stopped at s). Therefore, their product $U\widetilde{Y}$ is a $\widetilde{\mathbf{G}}$-martingale. Consequently, for any $t \leq s$

$$\mathbb{E}(1\!\!1_{\{\widetilde{\tau}>s\}} Y | \widetilde{\mathcal{G}}_t) = \mathbb{E}(U_T \widetilde{Y}_T | \widetilde{\mathcal{G}}_t) = U_t \widetilde{Y}_t = (1 - \widetilde{H}_t) e^{\widetilde{\Lambda}_t} \mathbb{E}(Y e^{-\widetilde{\Lambda}_s} | \mathcal{F}_t)$$

as expected. It is clear that we may replace $\widetilde{\mathbf{G}}$ by $\mathbf{G}$ in the reasoning above. □

Let us set $\widetilde{\mathbf{F}} := \mathbf{H}^{k+1} \vee \ldots \vee \mathbf{H}^n \vee \mathbf{F}$ for a fixed, but arbitrary, $k = 0, \ldots, n$. The next result generalizes Proposition **??**.

Proposition 4.5.5 *Assume that:*
(i) *the Brownian motion W remains a Brownian motion with respect to* $\mathbf{G}$,
(ii) *for each $i = 1, \ldots, n$ the $\mathbf{F}$-martingale hazard process Λ^i is continuous. Consider a $\widetilde{\mathbf{F}}$-martingale $M_t = E(X | \widetilde{\mathcal{F}}_t)$, $t \in [0, T]$, where X is a $\widetilde{\mathcal{F}}_T$-measurable random variable, integrable with respect to $\mathbb{P}$. Then M admits the following integral representation*

$$M_t = M_0 + \int_0^t \xi_u \, dW_u + \sum_{i=k+1}^n \int_{]0,t]} \zeta^i_u \, d\widetilde{M}^i_u, \tag{4.66}$$

where ξ and ζ^i, $i = k+1, \ldots, n$ are $\widetilde{\mathbf{F}}$-predictable processes.

Proof: The proof is similar to the proof of Proposition **??**. We start by noticing that it is enough to consider a random variable X of the form $X = Y\prod_{j=1}^{r}(1-H^{i_j}_{s_j})$ for some $r \le n-k$, where $0 < s_1 < \cdots < s_r \le T$, and $k+1 \le i_1 < \cdots < i_r \le n$. Finally, Y is assumed to be a $\mathcal{F}_T$-measurable integrable random variable. We introduce the $\mathbf{F}$-martingale

$$\widetilde{Y}_t = \mathbb{E}\Big(Y \exp\Big(\sum_{i=1}^{r}\Lambda^i_{s_i}\Big)\,\Big|\,\mathcal{F}_t\Big).$$

Since $\mathbf{F}$ is generated by a Brownian motion W, invoking the martingale representation property of the Brownian motion, we conclude that $\widetilde{Y}$ follows a continuous process that admits the integral representation

$$\widetilde{Y}_t = \widetilde{Y}_0 + \int_0^t \widetilde{\xi}_u\,dW_u, \quad \forall\, t\in[0,T],$$

for some $\mathbf{F}$-predictable process $\widetilde{\xi}$. Furthermore, W remains a martingale with respect to $\mathbf{G}$ and thus $\widetilde{Y}$ is also a $\mathbf{G}$-martingale. (Since $\widetilde{Y}$ is manifestly $\widetilde{\mathbf{F}}$-adapted, it follows also a martingale with respect to $\widetilde{\mathbf{F}}$.) Therefore, $\widetilde{Y}$ is orthogonal to each $\mathbf{G}$-martingale of finite variation $\widetilde{M}^i$. Using Itô's formula and (4.61), we obtain

$$Y\prod_{j=1}^{r}(1-H^{i_j}_{s_j}) = \widetilde{Y}_T\prod_{j=1}^{r}\widetilde{L}^{i_j}_{s_j} = \widetilde{Y}_0 + \int_0^T\prod_{j=1}^{r}\widetilde{L}^{i_j}_{(u\wedge s_j)-}\,d\widetilde{Y}_u + \sum_{l=1}^{r}\int_{]0,s_l]}\widetilde{Y}_{u-}\prod_{j=1}^{r}\widetilde{L}^{i_j}_{(u\wedge s_j)-}\,d\widetilde{M}^{i_l}_u.$$

The last formula leads to (4.66). □

Remark 4.5.2 If the random variable X is merely $\mathcal{G}_T$-measurable, we may still apply Proposition 4.5.5 to the $\widetilde{\mathbf{F}}$-martingale $M_t = E(X|\widetilde{\mathcal{F}}_t)$ since clearly $M_t = E(\widetilde{X}|\widetilde{\mathcal{F}}_t)$, where $\widetilde{X} := E(X|\widetilde{\mathcal{F}}_T)$ is a $\widetilde{\mathcal{F}}_T$-measurable random variable. This shows that representation (4.66) holds for any $\widetilde{\mathbf{F}}$-martingale.

It is also interesting to observe that we may in fact substitute in Proposition 4.5.4 the Brownian filtration $\mathbf{F}$ with the filtration $\widetilde{\mathbf{F}} := \mathbf{H}^{k+1}\vee\ldots\vee\mathbf{H}^n\vee\mathbf{F}$. First, it is clear that $\widetilde{\Lambda} = \sum_{i=1}^{k}\Lambda^i$ is also the $(\widetilde{\mathbf{F}},\mathbf{G})$-martingale hazard process of $\widetilde{\tau}$. Second, Proposition 4.5.5 shows that the process

$$\widehat{Y}_t := \mathbb{E}(Ye^{-\widetilde{\Lambda}_s}|\widetilde{\mathcal{F}}_t), \quad \forall\, t\in[0,T],$$

where Y is a $\widetilde{\mathcal{F}}_T$-measurable random variable, admits the following integral representation

$$\widehat{Y}_t = \widehat{Y}_0 + \int_0^t \xi_u\,dW_u + \sum_{i=k+1}^{n}\int_{]0,t]}\zeta^i_u\,d\widetilde{M}^i_u, \tag{4.67}$$

where ξ and ζ^i, $i = k+1,\ldots,n$ are $\widetilde{\mathbf{F}}$-predictable processes. Therefore, $\widehat{Y}$ is a $\mathbf{G}$-martingale orthogonal to the $\mathbf{G}$-martingale U given by (4.65). Arguing in a much the same way as in the proof of Proposition 4.5.4 we thus obtain the following result.

Corollary 4.5.1 *Let Y be a bounded $\widetilde{\mathcal{F}}_T$-measurable random variable. Let $\widetilde{\tau} = \tau_1\wedge\ldots\wedge\tau_k$. Then for any $t\le s\le T$ we have*

$$\mathbb{E}(\mathbb{1}_{\{\widetilde{\tau}>s\}}Y|\mathcal{G}_t) = \mathbb{1}_{\{\widetilde{\tau}>t\}}\,\mathbb{E}\big(Ye^{\widetilde{\Lambda}_t-\widetilde{\Lambda}_s}|\widetilde{\mathcal{F}}_t\big). \tag{4.68}$$

In particular, we have

$$\mathbb{P}(\widetilde{\tau}>s|\mathcal{G}_t) = \mathbb{1}_{\{\widetilde{\tau}>t\}}\mathbb{E}(e^{\widetilde{\Lambda}_t-\widetilde{\Lambda}_s}|\widetilde{\mathcal{F}}_t). \tag{4.69}$$

4.5.4 Change of a Probability Measure

In this section, in which we follow Kusuoka [140], we shall extend the results of Section **??** to the case of several random times. We preserve the assumptions of Section 4.5.3, in particular, the filtration $\mathbf{F}$ is generated by a Brownian motion W which is also a $\mathbf{G}$-martingale (the case of a trivial filtration $\mathbf{F}$ is also covered by the results of this section though).

For a fixed $T > 0$, we shall examine the properties of $\widetilde{\tau}$ under a probability measure $\mathbb{Q}$, which is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$. To this end, we introduce the associated $\mathbf{G}$-martingale η by setting, for $t \in [0, T]$,

$$\eta_t := \frac{d\mathbb{Q}}{d\mathbb{P}}_{\,|\mathcal{G}_t} = \mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t), \quad \mathbb{P}\text{-a.s.}, \tag{4.70}$$

where X is a $\mathcal{G}_T$-measurable random variable, integrable with respect to $\mathbb{P}$, and such that $\mathbb{P}(X > 0) = 1$. By virtue of Proposition 4.5.5 (with $k = 0$), the Radon-Nikodým density process η admits the following representation

$$\eta_t = 1 + \int_0^t \xi_u \, dW_u + \sum_{i=1}^n \int_{]0,t]} \zeta^i_u \, d\widetilde{M}^i_u, \tag{4.71}$$

where ξ and ζ^i, $i = 1, \ldots, n$ are $\mathbf{G}$-predictable stochastic processes. It can be shown that η is a strictly positive process, so that we may rewrite (4.71) as follows

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-} \big(\beta_u \, dW_u + \sum_{i=1}^n \kappa^i_u \, d\widetilde{M}^i_u \big), \tag{4.72}$$

where β and κ^i, $i = 1, \ldots, n$ are $\mathbf{G}$-predictable processes, with $\kappa^i > -1$. The following result extends Proposition 4.2.3 (its proof goes along exactly the same lines as the proof of Proposition 4.2.3 and thus it is left to the reader).

Proposition 4.5.6 *Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{G}_T)$ equivalent to $\mathbb{P}$. If the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by* (4.72) *then the process*

$$W^*_t = W_t - \int_0^t \beta_u \, du, \quad \forall \, t \in [0, T], \tag{4.73}$$

follows a $\mathbf{G}$-Brownian motion under $\mathbb{Q}$, and for each $i = 1, \ldots, n$ the process

$$M^{i*}_t := \widetilde{M}^i_t - \int_{]0,t\wedge\tau_i]} \kappa^i_u \, d\Lambda^i_u = H^i_t - \int_{]0,t\wedge\tau_i]} (1 + \kappa^i_u) \, d\Lambda^i_u, \quad \forall \, t \in [0, T], \tag{4.74}$$

is a $\mathbf{G}$-martingale orthogonal to W^* under $\mathbb{Q}$. Moreover, processes M^{i*} and M^{j*} follow mutually orthogonal $\mathbf{G}$-martingales under $\mathbb{Q}$ for any $i \neq j$.

Though the process M^{i*} follows a $\mathbf{G}$-martingale under $\mathbb{Q}$, it should be stressed that the process $\int_{]0,t]}(1+\kappa^i_u) \, d\Lambda^i_u$ is not necessarily the $(\mathbf{F}, \mathbf{G})$-martingale hazard process of τ_i under $\mathbb{Q}$, since it is not $\mathbf{F}$-adapted but merely $\mathbf{G}$-adapted, in general. This lack of measurability can be partially improved, however. For instance, for any fixed i we can choose a suitable version of the process κ^i, namely, a process κ^{i*} that coincides with κ^i on a random interval $[0, \tau_i]$, and such that κ^{i*} is a predictable process with respect to the enlarged filtration $\mathbf{F}^{i*} := \mathbf{H}^1 \vee \ldots \vee \mathbf{H}^{i-1} \vee \mathbf{H}^{i+1} \vee \ldots \mathbf{H}^n \vee \mathbf{F}$. It is obvious that the process

$$M^{i*}_t = H^i_t - \int_{]0,t\wedge\tau_i]} (1 + \kappa^{i*}_u) \, d\Lambda^i_u = H^i_t - \int_{]0,t\wedge\tau_i]} (1 + \kappa^i_u) \, d\Lambda^i_u,$$

follows a $\mathbf{G}$-martingale under $\mathbb{Q}$. We conclude that for each fixed i the process

$$\Lambda^{i*}_t = \int_{]0,t]} (1 + \kappa^{i*}_u) \, d\Lambda^i_u$$

represents the $(\mathbf{F}^{i*}, \mathbf{G})$-martingale hazard process of τ_i under $\mathbb{Q}$. This does not mean, however, that the following equality holds for $s \le t \le T$

$$\mathbb{Q}(\tau_i > s|\mathcal{G}_t) = 1\!\!1_{\{\tau_i>t\}}\mathbb{E}_{\mathbb{P}^*}(e^{\Lambda_t^{i*}-\Lambda_s^{i*}}|\mathcal{F}_t^{i*}).$$

We prefer to examine the last question in a slightly more general setting. For a fixed $k \le n$ let us consider the random time (Since the order of random times is not essential here, the analysis below covers also the case of a single random time τ_i for any $i = 1, \dots, n$.) $\widetilde{\tau} = \tau_1 \wedge \ldots \wedge \tau_k$. As in Section 4.5.3, we shall write $\widetilde{\mathbf{F}} = \mathbf{H}^{k+1} \vee \ldots \vee \mathbf{H}^n \vee \mathbf{F}$. For any $i = 1, \dots, n$ we denote by $\widetilde{\kappa}^i$ ($\widetilde{\beta}$, resp.) the $\widetilde{\mathbf{F}}$-predictable process such that $\widetilde{\kappa}^i = \kappa^i$ ($\widetilde{\beta} = \beta$, resp.) on the random set $[0, \widetilde{\tau}]$. Let us set

$$\widetilde{W}_t^* = W_t - \int_0^t \widetilde{\beta}_u \, du,$$

and

$$\widetilde{M}_t^{i*} = H_t^i - \int_{]0,t\wedge\tau_i]} (1 + \widetilde{\kappa}_u^i) \, d\Lambda_u^i$$

for $i = 1, \dots, n$. Notice that processes $\widetilde{W}^*$ and $\widetilde{M}^{i*}$ follow $\mathbf{G}$-martingales under $\mathbb{Q}$, provided that they are stopped at the random time $\widetilde{\tau}$ (since clearly $\widetilde{W}^*_{t\wedge\widetilde{\tau}} = W^*_{t\wedge\widetilde{\tau}}$ and $\widetilde{M}^{i*}_{t\wedge\widetilde{\tau}} = M^{i*}_{t\wedge\widetilde{\tau}}$). More importantly, the process

$$\widetilde{H}_t - \sum_{i=1}^k \int_{]0,t\wedge\widetilde{\tau}]} (1 + \widetilde{\kappa}_u^i) \, d\Lambda_u^i$$

also follows a $\mathbf{G}$-martingale. This shows that the $\widetilde{\mathbf{F}}$-predictable process Λ^* given by the formula

$$\Lambda_t^* = \sum_{i=1}^k \int_{]0,t]} (1 + \widetilde{\kappa}_u^i) \, d\Lambda_u^i \tag{4.75}$$

represents the $(\widetilde{\mathbf{F}}, \mathbf{G})$-martingale hazard process of the random time $\widetilde{\tau}$ under $\mathbb{Q}$. In view of Corollary 4.5.1, it would be tempting to conjecture that for any $t \le s \le T$ we have

$$\mathbb{E}_{\mathbb{P}^*}(1\!\!1_{\{\widetilde{\tau}>s\}}Y|\mathcal{G}_t) = 1\!\!1_{\{\widetilde{\tau}>t\}}\, \mathbb{E}_{\mathbb{P}^*}\big(Ye^{\Lambda_t^*-\Lambda_s^*}|\widetilde{\mathcal{F}}_t\big), \tag{4.76}$$

where Y is a bounded $\widetilde{\mathcal{F}}_T$-measurable random variable. It appears that in order to make the last formula true, we need to substitute the probability measure $\mathbb{Q}$ in the right-hand side of (4.76) with a related probability measure. To this end, we introduce the following auxiliary density processes $\widehat{\eta}^\ell$ for $\ell = 1, 2, 3$

$$\widehat{\eta}_t^1 = 1 + \int_{]0,t]} \widehat{\eta}_{u-}^1 \big(\widetilde{\beta}_u \, dW_u + \sum_{i=k+1}^n \widetilde{\kappa}_u^i \, d\widetilde{M}_u^i\big), \tag{4.77}$$

$$\widehat{\eta}_t^2 = 1 + \int_{]0,t]} \widehat{\eta}_{u-}^2 \big(\widetilde{\beta}_u \, dW_u + \sum_{i=1}^n \widetilde{\kappa}_u^i \, d\widetilde{M}_u^i\big), \tag{4.78}$$

and

$$\widehat{\eta}_t^3 = 1 + \int_{]0,t]} \widehat{\eta}_{u-}^3 \big(\widetilde{\beta}_u \, dW_u + \sum_{i=1}^k \kappa_u^i \, d\widetilde{M}_u^i + \sum_{i=k+1}^n \widetilde{\kappa}_u^i \, d\widetilde{M}_u^i\big). \tag{4.79}$$

It is useful to observe that the process $\widehat{\eta}^1$ is $\widetilde{\mathbf{F}}$-adapted (since, in particular, each process $\widetilde{M}^i$ is adapted to the filtration $\mathbf{H}^i \vee \mathbf{F}$). On the other hand, processes $\widehat{\eta}^2$ and $\widehat{\eta}^3$ are merely $\mathbf{G}$-adapted, but not necessarily $\widetilde{\mathbf{F}}$-adapted, in general. We find in convenient to introduce the $\widetilde{\mathbf{F}}$-adapted modifications $\widetilde{\eta}^2$, $\widetilde{\eta}^3$ of $\widehat{\eta}^2$, $\widehat{\eta}^3$ by setting $\widetilde{\eta}_t^\ell = \mathbb{E}(\widehat{\eta}_T^\ell|\widetilde{\mathcal{F}}_t)$ for $t \le T$, $\ell = 2, 3$. ¿From the uniqueness of the martingale representation property established in Proposition 4.5.5 we deduce that for $\ell = 2,3$ we have (for $\ell = 1$, (4.80) simply coincides with (4.77))

$$\widetilde{\eta}_t^\ell = 1 + \int_{]0,t]} \widetilde{\eta}_{u-}^\ell \big(\widetilde{\beta}_u \, dW_u + \sum_{i=k+1}^{n} \widetilde{\kappa}_u^i \, d\widetilde{M}_u^i\big). \tag{4.80}$$

We define a probability measure $\widetilde{\mathbb{P}}_\ell$ on $(\Omega, \mathcal{G}_T)$ by setting, for $\ell = 1,2,3$

$$\widehat{\eta}_t^\ell := \frac{d\widetilde{\mathbb{P}}_\ell}{d\mathbb{P}}_{|\mathcal{G}_t}, \quad \mathbb{P}\text{-a.s.}, \tag{4.81}$$

for $t \in [0,T]$. It is thus clear that

$$\widetilde{\eta}_t^\ell = \mathbb{E}(\eta_T^\ell | \widetilde{\mathcal{F}}_t) = \frac{d\widetilde{\mathbb{P}}_\ell}{d\mathbb{P}}_{|\widetilde{\mathcal{F}}_t}, \quad \mathbb{P}\text{-a.s.} \tag{4.82}$$

The following result, due to Kusuoka [140], is a counterpart of Corollary 4.5.1.

Proposition 4.5.7 *Let $\ell \in \{1,2,3\}$ and let Y be a bounded $\widetilde{\mathcal{F}}_T$-measurable random variable. Then for any $t \le s \le T$ we have*

$$\mathbb{E}_{\mathbb{P}^*}(\mathbb{1}_{\{\widetilde{\tau}>s\}} Y | \mathcal{G}_t) = \mathbb{1}_{\{\widetilde{\tau}>t\}} \, \mathbb{E}_{\widetilde{\mathbb{P}}_\ell}\big(Y e^{\Lambda_t^* - \Lambda_s^*} | \widetilde{\mathcal{F}}_t\big), \tag{4.83}$$

where the process Λ^* is given by formula (4.75). *In particular, we have*

$$\mathbb{Q}(\widetilde{\tau} > s | \mathcal{G}_t) = \mathbb{1}_{\{\widetilde{\tau}>t\}} \mathbb{E}_{\widetilde{\mathbb{P}}_\ell}(e^{\Lambda_t^* - \Lambda_s^*} | \widetilde{\mathcal{F}}_t). \tag{4.84}$$

The proof of Proposition 4.5.7 parallels the demonstration of Proposition 4.5.4 (see also remarks preceding Corollary 4.5.1). We need some preliminary results, however. First, we shall establish a counterpart of the integral representation (4.66).

Lemma 4.5.7 *Let Y be a $\widetilde{\mathbf{F}}$-martingale under $\widetilde{\mathbb{P}}_\ell$ for some $\ell \in \{1,2,3\}$. Then there exist $\widetilde{\mathbf{F}}$-predictable processes $\widetilde{\xi}$ and $\widetilde{\zeta}^i$, $i = k+1, \dots, n$, such that*

$$Y_t = Y_0 + \int_0^t \widetilde{\xi}_u \, d\widetilde{W}_u^* + \sum_{i=k+1}^{n} \int_{]0,t]} \widetilde{\zeta}_u^i \, d\widetilde{M}_u^{i*}. \tag{4.85}$$

Proof: The proof combines the calculations already employed in the proof of Corollary 4.2.1 with the martingale representation property under $\mathbb{P}$ established in Proposition 4.5.5. We fix ℓ, and we write $\widetilde{\eta}_t = \mathbb{E}(\eta_T^\ell | \widetilde{\mathcal{F}}_t)$ (of course, $\widetilde{\eta}_t = \widehat{\eta}_t^1$ if we take $\ell = 1$). We introduce an auxiliary process $\widetilde{Y}$, which follows a $\widetilde{\mathbf{F}}$-martingale under $\mathbb{P}$, by setting

$$\widetilde{Y}_t = \int_{]0,t]} \widetilde{\eta}_{u-}^{-1} \, d(\widetilde{\eta}_u Y_u) - \int_{]0,t]} \widetilde{\eta}_{u-}^{-1} Y_{u-} \, d\widetilde{\eta}_u$$

for $t \in [0,T]$. Since Itô's formula yields

$$\widetilde{\eta}_{u-}^{-1} \, d(\widetilde{\eta}_u Y_u) = dY_u + \widetilde{\eta}_{u-}^{-1} Y_{u-} \, d\widetilde{\eta}_u + \widetilde{\eta}_{u-}^{-1} \, d[Y, \widetilde{\eta}]_u,$$

the process Y admits the following useful representation

$$Y_t = Y_0 + \widetilde{Y}_t - \int_{]0,t]} \widetilde{\eta}_{u-}^{-1} \, d[Y, \widetilde{\eta}]_u. \tag{4.86}$$

On the other hand, invoking Proposition 4.5.5, we obtain the following integral representation for the process $\widetilde{Y}$

$$\widetilde{Y}_t = \int_0^t \xi_u \, dW_u + \sum_{i=k+1}^n \int_{]0,t]} \zeta_u^i \, d\widetilde{M}_u^i,$$

where ξ and ζ^i, $i = 1, \ldots, k$ are $\widetilde{\mathbf{F}}$-predictable processes. Consequently, we have

$$\begin{aligned} dY_t &= \xi_t \, dW_t + \sum_{i=k+1}^n \zeta_t^i \, d\widetilde{M}_t^i - \widetilde{\eta}_{t-}^{-1} \, d[Y, \widetilde{\eta}]_t \\ &= \xi_t \, d\widetilde{W}_t^* + \sum_{i=k+1}^n \zeta_t^i (1 + \widetilde{\kappa}_t^i)^{-1} \, d\widetilde{M}_t^{i*}. \end{aligned}$$

To establish the last equality, notice that (4.80) combined with (4.74) yield

$$\widetilde{\eta}_{t-}^{-1} \, d[Y, \widetilde{\eta}]_t = \xi_t \widetilde{\beta}_t \, dt + \sum_{i=k+1}^n \zeta_t^i \widetilde{\kappa}_t^i (1 + \widetilde{\kappa}_t^i)^{-1} \, dH_t^i,$$

where the last equality follows in turn from the following relationship

$$\Delta[Y, \widetilde{\eta}]_t = \widetilde{\eta}_{t-} \sum_{i=k+1}^n \big(\zeta_t^i \widetilde{\kappa}_t^i \, dH_t^i - \widetilde{\kappa}_t^i \Delta[Y, \widetilde{\eta}]_t \big).$$

We conclude that Y satisfies (4.12) with $\widetilde{\xi} = \xi$ and $\widetilde{\zeta}^i = \zeta^i (1 + \widetilde{\kappa}^i)^{-1}$ for $i = k+1, \ldots, n$. □

Corollary 4.5.2 *Let Y be a bounded $\widetilde{\mathcal{F}}_T$-measurable random variable. For a fixed $s \leq T$ we define the process $\widehat{Y}$ by setting*

$$\widehat{Y}_t = \mathbb{E}_{\widetilde{\mathbb{P}}_\ell} (Y e^{-\Lambda_s^*} | \widetilde{\mathcal{F}}_t), \quad \forall \, t \in [0, T]. \tag{4.87}$$

The process $\widehat{Y}$ admits the following integral representation under $\widetilde{\mathbb{P}}_\ell$

$$\widehat{Y}_t = \widehat{Y}_0 + \int_0^t \widehat{\xi}_u \, d\widetilde{W}_u^* + \sum_{i=k+1}^n \int_{]0,t]} \widehat{\zeta}_u^i \, d\widetilde{M}_u^{i*}, \tag{4.88}$$

where $\widehat{\xi}$ and $\widehat{\zeta}^i$, $i = k+1, \ldots, n$ are $\widetilde{\mathbf{F}}$-predictable processes. The stopped process $\widehat{Y}_{t \wedge \widetilde{\tau}}$ follows a $\mathbf{G}$-martingale orthogonal under $\mathbb{Q}$ to $\mathbf{G}$-martingales M^{i*}, $i = 1, \ldots, k$.

Proof: It is enough to apply Lemma 4.5.7 and to notice that the stopped process $\widehat{Y}_{t \wedge \widetilde{\tau}}$ satisfies

$$\widehat{Y}_{t \wedge \widetilde{\tau}} = \widehat{Y}_0 + \int_0^t \widehat{\xi}_u \, dW_{u \wedge \widetilde{\tau}}^* + \sum_{i=k+1}^n \int_{]0,t]} \widehat{\zeta}_u^i \, dM_{u \wedge \widetilde{\tau}}^{i*},$$

and to recall that $\widetilde{W}_{t \wedge \widetilde{\tau}}^* = W_{t \wedge \widetilde{\tau}}^*$ and $\widetilde{M}_{t \wedge \widetilde{\tau}}^{i*} = M_{t \wedge \widetilde{\tau}}^{i*}$. □

We are in the position to establish Proposition 4.5.7.

Proof of Proposition 4.5.7. For a fixed $s \leq T$, let $\widehat{Y}$ be the process defined through formula (4.87). Furthermore, let U be the process given by the expression (notice that the process U is stopped at $\widetilde{\tau} \wedge s$)

$$\begin{aligned} U_t &= (1 - \widetilde{H}_{t \wedge s}) e^{\Lambda_{t \wedge s}^*} = (1 - \widetilde{H}_{t \wedge s}) \prod_{i=1}^k e^{\int_0^{t \wedge s} (1 + \widetilde{\kappa}_u^i) d\Lambda_u^i} \\ &= \prod_{i=1}^k (1 - H_{t \wedge s}^i) e^{\int_0^t (1 + \kappa_u^i) d\Lambda_u^i} = \prod_{i=1}^k L_{t \wedge s}^{i*}, \end{aligned}$$

where

$$L_t^{i*} = (1 - H_t^i)e^{\int_0^t (1+\kappa_u^i)\,d\Lambda_u^i}$$

so that (cf. (4.61))

$$L_t^{i*} = 1 - \int_{]0,t]} L_{u-}^{i*}\, dM_u^{i*}. \tag{4.89}$$

In view of (4.88), the above representation of the process U and (4.89), the processes U and $\widehat{Y}_{t\wedge\widetilde{\tau}}$ are mutually orthogonal $\mathbf{G}$-martingales under $\mathbb{Q}$. Therefore,

$$\mathbb{E}_{\mathbb{P}^*}(\mathbb{1}_{\{\widetilde{\tau}>s\}}Y|\mathcal{G}_t) = \mathbb{E}_{\mathbb{P}^*}(U_T\widehat{Y}_T|\mathcal{G}_t) = U_t\widehat{Y}_t = (1-\widetilde{H}_t)e^{\Lambda_t^*}\mathbb{E}_{\widetilde{\mathbb{P}}_\ell}(Ye^{-\Lambda_s^*}|\widetilde{\mathcal{F}}_t).$$

The last expression yields the asserted formulae (4.83)-(4.84). □

Remark 4.5.3 It is also possible to consider the filtration generated by the random time $\widetilde{\tau}$, rather then the filtration $\mathbf{H}^1\vee\ldots\vee\mathbf{H}^k$. Consequently, instead of the filtration $\mathbf{G}$ we would have $\widehat{\mathbf{G}} = \widetilde{\mathbf{H}}\vee\widetilde{\mathbf{F}}$. Since the stopped process $\widetilde{H}_t - \Lambda^*_{t\wedge\widetilde{\tau}}$ (as usual, $\widetilde{H}_t = \mathbb{1}_{\{t\le\widetilde{\tau}\}}$) is a $\mathbf{G}$-martingale, and it is manifestly a $\widetilde{\mathbf{G}}$-adapted process, it follows also a $\widetilde{\mathbf{G}}$-martingale. Let us consider the following property: any $\widetilde{\mathbf{F}}$-martingale remains a $\mathbf{G}$-martingale (or a $\widehat{\mathbf{G}}$-martingale). It seems plausible to conjecture that this property is not valid under $\mathbb{Q}$, in general, but it holds under $\widetilde{\mathbb{P}}_\ell$ for $\ell = 1,2,3$.

4.5.5 Kusuoka's Example

The following example, borrowed from Kusuoka [140], shows that formula (4.76) does not hold, in general. We assume that under the original probability measure $\mathbb{P}$ the random times τ_i, $i = 1,2$ are mutually independent random variables, with exponential laws with the parameters λ_1 and λ_2, respectively. The joint probability law of the pair (τ_1,τ_2) under $\mathbb{P}$ admits the density function $f(x,y) = \lambda_1\lambda_2 e^{-(\lambda_1 x+\lambda_2 y)}$ for $(x,y)\in I\!R^2_+$. Our goal is to examine these random times under a specific equivalent change of probability measure $\mathbb{Q}$. In words, under $\mathbb{Q}$ the original intensity λ_1 of the random time τ_1 jumps to some fixed value α_1 after the occurence of τ_2 (the behaviour of the intensity of τ_2 is analogous). Such a specification of the stochastic intensity of dependent random times appears in a natural way in certain practical applications related to the valuation of defaultable claims. Notice that the filtration $\mathbf{F}$ is here assumed to be a trivial filtration.

We shall now formally define the probability measure $\mathbb{Q}$. Let α_1 and α_2 be positive real numbers. For a fixed $T>0$, we introduce an equivalent probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{G})$ by setting

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \eta_T, \quad \mathbb{P}\text{-a.s.}, \tag{4.90}$$

where $\eta_t,\ t\in[0,T]$, satisfies

$$\eta_t = 1 + \sum_{i=1}^2 \int_{]0,t]} \eta_{u-}\kappa_u^i\, d\widetilde{M}_u^i, \tag{4.91}$$

where

$$\kappa_t^1 = \mathbb{1}_{\{\tau_2<t\}}\Big(\frac{\alpha_1}{\lambda_1}-1\Big), \quad \kappa_t^2 = \mathbb{1}_{\{\tau_1<t\}}\Big(\frac{\alpha_2}{\lambda_2}-1\Big).$$

It is useful to notice that $\eta_T = \eta_T^1\eta_T^2$, where for every $t\in[0,T]$

$$\eta_t^i = 1 + \int_{]0,t]} \eta_{u-}^i\kappa_u^i\, d\widetilde{M}_u^i, \tag{4.92}$$

for $i=1,2$, or more explicitly, for every $t\in[0,T]$

$$\eta_t^1 = \mathbb{1}_{\{\tau_1\le\tau_2\}} + \mathbb{1}_{\{\tau_2\le t<\tau_1\}}e^{-(\alpha_1-\lambda_1)(t-\tau_2)} + \mathbb{1}_{\{\tau_2<\tau_1\le t\}}\frac{\alpha_1}{\lambda_1}e^{-(\alpha_1-\lambda_1)(\tau_1-\tau_2)}, \tag{4.93}$$

$$\eta_t^2 = \mathbb{1}_{\{\tau_2 \le \tau_1\}} + \mathbb{1}_{\{\tau_1 \le t < \tau_2\}} e^{-(\alpha_2-\lambda_2)(t-\tau_1)} + \mathbb{1}_{\{\tau_1 < \tau_2 \le t\}} \frac{\alpha_2}{\lambda_2} e^{-(\alpha_2-\lambda_2)(\tau_2-\tau_1)}. \tag{4.94}$$

It is obvious that the process κ^1 (κ^2, resp.) is $\mathbf{H}^2$-predictable ($\mathbf{H}^1$-predictable, resp.) Then $\Lambda_t^{i*} = \int_0^t \lambda_u^{i*}\,du$, where the processes λ^{i*}, $i = 1, 2$ are given by the formulae

$$\lambda_t^{*1} = \lambda_1(1 - H_t^2) + \alpha_1 H_t^2 = \lambda_1 \mathbb{1}_{\{\tau_2 > t\}} + \alpha_1 \mathbb{1}_{\{\tau_2 \le t\}},$$

and

$$\lambda_t^{*2} = \lambda_2(1 - H_t^1) + \alpha_2 H_t^1 = \lambda_2 \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \le t\}}.$$

This means that the processes

$$H_t^1 - \int_0^{t\wedge\tau_1} \big(\lambda_1 \mathbb{1}_{\{\tau_2 > u\}} + \alpha_1 \mathbb{1}_{\{\tau_2 \le u\}}\big)\,du = H_t^1 - \lambda_1(t \wedge \tau_1 \wedge \tau_2) - \alpha_1((t\wedge\tau_1) \vee \tau_2 - \tau_2)$$

and

$$H_t^2 - \int_0^{t\wedge\tau_2} \big(\lambda_2 \mathbb{1}_{\{\tau_1 > u\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \le u\}}\big)\,du = H_t^2 - \lambda_2(t \wedge \tau_1 \wedge \tau_2) - \alpha_2((t\wedge\tau_2) \vee \tau_1 - \tau_1)$$

are $\mathbb{Q}$-martingales with respect to the joint filtration $\mathbf{G} = \mathbf{H}^1 \vee \mathbf{H}^2$.

In view of the assumed symmetry, it is enough to consider the random time $\widetilde{\tau} = \tau_1$ (i.e., we have $n = 2$ and $k = 1$). Notice that in the present setup we have $\widetilde{\kappa}_t^2 = 0$ since obviously $\kappa_t^2 = 0$ on the random interval $[0, \tau_1]$. Therefore, the probability measure $\widetilde{\mathbb{P}}_1$ given by formulae (4.77)-(4.81) coincides with the original probability measure $\mathbb{P}$. It is useful to notice that κ^1 is $\mathbf{H}^1$-predictable so that $\widetilde{\kappa}_t^1 = \kappa_t^1$ for every t. Consequently, the probability measures $\widetilde{\mho P}_2$ and $\widetilde{\mathbb{P}}_3$ coincide with the probability measure $\mathbb{P}_1^*$ given by formulae (4.92) and (4.100) below.

Our aim is to evaluate directly the conditional expectation $\mathbb{Q}(\tau_1 > s | \mathcal{H}_t^1 \vee \mathcal{H}_t^2)$ for $t \le s \le T$, and subsequently to verify that

$$\mathbb{Q}(\tau_1 > s | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) = \mathbb{1}_{\{\tau_1 > t\}} \mathbb{E}_{\widetilde{\mathbb{P}}_1}(e^{\Lambda_t^{i*} - \Lambda_s^{i*}} | \mathcal{H}_t^2) = \mathbb{1}_{\{\tau_1 > t\}} \mathbb{E}_{\mathbb{P}}(e^{\Lambda_t^{i*} - \Lambda_s^{i*}} | \mathcal{H}_t^2),$$

where the second equality is an obvious consequence of the equality $\widetilde{\mathbb{P}}_1 = \mathbb{P}$. Finally, we shall check that, in general,

$$\mathbb{Q}(\tau_1 > s | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) \neq \mathbb{1}_{\{\tau_1 > t\}} \mathbb{E}_{\mathbb{P}^*}(e^{\Lambda_t^{i*} - \Lambda_s^{i*}} | \mathcal{H}_t^2).$$

Unconditional Law of τ_1 under $\mathbb{Q}$

We find it convenient to derive first the unconditional law of τ_1 under $\mathbb{Q}$. In view of (4.93)-(4.94), the marginal density $f_{\tau_1}^*$ of the random time τ_1 under $\mathbb{Q}$ equals

For simplicity of exposition, we shall assume that $\lambda_1 + \lambda_2 - \alpha_1 \neq 0$ and $\lambda_1 + \lambda_2 - \alpha_2 \neq 0$.

$$\begin{aligned} f_{\tau_1}^*(t) &= \int_0^t \lambda_1\lambda_2 \frac{\alpha_1}{\lambda_1} e^{-(\alpha_1-\lambda_1)(t-y)} e^{-(\lambda_1 t + \lambda_2 y)}\,dy \\ &\quad + \int_t^T \lambda_1\lambda_2 \frac{\alpha_2}{\lambda_2} e^{-(\alpha_2-\lambda_2)(y-t)} e^{-(\lambda_1 t + \lambda_2 y)}\,dy \\ &\quad + \int_t^\infty \lambda_1\lambda_2 e^{-(\alpha_2-\lambda_2)(T-t)} e^{-(\lambda_1 t + \lambda_2 y)}\,dy \\ &= \frac{1}{\lambda_1+\lambda_2-\alpha_1}\Big(\alpha_1\lambda_2 e^{-\alpha_1 t} + (\lambda_1-\alpha_1)(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}\Big) \end{aligned}$$

for every $t \le T$. For $t > T$, we have

$$\begin{aligned} f_{\tau_1}^*(t) &= \int_0^T \lambda_1\lambda_2 e^{-(\alpha_1-\lambda_1)(T-y)} e^{-(\lambda_1 t + \lambda_2 y)}\,dy + \int_T^\infty \lambda_1\lambda_2 e^{-(\lambda_1 t + \lambda_2 y)}\,dy \\ &= \frac{\lambda_1 e^{-\lambda_1 t}}{\lambda_1+\lambda_2-\alpha_1}\Big(\lambda_2 e^{-(\alpha_1-\lambda_1)T} + (\lambda_1-\alpha_1)e^{-\lambda_2 T}\Big). \end{aligned}$$

Consequently, for any $s \in [0,T]$, we get

$$\mathbb{Q}(\tau_1 > s) = \frac{1}{\lambda_1 + \lambda_2 - \alpha_1}\Big(\lambda_2 e^{-\alpha_1 s} + (\lambda_1 - \alpha_1)e^{-(\lambda_1+\lambda_2)s}\Big). \tag{4.95}$$

For $s > T$, we have

$$\mathbb{Q}(\tau_1 > s) = \frac{e^{-\lambda_1 s}}{\lambda_1 + \lambda_2 - \alpha_1}\Big(\lambda_2 e^{-(\alpha_1-\lambda_1)T} + (\lambda_1 - \alpha_1)e^{-\lambda_2 T}\Big). \tag{4.96}$$

Conditional Law of τ_1 under $\mathbb{Q}$

Our next goal is to derive an explicit formula for the conditional probability $I := \mathbb{Q}(\tau_1 > s | \mathcal{H}^1_t \vee \mathcal{H}^2_t)$.

Lemma 4.5.8 *For every $t \le s \le T$ we have*

$$I = 1\!\!1_{\{\tau_1 > t, \tau_2 > t\}} \frac{1}{\lambda_1 + \lambda_2 - \alpha_1}\Big(\lambda_2 e^{-\alpha_1(s-t)} + (\lambda_1 - \alpha_1)e^{-(\lambda_1+\lambda_2)(s-t)}\Big) + 1\!\!1_{\{\tau_2 \le t < \tau_1\}}\, e^{-\alpha_1(s-t)}.$$

Proof: In view of results of Section **??**, for arbitrary $t \le s$ we have

$$I = \mathbb{Q}(\tau_1 > s | \mathcal{H}^1_t \vee \mathcal{H}^2_t) = (1 - H^1_t)\, \frac{\mathbb{Q}(\tau_1 > s | \mathcal{H}^2_t)}{\mathbb{Q}(\tau_1 > t | \mathcal{H}^2_t)}, \tag{4.97}$$

where in turn

$$\mathbb{Q}(\tau_1 > s | \mathcal{H}^2_t) = (1 - H^2_t)\, \frac{\mathbb{Q}(\tau_1 > s, \tau_2 > t)}{\mathbb{Q}(\tau_2 > t)} + H^2_t\, \mathbb{Q}(\tau_1 > s | \tau_2). \tag{4.98}$$

Combining (4.97) with (4.98), we obtain

$$I = (1 - H^1_t)(1 - H^2_t)\, \frac{\mathbb{Q}(\tau_1 > s, \tau_2 > t)}{\mathbb{Q}(\tau_1 > t, \tau_2 > t)} + (1 - H^1_t) H^2_t\, \frac{\mathbb{Q}(\tau_1 > s | \tau_2)}{\mathbb{Q}(\tau_1 > t | \tau_2)},$$

or more explicitly,

$$I = 1\!\!1_{\{\tau_1 > t, \tau_2 > t\}} \frac{\mathbb{Q}(\tau_1 > s, \tau_2 > t)}{\mathbb{Q}(\tau_1 > t, \tau_2 > t)} + 1\!\!1_{\{\tau_2 \le t < \tau_1\}} \frac{\mathbb{Q}(\tau_1 > s | \tau_2)}{\mathbb{Q}(\tau_1 > t | \tau_2)} = 1\!\!1_{\{\tau_1 > t, \tau_2 > t\}} I_1 + 1\!\!1_{\{\tau_2 \le t < \tau_1\}} I_2.$$

In order to evaluate I_1, observe first that

$$\begin{aligned}
\mathbb{Q}(\tau_1 > s, \tau_2 \le t) &= \int_s^T \int_0^t \lambda_1 \lambda_2 \frac{\alpha_1}{\lambda_1} e^{-(\alpha_1 - \lambda_1)(x-y)} e^{-(\lambda_1 x + \lambda_2 y)}\, dx dy \\
&\quad + \int_T^\infty \int_0^t \lambda_1 \lambda_2 e^{-(\alpha_1 - \lambda_1)(T-y)} e^{-(\lambda_1 x + \lambda_2 y)}\, dx dy \\
&= \frac{\lambda_2 e^{-\alpha_1 s}}{\lambda_1 + \lambda_2 - \alpha_1}\Big(1 - e^{-(\lambda_1 + \lambda_2 - \alpha_1)t}\Big).
\end{aligned}$$

Combining the last formula with (4.95), we obtain

$$\begin{aligned}
\mathbb{Q}(\tau_1 > s, \tau_2 > t) &= \mathbb{Q}(\tau_1 > s) - \mathbb{Q}(\tau_1 > s, \tau_2 \le t) \\
&= \frac{\lambda_1 - \alpha_1}{\lambda_1 + \lambda_2 - \alpha_1} e^{-(\lambda_1+\lambda_2)s} + \frac{\lambda_2}{\lambda_1 + \lambda_2 - \alpha_1} e^{-\alpha_1 s - (\lambda_1 + \lambda_2 - \alpha_1)t}.
\end{aligned}$$

We conclude that

$$I_1 = \frac{\mathbb{Q}(\tau_1 > s, \tau_2 > t)}{\mathbb{Q}(\tau_1 > t, \tau_2 > t)} = \frac{1}{\lambda_1 + \lambda_2 - \alpha_1}\Big(\lambda_2 e^{-\alpha_1(s-t)} + (\lambda_1 - \alpha_1)e^{-(\lambda_1+\lambda_2)(s-t)}\Big).$$

It remains to evaluate I_2. To this end, it is enough to check that for $t \le s \le T$ we have

$$I_3 = \mathbb{1}_{\{\tau_2 \le t\}} \mathbb{Q}(\tau_1 > s | \tau_2) = \mathbb{1}_{\{\tau_2 \le t\}} \frac{(\lambda_1 + \lambda_2 - \alpha_2)\lambda_2 e^{-\alpha_1(s-\tau_2)}}{\lambda_1 \alpha_2 e^{(\lambda_1+\lambda_2-\alpha_2)\tau_2} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)}. \tag{4.99}$$

Indeed, the last formula would thus yield immediately

$$\mathbb{1}_{\{\tau_2 \le t < \tau_1\}} \frac{\mathbb{Q}(\tau_1 > s|\tau_2)}{\mathbb{Q}(\tau_1 > t|\tau_2)} = \mathbb{1}_{\{\tau_2 \le t < \tau_1\}}\, e^{-\alpha_1(s-t)},$$

the desired result. To evaluate I_3, we may, for instance, notice that for any $u \le s$

$$\begin{aligned}
\mathbb{Q}(\tau_1 > s|\tau_2 = u) &= \frac{1}{f^*_{\tau_2}(u)} \Big(\int_s^T \frac{\alpha_1}{\lambda_1} e^{-(\alpha_1-\lambda_1)(x-u)} f(x,u)\,dx + \int_T^\infty e^{-(\alpha_1-\lambda_1)(T-u)} f(x,u)\,dx \Big) \\
&= \frac{(\lambda_1+\lambda_2-\alpha_2)\lambda_2 e^{-(\lambda_1+\lambda_2-\alpha_1)u} e^{-\alpha_1 s}}{\lambda_1\alpha_2 e^{-\alpha_2 u} + (\lambda_2-\alpha_2)(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)u}},
\end{aligned}$$

which gives (4.99) upon simplification. An alternative, though somewhat lengthy, way do to the calculations for I_3 would be to use directly the Bayes formula

$$\mathbb{1}_{\{\tau_2 \le t\}} \mathbb{Q}(\tau_1 > s|\tau_2) = \mathbb{1}_{\{\tau_2 \le t\}} \frac{\mathbb{E}_{\mathbb{P}}(\eta_s \mathbb{1}_{\{\tau_1 > s\}}|\tau_2)}{\mathbb{E}_{\mathbb{P}}(\eta_s|\tau_2)}$$

and to check that for arbitrary $t \le s \le T$ we have

$$\mathbb{1}_{\{\tau_2 \le t\}} \mathbb{E}_{\mathbb{P}}(\eta_s \mathbb{1}_{\{\tau_1 > s\}}|\tau_2) = \mathbb{1}_{\{\tau_2 \le t\}} e^{-\alpha_1 s} e^{-(\lambda_1-\alpha_1)\tau_2}$$

and

$$\mathbb{1}_{\{\tau_2 \le t\}} \mathbb{E}_{\mathbb{P}}(\eta_s|\tau_2) = \mathbb{1}_{\{\tau_2 \le t\}} \frac{f^*_{\tau_2}(\tau_2)}{f_{\tau_2}(\tau_2)},$$

where $f_{\tau_2}(u) = \lambda_2 e^{-\lambda_2 u}$. Details are left to the reader. $\square$

Remark 4.5.4 Observe that to find $\mathbb{Q}(\tau_1 > s, \tau_2 > t)$ for $t \le s \le T$, it suffices in fact to notice that

$$J := \mathbb{Q}(\tau_1 > s, \tau_2 > t) = \mathbb{E}_{\mathbb{P}}(\eta_T \mathbb{1}_{\{\tau_1 > s, \tau_2 > t\}}) = \mathbb{E}_{\mathbb{P}}(\eta_s \mathbb{1}_{\{\tau_1 > s, \tau_2 > t\}})$$

and

$$\eta_s \mathbb{1}_{\{\tau_1 > s, \tau_2 > t\}} = \eta^1_s \mathbb{1}_{\{\tau_1 > s, \tau_2 > t\}} = \mathbb{1}_{\{\tau_1 > s, \tau_2 > s\}} + \mathbb{1}_{\{t < \tau_2 < s < \tau_1\}} e^{-(\alpha_1-\lambda_1)(s-\tau_2)}.$$

Therefore,

$$\begin{aligned}
J &= \int_s^\infty \int_s^\infty \lambda_1\lambda_2 e^{-(\lambda_1 x + \lambda_2 y)}\,dxdy + \int_s^\infty \int_t^s \lambda_1\lambda_2 e^{-(\alpha_1-\lambda_1)(s-y)} e^{-(\lambda_1 x+\lambda_2 y)}\,dxdy \\
&= \frac{\lambda_1-\alpha_1}{\lambda_1+\lambda_2-\alpha_1} e^{-(\lambda_1+\lambda_2)s} + \frac{\lambda_2}{\lambda_1+\lambda_2-\alpha_1} e^{-\alpha_1 s-(\lambda_1+\lambda_2-\alpha_1)t}.
\end{aligned}$$

Let us introduce a probability measure $\mathbb{P}^*_1$ by setting

$$\frac{d\mathbb{P}^*_1}{d\mathbb{P}} = \eta^1_T, \quad \mathbb{P}\text{-a.s.} \tag{4.100}$$

It is interesting to observe that the marginal density $\widetilde{f}_{\tau_1}$ of τ_1 under $\mathbb{P}^*_1$ coincides with $f^*_{\tau_1}$, since

$$\begin{aligned}
\widetilde{f}_{\tau_1}(t) &= \int_0^t \lambda_1\lambda_2 \frac{\alpha_1}{\lambda_1} e^{-(\alpha_1-\lambda_1)(t-y)} e^{-(\lambda_1 t+\lambda_2 y)}\,dxdy + \int_t^\infty \lambda_1\lambda_2 e^{-(\lambda_1 t+\lambda_2 y)}\,dxdy \\
&= \frac{1}{\lambda_1+\lambda_2-\alpha_1}\Big(\alpha_1\lambda_2 e^{-\alpha_1 t} + (\lambda_1-\alpha_1)(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}\Big)
\end{aligned}$$

for every $t \le T$. It is also obvious that $\widetilde{f}_{\tau_1} = f^*_{\tau_1}$ for $t > T$. In fact, one may also deduce easily from the calculations in the proof of Lemma 4.5.8 and remarks above that

$$\begin{aligned}
I &= \mathbb{Q}(\tau_1 > s|\mathcal{H}^1_t \vee \mathcal{H}^2_t) = \mathbb{P}^*_1(\tau_1 > s|\mathcal{H}^1_t \vee \mathcal{H}^2_t) \\
&= \mathbb{1}_{\{\tau_1 > t, \tau_2 > t\}} \frac{\mathbb{P}^*_1(\tau_1 > s, \tau_2 > t)}{\mathbb{P}^*_1(\tau_1 > t, \tau_2 > t)} + \mathbb{1}_{\{\tau_2 \le t < \tau_1\}} \frac{\mathbb{P}^*_1(\tau_1 > s|\tau_2)}{\mathbb{P}^*_1(\tau_1 > t|\tau_2)}.
\end{aligned}$$

Intensity of τ_1 under $\mathbb{Q}$

We shall now focus on the intensity process of τ_1 under $\mathbb{Q}$. We have

$$\Lambda_t^{1*} = \int_0^t \big(\lambda_1 1\!\!1_{\{\tau_2>u\}} + \alpha_1 1\!\!1_{\{\tau_2\le u\}}\big)\,du = \lambda_1(t\wedge\tau_2) + \alpha_1(t\vee\tau_2 - \tau_2). \tag{4.101}$$

The first equality in next result is merely a special case of Proposition 4.5.7. In particular, equality (4.102) shows that we may apply Λ^{1*} to evaluate the conditional probability. Inequality (4.103) makes it clear that the process Λ^{1*} does not coincide with the $\mathbf{H}^2$-hazard process of τ_1 under $\mathbb{Q}$, however.

Lemma 4.5.9 *Let $I := \mathbb{Q}(\tau_1 > s|\mathcal{H}_t^1 \vee \mathcal{H}_t^2)$. For every $t \le s \le T$ we have*

$$I = 1\!\!1_{\{\tau_1>t\}}\mathbb{E}_{\mathbb{P}}(e^{\Lambda_t^{1*}-\Lambda_s^{1*}}|\mathcal{H}_t^2) = 1\!\!1_{\{\tau_1>t\}}\mathbb{E}_{\mathbb{P}_1^*}(e^{\Lambda_t^{1*}-\Lambda_s^{1*}}|\mathcal{H}_t^2) = 1\!\!1_{\{\tau_1>t\}}\mathbb{E}_{\widetilde{\mathbb{P}}_1}(e^{\Lambda_t^{1*}-\Lambda_s^{1*}}|\mathcal{H}_t^2) \tag{4.102}$$

and

$$\mathbb{Q}(\tau_1 > s|\mathcal{H}_t^1 \vee \mathcal{H}_t^2) \neq 1\!\!1_{\{\tau_1>t\}}\mathbb{E}_{\mathbb{P}^*}(e^{\Lambda_t^{1*}-\Lambda_s^{1*}}|\mathcal{H}_t^2). \tag{4.103}$$

Proof: Let us check that $\widetilde{I} = I$, where I is given by Lemma 4.5.8, and

$$\widetilde{I} = 1\!\!1_{\{\tau_1>t\}}\mathbb{E}_{\mathbb{P}}(e^{\Lambda_t^{1*}-\Lambda_s^{1*}}|\mathcal{H}_t^2).$$

It is enough to verify that

$$\mathbb{E}_{\mathbb{P}}\big(e^{\Lambda_t^{1*}-\Lambda_s^{1*}}\,\big|\,\mathcal{H}_t^2\big) = 1\!\!1_{\{\tau_2>t\}}\,\frac{1}{\lambda_1+\lambda_2-\alpha_1}\Big(\lambda_2 e^{-\alpha_1(s-t)} + (\lambda_1-\alpha_1)e^{-(\lambda_1+\lambda_2)(s-t)}\Big) + 1\!\!1_{\{\tau_2\le t\}}\,e^{-\alpha_1(s-t)}.$$

If we denote $Y = e^{\Lambda_t^{1*}-\Lambda_s^{1*}}$ then the general formula yields

$$\mathbb{E}_{\mathbb{P}}\big(e^{\Lambda_t^{1*}-\Lambda_s^{1*}}\,\big|\,\mathcal{H}_t^2\big) = 1\!\!1_{\{\tau_2>t\}}\,\frac{\mathbb{E}_{\mathbb{P}}(Y 1\!\!1_{\{\tau_2>t\}})}{\mathbb{P}(\tau_2>t)} + 1\!\!1_{\{\tau_2\le t\}}\,\mathbb{E}_{\mathbb{P}}(Y|\tau_2). \tag{4.104}$$

Standard calculations show that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\big(Y 1\!\!1_{\{\tau_2>t\}}\big) &= \mathbb{E}_{\mathbb{P}}\Big(1\!\!1_{\{\tau_2>t\}}\,e^{\lambda_1(t-s\wedge\tau_2)+\alpha_1(\tau_2-s\vee\tau_2)}\Big)\\
&= \int_t^s e^{\lambda_1(t-u)+\alpha_1(u-s)}\lambda_2 e^{-\lambda_2 u}\,du + \int_s^\infty e^{\lambda_1(t-s)}\lambda_2 e^{-\lambda_2 u}\,du\\
&= \frac{\lambda_2 e^{\lambda_1 t-\alpha_1 s}}{\lambda_1+\lambda_2-\alpha_1}\Big(e^{-(\lambda_1+\lambda_2-\alpha_1)t} - e^{-(\lambda_1+\lambda_2-\alpha_1)s}\Big) + e^{\lambda_1 t}e^{-(\lambda_1+\lambda_2)s}
\end{aligned}$$

and, of course, $\mathbb{P}(\tau_2 > t) = e^{-\lambda_2 t}$. Consequently, we obtain

$$\frac{\mathbb{E}_{\mathbb{P}}(Y 1\!\!1_{\{\tau_2>t\}})}{\mathbb{P}(\tau_2>t)} = \frac{1}{\lambda_1+\lambda_2-\alpha_1}\Big(\lambda_2 e^{-\alpha_1(s-t)} + (\lambda_1-\alpha_1)e^{-(\lambda_1+\lambda_2)(s-t)}\Big),$$

as expected. Furthermore, it follows easily from (4.101) that

$$1\!\!1_{\{\tau_2\le t\}}\,\mathbb{E}_{\mathbb{P}}(e^{\Lambda_t^{1*}-\Lambda_s^{1*}}|\tau_2) = 1\!\!1_{\{\tau_2\le t\}}\,e^{-\alpha_1(s-t)}.$$

This ends the proof of the first equality in (4.102). The second equality in (4.102) follows from the calculations above and the fact that the law of τ_2 under $\mathbb{P}_1^*$ is identical with its law under $\mathbb{Q}$. The last equality in (4.102) is trivial since $\widetilde{\mathbb{P}}_1 = \mathbb{P}$.

We shall now consider (4.103) for $t = 0$ (the general case is left to the reader as exercise). More precisely, we wish to show that for $s \le T$

$$\mathbb{Q}(\tau_1 > s) \neq \mathbb{E}_{\mathbb{P}^*}(e^{-\Lambda_s^{1*}}), \tag{4.105}$$

where the left-hand side is given by (4.95). We have

$$\mathbb{E}_{\mathbb{P}^*}\big(e^{-\Lambda^{1*}_s}\big) = \mathbb{E}_{\mathbb{P}^*}\big(e^{-\lambda_1(s\wedge\tau_2)-\alpha_1(s\vee\tau_2-\tau_2)}\big) = \int_0^s e^{-\lambda_1 u-\alpha_1(s-u)} f^*_{\tau_2}(u)\,du + \int_s^\infty e^{-\lambda_1 s} f^*_{\tau_2}(u)\,du.$$

Consequently

$$\mathbb{E}_{\mathbb{P}^*}\big(e^{-\Lambda^{1*}_s}\big) = \int_0^s e^{-\lambda_1 u-\alpha_1(s-u)} f^*_{\tau_2}(u)\,du + e^{-\lambda_1 s}\mathbb{Q}(\tau_2 > s),$$

where (cf. Section 4.5.5)

$$f^*_{\tau_2}(u) = \frac{1}{\lambda_1+\lambda_2-\alpha_2}\Big(\alpha_2\lambda_1 e^{-\alpha_2 u} + (\lambda_2-\alpha_2)(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)u}\Big)$$

for $u \le s \le T$, and

$$\mathbb{Q}(\tau_2 > s) = \frac{1}{\lambda_1+\lambda_2-\alpha_2}\Big(\lambda_1 e^{-\alpha_2 s} + (\lambda_2-\alpha_2)e^{-(\lambda_1+\lambda_2)s}\Big).$$

Straightforward calculations yield

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*}\big(e^{-\Lambda^{1*}_s}\big) \;=\; & \frac{1}{\lambda_1+\lambda_2-\alpha_2}\Big[\frac{\lambda_1\alpha_2}{\lambda_1-\alpha_1+\alpha_2}\Big(e^{-\alpha_1 s}+e^{-(\lambda_1+\alpha_2)s}\Big)\\ &+\frac{(\lambda_2-\alpha_2)(\lambda_1+\lambda_2)}{2\lambda_1+\lambda_2-\alpha_1}\Big(e^{-\alpha_1 s}+e^{-(2\lambda_1+\lambda_2)s}\Big)\\ &+\Big(\lambda_1 e^{-(\lambda_1+\alpha_2)s}+(\lambda_2-\alpha_2)e^{-(2\lambda_1+\lambda_2)s}\Big)\Big]\end{aligned}$$

which shows, when combined with (4.95), that inequality (4.105) is valid. □

Remark 4.5.5 If $\lambda_i = \alpha_i$ for $i = 1, 2$, the last formula gives, as it should

$$\mathbb{E}_{\mathbb{P}^*}\big(e^{-\Lambda^{1*}_s}\big) = \mathbb{P}(\tau_1 > s) = e^{-\lambda_1 s}.$$

Let us now put $\lambda_2 = \alpha_2$, but $\lambda_2 \ne \alpha_1$ (this corresponds to the equality of probability measures $\mathbb{Q} = \mathbb{P}^*_1$). Then we get

$$\mathbb{E}_{\mathbb{P}^*_1}\big(e^{-\Lambda^{1*}_s}\big) = \frac{1}{\lambda_1+\lambda_2-\alpha_1}\Big(\lambda_2 e^{-\alpha_1 s} + (\lambda_1-\alpha_1)e^{-(\lambda_1+\lambda_2)s}\Big) = \mathbb{P}^*_1(\tau_1 > s) = \mathbb{Q}(\tau_1 > s).$$

This coincides with the second equality in (4.102), in the special case of $t = 0$.

Validity of Hypothesis (G) under $\mathbb{Q}$

We shall now check the validity of hypothesis (G). In the present context, we consider the random time $\tau = \tau_1$, we take $\mathbf{F} = \mathbf{H}^2$, and $t \le T$. Therefore, condition (G) reads as follows.

(G) The process $F_t = \mathbb{Q}(\tau_1 \le t \mid \mathcal{H}^2_t)$, $t \in [0,T]$, admits a modification with increasing sample paths.

Let us denote $G_t = 1 - F_t = \mathbb{Q}(\tau_1 > t \mid \mathcal{H}^2_t)$. It follows from the proof of Lemma 4.5.8 that

$$G_t = (1-H^2_t)\,\frac{\mathbb{Q}(\tau_1 > t, \tau_2 > t)}{\mathbb{Q}(\tau_2 > t)} + H^2_t\,\mathbb{Q}(\tau_1 > t \mid \tau_2), \tag{4.106}$$

where

$$\mathbb{Q}(\tau_1 > t, \tau_2 > t) = \frac{\lambda_1-\alpha_1}{\lambda_1+\lambda_2-\alpha_1}e^{-(\lambda_1+\lambda_2)t} + \frac{\lambda_2}{\lambda_1+\lambda_2-\alpha_1}e^{-\alpha_1 t-(\lambda_1+\lambda_2-\alpha_1)t} = e^{-(\lambda_1+\lambda_2)t}$$

and (cf. (4.95)-(4.96))

$$\mathbb{Q}(\tau_2 > t) = \frac{1}{\lambda_1 + \lambda_2 - \alpha_2}\Big(\lambda_1 e^{-\alpha_2 t} + (\lambda_2 - \alpha_2) e^{-(\lambda_1+\lambda_2)t}\Big), \quad \forall\, t \le T,$$

$$\mathbb{Q}(\tau_2 > t) = \frac{e^{-\lambda_2 t}}{\lambda_1 + \lambda_2 - \alpha_2}\Big(\lambda_1 e^{-(\alpha_2-\lambda_2)T} + (\lambda_2 - \alpha_2) e^{-\lambda_1 T}\Big), \quad \forall\, t > T.$$

On the other hand, for every $u \le t \le T$ we have (cf. (4.99))

$$\mathbb{Q}(\tau_1 > t | \tau_2 = u) = \frac{(\lambda_1 + \lambda_2 - \alpha_2)\lambda_2 e^{-\alpha_1(t-u)}}{\lambda_1 \alpha_2 e^{(\lambda_1+\lambda_2-\alpha_2)u} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)}.$$

Similar calculations yield

$$\mathbb{Q}(\tau_1 > t | \tau_2 = u) = \frac{(\lambda_1 + \lambda_2 - \alpha_2)\lambda_2 e^{-(\alpha_1-\lambda_1)T} e^{\alpha_1 u - \lambda_1 t}}{\lambda_1 \alpha_2 e^{(\lambda_1+\lambda_2-\alpha_2)u} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)}$$

for $u \le T < t,$ and finally

$$\mathbb{Q}(\tau_1 > t | \tau_2 = u) = \frac{(\lambda_1 + \lambda_2 - \alpha_2) e^{\lambda_2 u - \lambda_1 t}}{\lambda_1 \lambda_2 e^{-(\alpha_2-\lambda_2)T} + (\lambda_2 - \alpha_2) e^{-\lambda_1 T}}$$

for $T < u \le t$. Combining the formulae above, we obtain

$$\begin{aligned} G_t \;\; = \;\; & \mathbb{1}_{\{t<\tau_2\le T\}} \frac{c}{\lambda_1 e^{ct} + \lambda_2 - \alpha_2} + \mathbb{1}_{\{T<t<\tau_2\}} \frac{c e^{\lambda_1(T-t)}}{\lambda_1 e^{cT} + \lambda_2 - \alpha_2} \\ & + \mathbb{1}_{\{\tau_2=u\le t\le T\}} \frac{c\lambda_2 e^{\alpha_1(u-t)}}{\lambda_1 \alpha_2 e^{cu} + (\lambda_2 - \alpha_2)(\lambda_1+\lambda_2)} \\ & + \mathbb{1}_{\{\tau_2=u\le T<t\}} \frac{c\lambda_2 e^{(\lambda_1-\alpha_1)T} e^{\alpha_1 u - \lambda_1 t}}{\lambda_1 \alpha_2 e^{cu} + (\lambda_2 - \alpha_2)(\lambda_1+\lambda_2)} \\ & + \mathbb{1}_{\{T<\tau_2=u\le t\}} \frac{c e^{\lambda_1 T} e^{\lambda_2 u - \lambda_1 t}}{\lambda_1 \lambda_2 e^{cT} + (\lambda_2 - \alpha_2)}, \end{aligned}$$

where we denote $c = \lambda_1 + \lambda_2 - \alpha_2$. In particular, for every $t \in [0, T]$ we have

$$G_t = \mathbb{1}_{\{t<\tau_2\le T\}} \frac{c}{\lambda_1 e^{ct} + \lambda_2 - \alpha_2} + \mathbb{1}_{\{\tau_2\le t\le T\}} \frac{c\lambda_2 e^{-\alpha_1(t-\tau_2)}}{\lambda_1 \alpha_2 e^{c\tau_2} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)}.$$

Since both terms in the right-hand side of the last formula can be shown to follow decreasing functions, it is enough to examine the jump at $\tau_2,$ which equals

$$\Delta = \frac{c}{\lambda_1 e^{ct} + \lambda_2 - \alpha_2} - \frac{c\lambda_2}{\lambda_1 \alpha_2 e^{c\tau_2} + (\lambda_2 - \alpha_2)(\lambda_1 + \lambda_2)}.$$

Straightforward calculations show that $\Delta \le 0$ if and only if $\lambda_2 \le \alpha_2$.

Validity of Hypothesis (H) under $\mathbb{Q}$

Recall that we have $\mathbf{G} = \mathbf{H}^1 \vee \mathbf{H}^2$. If we choose $\tau = \tau_1$ then the filtration $\widetilde{\mathbf{F}} = \mathbf{H}^2$ generated by τ_2 plays the role of the additional filtration $\mathbf{F}$. Therefore, condition (H) introduced in Section **??** can be restated as follows.

(H) Any $\mathbf{H}^1$-martingale under $\mathbb{Q}$ follows also a $\mathbf{G}$-martingale under $\mathbb{Q}$.

Condition (H) is equivalent to the following condition (cf. hypothesis (H3) in Section **??**).

(H3) Equality $\mathbb{E}_{\mathbb{P}^*}(\xi|\mathcal{H}_t^1 \vee \mathcal{H}_t^2) = \mathbb{E}_{\mathbb{P}^*}(\xi|\mathcal{H}_t^2)$ holds for any bounded $\mathcal{H}_\infty^2$-measurable random variable ξ, and any $t \in \mathbb{R}_+$.

From the calculations done in preceding sections, it is clear that the last condition is not satisfied in Kusuoka's example. For instance, we may take $t < s$ and $\xi = \mathbb{1}_{\{\tau_2 > s\}}$. Using a suitable modification of formula (4.95), we get

$$\mathbb{E}_{\mathbb{P}^*}(\xi|\mathcal{H}_t^2) = \mathbb{Q}(\tau_2 > s|\mathcal{H}_t^2) = \mathbb{1}_{\{\tau_2 > t\}} \frac{\mathbb{Q}(\tau_2 > s)}{\mathbb{Q}(\tau_2 > t)} = \mathbb{1}_{\{\tau_2 > t\}} \frac{\lambda_1 e^{-\alpha_2 s} + (\lambda_2 - \alpha_2) e^{-(\lambda_1+\lambda_2)s}}{\lambda_1 e^{-\alpha_2 t} + (\lambda_2 - \alpha_2) e^{-(\lambda_1+\lambda_2)t}}.$$

On the other hand, Lemma 4.5.8 yields

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}(\xi|\mathcal{H}_t^1 \vee \mathcal{H}_t^2) &= \mathbb{Q}(\tau_2 > s|\mathcal{H}_t^1 \vee \mathcal{H}_t^2) \\
&= \mathbb{1}_{\{\tau_1 > t, \tau_2 > t\}} \frac{\mathbb{Q}(\tau_1 > t, \tau_2 > s)}{\mathbb{Q}(\tau_1 > t, \tau_2 > t)} + \mathbb{1}_{\{\tau_1 \le t < \tau_2\}} \frac{\mathbb{Q}(\tau_2 > s|\tau_1)}{\mathbb{Q}(\tau_2 > t|\tau_1)} \\
&= \mathbb{1}_{\{\tau_1 > t, \tau_2 > t\}} \frac{1}{\lambda_1 + \lambda_2 - \alpha_2} \Big(\lambda_1 e^{-\alpha_2(s-t)} + (\lambda_2 - \alpha_2) e^{-(\lambda_1+\lambda_2)(s-t)} \Big) \\
&\quad + \mathbb{1}_{\{\tau_1 \le t < \tau_1\}} e^{-\alpha_2(s-t)}.
\end{aligned}$$

It is thus clear that $\mathbb{E}_{\mathbb{P}^*}(\xi|\mathcal{H}_t^1 \vee \mathcal{H}_t^2) \neq \mathbb{E}_{\mathbb{P}^*}(\xi|\mathcal{H}_t^2)$, and thus the martingale invariance property (H) does not hold under $\mathbb{Q}$ (it is obvious that it holds under the original probability measure $\mathbb{P}$).

Chapter 5

Hedging

5.1 Semimartingale Model with a Common Default

In what follows, we fix a finite horizon date $T > 0$. For the purpose of this work, it is enough to formally define a generic defaultable claim through the following definition.

Definition 5.1.1 *A* defaultable claim *with maturity date T is represented by a triplet (X, Z, τ), where:*
(i) the default time τ specifies the random time of default, and thus also the default events $\{\tau \leq t\}$ for every $t \in [0, T]$,
(ii) the promised payoff $X \in \mathcal{F}_T$ represents the random payoff received by the owner of the claim at time T, provided that there was no default prior to or at time T; the actual payoff at time T associated with X thus equals $X \mathbb{1}_{\{T<\tau\}}$,
(iii) the $\mathbf{F}$-adapted recovery process Z specifies the recovery payoff Z_τ received by the owner of a claim at time of default (or at maturity), provided that the default occurred prior to or at maturity date T.

In practice, hedging of a credit derivative after default time is usually of minor interest. Also, in a model with a single default time, hedging after default reduces to replication of a non-defaultable claim. It is thus natural to define the replication of a defaultable claim in the following way.

Definition 5.1.2 *We say that a self-financing strategy ϕ replicates a defaultable claim (X, Z, τ) if its wealth process $V(\phi)$ satisfies $V_T(\phi)\mathbb{1}_{\{T<\tau\}} = X \mathbb{1}_{\{T<\tau\}}$ and $V_\tau(\phi)\mathbb{1}_{\{T\geq\tau\}} = Z_\tau \mathbb{1}_{\{T\geq\tau\}}$.*

When dealing with replicating strategies, in the sense of Definition 5.1.2, we will always assume, without loss of generality, that the components of the process ϕ are $\mathbf{F}$-predictable processes.

5.1.1 Dynamics of asset prices

We assume that we are given a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a (possibly multi-dimensional) standard Brownian motion W and a random time τ admitting an $\mathbf{F}$-intensity γ under $\mathbb{P}$, where $\mathbf{F}$ is the filtration generated by W. In addition, we assume that τ satisfies (4.7), so that the hypothesis (H) is valid under $\mathbb{P}$ for filtrations $\mathbf{F}$ and $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$. Since the default time admits an $\mathbf{F}$-intensity, it is not an $\mathbf{F}$-stopping time. Indeed, any stopping time with respect to a Brownian filtration is known to be predictable.

We interpret τ as the common default time for all defaultable assets in our model. For simplicity, we assume that only three primary assets are traded in the market, and the dynamics under the historical probability $\mathbb{P}$ of their prices are, for $i = 1, 2, 3$ and $t \in [0, T]$,

$$dY^i_t = Y^i_{t-}\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t + \kappa_{i,t}\,dM_t\big), \tag{5.1}$$

where $M_t = H_t - \int_0^t o(1-H_s)\gamma_s ds$ is a martingale, or equivalently,

$$dY^i_t = Y^i_{t-}\big((\mu_{i,t} - \kappa_{i,t}\gamma_t 1\!\!1_{\{t<\tau\}})\,dt + \sigma_{i,t}\,dW_t + \kappa_{i,t}\,dH_t\big). \tag{5.2}$$

The processes $(\mu_i, \sigma_i, \kappa_i) = (\mu_{i,t}, \sigma_{i,t}, \kappa_{i,t},\, t \geq 0)$, $i = 1, 2, 3$, are assumed to be $\mathbf{G}$-adapted, where $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$. In addition, we assume that $\kappa_i \geq -1$ for any $i = 1, 2, 3$, so that Y^i are nonnegative processes, and they are strictly positive prior to τ. Note that,in the case of constant coefficients

$$Y^i_t = Y^i_0 e^{\mu_i t} e^{\sigma_i W_t - \sigma_i^2 t/2} e^{-\kappa_i \gamma_i (t\wedge\tau)} (1+\kappa_i)^{H_t}$$

Note that, according to Definition 5.1.2, replication refers to the behavior of the wealth process $V(\phi)$ on the random interval $[\![0, \tau \wedge T]\!]$ only. Hence, for the purpose of replication of defaultable claims of the form (X, Z, τ), it is sufficient to consider prices of primary assets stopped at $\tau \wedge T$. This implies that instead of dealing with $\mathbf{G}$-adapted coefficients in (5.1), it suffices to focus on $\mathbf{F}$-adapted coefficients of stopped price processes. However, for the sake of completeness, we shall also deal with T-maturity claims of the form $Y = G(Y^1_T, Y^2_T, Y^3_T, H_T)$ (see Section 5.4 below).

Pre-default values

As will become clear in what follows, when dealing with defaultable claims of the form (X, Z, τ), we will be mainly concerned with the so-called pre-default prices. The *pre-default price* $\widetilde{Y}^i$ of the ith asset is an $\mathbf{F}$-adapted, continuous process, given by the equation, for $i = 1, 2, 3$ and $t \in [0, T]$,

$$d\widetilde{Y}^i_t = \widetilde{Y}^i_t\big((\mu_{i,t} - \kappa_{i,t}\gamma_t)\,dt + \sigma_{i,t}\,dW_t\big) \tag{5.3}$$

with $\widetilde{Y}^i_0 = Y^i_0$. Put another way, $\widetilde{Y}^i$ is the unique $\mathbf{F}$-predictable process such that $\widetilde{Y}^i_t 1\!\!1_{\{t\leq\tau\}} = Y^i_t 1\!\!1_{\{t\leq\tau\}}$ for $t \in I\!\!R_+$. When dealing with the pre-default prices, we may and do assume, without loss of generality, that the processes μ_i, σ_i and κ_i are $\mathbf{F}$-predictable.

It is worth stressing that the historically observed drift coefficient equals $\mu_{i,t} - \kappa_{i,t}\gamma_t$, rather than $\mu_{i,t}$. The drift coefficient denoted by $\mu_{i,t}$ is already credit-risk adjusted in the sense of our model, and it is not directly observed. This convention was chosen here for the sake of simplicity of notation. It also lends itself to the following intuitive interpretation: if ϕ^i is the number of units of the ith asset held in our portfolio at time t then the gains/losses from trades in this asset, prior to default time, can be represented by the differential

$$\phi^i_t\,d\widetilde{Y}^i_t = \phi^i_t\widetilde{Y}^i_t\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t\big) - \phi^i_t\widetilde{Y}^i_t\kappa_{i,t}\gamma_t\,dt.$$

The last term may be here separated, and formally treated as an effect of continuously paid dividends at the dividend rate $\kappa_{i,t}\gamma_t$. However, this interpretation may be misleading, since this quantity is not directly observed. In fact, the mere estimation of the drift coefficient in dynamics (5.3) is not practical.

Still, if this formal interpretation is adopted, it is sometimes possible make use of the standard results concerning the valuation of derivatives of dividend-paying assets. It is, of course, a delicate issue how to separate in practice both components of the drift coefficient. We shall argue below that although the dividend-based approach is formally correct, a more pertinent and simpler way of dealing with hedging relies on the assumption that only the effective drift $\mu_{i,t} - \kappa_{i,t}\gamma_t$ is observable. In practical approach to hedging, the values of drift coefficients in dynamics of asset prices play no essential role, so that they are considered as market observables.

Market observables

To summarize, we assume throughout that the *market observables* are: the pre-default market prices of primary assets, their volatilities and correlations, as well as the jump coefficients $\kappa_{i,t}$ (the financial interpretation of jump coefficients is examined in the next subsection). To summarize we postulate that under the statistical probability $\mathbb{P}$ we have

$$dY_t^i = Y_{t-}^i \big(\widetilde{\mu}_{i,t}\, dt + \sigma_{i,t}\, dW_t + \kappa_{i,t}\, dH_t \big) \tag{5.4}$$

where the drift terms $\widetilde{\mu}_{i,t}$ are not observable, but we can observe the volatilities $\sigma_{i,t}$ (and thus the assets correlations), and we have an a priori assessment of jump coefficients $\kappa_{i,t}$. In this general set-up, the most natural assumption is that the dimension of a driving Brownian motion W equals the number of tradable assets. However, for the sake of simplicity of presentation, we shall frequently assume that W is one-dimensional. One of our goals will be to derive closed-form solutions for replicating strategies for derivative securities in terms of market observables only (whenever replication of a given claim is actually feasible). To achieve this goal, we shall combine a general theory of hedging defaultable claims within a continuous semimartingale set-up, with a judicious specification of particular models with deterministic volatilities and correlations.

Recovery schemes

It is clear that the sample paths of price processes Y^i are continuous, except for a possible discontinuity at time τ. Specifically, we have that

$$\Delta Y_\tau^i := Y_\tau^i - Y_{\tau-}^i = \kappa_{i,\tau} Y_{\tau-}^i,$$

so that $Y_\tau^i = Y_{\tau-}^i(1+\kappa_{i,\tau}) = \widetilde{Y}_{\tau-}^i(1+\kappa_{i,\tau})$.

A primary asset Y^i is termed a *default-free asset* (*defaultable asset*, respectively) if $\kappa_i = 0$ ($\kappa_i \neq 0$, respectively). In the special case when $\kappa_i = -1$, we say that a defaultable asset Y^i is subject to a *total default*, since its price drops to zero at time τ and stays there forever. Such an asset ceases to exist after default, in the sense that it is no longer traded after default. This feature makes the case of a total default quite different from other cases, as we shall see in our study below.

In market practice, it is common for a credit derivative to deliver a positive recovery (for instance, a *protection payment*) in case of default. Formally, the value of this recovery at default is determined as the value of some underlying process, that is, it is equal to the value at time τ of some $\mathbf{F}$-adapted recovery process Z.

For example, the process Z can be equal to δ, where δ is a constant, or to $g(t, \delta Y_t)$ where g is a deterministic function and $(Y_t,\, t \geq 0)$ is the price process of some default-free asset. Typically, the recovery is paid at default time, but it may also happen that it is postponed to the maturity date.

Let us observe that the case where a defaultable asset Y^i pays a pre-determined recovery at default is covered by our set-up defined in (5.1). For instance, the case of a constant recovery payoff $\delta_i \geq 0$ at default time τ corresponds to the process $\kappa_{i,t} = \delta_i (Y_{t-}^i)^{-1} - 1$. Under this convention, the price Y^i is governed under $\mathbb{P}$ by the SDE

$$dY_t^i = Y_{t-}^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t + (\delta_i (Y_{t-}^i)^{-1} - 1)\, dM_t \big). \tag{5.5}$$

If the recovery is proportional to the pre-default value $Y_{\tau-}^i$, and is paid at default time τ (this scheme is known as the *fractional recovery of market value*), we have $\kappa_{i,t} = \delta_i - 1$ and

$$dY_t^i = Y_{t-}^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t + (\delta_i - 1)\, dM_t \big). \tag{5.6}$$

5.2 Trading Strategies in a Semimartingale Set-up

We consider trading within the time interval $[0, T]$ for some finite horizon date $T > 0$. For the sake of expositional clarity, we restrict our attention to the case where only three primary assets are traded. The general case of k traded assets was examined by Bielecki *et al.* [14]. We first recall some general properties, which do not depend on the choice of specific dynamics of asset prices.

In this section, we consider a fairly general set-up. In particular, processes Y^i, $i = 1, 2, 3$, are assumed to be nonnegative semi-martingales on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with some filtration $\mathbf{G}$. We assume that they represent spot prices of traded assets in our model of the financial market. Neither the existence of a savings account, nor the market completeness are assumed, in general.

Our goal is to characterize contingent claims which are *hedgeable*, in the sense that they can be replicated by continuously rebalanced portfolios consisting of primary assets. Here, by a contingent claim we mean an arbitrary $\mathcal{G}_T$-measurable random variable. We work under the standard assumptions of a frictionless market.

5.2.1 Unconstrained strategies

Let $\phi = (\phi^1, \phi^2, \phi^3)$ be a trading strategy; in particular, each process ϕ^i is predictable with respect to the filtration $\mathbf{G}$. The wealth of ϕ equals

$$V_t(\phi) = \sum_{i=1}^{3} \phi^i_t Y^i_t, \quad \forall\, t \in [0, T],$$

and a trading strategy ϕ is said to be *self-financing* if

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{3} \int_0^t \phi^i_u \, dY^i_u, \quad \forall\, t \in [0, T].$$

Let Φ stand for the class of all self-financing trading strategies. We shall first prove that a self-financing strategy is determined by its initial wealth and the two components ϕ^2, ϕ^3. To this end, we postulate that the price of Y^1 follows a strictly positive process, and we choose Y^1 as a numéraire asset. We shall now analyze the relative values:

$$V^1_t(\phi) := V_t(\phi)(Y^1_t)^{-1}, \quad Y^{i,1}_t := Y^i_t (Y^1_t)^{-1}.$$

Lemma 5.2.1 (i) *For any $\phi \in \Phi$, we have*

$$V^1_t(\phi) = V^1_0(\phi) + \sum_{i=2}^{3} \int_0^t \phi^i_u \, dY^{i,1}_u, \quad \forall\, t \in [0, T].$$

(ii) *Conversely, let X be a $\mathcal{G}_T$-measurable random variable, and let us assume that there exists $x \in I\!R$ and $\mathbf{G}$-predictable processes ϕ^i, $i = 2, 3$ such that*

$$X = Y^1_T \left(x + \sum_{i=2}^{3} \int_0^T \phi^i_u \, dY^{i,1}_u \right). \tag{5.7}$$

Then there exists a $\mathbf{G}$-predictable process ϕ^1 such that the strategy $\phi = (\phi^1, \phi^2, \phi^3)$ is self-financing and replicates X. Moreover, the wealth process of ϕ (i.e. the time-t price of X) satisfies $V_t(\phi) = V^1_t Y^1_t$, where

$$V^1_t = x + \sum_{i=2}^{3} \int_0^t \phi^i_u \, dY^{i,1}_u, \quad \forall\, t \in [0, T]. \tag{5.8}$$

Proof: In the case of continuous semimartingales, (it is a well-known result; for discontinuous processes, the proof is not much different. We reproduce it here for the reader's convenience.

Let us first introduce some notation. As usual, $[X,Y]$ stands for the *quadratic covariation* of the two semi-martingales X and Y, as defined by the integration by parts formula:

$$X_tY_t = X_0Y_0 + \int_0^t X_{u-}\,dY_u + \int_0^t Y_{u-}\,dX_u + [X,Y]_t.$$

For any càdlàg (i.e., RCLL) process Y, we denote by $\Delta Y_t = Y_t - Y_{t-}$ the size of the jump at time t. Let $V = V(\phi)$ be the value of a self-financing strategy, and let $V^1 = V^1(\phi) = V(\phi)(Y^1)^{-1}$ be its value relative to the numéraire Y^1. The integration by parts formula yields

$$dV_t^1 = V_{t-}d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}dV_t + d[(Y^1)^{-1},V]_t.$$

From the self-financing condition, we have $dV_t = \sum_{i=1}^3 \phi_t^i\,dY_t^i$. Hence, using elementary rules to compute the quadratic covariation $[X,Y]$ of the two semi-martingales X,Y, we obtain

$$\begin{aligned}
dV_t^1 &= \phi_t^1 Y_{t-}^1\,d(Y_t^1)^{-1} + \phi_t^2 Y_{t-}^2\,d(Y_t^1)^{-1} + \phi_t^3 Y_{t-}^3\,d(Y_t^1)^{-1} \\
&\quad + (Y_{t-}^1)^{-1}\phi_t^1\,dY_t^1 + (Y_{t-}^1)^{-1}\phi_t^2\,dY_t^1 + (Y_{t-}^1)^{-1}\phi_t^3\,dY_t^1 \\
&\quad + \phi_t^1\,d[(Y^1)^{-1},Y^1]_t + \phi_t^2\,d[(Y^1)^{-1},Y^2]_t + \phi_t^3\,d[(Y^1)^{-1},Y^1]_t \\
&= \phi_t^1\big(Y_{t-}^1\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_t^1 + d[(Y^1)^{-1},Y^1]_t\big) \\
&\quad + \phi_t^2\big(Y_{t-}^2\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_{t-}^1 + d[(Y^1)^{-1},Y^2]_t\big) \\
&\quad + \phi_t^3\big(Y_{t-}^3\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_{t-}^1 + d[(Y^1)^{-1},Y^3]_t\big).
\end{aligned}$$

We now observe that

$$Y_{t-}^1\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_t^1 + d[(Y^1)^{-1},Y^1]_t = d(Y_t^1(Y_t^1)^{-1}) = 0$$

and

$$Y_{t-}^i\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_t^i + d[(Y^1)^{-1},Y^i]_t = d((Y_t^1)^{-1}Y_t^i).$$

Consequently,

$$dV_t^1 = \phi_t^2\,dY_t^{2,1} + \phi_t^3\,dY_t^{3,1},$$

as was claimed in part (i). We now proceed to the proof of part (ii). We assume that (5.7) holds for some constant x and processes ϕ^2,ϕ^3, and we define the process V^1 by setting (cf. (5.8))

$$V_t^1 = x + \sum_{i=2}^3 \int_0^t \phi_u^i\,dY_u^{i,1}, \quad \forall\, t\in[0,T].$$

Next, we define the process ϕ^1 as follows:

$$\phi_t^1 = V_t^1 - \sum_{i=2}^3 \phi_t^i Y_t^{i,1} = (Y_t^1)^{-1}\Big(V_t - \sum_{i=2}^3 \phi_t^i Y_t^i\Big),$$

where $V_t = V_t^1 Y_t^1$. Since $dV_t^1 = \sum_{i=2}^3 \phi_t^i\,dY_t^{i,1}$, we obtain

$$\begin{aligned}
dV_t &= d(V_t^1Y_t^1) = V_{t-}^1 dY_t^1 + Y_{t-}^1 dV_t^1 + d[Y^1,V^1]_t \\
&= V_{t-}^1 dY_t^1 + \sum_{i=2}^3 \phi_t^i\big(Y_{t-}^1\,dY_t^{i,1} + d[Y^1,Y^{i,1}]_t\big).
\end{aligned}$$

From the equality

$$dY_t^i = d(Y_t^{i,1}Y_t^1) = Y_{t-}^{i,1}dY_t^1 + Y_{t-}^1 dY_t^{i,1} + d[Y^1,Y^{i,1}]_t,$$

it follows that

$$dV_t = V_{t-}^1 dY_t^1 + \sum_{i=2}^3 \phi_t^i\big(dY_t^i - Y_{t-}^{i,1}dY_t^1\big) = \Big(V_{t-}^1 - \sum_{i=2}^3 \phi_t^i Y_{t-}^{i,1}\Big)dY_t^1 + \sum_{i=2}^3 \phi_t^i\,dY_t^i,$$

and our aim is to prove that $dV_t = \sum_{i=1}^3 \phi_t^i \, dY_t^i$. The last equality holds if

$$\phi_t^1 = V_t^1 - \sum_{i=2}^3 \phi_t^i Y_t^{i,1} = V_{t-}^1 - \sum_{i=2}^3 \phi_t^i Y_{t-}^{i,1}, \tag{5.9}$$

i.e., if $\Delta V_t^1 = \sum_{i=2}^3 \phi_t^i \Delta Y_t^{i,1}$, which is the case from the definition (5.8) of V^1. Note also that from the second equality in (5.9) it follows that the process ϕ^1 is indeed **G**-predictable. Finally, the wealth process of ϕ satisfies $V_t(\phi) = V_t^1 Y_t^1$ for every $t \in [0, T]$, and thus $V_T(\phi) = X$. $\square$

We say that a self-financing strategy ϕ *replicates* a claim $X \in \mathcal{G}_T$ if

$$X = \sum_{i=1}^3 \phi_T^i Y_T^i = V_T(\phi),$$

or equivalently,

$$X = V_0(\phi) + \sum_{i=1}^3 \int_0^T \phi_t^i \, dY_t^i.$$

Suppose that there exists an e.m.m. for some choice of a numéraire asset, and let us restrict our attention to the class of all *admissible* trading strategies, so that our model is arbitrage-free.

Assume that a claim X can be replicated by some admissible trading strategy, so that it is *attainable* (or *hedgeable*). Then, by definition, the *arbitrage price* at time t of X, denoted as $\pi_t(X)$, equals $V_t(\phi)$ for any admissible trading strategy ϕ that replicates X.

In the context of Lemma 5.2.1, it is natural to choose as an e.m.m. a probability measure $\mathbb{Q}^1$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ and such that the prices $Y^{i,1}$, $i = 2, 3$, are **G**-martingales under $\mathbb{Q}^1$. If a contingent claim X is hedgeable, then its arbitrage price satisfies

$$\pi_t(X) = Y_t^1 \mathbb{E}_{\mathbb{Q}^1}(X (Y_T^1)^{-1} \,|\, \mathcal{G}_t).$$

We emphasize that even if an e.m.m. $\mathbb{Q}^1$ is not unique, the price of any hedgeable claim X is given by this conditional expectation. That is to say, in case of a hedgeable claim these conditional expectations under various equivalent martingale measures coincide.

In the special case where $Y_t^1 = B(t, T)$ is the price of a default-free zero-coupon bond with maturity T (abbreviated as ZC-bond in what follows), $\mathbb{Q}^1$ is called *T-forward martingale measure*, and it is denoted by $\mathbb{Q}_T$. Since $B(T, T) = 1$, the price of any hedgeable claim X now equals $\pi_t(X) = B(t, T) \, \mathbb{E}_{\mathbb{Q}_T}(X \,|\, \mathcal{G}_t)$.

5.2.2 Constrained strategies

In this section, we make an additional assumption that the price process Y^3 is strictly positive. Let $\phi = (\phi^1, \phi^2, \phi^3)$ be a self-financing trading strategy satisfying the following constraint:

$$\sum_{i=1}^2 \phi_t^i Y_{t-}^i = Z_t, \quad \forall \, t \in [0, T], \tag{5.10}$$

for a predetermined, **G**-predictable process Z. In the financial interpretation, equality (5.10) means that a portfolio ϕ is rebalanced in such a way that the total wealth invested in assets Y^1, Y^2 matches a predetermined stochastic process Z. For this reason, the constraint given by (5.10) is referred to as the *balance condition*.

Our first goal is to extend part (i) in Lemma 5.2.1 to the case of constrained strategies. Let $\Phi(Z)$ stand for the class of all (admissible) self-financing trading strategies satisfying the balance condition (5.10). They will be sometimes referred to as *constrained strategies*. Since any strategy $\phi \in \Phi(Z)$ is self-financing, from $dV_t(\phi) = \sum_{i=1}^3 \phi_t^i \, dY_t^i$, we obtain

$$\Delta V_t(\phi) = \sum_{i=1}^3 \phi_t^i \Delta Y_t^i = V_t(\phi) - \sum_{i=1}^3 \phi_t^i Y_{t-}^i.$$

By combining this equality with (5.10), we deduce that

$$V_{t-}(\phi) = \sum_{i=1}^3 \phi_t^i Y_{t-}^i = Z_t + \phi_t^3 Y_{t-}^i.$$

Let us write $Y_t^{i,3} = Y_t^i (Y_t^3)^{-1}$, $Z_t^3 = Z_t (Y_t^3)^{-1}$. The following result extends Lemma 1.7 in Bielecki *et al.* [15] from the case of continuous semi-martingales to the general case (see also [14]). It is apparent from Proposition 5.2.1 that the wealth process $V(\phi)$ of a strategy $\phi \in \Phi(Z)$ depends only on a single component of ϕ, namely, ϕ^2.

Proposition 5.2.1 *The relative wealth $V_t^3(\phi) = V_t(\phi)(Y_t^3)^{-1}$ of any trading strategy $\phi \in \Phi(Z)$ satisfies*

$$V_t^3(\phi) = V_0^3(\phi) + \int_0^t \phi_u^2 \left(dY_u^{2,3} - \frac{Y_{u-}^{2,3}}{Y_{u-}^{1,3}} \, dY_u^{1,3} \right) + \int_0^t \frac{Z_u^3}{Y_{u-}^{1,3}} \, dY_u^{1,3}. \tag{5.11}$$

Proof: Let us consider discounted values of price processes Y^1, Y^2, Y^3, with Y^3 taken as a numéraire asset. By virtue of part (i) in Lemma 5.2.1, we thus have

$$V_t^3(\phi) = V_0^3(\phi) + \sum_{i=1}^2 \int_0^t \phi_u^i \, dY_u^{i,3}. \tag{5.12}$$

The balance condition (5.10) implies that

$$\sum_{i=1}^2 \phi_t^i Y_{t-}^{i,3} = Z_t^3,$$

and thus

$$\phi_t^1 = (Y_{t-}^{1,3})^{-1} \Big(Z_t^3 - \phi_t^2 Y_{t-}^{2,3} \Big). \tag{5.13}$$

By inserting (5.13) into (5.12), we arrive at the desired formula (5.11). □

The next result will prove particularly useful for deriving replicating strategies for defaultable claims.

Proposition 5.2.2 *Let a $\mathcal{G}_T$-measurable random variable X represent a contingent claim that settles at time T. We set*

$$dY_t^* = dY_t^{2,3} - \frac{Y_{t-}^{2,3}}{Y_{t-}^{1,3}} \, dY_t^{1,3} = dY_t^{2,3} - Y_{t-}^{2,1} \, dY_t^{1,3}, \tag{5.14}$$

where, by convention, $Y_0^* = 0$. Assume that there exists a $\mathbf{G}$-predictable process ϕ^2, such that

$$X = Y_T^3 \left(x + \int_0^T \phi_t^2 \, dY_t^* + \int_0^T \frac{Z_t^3}{Y_{t-}^{1,3}} \, dY_t^{1,3} \right). \tag{5.15}$$

Then there exist $\mathbf{G}$-predictable processes ϕ^1 and ϕ^3 such that the strategy $\phi = (\phi^1, \phi^2, \phi^3)$ belongs to $\Phi(Z)$ and replicates X. The wealth process of ϕ equals, for every $t \in [0, T]$,

$$V_t(\phi) = Y_t^3 \left(x + \int_0^t \phi_u^2 \, dY_u^* + \int_0^t \frac{Z_u^3}{Y_{u-}^{1,3}} \, dY_u^{1,3} \right). \tag{5.16}$$

PROOF: As expected, we first set (note that the process ϕ^1 is a $\mathbf{G}$-predictable process)

$$\phi^1_t = \frac{1}{Y^1_{t-}}\Big(Z_t - \phi^2_t Y^2_{t-}\Big) \tag{5.17}$$

and

$$V^3_t = x + \int_0^t \phi^2_u \, dY^*_u + \int_0^t \frac{Z^3_u}{Y^{1,3}_{u-}} \, dY^{1,3}_u.$$

Arguing along the same lines as in the proof of Proposition 5.2.1, we obtain

$$V^3_t = V^3_0 + \sum_{i=1}^2 \int_0^t \phi^i_u \, dY^{i,3}_u.$$

Now, we define

$$\phi^3_t = V^3_t - \sum_{i=1}^2 \phi^i_t Y^{i,3}_t = (Y^3_t)^{-1}\Big(V_t - \sum_{i=1}^2 \phi^i_t Y^i_t\Big),$$

where $V_t = V^3_t Y^3_t$. As in the proof of Lemma 5.2.1, we check that

$$\phi^3_t = V^3_{t-} - \sum_{i=1}^2 \phi^i_t Y^{i,3}_{t-},$$

and thus the process ϕ^3 is $\mathbf{G}$-predictable. It is clear that the strategy $\phi = (\phi^1, \phi^2, \phi^3)$ is self-financing and its wealth process satisfies $V_t(\phi) = V_t$ for every $t \in [0, T]$. In particular, $V_T(\phi) = X$, so that ϕ replicates X. Finally, equality (5.17) implies (5.10), and thus ϕ belongs to the class $\Phi(Z)$. □

Note that equality (5.15) is a necessary (by Lemma 5.2.1) and sufficient (by Proposition 5.2.2) condition for the existence of a constrained strategy that replicates a given contingent claim X.

Synthetic asset

Let us take $Z = 0$, so that $\phi \in \Phi(0)$. Then the balance condition becomes $\sum_{i=1}^2 \phi^i_t Y^i_{t-} = 0$, and formula (5.11) reduces to

$$dV^3_t(\phi) = \phi^2_t \left(dY^{2,3}_t - \frac{Y^{2,3}_{t-}}{Y^{1,3}_{t-}} \, dY^{1,3}_t \right). \tag{5.18}$$

The process $\bar{Y}^2 = Y^3 Y^*$, where Y^* is defined in (5.14) is called a *synthetic asset*. It corresponds to a particular self-financing portfolio, with the long position in Y^2 and the short position of $Y^{2,1}_{t-}$ number of shares of Y^1, and suitably re-balanced positions in the third asset so that the portfolio is self-financing, as in Lemma 5.2.1.

It can be shown (see Bielecki *et al.* [16]) that trading in primary assets Y^1, Y^2, Y^3 is formally equivalent to trading in assets $Y^1, \bar{Y}^2, Y^3$. This observation supports the name synthetic asset attributed to the process $\bar{Y}^2$. Note, however, that the synthetic asset process may take negative values.

Case of continuous asset prices

In the case of continuous asset prices, the relative price $Y^* = \bar{Y}^2 (Y^3)^{-1}$ of the synthetic asset can be given an alternative representation, as the following result shows. Recall that the *predictable bracket* of the two continuous semi-martingales X and Y, denoted as $\langle X, Y\rangle$, coincides with their quadratic covariation $[X, Y]$.

Proposition 5.2.3 *Assume that the price processes Y^1 and Y^2 are continuous. Then the relative price of the synthetic asset satisfies*

$$Y_t^* = \int_0^t (Y_u^{3,1})^{-1} e^{\alpha_u}\, d\widehat{Y}_u,$$

where $\widehat{Y}_t := Y_t^{2,1} e^{-\alpha_t}$ and

$$\alpha_t := \langle \ln Y^{2,1}, \ln Y^{3,1} \rangle_t = \int_0^t (Y_u^{2,1})^{-1} (Y_u^{3,1})^{-1}\, d\langle Y^{2,1}, Y^{3,1} \rangle_u. \tag{5.19}$$

In terms of the auxiliary process $\widehat{Y}$, formula (5.11) *becomes*

$$V_t^3(\phi) = V_0^3(\phi) + \int_0^t \widehat{\phi}_u\, d\widehat{Y}_u + \int_0^t \frac{Z_u^3}{Y_{u-}^{1,3}}\, dY_u^{1,3}, \tag{5.20}$$

where $\widehat{\phi}_t = \phi_t^2 (Y_t^{3,1})^{-1} e^{\alpha_t}$.

PROOF: It suffices to give the proof for $Z = 0$. The proof relies on the integration by parts formula stating that for any two continuous semi-martingales, say X and Y, we have

$$Y_t^{-1}\big(dX_t - Y_t^{-1} d\langle X, Y \rangle_t\big) = d(X_t Y_t^{-1}) - X_t\, dY_t^{-1},$$

provided that Y is strictly positive. An application of this formula to processes $X = Y^{2,1}$ and $Y = Y^{3,1}$ leads to

$$(Y_t^{3,1})^{-1}\big(dY_t^{2,1} - (Y_t^{3,1})^{-1} d\langle Y^{2,1}, Y^{3,1} \rangle_t\big) = d(Y_t^{2,1}(Y_t^{3,1})^{-1}) - Y_t^{2,1}\, d(Y^{3,1})_t^{-1}.$$

The relative wealth $V_t^3(\phi) = V_t(\phi)(Y_t^3)^{-1}$ of a strategy $\phi \in \Phi(0)$ satisfies

$$\begin{aligned} V_t^3(\phi) &= V_0^3(\phi) + \int_0^t \phi_u^2\, dY_u^* \\ &= V_0^3(\phi) + \int_0^t \phi_u^2 (Y_u^{3,1})^{-1} e^{\alpha_u}\, d\widehat{Y}_u, \\ &= V_0^3(\phi) + \int_0^t \widehat{\phi}_u\, d\widehat{Y}_u \end{aligned}$$

where we denote $\widehat{\phi}_t = \phi_t^2 (Y_t^{3,1})^{-1} e^{\alpha_t}$.

Remark 5.2.1 The financial interpretation of the auxiliary process $\widehat{Y}$ will be studied below. Let us only observe here that if Y^* is a local martingale under some probability $\mathbb{Q}$ then $\widehat{Y}$ is a $\mathbb{Q}$-local martingale (and vice versa, if $\widehat{Y}$ is a $\widehat{\mathbb{Q}}$-local martingale under some probability $\widehat{\mathbb{Q}}$ then Y^* is a $\widehat{\mathbb{Q}}$-local martingale). Nevertheless, for the reader's convenience, we shall use two symbols $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$, since this equivalence holds for continuous processes only.
It is thus worth stressing that we will apply Proposition 5.2.3 to pre-default values of assets, rather than directly to asset prices, within the set-up of a semimartingale model with a common default, as described in Section 5.1.1. In this model, the asset prices may have discontinuities, but their pre-default values follow continuous processes.

5.3 Martingale Approach to Valuation and Hedging

Our goal is to derive quasi-explicit conditions for replicating strategies for a defaultable claim in a fairly general set-up introduced in Section 5.1.1. In this section, we only deal with trading strategies based on the reference filtration $\mathbf{F}$, and the underlying price processes (that is, prices of default-free assets and pre-default values of defaultable assets) are assumed to be continuous. Hence, our arguments will hinge on Proposition 5.2.3, rather than on a more general Proposition 5.2.1. We shall also adapt Proposition 5.2.2 to our current purposes.

To simplify the presentation, we make a standing assumption that all coefficient processes are such that the SDEs appearing below admit unique strong solutions, and all stochastic exponentials (used as Radon-Nikodým derivatives) are true martingales under respective probabilities.

5.3.1 Defaultable asset with total default

In this section, we shall examine in some detail a particular model where the two assets, Y^1 and Y^2, are default-free and satisfy

$$dY_t^i = Y_t^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t\big), \quad i = 1, 2,$$

where W is a one-dimensional Brownian motion. The third asset is a defaultable asset with total default, so that

$$dY_t^3 = Y_{t-}^3 \big(\mu_{3,t}\, dt + \sigma_{3,t}\, dW_t - dM_t\big).$$

Since we will be interested in replicating strategies in the sense of Definition 5.1.2, we may and do assume, without loss of generality, that the coefficients $\mu_{i,t}$, $\sigma_{i,t}$, $i = 1, 2$, are $\mathbf{F}$-predictable, rather than $\mathbf{G}$-predictable. Recall that, in general, there exist $\mathbf{F}$-predictable processes $\widetilde{\mu}_3$ and $\widetilde{\sigma}_3$ such that

$$\widetilde{\mu}_{3,t} \mathbb{1}_{\{t \le \tau\}} = \mu_{3,t} \mathbb{1}_{\{t \le \tau\}}, \quad \widetilde{\sigma}_{3,t} \mathbb{1}_{\{t \le \tau\}} = \sigma_{3,t} \mathbb{1}_{\{t \le \tau\}}. \tag{5.21}$$

We assume throughout that $Y_0^i > 0$ for every i, so that the price processes Y^1, Y^2 are strictly positive, and the process Y^3 is nonnegative, and has strictly positive pre-default value.

Default-free market

It is natural to postulate that the default-free market with the two traded assets, Y^1 and Y^2, is arbitrage-free. More precisely, we choose Y^1 as a numéraire, and we require that there exists a probability measure $\mathbb{P}^1$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, and such that the process $Y^{2,1}$ is a $\mathbb{P}^1$-martingale. The dynamics of processes $(Y^1)^{-1}$ and $Y^{2,1}$ are

$$d(Y_t^1)^{-1} = (Y_t^1)^{-1}\big((\sigma_{1,t}^2 - \mu_{1,t})\, dt - \sigma_{1,t}\, dW_t\big), \tag{5.22}$$

and

$$dY_t^{2,1} = Y_t^{2,1}\big((\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t}))\, dt + (\sigma_{2,t} - \sigma_{1,t})\, dW_t\big),$$

respectively. Hence, the necessary condition for the existence of an e.m.m. $\mathbb{P}^1$ is the inclusion $A \subseteq B$, where $A = \{(t,\omega) \in [0,T] \times \Omega :\, \sigma_{1,t}(\omega) = \sigma_{2,t}(\omega)\}$ and $B = \{(t,\omega) \in [0,T] \times \Omega :\, \mu_{1,t}(\omega) = \mu_{2,t}(\omega)\}$. The necessary and sufficient condition for the existence and uniqueness of an e.m.m. $\mathbb{P}^1$ reads

$$\mathbb{E}_{\mathbb{P}} \left\{ \mathcal{E}_T \left(\int_0^{\cdot} \theta_u\, dW_u \right) \right\} = 1 \tag{5.23}$$

where the process θ is given by the formula (by convention, $0/0 = 0$)

$$\theta_t = \sigma_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}, \quad \forall\, t \in [0,T]. \tag{5.24}$$

Note that in the case of constant coefficients, if $\sigma_1 = \sigma_2$ then the model is arbitrage-free only in the trivial case when $\mu_2 = \mu_1$.

Remark 5.3.1 Since the martingale measure $\mathbb{P}^1$ is unique, the default-free model (Y^1, Y^2) is complete. However, this is not a necessary assumption and thus it can be relaxed. As we shall see in what follows, it is typically more natural to assume that the driving Brownian motion W is multi-dimensional.

Arbitrage-free property

Let us now consider also a defaultable asset Y^3. Our goal is now to find a martingale measure $\mathbb{Q}^1$ (if it exists) for relative prices $Y^{2,1}$ and $Y^{3,1}$. Recall that we postulate that the hypothesis (H) holds under $\mathbb{P}$ for filtrations $\mathbf{F}$ and $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$. The dynamics of $Y^{3,1}$ under $\mathbb{P}$ are

$$dY_t^{3,1} = Y_{t-}^{3,1}\Big\{\big(\mu_{3,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{3,t})\big)dt + (\sigma_{3,t} - \sigma_{1,t})\,dW_t - dM_t\Big\}.$$

Let $\mathbb{Q}^1$ be any probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, and let η be the associated Radon-Nikodým density process, so that

$$d\mathbb{Q}^1\,|\,_{\mathcal{G}_t} = \eta_t\, d\mathbb{P}\,|\,_{\mathcal{G}_t}, \tag{5.25}$$

where the process η satisfies

$$d\eta_t = \eta_{t-}(\theta_t\, dW_t + \zeta_t\, dM_t) \tag{5.26}$$

for some $\mathbf{G}$-predictable processes θ and ζ, and η is a $\mathbf{G}$-martingale under $\mathbb{P}$.

From Girsanov's theorem, the processes $\widehat{W}$ and $\widehat{M}$, given by

$$\widehat{W}_t = W_t - \int_0^t \theta_u\, du, \quad \widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}}\gamma_u\zeta_u\, du, \tag{5.27}$$

are $\mathbf{G}$-martingales under $\mathbb{Q}^1$. To ensure that $Y^{2,1}$ is a $\mathbb{Q}^1$-martingale, we postulate that (5.23) and (5.24) are valid. Consequently, for the process $Y^{3,1}$ to be a $\mathbb{Q}^1$-martingale, it is necessary and sufficient that ζ satisfies

$$\gamma_t\zeta_t = \mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}(\sigma_{3,t} - \sigma_{1,t}).$$

To ensure that $\mathbb{Q}^1$ is a probability measure equivalent to $\mathbb{P}$, we require that $\zeta_t > -1$. The unique martingale measure $\mathbb{Q}^1$ is then given by the formula (5.25) where η solves (5.26), so that

$$\eta_t = \mathcal{E}_t\left(\int_0^{\cdot} \theta_u\, dW_u\right)\mathcal{E}_t\left(\int_0^{\cdot} \zeta_u\, dM_u\right).$$

We are in a position to formulate the following result.

Proposition 5.3.1 *Assume that the process θ given by (5.24) satisfies (5.23), and*

$$\zeta_t = \frac{1}{\gamma_t}\left(\mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}(\sigma_{3,t} - \sigma_{1,t})\right) > -1. \tag{5.28}$$

Then the model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is arbitrage-free and complete. The dynamics of relative prices under the unique martingale measure $\mathbb{Q}^1$ are

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1}(\sigma_{2,t} - \sigma_{1,t})\, d\widehat{W}_t, \\ dY_t^{3,1} &= Y_{t-}^{3,1}\big((\sigma_{3,t} - \sigma_{1,t})\, d\widehat{W}_t - d\widehat{M}_t\big). \end{aligned}$$

Since the coefficients $\mu_{i,t}$, $\sigma_{i,t}$, $i = 1, 2$, are $\mathbf{F}$-adapted, the process $\widehat{W}$ is an $\mathbf{F}$-martingale (hence, a Brownian motion) under $\mathbb{Q}^1$. Hence, by virtue of Proposition 4.2.4, the hypothesis (H) holds under $\mathbb{Q}^1$, and the $\mathbf{F}$-intensity of default under $\mathbb{Q}^1$ equals

$$\widehat{\gamma}_t = \gamma_t(1 + \zeta_t) = \gamma_t + \left(\mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}(\sigma_{3,t} - \sigma_{1,t})\right).$$

Example 5.3.1 We present an example where the condition (5.28) does not hold, and thus arbitrage opportunities arise. Assume the coefficients are constant and satisfy: $\mu_1 = \mu_2 = \sigma_1 = 0$, $\mu_3 < -\gamma$ for a constant default intensity $\gamma > 0$. Then

$$Y_t^3 = 1\!\!1_{\{t<\tau\}} Y_0^3 \exp\left(\sigma_3 W_t - \frac{1}{2}\sigma_3^2 t + (\mu_3 + \gamma)t\right) \leq Y_0^3 \exp\left(\sigma_3 W_t - \frac{1}{2}\sigma_3^2 t\right) = V_t(\phi),$$

where $V(\phi)$ represents the wealth of a self-financing strategy $(\phi^1, \phi^2, 0)$ with $\phi^2 = \frac{\sigma_3}{\sigma_2}$. Hence, the arbitrage strategy would be to sell the asset Y^3, and to follow the strategy ϕ.

Remark 5.3.2 Let us stress once again, that the existence of an e.m.m. is a necessary condition for viability of a financial model, but the uniqueness of an e.m.m. is not always a convenient condition to impose on a model. In fact, when constructing a model, we should be mostly concerned with its flexibility and ability to reflect the pertinent risk factors, rather than with its mathematical completeness. In the present context, it is natural to postulate that the dimension of the underlying Brownian motion equals the number of tradeable risky assets. In addition, each particular model should be tailored to provide intuitive and handy solutions for a predetermined family of contingent claims that will be priced and hedged within its framework.

Hedging a survival claim

We first focus on replication of a *survival claim* $(X, 0, \tau)$, that is, a defaultable claim represented by the terminal payoff $X 1\!\!1_{\{T<\tau\}}$, where X is an $\mathcal{F}_T$-measurable random variable. For the moment, we maintain the simplifying assumption that W is one-dimensional. As we shall see in what follows, it may lead to certain pathological features of a model. If, on the contrary, the driving noise is multi-dimensional, most of the analysis remains valid, except that the model completeness is no longer ensured, in general.

Recall that $\widetilde{Y}^3$ stands for the pre-default price of Y^3, defined as (see (5.3))

$$d\widetilde{Y}_t^3 = \widetilde{Y}_t^3 \big((\widetilde{\mu}_{3,t} + \gamma_t)\, dt + \widetilde{\sigma}_{3,t}\, dW_t \big) \tag{5.29}$$

with $\widetilde{Y}_0^3 = Y_0^3$. This strictly positive, continuous, **F**-adapted process enjoys the property that $Y_t^3 = 1\!\!1_{\{t<\tau\}} \widetilde{Y}_t^3$. Let us denote the pre-default values in the numéraire $\widetilde{Y}^3$ by $\widetilde{Y}_t^{i,3} = Y_t^i (\widetilde{Y}_t^3)^{-1}$, $i = 1, 2$, and let us introduce the pre-default relative price $\widetilde{Y}^*$ of the synthetic asset $\bar{Y}^2$ by setting

$$d\widetilde{Y}_t^* := d\widetilde{Y}_t^{2,3} - \frac{\widetilde{Y}_t^{2,3}}{\widetilde{Y}_t^{1,3}}\, d\widetilde{Y}_t^{1,3} = \widetilde{Y}_t^{2,3} \Big(\big(\mu_{2,t} - \mu_{1,t} + \widetilde{\sigma}_{3,t}(\sigma_{1,t} - \sigma_{2,t})\big)\, dt + (\sigma_{2,t} - \sigma_{1,t})\, dW_t \Big),$$

and let us assume that $\sigma_{1,t} - \sigma_{2,t} \neq 0$. It is also useful to note that the process $\widehat{Y}$, defined in Proposition 5.2.3, satisfies

$$d\widehat{Y}_t = \widehat{Y}_t \Big(\big(\mu_{2,t} - \mu_{1,t} + \widetilde{\sigma}_{3,t}(\sigma_{1,t} - \sigma_{2,t})\big)\, dt + (\sigma_{2,t} - \sigma_{1,t})\, dW_t \Big).$$

We shall show that in the case, where α given by (5.19) is deterministic, the process $\widehat{Y}$ has a nice financial interpretation as a credit-risk adjusted forward price of Y^2 relative to Y^1. Therefore, it is more convenient to work with the process $\widetilde{Y}^*$ when dealing with the general case, but to use the process $\widehat{Y}$ when analyzing a model with deterministic volatilities.

Consider an **F**-predictable self-financing strategy ϕ satisfying the balance condition $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0$, and the corresponding wealth process

$$V_t(\phi) := \sum_{i=1}^{3} \phi_t^i Y_t^i = \phi_t^3 Y_t^3.$$

Let $\widetilde{V}_t(\phi) := \phi^3_t \widetilde{Y}^3_t$. Since the process $\widetilde{V}(\phi)$ is $\mathbf{F}$-adapted, we see that this is the *pre-default price* process of the portfolio ϕ, that is, we have $1\!\!1_{\{\tau>t\}} V_t(\phi) = 1\!\!1_{\{\tau>t\}} \widetilde{V}_t(\phi)$; we shall call this process the *pre-default wealth* of ϕ. Consequently, the process $\widetilde{V}^3_t(\phi) := \widetilde{V}_t(\phi)(\widetilde{Y}^3_t)^{-1} = \phi^3_t$ is termed the relative pre-default wealth.

Using Proposition 5.2.1, with suitably modified notation, we find that the $\mathbf{F}$-adapted process $\widetilde{V}^3(\phi)$ satisfies, for every $t \in [0,T]$,

$$\widetilde{V}^3_t(\phi) = \widetilde{V}^3_0(\phi) + \int_0^t \phi^2_u \, d\widetilde{Y}^*_u.$$

Define a new probability $\mathbb{Q}^*$ on $(\Omega, \mathcal{F}_T)$ by setting

$$d\mathbb{Q}^* = \eta^*_T \, d\mathbb{P},$$

where $d\eta^*_t = \eta^*_t \theta^*_t \, dW_t$, and

$$\theta^*_t = \frac{\mu_{2,t} - \mu_{1,t} + \widetilde{\sigma}_{3,t}(\sigma_{1,t} - \sigma_{2,t})}{\sigma_{1,t} - \sigma_{2,t}}. \tag{5.30}$$

The process $\widetilde{Y}^*_t$, $t \in [0,T]$, is a (local) martingale under $\mathbb{Q}^*$ driven by a Brownian motion. We shall require that this process is in fact a true martingale; a sufficient condition for this is that

$$\int_0^T \mathbb{E}_{\mathbb{Q}^*} \left(\widetilde{Y}^{2,3}_t (\sigma_{2,t} - \sigma_{1,t}) \right)^2 dt < \infty.$$

From the predictable representation theorem, it follows that for any $X \in \mathcal{F}_T$, such that $X(\widetilde{Y}^3_T)^{-1}$ is square-integrable under $\mathbb{Q}$, there exists a constant x and an $\mathbf{F}$-predictable process ϕ^2 such that

$$X = \widetilde{Y}^3_T \left(x + \int_0^T \phi^2_u \, d\widetilde{Y}^*_u \right). \tag{5.31}$$

We now deduce from Proposition 5.2.2 that there exists a self-financing strategy ϕ with the pre-default wealth $\widetilde{V}_t(\phi) = \widetilde{Y}^3_t \widetilde{V}^3_t$ for every $t \in [0,T]$, where we set

$$\widetilde{V}^3_t = x + \int_0^t \phi^2_u \, d\widetilde{Y}^*_u. \tag{5.32}$$

Moreover, it satisfies the balance condition $\phi^1_t Y^1_t + \phi^2_t Y^2_t = 0$ for every $t \in [0,T]$. Since clearly $\widetilde{V}_T(\phi) = X$, we have that

$$V_T(\phi) = \phi^3_T Y^3_T = 1\!\!1_{\{T<\tau\}} \phi^3_T \widetilde{Y}^3_T = 1\!\!1_{\{T<\tau\}} \widetilde{V}_T(\phi) = 1\!\!1_{\{T<\tau\}} X,$$

and thus this strategy replicates the survival claim $(X, 0, \tau)$. In fact, we have that $V_t(\phi) = 0$ on the random interval $[\![\tau, T]\!]$.

Definition 5.3.1 *We say that a survival claim $(X, 0, \tau)$ is attainable if the process $\widetilde{V}^3$ given by (5.32) is a martingale under $\mathbb{Q}^*$.*

The following result is an immediate consequence of (5.31) and (5.32).

Corollary 5.3.1 *Let $X \in \mathcal{F}_T$ be such that $X(\widetilde{Y}^3_T)^{-1}$ is square-integrable under $\mathbb{Q}^*$. Then the survival claim $(X, 0, \tau)$ is attainable. Moreover, the pre-default price $\widetilde{\pi}_t(X, 0, \tau)$ of the claim $(X, 0, \tau)$ is given by the conditional expectation*

$$\widetilde{\pi}_t(X, 0, \tau) = \widetilde{Y}^3_t \, \mathbb{E}_{\mathbb{Q}^*}(X(\widetilde{Y}^3_T)^{-1} \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T]. \tag{5.33}$$

The process $\widetilde{\pi}(X, 0, \tau)(\widetilde{Y}^3)^{-1}$ is an $\mathbf{F}$-martingale under $\mathbb{Q}$.

Proof: Since $X(\widetilde{Y}^3_T)^{-1}$ is square-integrable under $\mathbb{Q}$, we know from the predictable representation theorem that ϕ^2 in (5.31) is such that $\mathbb{E}_{\mathbb{Q}^*}\left(\int_0^T (\phi^2_t)^2 \, d\langle \widetilde{Y}^* \rangle_t\right) < \infty$, so that the process $\widetilde{V}^3$ given by (5.32) is a true martingale under $\mathbb{Q}$. We conclude that $(X, 0, \tau)$ is attainable.

Now, let us denote by $\pi_t(X, 0, \tau)$ the time-t price of the claim $(X, 0, \tau)$. Since ϕ is a hedging portfolio for $(X, 0, \tau)$ we thus have $V_t(\phi) = \pi_t(X, 0, \tau)$ for each $t \in [0, T]$. Consequently,

$$
\begin{aligned}
\mathbb{1}_{\{\tau>t\}}\widetilde{\pi}_t(X,0,\tau) &= \mathbb{1}_{\{\tau>t\}}\widetilde{V}_t(\phi) = \mathbb{1}_{\{\tau>t\}}\widetilde{Y}^3_t \, \mathbb{E}_{\mathbb{Q}^*}(\widetilde{V}^3_T \,|\, \mathcal{F}_t) \\
&= \mathbb{1}_{\{\tau>t\}}\widetilde{Y}^3_t \, \mathbb{E}_{\mathbb{Q}^*}(X(\widetilde{Y}^3_T)^{-1} \,|\, \mathcal{F}_t)
\end{aligned}
$$

for each $t \in [0, T]$. This proves equality (5.33). □

In view of the last result, it is justified to refer to $\mathbb{Q}$ as the *pricing measure relative to* Y^3 for attainable survival claims.

Remark 5.3.3 It can be proved that there exists a unique absolutely continuous probability measure $\bar{\mathbb{Q}}$ on $(\Omega, \mathcal{G}_T)$ such that we have

$$
Y^3_t \, \mathbb{E}_{\bar{\mathbb{Q}}}\left(\frac{\mathbb{1}_{\{\tau>T\}}X}{Y^3_T} \,\Big|\, \mathcal{G}_t\right) = \mathbb{1}_{\{\tau>t\}}\widetilde{Y}^3_t \, \mathbb{E}_{\mathbb{Q}^*}\left(\frac{X}{\widetilde{Y}^3_T} \,\Big|\, \mathcal{F}_t\right).
$$

However, this probability measure is not equivalent to $\mathbb{Q}$, since its Radon-Nikodým density vanishes after τ (for a related result, see Collin-Dufresne *et al.* [48]).

Example 5.3.2 We provide here an explicit calculation of the pre-default price of a survival claim. For simplicity, we assume that $X = 1$, so that the claim represents a defaultable zero-coupon bond. Also, we set $\gamma_t = \gamma = const$, $\mu_{i,t} = 0$, and $\sigma_{i,t} = \sigma_i$, $i = 1, 2, 3$. Straightforward calculations yield the following pricing formula

$$
\widetilde{\pi}_0(1, 0, \tau) = Y^3_0 e^{-(\gamma + \frac{1}{2}\sigma^2_3)T}.
$$

We see that here the pre-default price $\widetilde{\pi}_0(1, 0, \tau)$ depends explicitly on the intensity γ, or rather, on the drift term in dynamics of pre-default value of defaultable asset. Indeed, from the practical viewpoint, the interpretation of the drift coefficient in dynamics of Y^2 as the real-world default intensity is questionable, since within our set-up the default intensity never appears as an independent variable, but is merely a component of the drift term in dynamics of pre-default value of Y^3.

Note also that we deal here with a model with three tradeable assets driven by a one-dimensional Brownian motion. No wonder that the model enjoys completeness, but as a downside, it has an undesirable property that the pre-default values of all three assets are perfectly correlated. Consequently, the drift terms in dynamics of traded assets are closely linked to each other, in the sense, that their behavior under an equivalent change of a probability measure is quite specific.

As we shall see later, if traded primary assets are judiciously chosen then, typically, the pre-default price (and hence the price) of a survival claim will not explicitly depend on the intensity process.

Remark 5.3.4 Generally speaking, we believe that one can classify a financial model as 'realistic' if its implementation does not require estimation of drift parameters in (pre-default) prices, at least for the purpose of hedging and valuation of a sufficiently large class of (defaultable) contingent claims of interest. It is worth recalling that the drift coefficients are not assumed to be market observables. Since the default intensity can formally interpreted as a component of the drift term in dynamics of pre-default prices, in a realistic model there is no need to estimate this quantity. From this perspective, the model considered in Example 5.3.2 may serve as an example of an 'unrealistic' model, since its implementation requires the knowledge of the drift parameter in the dynamics of Y^3. We do not pretend here that it is always possible to hedge derivative assets without using the drift coefficients in dynamics of tradeable assets, but it seems to us that a good idea is to develop models in which this knowledge is not essential.

Of course, a generic semimartingale model considered until now provides only a framework for a construction of realistic models for hedging of default risk. A choice of tradeable assets and specification of their dynamics should be examined on a case-by-case basis, rather than in a general semimartingale set-up. We shall address this important issue in the foregoing sections, in which we shall deal with particular examples of practically interesting defaultable claims.

Hedging a recovery process

Let us now briefly study the situation where the promised payoff equals zero, and the recovery payoff is paid at time τ and equals Z_τ for some $\mathbf{F}$-adapted process Z. Put another way, we consider a defaultable claim of the form $(0, Z, \tau)$. Once again, we make use of Propositions 5.2.1 and 5.2.2. In view of (5.15), we need to find a constant x and an $\mathbf{F}$-predictable process ϕ^2 such that

$$\psi_T := -\int_0^T \frac{Z_t}{Y_t^1}\, d\widetilde{Y}_t^{1,3} = x + \int_0^T \phi_t^2\, d\widetilde{Y}_t^*. \tag{5.34}$$

Similarly as before, we conclude that, under suitable integrability conditions on ψ_T, there exists ϕ^2 such that $d\psi_t = \phi_t^2\, dY_t^*$, where $\psi_t = \mathbb{E}_{\mathbb{Q}^*}(\psi_T \,|\, \mathcal{F}_t)$. We now set

$$\widetilde{V}_t^3 = x + \int_0^t \phi_u^2\, dY_u^* + \int_0^T \frac{\widetilde{Z}_u^3}{\widetilde{Y}_u^{1,3}}\, d\widetilde{Y}_u^{1,3},$$

so that, in particular, $\widetilde{V}_T^3 = 0$. Then it is possible to find processes ϕ^1 and ϕ^3 such that the strategy ϕ is self-financing and it satisfies: $\widetilde{V}_t(\phi) = \widetilde{V}_t^3 \widetilde{Y}_t^3$ and $V_t(\phi) = Z_t + \phi_t^3 Y_t^3$ for every $t \in [0, T]$. It is thus clear that $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \leq T\}$ and $V_T(\phi) = 0$ on the set $\{\tau > T\}$.

Bond market

For the sake of concreteness, we assume that $Y_t^1 = B(t, T)$ is the price of a default-free ZC-bond with maturity T, and $Y_t^3 = D(t, T)$ is the price of a defaultable ZC-bond with zero recovery, that is, an asset with the terminal payoff $Y_T^3 = \mathbb{1}_{\{T<\tau\}}$. We postulate that the dynamics under $\mathbb{P}$ of the default-free ZC-bond are

$$dB(t, T) = B(t, T)\big(\mu(t, T)\, dt + b(t, T)\, dW_t\big) \tag{5.35}$$

for some $\mathbf{F}$-predictable processes $\mu(t, T)$ and $b(t, T)$. We choose the process $Y_t^1 = B(t, T)$ as a numéraire. Since the prices of the other two assets are not given a priori, we may choose any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ to play the role of $\mathbb{Q}^1$.

In such a case, an e.m.m. $\mathbb{Q}^1$ is referred to as the *forward martingale measure* for the date T, and is denoted by $\mathbb{Q}_T$. Hence, the Radon-Nikodým density of $\mathbb{Q}_T$ with respect to $\mathbb{P}$ is given by (5.26) for some $\mathbf{F}$-predictable processes θ and ζ, and the process

$$W_t^T = W_t - \int_0^t \theta_u\, du, \quad \forall\, t \in [0, T],$$

is a Brownian motion under $\mathbb{Q}_T$. Under $\mathbb{Q}_T$ the default-free ZC-bond is governed by

$$dB(t, T) = B(t, T)\big(\widehat{\mu}(t, T)\, dt + b(t, T)\, dW_t^T\big)$$

where $\widehat{\mu}(t, T) = \mu(t, T) + \theta_t b(t, T)$. Let $\widehat{\Gamma}$ stand for the $\mathbf{F}$-hazard process of τ under $\mathbb{Q}_T$, so that $\widehat{\Gamma}_t = -\ln(1 - \widehat{F}_t)$, where $\widehat{F}_t = \mathbb{Q}_T(\tau \leq t \,|\, \mathcal{F}_t)$. Assume that the hypothesis (H) holds under $\mathbb{Q}_T$ so that, in particular, the process $\widehat{\Gamma}$ is increasing. We define the price process of a defaultable ZC-bond with zero recovery by the formula

$$D(t, T) := B(t, T)\mathbb{E}_{\mathbb{Q}_T}(\mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} B(t, T)\, \mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T} \,|\, \mathcal{F}_t\big),$$

It is then clear that $Y_t^{3,1} = D(t,T)(B(t,T))^{-1}$ is a $\mathbb{Q}_T$-martingale, and the pre-default price $\widetilde{D}(t,T)$ equals

$$\widetilde{D}(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T}\,\big|\,\mathcal{F}_t\big).$$

The next result examines the basic properties of the auxiliary process $\widehat{\Gamma}(t,T)$ given as, for every $t \in [0,T]$,

$$\widehat{\Gamma}(t,T) = \widetilde{Y}_t^{3,1} = \widetilde{D}(t,T)(B(t,T))^{-1} = \mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T}\,\big|\,\mathcal{F}_t\big).$$

The quantity $\widehat{\Gamma}(t,T)$ can be interpreted as the conditional probability (under $\mathbb{Q}_T$) that default will not occur prior to the maturity date T, given that we observe $\mathcal{F}_t$ and we know that the default has not yet happened. We will be more interested, however, in its volatility process $\beta(t,T)$ as defined in the following result.

Lemma 5.3.1 *Assume that the* $\mathbf{F}$*-hazard process* $\widehat{\Gamma}$ *of* τ *under* $\mathbb{Q}_T$ *is continuous. Then the process* $\widehat{\Gamma}(t,T)$, $t \in [0,T]$, *is a continuous* $\mathbf{F}$*-submartingale and*

$$d\widehat{\Gamma}(t,T) = \widehat{\Gamma}(t,T)\big(d\widehat{\Gamma}_t + \beta(t,T)\,dW_t^T\big) \tag{5.36}$$

for some $\mathbf{F}$*-predictable process* $\beta(t,T)$*. The process* $\widehat{\Gamma}(t,T)$ *is of finite variation if and only if the hazard process* $\widehat{\Gamma}$ *is deterministic. In this case, we have* $\widehat{\Gamma}(t,T) = e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T}$.

Proof: We have

$$\widehat{\Gamma}(t,T) = \mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T}\,|\,\mathcal{F}_t\big) = e^{\widehat{\Gamma}_t} L_t,$$

where we set $L_t = \mathbb{E}_{\mathbb{Q}_T}\big(e^{-\widehat{\Gamma}_T}\,|\,\mathcal{F}_t\big)$. Hence, $\widehat{\Gamma}(t,T)$ is equal to the product of a strictly positive, increasing, right-continuous, $\mathbf{F}$-adapted process $e^{\widehat{\Gamma}_t}$, and a strictly positive, continuous $\mathbf{F}$-martingale L. Furthermore, there exists an $\mathbf{F}$-predictable process $\widehat{\beta}(t,T)$ such that L satisfies

$$dL_t = L_t \widehat{\beta}(t,T)\,dW_t^T$$

with the initial condition $L_0 = \mathbb{E}_{\mathbb{Q}_T}\big(e^{-\widehat{\Gamma}_T}\big)$. Formula (5.36) now follows by an application of Itô's formula, by setting $\beta(t,T) = e^{-\widehat{\Gamma}_t}\widehat{\beta}(t,T)$. To complete the proof, it suffices to recall that a continuous martingale is never of finite variation, unless it is a constant process. □

Remark 5.3.5 It can be checked that $\beta(t,T)$ is also the volatility of the process

$$\Gamma(t,T) = \mathbb{E}_{\mathbb{P}}\big(e^{\Gamma_t - \Gamma_T}\,\big|\,\mathcal{F}_t\big).$$

Assume that $\widehat{\Gamma}_t = \int_0^t \widehat{\gamma}_u\,du$ for some $\mathbf{F}$-predictable, nonnegative process $\widehat{\gamma}$. Then we have the following auxiliary result, which gives, in particular, the volatility of the defaultable ZC-bond.

Corollary 5.3.2 *The dynamics under* $\mathbb{Q}_T$ *of the pre-default price* $\widetilde{D}(t,T)$ *equals*

$$d\widetilde{D}(t,T) = \widetilde{D}(t,T)\Big(\big(\widehat{\mu}(t,T) + b(t,T)\beta(t,T) + \widehat{\gamma}_t\big)\,dt + \big(b(t,T) + \beta(t,T)\big)\widetilde{d}(t,T)\,dW_t^T\Big).$$

Equivalently, the price $D(t,T)$ *of the defaultable ZC-bond satisfies under* $\mathbb{Q}_T$

$$dD(t,T) = D(t,T)\Big(\big(\widehat{\mu}(t,T) + b(t,T)\beta(t,T)\big)\,dt + \widetilde{d}(t,T)\,dW_t^T - dM_t\Big).$$

where we set $\widetilde{d}(t,T) = b(t,T) + \beta(t,T)$.

Note that the process $\beta(t,T)$ can be expressed in terms of market observables, since it is simply the difference of volatilities $\widetilde{d}(t,T)$ and $b(t,T)$ of pre-default prices of tradeable assets.

Credit-risk-adjusted forward price

Assume that the price Y^2 satisfies under the statistical probability $\mathbb{P}$

$$dY_t^2 = Y_t^2\big(\mu_{2,t}\,dt + \sigma_t\,dW_t\big) \tag{5.37}$$

with $\mathbf{F}$-predictable coefficients μ and σ. Let $F_{Y^2}(t,T) = Y_t^2(B(t,T))^{-1}$ be the forward price of Y_T^2. For an appropriate choice of θ (see 5.30), we shall have that

$$dF_{Y^2}(t,T) = F_{Y^2}(t,T)\big(\sigma_t - b(t,T)\big)\,dW_t^T.$$

Therefore, the dynamics of the pre-default synthetic asset $\widetilde{Y}_t^*$ under $\mathbb{Q}^T$ are

$$d\widetilde{Y}_t^* = \widetilde{Y}_t^{2,3}\big(\sigma_t - b(t,T)\big)\,\big(dW_t^T - \beta(t,T)\,dt\big),$$

and the process $\widehat{Y}_t = Y_t^{2,1}e^{-\alpha_t}$ (see Proposition 5.2.3 for the definition of α) satisfies

$$d\widehat{Y}_t = \widehat{Y}_t\big(\sigma_t - b(t,T)\big)\,\big(dW_t^T - \beta(t,T)\,dt\big).$$

Let $\widehat{\mathbb{Q}}$ be an equivalent probability measure on $(\Omega,\mathcal{G}_T)$ such that $\widehat{Y}$ (or, equivalently, $\widetilde{Y}^*$) is a $\widehat{\mathbb{Q}}$-martingale. By virtue of Girsanov's theorem, the process $\widehat{W}$ given by the formula

$$\widehat{W}_t = W_t^T - \int_0^t \beta(u,T)\,du, \quad \forall\, t\in[0,T],$$

is a Brownian motion under $\widehat{\mathbb{Q}}$. Thus, the forward price $F_{Y^2}(t,T)$ satisfies under $\widehat{\mathbb{Q}}$

$$dF_{Y^2}(t,T) = F_{Y^2}(t,T)\big(\sigma_t - b(t,T)\big)\big(d\widehat{W}_t + \beta(t,T)\,dt\big). \tag{5.38}$$

It appears that the valuation results are easier to interpret when they are expressed in terms of forward prices associated with vulnerable forward contracts, rather than in terms of spot prices of primary assets. For this reason, we shall now examine credit-risk-adjusted forward prices of default-free and defaultable assets.

Definition 5.3.2 *Let Y be a $\mathcal{G}_T$-measurable claim. An $\mathcal{F}_t$-measurable random variable K is called the credit-risk-adjusted forward price of Y if the pre-default value at time t of the vulnerable forward contract represented by the claim $\mathbb{1}_{\{T<\tau\}}(Y-K)$ equals 0.*

Lemma 5.3.2 *The credit-risk-adjusted forward price $\widehat{F}_Y(t,T)$ of an attainable survival claim $(X,0,\tau)$, represented by a $\mathcal{G}_T$-measurable claim $Y = X\mathbb{1}_{\{T<\tau\}}$, equals $\widetilde{\pi}_t(X,0,\tau)(\widetilde{D}(t,T))^{-1}$, where $\widetilde{\pi}_t(X,0,\tau)$ is the pre-default price of $(X,0,\tau)$. The process $\widehat{F}_Y(t,T)$, $t\in[0,T]$, is an $\mathbf{F}$-martingale under $\widehat{\mathbb{Q}}$.*

Proof: The forward price is defined as an $\mathcal{F}_t$-measurable random variable K such that the claim

$$\mathbb{1}_{\{T<\tau\}}(X\mathbb{1}_{\{T<\tau\}} - K) = X\mathbb{1}_{\{T<\tau\}} - KD(T,T)$$

is worthless at time t on the set $\{t<\tau\}$. It is clear that the pre-default value at time t of this claim equals $\widetilde{\pi}_t(X,0,\tau) - K\widetilde{D}(t,T)$. Consequently, we obtain $\widetilde{F}_Y(t,T) = \widetilde{\pi}_t(X,0,\tau)(\widetilde{D}(t,T))^{-1}$. □

Let us now focus on default-free assets. Manifestly, the credit-risk-adjusted forward price of the bond $B(t,T)$ equals 1. To find the credit-risk-adjusted forward price of Y^2, let us write

$$\widehat{F}_{Y^2}(t,T) := F_{Y^2}(t,T)\,e^{\alpha_T-\alpha_t} = Y_t^{2,1}e^{\alpha_T-\alpha_t}, \tag{5.39}$$

where α is given by (see (5.19))

$$\alpha_t = \int_0^t \big(\sigma_u - b(u,T)\big)\beta(u,T)\,du = \int_0^t \big(\sigma_u - b(u,T)\big)\big(\widetilde{d}(u,T) - b(u,T)\big)\,du. \tag{5.40}$$

Lemma 5.3.3 *Assume that α given by* (5.40) *is a deterministic function. Then the credit-risk-adjusted forward price of Y^2 equals $\widehat{F}_{Y^2}(t,T)$ (defined in 5.39) for every $t \in [0,T]$.*

PROOF: According to Definition 5.3.2, the price $\widehat{F}_{Y^2}(t,T)$ is an $\mathcal{F}_t$-measurable random variable K, which makes the forward contract represented by the claim $D(T,T)(Y_T^2 - K)$ worthless on the set $\{t < \tau\}$. Assume that the claim $Y_T^2 - K$ is attainable. Since $\widetilde{D}(T,T) = 1$, from equation (5.33) it follows that the pre-default value of this claim is given by the conditional expectation

$$\widetilde{D}(t,T)\, \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^2 - K \,\big|\, \mathcal{F}_t\big).$$

Consequently,

$$\widehat{F}_{Y^2}(t,T) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^2 \,\big|\, \mathcal{F}_t\big) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(F_{Y^2}(T,T) \,\big|\, \mathcal{F}_t\big) = F_{Y^2}(t,T)\, e^{\alpha_T - \alpha_t},$$

as was claimed. □

It is worth noting that the process $\widehat{F}_{Y^2}(t,T)$ is a (local) martingale under the pricing measure $\widehat{\mathbb{Q}}$, since it satisfies

$$d\widehat{F}_{Y^2}(t,T) = \widehat{F}_{Y^2}(t,T)(\sigma_t - b(t,T))\, d\widehat{W}_t. \tag{5.41}$$

Under the present assumptions, the auxiliary process $\widehat{Y}$ introduced in Proposition 5.2.3 and the credit-risk-adjusted forward price $\widehat{F}_{Y^2}(t,T)$ are closely related to each other. Indeed, we have $\widehat{F}_{Y^2}(t,T) = \widehat{Y}_t e^{\alpha_T}$, so that the two processes are proportional.

Vulnerable option on a default-free asset

We shall now analyze a vulnerable call option with the payoff

$$C_T^d = \mathbb{1}_{\{T<\tau\}}(Y_T^2 - K)^+.$$

Here K is a constant. Our goal is to find a replicating strategy for this claim, interpreted as a survival claim $(X, 0, \tau)$ with the promised payoff $X = C_T = (Y_T^2 - K)^+$, where C_T is the payoff of an equivalent non-vulnerable option. The method presented below is quite general, however, so that it can be applied to any survival claim with the promised payoff $X = G(Y_T^2)$ for some function $G : \mathbb{R} \to \mathbb{R}$ satisfying the usual integrability assumptions.

We assume that $Y_t^1 = B(t,T)$, $Y_t^3 = D(t,T)$ and the price of a default-free asset Y^2 is governed by (5.37). Then

$$C_T^d = \mathbb{1}_{\{T<\tau\}}(Y_T^2 - K)^+ = \mathbb{1}_{\{T<\tau\}}(Y_T^2 - KY_T^1)^+.$$

We are going to apply Proposition 5.2.3. In the present set-up, we have $Y_t^{2,1} = F_{Y^2}(t,T)$ and $\widehat{Y}_t = F_{Y^2}(t,T)e^{-\alpha_t}$. Since a vulnerable option is an example of a survival claim, in view of Lemma 5.3.2, its credit-risk-adjusted forward price satisfies $\widehat{F}_{C^d}(t,T) = \widetilde{C}_t^d(\widetilde{D}(t,T))^{-1}$.

Proposition 5.3.2 *Suppose that the volatilities σ, b and β are deterministic functions. Then the credit-risk-adjusted forward price of a vulnerable call option written on a default-free asset Y^2 equals*

$$\widehat{F}_{C^d}(t,T) = \widehat{F}_{Y^2}(t,T)N(d_+(\widehat{F}_{Y^2}(t,T),t,T)) - KN(d_-(\widehat{F}_{Y^2}(t,T),t,T)) \tag{5.42}$$

where

$$d_\pm(z,t,T) = \frac{\ln z - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_u - b(u,T))^2\, du.$$

The replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi_t^1 B(t,T) = -\phi_t^2 Y_t^2,\ \ \phi_t^2 = \widetilde{D}(t,T)(B(t,T))^{-1}N(d_+(t,T))e^{\alpha_T - \alpha_t},\ \ \phi_t^3 \widetilde{D}(t,T) = \widetilde{C}_t^d,$$

where $d_+(t,T) = d_+(\widehat{F}_{Y^2}(t,T),t,T)$.

PROOF: In the first step, we establish the valuation formula. Assume for the moment that the option is attainable. Then the pre-default value of the option equals, for every $t \in [0,T]$,

$$\widetilde{C}^d_t = \widetilde{D}(t,T)\,\mathbb{E}_{\widehat{\mathbb{Q}}}\big((F_{Y^2}(T,T)-K)^+ \,\big|\, \mathcal{F}_t\big) = \widetilde{D}(t,T)\,\mathbb{E}_{\widehat{\mathbb{Q}}}\big((\widehat{F}_{Y^2}(T,T)-K)^+ \,\big|\, \mathcal{F}_t\big). \tag{5.43}$$

In view of (5.41), the conditional expectation above can be computed explicitly, yielding the valuation formula (5.42).

To find the replicating strategy, and establish attainability of the option, we consider the Itô differential $d\widehat{F}_{C^d}(t,T)$ and we identify terms in (5.32). It appears that

$$\begin{aligned} d\widehat{F}_{C^d}(t,T) &= N(d_+(t,T))\,d\widehat{F}_{Y^2}(t,T) = N(d_+(t,T))e^{\alpha_T}\,d\widehat{Y}_t \\ &= N(d_+(t,T))\widetilde{Y}^{3,1}_t e^{\alpha_T-\alpha_t}\,d\widetilde{Y}^*_t, \end{aligned} \tag{5.44}$$

so that the process ϕ^2 in (5.31) equals

$$\phi^2_t = \widetilde{Y}^{3,1}_t N(d_+(t,T))e^{\alpha_T-\alpha_t}.$$

Moreover, ϕ^1 is such that $\phi^1_t B(t,T) + \phi^2_t Y^2_t = 0$ and $\phi^3_t = \widetilde{C}^d_t(\widetilde{D}(t,T))^{-1}$. It is easily seen that this proves also the attainability of the option. □

Let us examine the financial interpretation of the last result.

First, equality (5.44) shows that it is easy to replicate the option using vulnerable forward contracts. Indeed, we have

$$\widehat{F}_{C^d}(T,T) = X = \frac{\widetilde{C}^d_0}{\widetilde{D}(0,T)} + \int_0^T N(d_+(t,T))\,d\widehat{F}_{Y^2}(t,T)$$

and thus it is enough to invest the premium $\widetilde{C}^d_0 = C^d_0$ in defaultable ZC-bonds of maturity T, and take at any instant t prior to default $N(d_+(t,T))$ positions in vulnerable forward contracts. It is understood that if default occurs prior to T, all outstanding vulnerable forward contracts become void.

Second, it is worth stressing that neither the arbitrage price, nor the replicating strategy for a vulnerable option, depend explicitly on the default intensity. This remarkable feature is due to the fact that the default risk of the writer of the option can be completely eliminated by trading in defaultable zero-coupon bond with the same exposure to credit risk as a vulnerable option.

In fact, since the volatility β is invariant with respect to an equivalent change of a probability measure, and so are the volatilities σ and $b(t,T)$, the formulae of Proposition 5.3.2 are valid for any choice of a forward measure $\mathbb{Q}_T$ equivalent to $\mathbb{P}$ (and, of course, they are valid under $\mathbb{P}$ as well). The only way in which the choice of a forward measure $\mathbb{Q}_T$ impacts these results is through the pre-default value of a defaultable ZC-bond.

We conclude that we deal here with the volatility based relative pricing a defaultable claim. This should be contrasted with more popular intensity-based risk-neutral pricing, which is commonly used to produce an arbitrage-free model of tradeable defaultable assets. Recall, however, that if tradeable assets are not chosen carefully for a given class of survival claims, then both hedging strategy and pre-default price may depend explicitly on values of drift parameters, which can be linked in our set-up to the default intensity (see Example 5.3.2).

Remark 5.3.6 Assume that $X = G(Y^2_T)$ for some function $G : I\!R \to I\!R$. Then the credit-risk-adjusted forward price of a survival claim satisfies $\widehat{F}_X(t,T) = v(t,\widehat{F}_{Y^2}(t,T))$, where the pricing function v solves the PDE

$$\partial_t v(t,z) + \frac{1}{2}(\sigma_t - b(t,T))^2 z^2 \partial_{zz} v(t,z) = 0$$

with the terminal condition $v(T,z) = G(z)$. The PDE approach is studied in Section 5.4 below.

Remark 5.3.7 Proposition 5.3.2 is still valid if the driving Brownian motion is two-dimensional, rather than one-dimensional. In an extended model, the volatilities $\sigma_t, b(t,T)$ and $\beta(t,T)$ take values in $I\!R^2$ and the respective products are interpreted as inner products in $I\!R^3$. Equivalently, one may prefer to deal with real-valued volatilities, but with correlated one-dimensional Brownian motions.

Vulnerable swaption

In this section, we relax the assumption that Y^1 is the price of a default-free bond. We now let Y^1 and Y^2 to be arbitrary default-free assets, with dynamics

$$dY_t^i = Y_t^i\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t\big), \quad i = 1, 2.$$

We still take $D(t,T)$ to be the third asset, and we maintain the assumption that the model is arbitrage-free, but we no longer postulate its completeness. In other words, we postulate the existence an e.m.m. $\mathbb{Q}^1$, as defined in subsection on arbitrage free property, but not the uniqueness of $\mathbb{Q}^1$.

We take the first asset as a numéraire, so that all prices are expressed in units of Y^1. In particular, $Y_t^{1,1} = 1$ for every $t \in I\!R_+$, and the relative prices $Y^{2,1}$ and $Y^{3,1}$ satisfy under $\mathbb{Q}^1$ (cf. Proposition 5.3.1)

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1}(\sigma_{2,t} - \sigma_{1,t})\,d\widehat{W}_t, \\ dY_t^{3,1} &= Y_{t-}^{3,1}\big((\sigma_{3,t} - \sigma_{1,t})\,d\widehat{W}_t - d\widehat{M}_t\big). \end{aligned}$$

It is natural to postulate that the driving Brownian noise is two-dimensional. In such a case, we may represent the joint dynamics of $Y^{2,1}$ and $Y^{3,1}$ under $\mathbb{Q}^1$ as follows

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1}(\sigma_{2,t} - \sigma_{1,t})\,dW_t^1, \\ dY_t^{3,1} &= Y_{t-}^{3,1}\big((\sigma_{3,t} - \sigma_{1,t})\,dW_t^2 - d\widehat{M}_t\big), \end{aligned}$$

where W^1, W^2 are one-dimensional Brownian motions under $\mathbb{Q}^1$, such that $d\langle W^1, W^2\rangle_t = \rho_t\,dt$ for a deterministic instantaneous correlation coefficient ρ taking values in $[-1,1]$.

We assume from now on that the volatilities σ_i, $i = 1,2,3$ are deterministic. Let us set

$$\alpha_t = \langle \ln \widetilde{Y}^{2,1}, \ln \widetilde{Y}^{3,1}\rangle_t = \int_0^t \rho_u(\sigma_{2,u} - \sigma_{1,u})(\sigma_{3,u} - \sigma_{1,u})\,du, \tag{5.45}$$

and let $\widehat{\mathbb{Q}}$ be an equivalent probability measure on $(\Omega, \mathcal{G}_T)$ such that the process $\widehat{Y}_t = Y_t^{2,1}e^{-\alpha_t}$ is a $\widehat{\mathbb{Q}}$-martingale. To clarify the financial interpretation of the auxiliary process $\widehat{Y}$ in the present context, we introduce the concept of credit-risk-adjusted forward price relative to the numéraire Y^1.

Definition 5.3.3 *Let Y be a $\mathcal{G}_T$-measurable claim. An $\mathcal{F}_t$-measurable random variable K is called the time-t credit-risk-adjusted Y^1-forward price of Y if the pre-default value at time t of a vulnerable forward contract, represented by the claim*

$$1\!\!1_{\{T<\tau\}}(Y_T^1)^{-1}(Y - KY_T^1) = 1\!\!1_{\{T<\tau\}}(Y(Y_T^1)^{-1} - K),$$

equals 0.

The credit-risk-adjusted Y^1-forward price of Y is denoted by $\widehat{F}_{Y|Y^1}(t,T)$, and it is also interpreted as an abstract defaultable swap rate. The following auxiliary results are easy to establish, along the same lines as Lemmas 5.3.2 and 5.3.3.

Lemma 5.3.4 *The credit-risk-adjusted Y^1-forward price of a survival claim $Y = (X, 0, \tau)$ equals*

$$\widehat{F}_{Y|Y^1}(t,T) = \widetilde{\pi}_t(X^1, 0, \tau)(\widetilde{D}(t,T))^{-1}$$

where $X^1 = X(Y_T^1)^{-1}$ is the price of X in the numéraire Y^1, and $\widetilde{\pi}_t(X^1, 0, \tau)$ is the pre-default value of a survival claim with the promised payoff X^1.

Proof: It suffices to note that for $Y = \mathbb{1}_{\{T<\tau\}}X$, we have

$$\mathbb{1}_{\{T<\tau\}}(Y(Y_T^1)^{-1} - K) = \mathbb{1}_{\{T<\tau\}}X^1 - KD(T,T),$$

where $X^1 = X(Y_T^1)^{-1}$, and to consider the pre-default values. □

Lemma 5.3.5 *The credit-risk-adjusted Y^1-forward price of the asset Y^2 equals*

$$\widehat{F}_{Y^2|Y^1}(t,T) = Y_t^{2,1}\, e^{\alpha_T - \alpha_t} = \widehat{Y}_t e^{\alpha_T}, \tag{5.46}$$

where α, assumed to be deterministic, is given by (5.45).

Proof: It suffices to find an $\mathcal{F}_t$-measurable random variable K for which

$$\widetilde{D}(t,T)\,\mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^2(Y_T^1)^{-1} - K \,\big|\, \mathcal{F}_t\big) = 0.$$

Consequently, $K = \widehat{F}_{Y^2|Y^1}(t,T)$, where

$$\widehat{F}_{Y^2|Y^1}(t,T) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^{2,1} \,\big|\, \mathcal{F}_t\big) = Y_t^{2,1}\, e^{\alpha_T - \alpha_t} = \widehat{Y}_t\, e^{\alpha_T},$$

where we have used the facts that $\widehat{Y}_t = Y_t^{2,1}e^{-\alpha_t}$ is a $\widehat{\mathbb{Q}}$-martingale, and α is deterministic. □

We are in a position to examine a vulnerable option to exchange default-free assets with the payoff

$$C_T^d = \mathbb{1}_{\{T<\tau\}}(Y_T^1)^{-1}(Y_T^2 - KY_T^1)^+ = \mathbb{1}_{\{T<\tau\}}(Y_T^{2,1} - K)^+. \tag{5.47}$$

The last expression shows that the option can be interpreted as a vulnerable swaption associated with the assets Y^1 and Y^2. It is useful to observe that

$$\frac{C_T^d}{Y_T^1} = \frac{\mathbb{1}_{\{T<\tau\}}}{Y_T^1}\left(\frac{Y_T^2}{Y_T^1} - K\right)^+,$$

so that, when expressed in the numéraire Y^1, the payoff becomes

$$C_T^{1,d} = D^1(T,T)(Y_T^{2,1} - K)^+,$$

where $C_t^{1,d} = C_t^d(Y_t^1)^{-1}$ and $D^1(t,T) = D(t,T)(Y_t^1)^{-1}$ stand for the prices relative to Y^1.

It is clear that we deal here with a model analogous to the model examined in previous subsections in which, however, all prices are now relative to the numéraire Y^1. This observation allows us to directly derive the valuation formula from Proposition 5.3.2.

Proposition 5.3.3 *Assume that the volatilities are deterministic. The credit-risk-adjusted Y^1-forward price of a vulnerable call option written with the payoff given by* (5.47) *equals*

$$\widehat{F}_{C^d|Y^1}(t,T) = \widehat{F}_{Y^2|Y^1}(t,T)N\big(d_+(\widehat{F}_{Y^2|Y^1}(t,T),t,T)\big) - KN\big(d_-(\widehat{F}_{Y^2|Y^1}(t,T),t,T)\big)$$

where

$$d_\pm(z,t,T) = \frac{\ln z - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_{2,u} - \sigma_{1,u})^2\, du.$$

The replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi_t^1 Y_t^1 = -\phi_t^2 Y_t^2, \quad \phi_t^2 = \widetilde{D}(t,T)(Y_t^1)^{-1}N(d_+(t,T))e^{\alpha_T - \alpha_t}, \quad \phi_t^3\widetilde{D}(t,T) = \widetilde{C}_t^d,$$

where $d_+(t,T) = d_+\big(\widehat{F}_{Y^2}(t,T),t,T\big)$.

PROOF: The proof is analogous to that of Proposition 5.3.2, and thus it is omitted. □

It is worth noting that the payoff (5.47) was judiciously chosen. Suppose instead that the option payoff is not defined by (5.47), but it is given by an apparently simpler expression

$$C^d_T = \mathbb{1}_{\{T<\tau\}}(Y^2_T - KY^1_T)^+. \tag{5.48}$$

Since the payoff C^d_T can be represented as follows

$$C^d_T = \widehat{G}(Y^1_T, Y^2_T, Y^3_T) = Y^3_T(Y^2_T - KY^1_T)^+,$$

where $\widehat{G}(y_1, y_2, y_3) = y_3(y_2 - Ky_1)^+$, the option can be seen an option to exchange the second asset for K units of the first asset, but with the payoff expressed in units of the defaultable asset. When expressed in relative prices, the payoff becomes

$$C^{1,d}_T = \mathbb{1}_{\{T<\tau\}}(Y^{2,1}_T - K)^+.$$

where $\mathbb{1}_{\{T<\tau\}} = D^1(T,T)Y^1_T$. It is thus rather clear that it is not longer possible to apply the same method as in the proof of Proposition 5.3.2.

5.3.2 Defaultable asset with non-zero recovery

We now assume that

$$dY^3_t = Y^3_{t-}(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t)$$

with $\kappa_3 > -1$ and $\kappa_3 \neq 0$. We assume that $Y^3_0 > 0$, so that $Y^3_t > 0$ for every $t \in \mathbb{R}_+$. We shall briefly describe the same steps as in the case of a defaultable asset with total default.

Arbitrage-free property

As usual, we need first to impose specific constraints on model coefficients, so that the model is arbitrage-free. Indeed, an e.m.m. $\mathbb{Q}^1$ exists if there exists a pair (θ, ζ) such that

$$\theta_t(\sigma_i - \sigma_1) + \zeta_t\xi_t\frac{\kappa_i - \kappa_1}{1+\kappa_1} = \mu_1 - \mu_i + \sigma_1(\sigma_i - \sigma_1) + \xi_t(\kappa_i - \kappa_1)\frac{\kappa_1}{1+\kappa_1}, \quad i = 2, 3.$$

To ensure the existence of a solution (θ, ζ) on the set $\tau < t$, we impose the condition

$$\sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3},$$

that is,

$$\mu_1(\sigma_3 - \sigma_2) + \mu_2(\sigma_1 - \sigma_3) + \mu_3(\sigma_2 - \sigma_1) = 0.$$

Now, on the set $\tau \geq t$, we have to solve the two equations

$$\begin{aligned} \theta_t(\sigma_2 - \sigma_1) &= \mu_1 - \mu_2 + \sigma_1(\sigma_2 - \sigma_1), \\ \theta_t(\sigma_3 - \sigma_1) + \zeta_t\gamma\kappa_3 &= \mu_1 - \mu_3 + \sigma_1(\sigma_3 - \sigma_1). \end{aligned}$$

If, in addition, $(\sigma_2 - \sigma_1)\kappa_3 \neq 0$, we obtain the unique solution

$$\begin{aligned} &\theta = \sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3}, \\ &\zeta = 0 > -1, \end{aligned}$$

so that the martingale measure $\mathbb{Q}^1$ exists and is unique.

5.3.3 Two defaultable assets with total default

We shall now assume that we have only two assets, and both are defaultable assets with total default. This case is also examined by Carr [41], who studies some imperfect hedging of digital options. Note that here we present results for perfect hedging.

We shall briefly outline the analysis of hedging of a survival claim. Under the present assumptions, we have, for $i = 1, 2$,

$$dY_t^i = Y_{t-}^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t - dM_t \big), \tag{5.49}$$

where W is a one-dimensional Brownian motion, so that

$$Y_t^1 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^1, \quad Y_t^2 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^2,$$

with the pre-default prices governed by the SDEs

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \big((\mu_{i,t} + \gamma_t)\, dt + \sigma_{i,t}\, dW_t \big). \tag{5.50}$$

The wealth process V associated with the self-financing trading strategy (ϕ^1, ϕ^2) satisfies, for every $t \in [0, T]$,

$$V_t = Y_t^1 \left(V_0^1 + \int_0^t \phi_u^2\, d\widetilde{Y}_u^{2,1} \right),$$

where $\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2 / \widetilde{Y}_t^1$. Since both primary traded assets are subject to total default, it is clear that the present model is incomplete, in the sense, that not all defaultable claims can be replicated. We shall check in the following subsection that, under the assumption that the driving Brownian motion W is one-dimensional, all survival claims satisfying natural technical conditions are hedgeable, however. In the more realistic case of a two-dimensional noise, we will still be able to hedge a large class of survival claims, including options on a defaultable asset and options to exchange defaultable assets.

Hedging a survival claim

For the sake of expositional simplicity, we assume in this section that the driving Brownian motion W is one-dimensional. This is definitely not the right choice, since we deal here with two risky assets, and thus they will be perfectly correlated. However, this assumption is convenient for the expositional purposes, since it will ensure the model completeness with respect to survival claims, and it will be later relaxed anyway.

We shall argue that in a model with two defaultable assets governed by (5.49), replication of a survival claim $(X, 0, \tau)$ is in fact equivalent to replication of the promised payoff X using the pre-default processes.

Lemma 5.3.6 *If a strategy ϕ^i, $i = 1, 2$, based on pre-default values $\widetilde{Y}^i$, $i = 1, 2$, is a replicating strategy for an $\mathcal{F}_T$-measurable claim X, that is, if ϕ is such that the process $\widetilde{V}_t(\phi) = \phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2$ satisfies, for every $t \in [0, T]$,*

$$\begin{aligned} d\widetilde{V}_t(\phi) &= \phi_t^1\, d\widetilde{Y}_t^1 + \phi_t^2\, d\widetilde{Y}_t^2, \\ \widetilde{V}_T(\phi) &= X, \end{aligned}$$

then for the process $V_t(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2$ we have, for every $t \in [0, T]$,

$$\begin{aligned} dV_t(\phi) &= \phi_t^1\, dY_t^1 + \phi_t^2\, dY_t^2, \\ V_T(\phi) &= X \mathbb{1}_{\{T<\tau\}}. \end{aligned}$$

This means that the strategy ϕ replicates the survival claim $(X, 0, \tau)$.

PROOF: It is clear that $V_t(\phi) = \mathbb{1}_{\{t<\tau\}} V_t(\phi) = \mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\phi)$. From

$$\phi_t^1\, dY_t^1 + \phi_t^2\, dY_t^2 = -(\phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2)\, dH_t + (1 - H_{t-})(\phi_t^1\, d\widetilde{Y}_t^1 + \phi_t^2\, d\widetilde{Y}_t^2),$$

it follows that

$$\phi_t^1\, dY_t^1 + \phi_t^2\, dY_t^2 = -\widetilde{V}_t(\phi)\, dH_t + (1 - H_{t-}) d\widetilde{V}_t(\phi),$$

that is,

$$\phi_t^1\, dY_t^1 + \phi_t^2\, dY_t^2 = d(\mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\phi)) = dV_t(\phi).$$

It is also obvious that $V_T(\phi) = X \mathbb{1}_{\{T<\tau\}}$. □

Combining the last result with Lemma 5.2.1, we see that a strategy (ϕ^1, ϕ^2) replicates a survival claim $(X, 0, \tau)$ whenever we have

$$\widetilde{Y}_T^1 \Big(x + \int_0^T \phi_t^2\, d\widetilde{Y}_t^{2,1} \Big) = X$$

for some constant x and some $\mathbf{F}$-predictable process ϕ^2, where, in view of (5.50),

$$d\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^{2,1} \Big(\big(\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t}) \big)\, dt + (\sigma_{2,t} - \sigma_{1,t})\, dW_t \Big).$$

We introduce a probability measure $\widetilde{\mathbb{Q}}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, and such that $\widetilde{Y}^{2,1}$ is an $\mathbf{F}$-martingale under $\widetilde{\mathbb{Q}}$. It is easily seen that the Radon-Nikodým density η satisfies, for $t \in [0,T]$,

$$d\widetilde{\mathbb{Q}} \,|\, \mathcal{G}_t = \eta_t\, d\mathbb{P} \,|\, \mathcal{G}_t = \mathcal{E}_t \left(\int_0^{\cdot} \theta_s\, dW_s \right) d\mathbb{P} \,|\, \mathcal{G}_t \tag{5.51}$$

with

$$\theta_t = \frac{\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t})}{\sigma_{1,t} - \sigma_{2,t}},$$

provided, of course, that the process θ is well defined and satisfies suitable integrability conditions. We shall show that a survival claim is attainable if the random variable $X(\widetilde{Y}_T^1)^{-1}$ is $\widetilde{\mathbb{Q}}$-integrable. Indeed, the pre-default value $\widetilde{V}_t$ at time t of a survival claim equals

$$\widetilde{V}_t = \widetilde{Y}_t^1\, \mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X(\widetilde{Y}_T^1)^{-1} \,|\, \mathcal{F}_t \big),$$

and from the predictable representation theorem, we deduce that there exists a process ϕ^2 such that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X(\widetilde{Y}_T^1)^{-1} \,|\, \mathcal{F}_t \big) = \mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X(\widetilde{Y}_T^1)^{-1} \big) + \int_0^t \phi_u^2\, d\widetilde{Y}_u^{2,1}.$$

The component ϕ^1 of the self-financing trading strategy $\phi = (\phi^1, \phi^2)$ is then chosen in such a way that

$$\phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2 = \widetilde{V}_t, \quad \forall\, t \in [0,T].$$

To conclude, by focusing on pre-default values, we have shown that the replication of survival claims can be reduced here to classic results on replication of (non-defaultable) contingent claims in a default-free market model.

Option on a defaultable asset

In order to get a complete model with respect to survival claims, we postulated in the previous section that the driving Brownian motion in dynamics (5.49) is one-dimensional. This assumption is questionable, since it implies the perfect correlation of risky assets. However, we may relax this restriction, and work instead with the two correlated one-dimensional Brownian motions. The model will no longer be complete, but options on a defaultable assets will be still attainable. The payoff of a (non-vulnerable) call option written on the defaultable asset Y^2 equals

$$C_T = (Y_T^2 - K)^+ = \mathbb{1}_{\{T<\tau\}}(\widetilde{Y}_T^2 - K)^+,$$

so that it is natural to interpret this contract as a survival claim with the promised payoff $X = (\widetilde{Y}_T^2 - K)^+$.

To deal with this option in an efficient way, we consider a model in which

$$dY_t^i = Y_{t-}^i\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t^i - dM_t\big), \tag{5.52}$$

where W^1 and W^2 are two one-dimensional correlated Brownian motions with the instantaneous correlation coefficient ρ_t. More specifically, we assume that $Y_t^1 = D(t,T) = \mathbb{1}_{\{t<\tau\}}\widetilde{D}(t,T)$ represents a defaultable ZC-bond with zero recovery, and $Y_t^2 = \mathbb{1}_{\{t<\tau\}}\widetilde{Y}_t^2$ is a generic defaultable asset with total default. Within the present set-up, the payoff can also be represented as follows

$$C_T = G(Y_T^1, Y_T^2) = (Y_T^2 - KY_T^1)^+,$$

where $g(y_1,y_2) = (y_2 - Ky_1)^+$, and thus it can also be seen as an option to exchange the second asset for K units of the first asset.

The requirement that the process $\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2(\widetilde{Y}_t^1)^{-1}$ follows an $\mathbf{F}$-martingale under $\widetilde{\mathbb{Q}}$ implies that

$$d\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^{2,1}\big((\sigma_{2,t}\rho_t - \sigma_{1,t})\,d\widetilde{W}_t^1 + \sigma_{2,t}\sqrt{1-\rho_t^2}\,d\widetilde{W}_t^2\big), \tag{5.53}$$

where $\widetilde{W} = (\widetilde{W}^1, \widetilde{W}^2)$ follows a two-dimensional Brownian motion under $\widetilde{\mathbb{Q}}$. Since $\widetilde{Y}_T^1 = 1$, replication of the option reduces to finding a constant x and an $\mathbf{F}$-predictable process ϕ^2 satisfying

$$x + \int_0^T \phi_t^2\,d\widetilde{Y}_t^{2,1} = (\widetilde{Y}_T^2 - K)^+.$$

To obtain closed-form expressions for the option price and replicating strategy, we postulate that the volatilities $\sigma_{1,t}, \sigma_{2,t}$ and the correlation coefficient ρ_t are deterministic. Let $\widehat{F}_{Y^2}(t,T) = \widetilde{Y}_t^2(\widetilde{D}(t,T))^{-1}$ ($\widehat{F}_C(t,T) = \widetilde{C}_t(\widetilde{D}(t,T))^{-1}$, respectively) stand for the credit-risk-adjusted forward price of the second asset (the option, respectively). The proof of the following valuation result is fairly standard, and thus it is omitted.

Proposition 5.3.4 *Assume that the volatilities are deterministic and that Y^1 is a DZC. The credit-risk-adjusted forward price of the option written on Y^2 equals*

$$\widehat{F}_C(t,T) = \widehat{F}_{Y^2}(t,T)N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big) - KN\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big).$$

Equivalently, the pre-default price of the option equals

$$\widetilde{C}_t = \widetilde{Y}_t^2 N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big) - K\widetilde{D}(t,T)N\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big),$$

where

$$d_\pm(z,t,T) = \frac{\ln zf - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_{1,u}^2 + \sigma_{2,u}^2 - 2\rho_u\sigma_{1,u}\sigma_{2,u})\,du.$$

Moreover the replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi_t^1 = -KN\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big), \quad \phi_t^2 = N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big).$$

5.4 PDE Approach to Valuation and Hedging

In the remaining part of the paper, we take a different perspective, and we assume that trading occurs on the time interval $[0,T]$ and our goal is to replicate a contingent claim of the form

$$Y = \mathbb{1}_{\{T\geq\tau\}} g_1(Y^1_T, Y^2_T, Y^3_T) + \mathbb{1}_{\{T<\tau\}} g_0(Y^1_T, Y^2_T, Y^3_T) = G(Y^1_T, Y^2_T, Y^3_T, H_T),$$

which settles at time T. We do not need to assume here that the coefficients in dynamics of primary assets are $\mathbf{F}$-predictable. Since our goal is to develop the PDE approach, it will be essential, however, to postulate a Markovian character of a model. For the sake of simplicity, we assume that the coefficients are constant, so that

$$dY^i_t = Y^i_{t-}\big(\mu_i\,dt + \sigma_i\,dW_t + \kappa_i\,dM_t\big), \quad i = 1,2,3.$$

The assumption of constancy of coefficients is rarely, if ever, satisfied in practically relevant models of credit risk. It is thus important to note that it was postulated here mainly for the sake of notational convenience, and the general results established in this section can be easily extended to a non-homogeneous Markov case in which $\mu_{i,t} = \mu_i(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}, H_{t-}), \sigma_{i,t} = \sigma_i(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}, H_{t-})$, etc.

5.4.1 Defaultable asset with total default

We first assume that Y^1 and Y^2 are default-free, so that $\kappa_1 = \kappa_2 = 0$, and the third asset is subject to total default, i.e. $\kappa_3 = -1$,

$$dY^3_t = Y^3_{t-}\big(\mu_3\,dt + \sigma_3\,dW_t - dM_t\big).$$

We work throughout under the assumptions of Proposition 5.3.1. This means that any $\mathbb{Q}^1$-integrable contingent claim $Y = G(Y^1_T, Y^2_T, Y^3_T; H_T)$ is attainable, and its arbitrage price equals

$$\pi_t(Y) = Y^1_t\,\mathbb{E}_{\mathbb{Q}^1}(Y(Y^1_T)^{-1}\,|\,\mathcal{G}_t), \quad \forall\, t \in [0,T]. \tag{5.54}$$

The following auxiliary result is thus rather obvious.

Lemma 5.4.1 *The process (Y^1, Y^2, Y^3, H) has the Markov property with respect to the filtration $\mathbf{G}$ under the martingale measure $\mathbb{Q}^1$. For any attainable claim $Y = G(Y^1_T, Y^2_T, Y^3_T; H_T)$ there exists a function $v : [0,T]\times \mathbb{R}^3 \times \{0,1\} \to \mathbb{R}$ such that $\pi_t(Y) = v(t, Y^1_t, Y^2_t, Y^3_t; H_t)$.*

We find it convenient to introduce the *pre-default* pricing function $v(\cdot\,;0) = v(t,y_1,y_2,y_3;0)$ and the *post-default* pricing function $v(\cdot\,;1) = v(t,y_1,y_2,y_3;1)$. In fact, since $Y^3_t = 0$ if $H_t = 1$, it suffices to study the post-default function $v(t,y_1,y_2;1) = v(t,y_1,y_2,0;1)$. Also, we write

$$\alpha_i = \mu_i - \sigma_i\frac{\mu_1-\mu_2}{\sigma_1-\sigma_2}, \quad b = (\mu_3-\mu_1)(\sigma_1-\sigma_2) - (\mu_1-\mu_3)(\sigma_1-\sigma_3).$$

Let $\gamma > 0$ be the constant default intensity under $\mathbb{P}$, and let $\zeta > -1$ be given by formula (5.28).

Proposition 5.4.1 *Assume that the functions $v(\cdot\,;0)$ and $v(\cdot\,;1)$ belong to the class $\mathrm{C}^{1,2}([0,T]\times \mathbb{R}^3_+, \mathbb{R})$. Then $v(t,y_1,y_2,y_3;0)$ satisfies the PDE*

$$\partial_t v(\cdot\,;0) + \sum_{i=1}^{2}\alpha_i y_i \partial_i v(\cdot\,;0) + (\alpha_3+\zeta)y_3\partial_3 v(\cdot\,;0) + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(\cdot\,;0)$$
$$- \alpha_1 v(\cdot\,;0) + \left(\gamma - \frac{b}{\sigma_1-\sigma_2}\right)\big[v(t,y_1,y_2;1) - v(t,y_1,y_2,y_3;0)\big] = 0$$

subject to the terminal condition $v(T, y_1, y_2, y_3; 0) = G(y_1, y_2, y_3; 0)$, *and* $v(t, y_1, y_2; 1)$ *satisfies the PDE*

$$\partial_t v(\cdot\,; 1) + \sum_{i=1}^{2} \alpha_i y_i \partial_i v(\cdot\,; 1) + \frac{1}{2} \sum_{i,j=1}^{2} \sigma_i \sigma_j y_i y_j \partial_{ij} v(\cdot\,; 1) - \alpha_1 v(\cdot\,; 1) = 0$$

subject to the terminal condition $v(T, y_1, y_2; 1) = G(y_1, y_2, 0; 1)$.

Proof: For simplicity, we write $C_t = \pi_t(Y)$. Let us define

$$\Delta v(t, y_1, y_2, y_3) = v(t, y_1, y_2; 1) - v(t, y_1, y_2, y_3; 0).$$

Then the jump $\Delta C_t = C_t - C_{t-}$ can be represented as follows:

$$\Delta C_t = 1\!\!1_{\{\tau = t\}} \big(v(t, Y_t^1, Y_t^2; 1) - v(t, Y_t^1, Y_t^2, Y_{t-}^3; 0) \big) = 1\!\!1_{\{\tau = t\}} \, \Delta v(t, Y_t^1, Y_t^2, Y_{t-}^3).$$

We write ∂_i to denote the partial derivative with respect to the variable y_i, and we typically omit the variables $(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3, H_{t-})$ in expressions $\partial_t v$, $\partial_i v$, Δv, etc. We shall also make use of the fact that for any Borel measurable function g we have

$$\int_0^t g(u, Y_u^2, Y_{u-}^3)\, du = \int_0^t g(u, Y_u^2, Y_u^3)\, du$$

since Y_u^3 and Y_{u-}^3 differ only for at most one value of u (for each ω). Let $\xi_t = 1\!\!1_{\{t < \tau\}} \gamma$. An application of Itô's formula yields

$$\begin{aligned} dC_t &= \partial_t v\, dt + \sum_{i=1}^{3} \partial_i v\, dY_t^i + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v\, dt \\ &\quad + \Big(\Delta v + Y_{t-}^3 \partial_3 v \Big)\, dH_t \\ &= \partial_t v\, dt + \sum_{i=1}^{3} \partial_i v\, dY_t^i + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v\, dt \\ &\quad + \Big(\Delta v + Y_{t-}^3 \partial_3 v \Big) \big(dM_t + \xi_t\, dt \big), \end{aligned}$$

and this in turn implies that

$$\begin{aligned} dC_t &= \partial_t v\, dt + \sum_{i=1}^{3} Y_{t-}^i \partial_i v \big(\mu_i\, dt + \sigma_i\, dW_t \big) + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v\, dt \\ &\quad + \Delta v\, dM_t + \Big(\Delta v + Y_{t-}^3 \partial_3 v \Big) \xi_t\, dt \\ &= \left\{ \partial_t v + \sum_{i=1}^{3} \mu_i Y_{t-}^i \partial_i v + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v + \Big(\Delta v + Y_{t-}^3 \partial_3 v \Big) \xi_t \right\} dt \\ &\quad + \Big(\sum_{i=1}^{3} \sigma_i Y_{t-}^i \partial_i v \Big)\, dW_t + \Delta v\, dM_t. \end{aligned}$$

We now use the integration by parts formula together with (5.22) to derive dynamics of the relative price $\widehat{C}_t = C_t (Y_t^1)^{-1}$. We find that

$$\begin{aligned} d\widehat{C}_t &= \widehat{C}_{t-} \Big((-\mu_1 + \sigma_1^2)\, dt - \sigma_1\, dW_t \Big) \\ &\quad + (Y_{t-}^1)^{-1} \left\{ \partial_t v + \sum_{i=1}^{3} \mu_i Y_{t-}^i \partial_i v + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v + \Big(\Delta v + Y_{t-}^3 \partial_3 v \Big) \xi_t \right\} dt \\ &\quad + (Y_{t-}^1)^{-1} \sum_{i=1}^{3} \sigma_i Y_{t-}^i \partial_i v\, dW_t + (Y_{t-}^1)^{-1} \Delta v\, dM_t - (Y_{t-}^1)^{-1} \sigma_1 \sum_{i=1}^{3} \sigma_i Y_{t-}^i \partial_i v\, dt. \end{aligned}$$

Hence, using (5.27), we obtain

$$
\begin{aligned}
d\widehat{C}_t &= \widehat{C}_{t-}\big(-\mu_1+\sigma_1^2\big)\,dt + \widehat{C}_{t-}\Big(-\sigma_1\,d\widehat{W}_t - \sigma_1\theta\,dt\Big)\\
&+ (Y^1_{t-})^{-1}\left\{\partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\right\}dt\\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v\,d\widehat{W}_t + (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i v\,dt\\
&+ (Y^1_{t-})^{-1}\Delta v\,d\widehat{M}_t + (Y^1_{t-})^{-1}\zeta\xi_t\Delta v\,dt - (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma^i Y^i_{t-}\partial_i v\,dt.
\end{aligned}
$$

This means that the process $\widehat{C}$ admits the following decomposition under $\mathbb{Q}^1$

$$
\begin{aligned}
d\widehat{C}_t &= \widehat{C}_{t-}\big(-\mu_1+\sigma_1^2-\sigma_1\theta\big)\,dt\\
&+ (Y^1_{t-})^{-1}\left\{\partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\right\}dt\\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i v\,dt + (Y^1_{t-})^{-1}\zeta\xi_t\Delta v\,dt\\
&- (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v\,dt + \text{a } \mathbb{Q}^1\text{-martingale.}
\end{aligned}
$$

From (5.54), it follows that the process $\widehat{C}$ is a martingale under $\mathbb{Q}^1$. Therefore, the continuous finite variation part in the above decomposition necessarily vanishes, and thus we get

$$
\begin{aligned}
0 &= C_{t-}(Y^1_{t-})^{-1}\big(-\mu_1+\sigma_1^2-\sigma_1\theta\big)\\
&+ (Y^1_{t-})^{-1}\left\{\partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \big(\Delta v + Y^3_{t-}\partial_3 v\big)\,\xi_t\right\}\\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i v + (Y^1_{t-})^{-1}\zeta\xi_t\Delta v - (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v.
\end{aligned}
$$

Consequently, we have that

$$
\begin{aligned}
0 &= C_{t-}\big(-\mu_1+\sigma_1^2-\sigma_1\theta\big)\\
&+ \partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \big(\Delta v + Y^3_{t-}\partial_3 v\big)\,\xi_t\\
&+ \sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i v + \zeta\xi_t\Delta v - \sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v.
\end{aligned}
$$

Finally, we conclude that

$$
\begin{aligned}
&\partial_t v + \sum_{i=1}^2 \alpha_i Y^i_{t-}\partial_i v + (\alpha_3+\xi_t)\,Y^3_{t-}\partial_3 v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v\\
&\quad - \alpha_1 C_{t-} + (1+\zeta)\xi_t\Delta v = 0.
\end{aligned}
$$

Recall that $\xi_t = \mathbb{1}_{\{t<\tau\}}\gamma$. It is thus clear that the pricing functions $v(\cdot,0)$ and $v(\cdot;1)$ satisfy the PDEs given in the statement of the proposition. □

The next result deals with a replicating strategy for Y.

Proposition 5.4.2 *The replicating strategy ϕ for the claim Y is given by formulae*

$$\begin{aligned}
\phi^3_t Y^3_{t-} &= -\Delta v(t, Y^1_t, Y^2_t, Y^3_{t-}) = v(t, Y^1_t, Y^2_t, Y^3_{t-}; 0) - v(t, Y^1_t, Y^2_t; 1), \\
\phi^2_t Y^2_t(\sigma_2 - \sigma_1) &= -(\sigma_1 - \sigma_3)\Delta v - \sigma_1 v + \sum_{i=1}^{3} Y^i_{t-}\sigma_i \partial_i v, \\
\phi^1_t Y^1_t &= v - \phi^2_t Y^2_t - \phi^3_t Y^3_t.
\end{aligned}$$

Proof: As a by-product of our computations, we obtain

$$d\widehat{C}_t = -(Y^1_t)^{-1}\sigma_1 v\, d\widehat{W}_t + (Y^1_t)^{-1}\sum_{i=1}^{3}\sigma_i Y^i_{t-}\partial_i v\, d\widehat{W}_t + (Y^1_t)^{-1}\Delta v\, d\widehat{M}_t.$$

The self-financing strategy that replicates Y is determined by two components ϕ^2, ϕ^3 and the following relationship:

$$d\widehat{C}_t = \phi^2_t\, dY^{2,1}_t + \phi^3_t\, dY^{3,1}_t = \phi^2_t Y^{2,1}_t(\sigma_2 - \sigma_1)\, d\widehat{W}_t + \phi^3_t Y^{3,1}_{t-}\Big((\sigma_3 - \sigma_1)\, d\widehat{W}_t - d\widehat{M}_t\Big).$$

By identification, we obtain $\phi^3_t Y^{3,1}_{t-} = (Y^1_t)^{-1}\Delta v$ and

$$\phi^2_t Y^2_t(\sigma_2 - \sigma_1) - (\sigma_3 - \sigma_1)\Delta v = -\sigma_1 C_t + \sum_{i=1}^{3} Y^i_{t-}\sigma_i\partial_i v.$$

This yields the claimed formulae. □

Corollary 5.4.1 *In the case of a total default claim, the hedging strategy satisfies the balance condition.*

Proof: A total default corresponds to the assumption that $G(y_1, y_2, y_3, 1) = 0$. We now have $v(t, y_1, y_2; 1) = 0$, and thus $\phi^3_t Y^3_{t-} = v(t, Y^1_t, Y^2_t, Y^3_{t-}; 0)$ for every $t \in [0, T]$. Hence, the equality $\phi^1_t Y^1_t + \phi^2_t Y^2_t = 0$ holds for every $t \in [0, T]$. The last equality is the balance condition for $Z = 0$. Recall that it ensures that the wealth of a replicating portfolio jumps to zero at default time. □

Hedging with the savings account

Let us now study the particular case where Y^1 is the savings account, i.e.,

$$dY^1_t = rY^1_t\, dt, \quad Y^1_0 = 1,$$

which corresponds to $\mu_1 = r$ and $\sigma_1 = 0$. Let us write $\widehat{r} = r + \widehat{\gamma}$, where

$$\widehat{\gamma} = \gamma(1 + \zeta) = \gamma + \mu_3 - r + \frac{\sigma_3}{\sigma_2}(r - \mu_2)$$

stands for the intensity of default under $\mathbb{Q}^1$. The quantity $\widehat{r}$ has a natural interpretation as the risk-neutral *credit-risk adjusted* short-term interest rate. Straightforward calculations yield the following corollary to Proposition 5.4.1.

Corollary 5.4.2 *Assume that $\sigma_2 \neq 0$ and*

$$\begin{aligned}
dY^1_t &= rY^1_t\, dt, \\
dY^2_t &= Y^2_t\big(\mu_2\, dt + \sigma_2\, dW_t\big), \\
dY^3_t &= Y^3_{t-}\big(\mu_3\, dt + \sigma_3\, dW_t - dM_t\big).
\end{aligned}$$

Then the function $v(\cdot\,;0)$ satisfies

$$\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + \widehat{r}y_3\partial_3 v(t,y_2,y_3;0) - \widehat{r}v(t,y_2,y_3;0)$$
$$+\frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;0) + \widehat{\gamma}v(t,y_2;1) = 0$$

with $v(T,y_2,y_3;0) = G(y_2,y_3;0)$, and the function $v(\cdot\,;1)$ satisfies

$$\partial_t v(t,y_2;1) + ry_2\partial_2 v(t,y_2;1) + \frac{1}{2}\sigma_2^2 y_2^2 \partial_{22} v(t,y_2;1) - rv(t,y_2;1) = 0$$

with $v(T,y_2;1) = G(y_2,0;1)$.

In the special case of a survival claim, the function $v(\cdot\,;1)$ vanishes identically, and thus the following result can be easily established.

Corollary 5.4.3 *The pre-default pricing function $v(\cdot\,;0)$ of a survival claim $Y = 1\!\!1_{\{T<\tau\}}G(Y_T^2,Y_T^3)$ is a solution of the following PDE:*

$$\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + \widehat{r}y_3\partial_3 v(t,y_2,y_3;0)$$
$$+\frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;0) - \widehat{r}v(t,y_2,y_3;0) = 0$$

with the terminal condition $v(T,y_2,y_3;0) = G(y_2,y_3)$. The components ϕ^2 and ϕ^3 of the replicating strategy satisfy

$$\phi_t^2\sigma_2 Y_t^2 = \sum_{i=2}^{3}\sigma_i Y_{t-}^i \partial_i v(t,Y_t^2,Y_{t-}^3;0) + \sigma_3 v(t,Y_t^2,Y_{t-}^3;0),$$
$$\phi_t^3 Y_{t-}^3 = v(t,Y_t^2,Y_{t-}^3;0).$$

Example 5.4.1 Consider a survival claim $Y = 1\!\!1_{\{T<\tau\}}g(Y_T^2)$, that is, a vulnerable claim with default-free underlying asset. Its pre-default pricing function $v(\cdot\,;0)$ does not depend on y_3, and satisfies the PDE (y stands here for y_2 and σ for σ_2)

$$\partial_t v(t,y;0) + ry\partial_2 v(t,y;0) + \frac{1}{2}\sigma^2 y^2 \partial_{22} v(t,y;0) - \widehat{r}v(t,y;0) = 0 \tag{5.55}$$

with the terminal condition $v(T,y;0) = 1\!\!1_{\{t<\tau\}}g(y)$. The solution to (5.55) is

$$v(t,y) = e^{(\widehat{r}-r)(t-T)}\, v^{r,g,2}(t,y) = e^{\widehat{\gamma}(t-T)}\, v^{r,g,2}(t,y),$$

where the function $v^{r,g,2}$ is the Black-Scholes price of $g(Y_T)$ in a Black-Scholes model for Y_t with interest rate r and volatility σ_2.

5.4.2 Defaultable asset with non-zero recovery

We now assume that

$$dY_t^3 = Y_{t-}^3(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t)$$

with $\kappa_3 > -1$ and $\kappa_3 \neq 0$. We assume that $Y_0^3 > 0$, so that $Y_t^3 > 0$ for every $t \in I\!\!R_+$. We shall briefly describe the same steps as in the case of a defaultable asset with total default.

Pricing PDE and replicating strategy

We are in a position to derive the pricing PDEs. For the sake of simplicity, we assume that Y^1 is the savings account, so that Proposition 5.4.3 is a counterpart of Corollary 5.4.2. For the proof of Proposition 5.4.3, the interested reader is referred to Bielecki *et al.* [17].

Proposition 5.4.3 *Let $\sigma_2 \neq 0$ and let Y^1, Y^2, Y^3 satisfy*

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(\mu_2\,dt + \sigma_2\,dW_t\big), \\ dY_t^3 &= Y_{t-}^3\big(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t\big). \end{aligned}$$

Assume, in addition, that $\sigma_2(r-\mu_3) = \sigma_3(r-\mu_2)$ and $\kappa_3 \neq 0$, $\kappa_3 > -1$. Then the price of a contingent claim $Y = G(Y_T^2, Y_T^3, H_T)$ can be represented as $\pi_t(Y) = v(t, Y_t^2, Y_t^3, H_t)$, where the pricing functions $v(\cdot\,;0)$ and $v(\cdot\,;1)$ satisfy the following PDEs

$$\begin{aligned} &\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + y_3\,(r-\kappa_3\gamma)\,\partial_3 v(t,y_2,y_3;0) - rv(t,y_2,y_3;0) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;0) + \gamma\big(v(t,y_2,y_3(1+\kappa_3);1) - v(t,y_2,y_3;0)\big) = 0 \end{aligned}$$

and

$$\begin{aligned} &\partial_t v(t,y_2,y_3;1) + ry_2\partial_2 v(t,y_2,y_3;1) + ry_3\partial_3 v(t,y_2,y_3;1) - rv(t,y_2,y_3;1) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;1) = 0 \end{aligned}$$

subject to the terminal conditions

$$v(T,y_2,y_3;0) = G(y_2,y_3;0), \quad v(T,y_2,y_3;1) = G(y_2,y_3;1).$$

The replicating strategy ϕ equals

$$\begin{aligned} \phi_t^2 &= \frac{1}{\sigma_2 Y_t^2}\sum_{i=2}^{3}\sigma_i y_i \partial_i v(t, Y_t^2, Y_{t-}^3, H_{t-}) \\ &\quad - \frac{\sigma_3}{\sigma_2\kappa_3 Y_t^2}\big(v(t,Y_t^2,Y_{t-}^3(1+\kappa_3);1) - v(t,Y_t^2,Y_{t-}^3;0)\big), \\ \phi_t^3 &= \frac{1}{\kappa_3 Y_{t-}^3}\big(v(t,Y_t^2,Y_{t-}^3(1+\kappa_3);1) - v(t,Y_t^2,Y_{t-}^3;0)\big), \end{aligned}$$

and ϕ_t^1 is given by $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 = C_t$.

Hedging of a survival claim

We shall illustrate Proposition 5.4.3 by means of examples. First, consider a survival claim of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T<\tau\}}\, g(Y_T^3).$$

Then the post-default pricing function $v^g(\cdot\,;1)$ vanishes identically, and the pre-default pricing function $v^g(\cdot\,;0)$ solves the PDE

$$\begin{aligned} &\partial_t v^g(\cdot\,;0) + ry_2\partial_2 v^g(\cdot\,;0) + y_3\,(r-\kappa_3\gamma)\,\partial_3 v^g(\cdot\,;0) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v^g(\cdot\,;0) - (r+\gamma)v^g(\cdot\,;0) = 0 \end{aligned}$$

with the terminal condition $v^g(T, y_2, y_3; 0) = g(y_3)$. Denote $\alpha = r - \kappa_3\gamma$ and $\beta = \gamma(1 + \kappa_3)$.

It is not difficult to check that $v^g(t, y_2, y_3; 0) = e^{\beta(T-t)}v^{\alpha,g,3}(t, y_3)$ is a solution of the above equation, where the function $w(t, y) = v^{\alpha,g,3}(t, y)$ is the solution of the standard Black-Scholes PDE equation

$$\partial_t w + y\alpha\partial_y w + \frac{1}{2}\sigma_3^2 y^2 \partial_{yy} w - \alpha w = 0$$

with the terminal condition $w(T, y) = g(y)$, that is, the price of the contingent claim $g(Y_T)$ in the Black-Scholes framework with the interest rate α and the volatility parameter equal to σ_3.

Let C_t be the current value of the contingent claim Y, so that

$$C_t = 1\!\!1_{\{t<\tau\}} e^{\beta(T-t)} v^{\alpha,g,3}(t, Y_t^3).$$

The hedging strategy of the survival claim is, on the event $\{t < \tau\}$,

$$\begin{aligned}
\phi_t^3 Y_t^3 &= -\frac{1}{\kappa_3} e^{-\beta(T-t)} v^{\alpha,g,3}(t, Y_t^3) = -\frac{1}{\kappa_3} C_t, \\
\phi_t^2 Y_t^2 &= \frac{\sigma_3}{\sigma_2}\left(Y_t^3 e^{-\beta(T-t)} \partial_y v^{\alpha,g,3}(t, Y_t^3) - \phi_t^3 Y_t^3\right).
\end{aligned}$$

Hedging of a recovery payoff

As another illustration of Proposition 5.4.3, we shall now consider the contingent claim $G(Y_T^2, Y_T^3, H_T) = 1\!\!1_{\{T\geq\tau\}} g(Y_T^2)$, that is, we assume that recovery is paid at maturity and equals $g(Y_T^2)$. Let v^g be the pricing function of this claim. The post-default pricing function $v^g(\cdot\,; 1)$ does not depend on y_3. Indeed, the equation (we write here $y_2 = y$)

$$\partial_t v^g(\cdot\,; 1) + ry\partial_y v^g(\cdot\,; 1) + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy} v^g(\cdot\,; 1) - rv^g(\cdot\,; 1) = 0,$$

with $v^g(T, y; 1) = g(y)$, admits a unique solution $v^{r,g,2}$, which is the price of $g(Y_T)$ in the Black-Scholes model with interest rate r and volatility σ_2.

Prior to default, the price of the claim can be found by solving the following PDE

$$\begin{aligned}
&\partial_t v^g(\cdot; 0) + ry_2\partial_2 v^g(\cdot; 0) + y_3\left(r - \kappa_3\gamma\right)\partial_3 v^g(\cdot; 0) \\
&\quad + \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} v^g(\cdot; 0) - (r + \gamma)v^g(\cdot; 0) = -\gamma v^g(t, y_2; 1)
\end{aligned}$$

with $v^g(T, y_2, y_3; 0) = 0$. It is not difficult to check that

$$v^g(t, y_2, y_3; 0) = (1 - e^{\gamma(t-T)}) v^{r,g,2}(t, y_2).$$

The reader can compare this result with the one of Example 5.4.1. e now assume that

$$dY_t^3 = Y_{t-}^3(\mu_3\, dt + \sigma_3\, dW_t + \kappa_3\, dM_t)$$

with $\kappa_3 > -1$ and $\kappa_3 \neq 0$. We assume that $Y_0^3 > 0$, so that $Y_t^3 > 0$ for every $t \in I\!\!R_+$. We shall briefly describe the same steps as in the case of a defaultable asset with total default.

Arbitrage-free property

As usual, we need first to impose specific constraints on model coefficients, so that the model is arbitrage-free. Indeed, an e.m.m. $\mathbb{Q}^1$ exists if there exists a pair (θ, ζ) such that

$$\theta_t(\sigma_i - \sigma_1) + \zeta_t\xi_t\frac{\kappa_i - \kappa_1}{1 + \kappa_1} = \mu_1 - \mu_i + \sigma_1(\sigma_i - \sigma_1) + \xi_t(\kappa_i - \kappa_1)\frac{\kappa_1}{1 + \kappa_1}, \quad i = 2, 3.$$

To ensure the existence of a solution (θ, ζ) on the set $\tau < t$, we impose the condition

$$\sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3},$$

that is,

$$\mu_1(\sigma_3 - \sigma_2) + \mu_2(\sigma_1 - \sigma_3) + \mu_3(\sigma_2 - \sigma_1) = 0.$$

Now, on the set $\tau \geq t$, we have to solve the two equations

$$\begin{aligned} \theta_t(\sigma_2 - \sigma_1) \quad &= \quad \mu_1 - \mu_2 + \sigma_1(\sigma_2 - \sigma_1), \\ \theta_t(\sigma_3 - \sigma_1) + \zeta_t \gamma \kappa_3 \quad &= \quad \mu_1 - \mu_3 + \sigma_1(\sigma_3 - \sigma_1). \end{aligned}$$

If, in addition, $(\sigma_2 - \sigma_1)\kappa_3 \neq 0$, we obtain the unique solution

$$\begin{aligned} &\theta = \sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3}, \\ &\zeta = 0 > -1, \end{aligned}$$

so that the martingale measure $\mathbb{Q}^1$ exists and is unique.

5.4.3 Two defaultable assets with total default

We shall now assume that we have only two assets, and both are defaultable assets with total default. We shall briefly outline the analysis of this case, leaving the details and the study of other relevant cases to the reader. We postulate that

$$dY_t^i = Y_{t-}^i \big(\mu_i \, dt + \sigma_i \, dW_t - dM_t\big), \ i = 1, 2, \tag{5.56}$$

so that

$$Y_t^1 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^1, \quad Y_t^2 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^2,$$

with the pre-default prices governed by the SDEs

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \big((\mu_i + \gamma) \, dt + \sigma_i \, dW_t\big), \ i = 1, 2.$$

In the case where the promised payoff X is path-independent, so that

$$X \mathbb{1}_{\{T<\tau\}} = G(Y_T^1, Y_T^2) \mathbb{1}_{\{T<\tau\}} = G(\widetilde{Y}_T^1, \widetilde{Y}_T^2) \mathbb{1}_{\{T<\tau\}}$$

for some function G, it is possible to use the PDE approach in order to value and replicate survival claims prior to default (needless to say that the valuation and hedging after default are trivial here).

We know already from the martingale approach that hedging of a survival claim $X\mathbb{1}_{\{T<\tau\}}$ is formally equivalent to replicating the promised payoff X using the pre-default values of tradeable assets

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \big((\mu_i + \gamma) \, dt + \sigma_i \, dW_t\big), \quad i = 1, 2.$$

We need not to worry here about the balance condition, since in case of default the wealth of the portfolio will drop to zero, as it should in view of the equality $Z = 0$.

We shall find the pre-default pricing function $v(t, y_1, y_2)$, which is required to satisfy the terminal condition $v(T, y_1, y_2) = G(y_1, y_2)$, as well as the hedging strategy (ϕ^1, ϕ^2). The replicating strategy ϕ is such that for the pre-default value $\widetilde{C}$ of our claim we have $\widetilde{C}_t := v(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2) = \phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2$, and

$$d\widetilde{C}_t = \phi_t^1 \, d\widetilde{Y}_t^1 + \phi_t^2 \, d\widetilde{Y}_t^2. \tag{5.57}$$

Proposition 5.4.4 *Assume that $\sigma_1 \neq \sigma_2$. Then the pre-default pricing function v satisfies the PDE*

$$\begin{aligned}\partial_t v + y_1\left(\mu_1+\gamma-\sigma_1\frac{\mu_2-\mu_1}{\sigma_2-\sigma_1}\right)\partial_1 v + y_2\left(\mu_2+\gamma-\sigma_2\frac{\mu_2-\mu_1}{\sigma_2-\sigma_1}\right)\partial_2 v\\ +\frac{1}{2}\Big(y_1^2\sigma_1^2\partial_{11}v + y_2^2\sigma_2^2\partial_{22}v + 2y_1y_2\sigma_1\sigma_2\partial_{12}v\Big) = \left(\mu_1+\gamma-\sigma_1\frac{\mu_2-\mu_1}{\sigma_2-\sigma_1}\right)v\end{aligned}$$

with the terminal condition $v(T,y_1,y_2)=G(y_1,y_2)$.

Proof: We shall merely sketch the proof. By applying Itô's formula to $v(t,\widetilde{Y}^1_t,\widetilde{Y}^2_t)$, and comparing the diffusion terms in (5.57) and in the Itô differential $dv(t,\widetilde{Y}^1_t,\widetilde{Y}^2_t)$, we find that

$$y_1\sigma_1\partial_1 v + y_2\sigma_2\partial_2 v = \phi^1 y_1\sigma_1 + \phi^2 y_2\sigma_2, \tag{5.58}$$

where $\phi^i=\phi^i(t,y_1,y_2)$. Since $\phi^1 y_1 = v(t,y_1,y_2)-\phi^2 y_2$, we deduce from (5.58) that

$$y_1\sigma_1\partial_1 v + y_2\sigma_2\partial_2 v = v\sigma_1 + \phi^2 y_2(\sigma_2-\sigma_1),$$

and thus

$$\phi^2 y_2 = \frac{y_1\sigma_1\partial_1 v + y_2\sigma_2\partial_2 v - v\sigma_1}{\sigma_2-\sigma_1}.$$

On the other hand, by identification of drift terms in (5.58), we obtain

$$\begin{aligned}\partial_t v \quad &+ \quad y_1(\mu_1+\gamma)\partial_1 v + y_2(\mu_2+\gamma)\partial_2 v\\ &\quad+\frac{1}{2}\Big(y_1^2\sigma_1^2\partial_{11}v + y_2^2\sigma_2^2\partial_{22}v + 2y_1y_2\sigma_1\sigma_2\partial_{12}v\Big)\\ &= \quad \phi^1 y_1(\mu_1+\gamma) + \phi^2 y_2(\mu_2+\gamma).\end{aligned}$$

Upon elimination of ϕ^1 and ϕ^2, we arrive at the stated PDE. □

Recall that the historically observed drift terms are $\widehat{\mu}_i = \mu_i+\gamma$, rather than μ_i. The pricing PDE can thus be simplified as follows:

$$\begin{aligned}\partial_t v + y_1\left(\widehat{\mu}_1-\sigma_1\frac{\widehat{\mu}_2-\widehat{\mu}_1}{\sigma_2-\sigma_1}\right)\partial_1 v + y_2\left(\widehat{\mu}_2-\sigma_2\frac{\widehat{\mu}_2-\widehat{\mu}_1}{\sigma_2-\sigma_1}\right)\partial_2 v\\ +\frac{1}{2}\Big(y_1^2\sigma_1^2\partial_{11}v + y_2^2\sigma_2^2\partial_{22}v + 2y_1y_2\sigma_1\sigma_2\partial_{12}v\Big) = v\left(\widehat{\mu}_1-\sigma_1\frac{\widehat{\mu}_2-\widehat{\mu}_1}{\sigma_2-\sigma_1}\right).\end{aligned}$$

The pre-default pricing function v depends on the market observables (drift coefficients, volatilities, and pre-default prices), but not on the (deterministic) default intensity.

To make one more simplifying step, we make an additional assumption about the payoff function. Suppose, in addition, that the payoff function is such that $G(y_1,y_2)=y_1 g(y_2/y_1)$ for some function $g:\mathbb{R}_+\to\mathbb{R}$ (or equivalently, $G(y_1,y_2)=y_2 h(y_1/y_2)$ for some function $h:\mathbb{R}_+\to\mathbb{R}$). Then we may focus on relative pre-default prices $\widehat{C}_t=\widetilde{C}_t(\widetilde{Y}^1_t)^{-1}$ and $\widetilde{Y}^{2,1}=\widetilde{Y}^2_t(\widetilde{Y}^1_t)^{-1}$. The corresponding pre-default pricing function $\widehat{v}(t,z)$, such that $\widehat{C}_t=\widehat{v}(t,Y^{2,1}_t)$ will satisfy the PDE

$$\partial_t\widehat{v} + \frac{1}{2}(\sigma_2-\sigma_1)^2 z^2\partial_{zz}\widehat{v} = 0$$

with terminal condition $\widehat{v}(T,z)=g(z)$. If the price processes Y^1 and Y^2 in (5.49) are driven by the correlated Brownian motions W and $\widehat{W}$ with the constant instantaneous correlation coefficient ρ, then the PDE becomes

$$\partial_t\widehat{v} + \frac{1}{2}(\sigma_2^2+\sigma_1^2-2\rho\sigma_1\sigma_2)z^2\partial_{zz}\widehat{v} = 0.$$

Consequently, the pre-default price $\widetilde{C}_t=\widetilde{Y}^1_t\widehat{v}(t,\widetilde{Y}^{2,1}_t)$ will not depend directly on the drift coefficients $\widehat{\mu}_1$ and $\widehat{\mu}_2$, and thus, in principle, we should be able to derive an expression the price of the claim in terms of market observables: the prices of the underlying assets, their volatilities and the correlation coefficient. Put another way, neither the default intensity nor the drift coefficients of the underlying assets appear as independent parameters in the pre-default pricing function.

Before we conclude this work, let us stress once again that the martingale approach can be used in a fairly general set-up. By contrast, the PDE methodology is only suitable when dealing with a Markovian framework.

Chapter 6

Indifference pricing

6.1 Defaultable Claims

A *defaultable* claim (X_1, X_2, τ) with maturity date T consists of:

- The *default time* τ specifying the random time of default and thus also the default events $\{\tau \le t\}$ for every $t \in [0, T]$. It is always assumed that τ is strictly positive with probability 1.
- The *promised payoff* X_1, which represents the random payoff received by the owner of the claim at time T, if there was no default prior to or at time T. The actual payoff at time T associated with X_1 thus equals $X_1 \mathbb{1}_{\{\tau > T\}}$. We assume that X_1 is an $\mathcal{F}_T$-measurable random variable.
- The *recovery payoff* X_2, where X_2 is an $\mathcal{F}_T$-measurable random variable which is received by the owner of the claim at maturity, provided that the default occurs prior to or at maturity date T.

In what follows, we shall denote by $X = X_1 \mathbb{1}_{T<\tau} + X_2 \mathbb{1}_{\tau \le T}$ the value of the defaultable contingent claim at maturity.

6.1.1 Hodges Indifference Price

In this section we discuss the concept of Hodges indifference price in our setup. When considering Hodges indifference prices one starts with a given utility function, say u. Typically, u is assumed to be strictly increasing and strictly concave. We shall also apply a similar methodology in the case where u is assumed to be strictly convex (namely $u(x) = x^2$) for quadratic hedging. In this case however one can not use the term indifference price and one solves a minimization problem.

Problem $(\mathcal{P})$: Optimization in the default-free market.

The agent invests his initial wealth $v > 0$ in the default-free financial market using a self-financing strategy. The associated optimization problem is,

$$(\mathcal{P}) \,:\, \mathcal{V}(v) := \sup_{\phi \in \Phi(F)} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^v(\phi)\big)\big\},$$

where the wealth process $(V_t = V_t^v(\phi),\, t \le T)$, is solution of

$$dV_t = rV_t\,dt + \phi_t(dS_t - rS_t dt), \quad V_0 = v. \tag{6.1}$$

Here $\Phi(F)$ is the class of all $\mathbf{F}$-adapted, self-financing trading strategies.

Problem ($\mathcal{P}_{\mathbf{F}}^X$): Optimization in the default-free market using F-adapted strategies and buying the defaultable claim.

The agent buys the defaultable claim X at price p, and invests his remaining wealth $v-p$ in the default-free financial market, using a trading strategy $\phi \in \Phi(F)$. The resulting *global terminal wealth* will be

$$V_T^{v-p,X}(\phi) = V_T^{v-p}(\phi) + X.$$

The associated optimization problem is

$$(\mathcal{P}_{\mathbf{F}}^X) \;:\; \mathcal{V}_X^{\mathbf{F}}(v-p) := \sup_{\phi\in\Phi(F)} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-p}(\phi)+X\big)\big\},$$

where the process $V^{v-p}(\phi)$ is solution of (6.1) with the initial condition $V_0^{v-p}(\phi) = v-p$. We emphasize that the class $\Phi(F)$ of admissible strategies is the same as in the problem $(\mathcal{P})$, that is, we restrict here our attention to trading strategies that are adapted to the reference filtration $\mathbf{F}$.

Problem ($\mathcal{P}_{\mathbf{G}}^X$): Optimization in the default-free market using G-adapted strategies and buying the defaultable claim.

The agent buys the defaultable contingent claim X at price p, and invests the remaining wealth $v-p$ in the financial market, using a strategy adapted to the enlarged filtration $\mathbf{G}$. The associated optimization problem is

$$(\mathcal{P}_{\mathbf{G}}^X) \;:\; \mathcal{V}_X^{\mathbf{G}}(v-p) := \sup_{\phi\in\Phi(G)} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-p}(\phi)+X\big)\big\},$$

where $\Phi(G)$ is the class of all $\mathbf{G}$-admissible trading strategies.

Remark. It is easy to check that the solution of

$$(\mathcal{P}_{\mathbf{G}}) \;:\; \sup_{\phi\in\Phi(G)} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v}(\phi)\big)\big\},$$

is the same as the solution of $(\mathcal{P})$.

Definition 6.1.1 *For a given initial endowment v, the* $\mathbf{F}$*-Hodges buying price of the defaultable claim X is the real number $p_{\mathbf{F}}^*(v)$ such that*

$$\mathcal{V}(v) = \mathcal{V}_X^{\mathbf{F}}\big(v - p_{\mathbf{F}}^*(v)\big)\,.$$

Similarly, the $\mathbf{G}$*-Hodges buying price of X is the real number $p_{\mathbf{G}}^*(v)$ such that $\mathcal{V}(v) = \mathcal{V}_X^{\mathbf{G}}\big(v - p_{\mathbf{G}}^*(v)\big)$.*

Remark 6.1.1 We can define the $\mathbf{F}$-*Hodges selling price* $p_*^{\mathbf{F}}(v)$ of X by considering $-p$, where p is the buying price of $-X$, as specified in Definition 6.1.1.

If the contingent claim X is $\mathcal{F}_T$-measurable, then (See Rouge and ElKaroui[79]) the $\mathbf{F}$- and the $\mathbf{G}$-Hodges selling and buying prices coincide with the hedging price of X, i.e.,

$$p_{\mathbf{F}}^*(v) = p_{\mathbf{G}}^*(v) = \mathbb{E}_{\mathbb{P}}(\zeta_T X) = \mathbb{E}_{\mathbb{Q}}(X) = p_*^{\mathbf{G}}(v) = p_*^{\mathbf{F}}(v)\,,$$

where we denote by ζ the deflator process $\zeta_t = \eta_t e^{-rt}$.

6.2 Hodges prices relative to the reference filtration

In this section, we study the problem $(\mathcal{P}_{\mathbf{F}}^X)$ (i.e., we use strategies adapted to the reference filtration). First, we compute the value function, i.e., $\mathcal{V}_X^{\mathbf{F}}(v-p)$. Next, we establish a quasi-explicit representation for the Hodges price of X in the case of exponential utility. Finally, we compare the spread obtained via the risk-neutral valuation with the spread determined by the Hodges price of a defaultable zero-coupon bond.

6.2.1 Solution of Problem $(\mathcal{P}^X_{\mathbf{F}})$

In view of the particular form of the defaultable claim X it follows that

$$V_T^{v-p,X}(\phi) = 1\!\!1_{\{\tau>T\}}(V_T^{v-p}(\phi) + X_1) + 1\!\!1_{\{\tau\le T\}}(V_T^{v-p}(\phi) + X_2).$$

Since the trading strategies are $\mathbf{F}$-adapted, the terminal wealth $V_T^{v-p}(\phi)$ is an $\mathcal{F}_T$-measurable random variable. Consequently, it holds that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[u\big(V_T^{v-p,X}(\phi)\big)\right] = \\
&= \mathbb{E}_{\mathbb{P}}\left(u\big(V_T^{v-p}(\phi)+X_1\big)1\!\!1_{\{\tau>T\}} + u\big(V_T^{v-p}(\phi)+X_2\big)1\!\!1_{\{\tau\le T\}}\right) \\
&= \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}\left[u\left(V_T^{v-p}(\phi)+X_1\right)1\!\!1_{\{\tau>T\}} + u\left(V_T^{v-p}(\phi)+X_2\right)1\!\!1_{\{\tau\le T\}}|\mathcal{F}_T\right]\right) \\
&= \mathbb{E}_{\mathbb{P}}\left[u\big(V_T^{v-p}(\phi)+X_1\big)(1-F_T) + u\big(V_T^{v-p}(\phi)+X_2\big)F_T\right],
\end{aligned}$$

where $F_T = \mathbb{P}\{\tau \le T \,|\, \mathcal{F}_T\}$. Thus, problem $(\mathcal{P}^X_{\mathbf{F}})$ is equivalent to the following problem:

$$(\mathcal{P}^X_{\mathbf{F}}) \;:\; \mathcal{V}^{\mathbf{F}}_X(v-p) := \sup_{\phi\in\Phi(F)} \mathbb{E}_{\mathbb{P}}\left(J_X\big(V_T^{v-p}(\phi),\cdot\big)\right),$$

where

$$J_X(y,\omega) = u(y + X_1(\omega))(1 - F_T(\omega)) + u(y + X_2(\omega))F_T(\omega),$$

for every $\omega \in \Omega$ and $y \in I\!R$. The real-valued mapping $J_X(\cdot,\omega)$ is strictly concave and increasing. Consequently, for any $\omega \in \Omega$, we can define the mapping $I_X(z,\omega)$ by setting $I_X(z,\omega) = \big(J'_X(\cdot,\omega)\big)^{-1}(z)$ for $z \in I\!R$, where $(J'_X(\cdot,\omega))^{-1}$ denotes the inverse mapping of the derivative of J_X with respect to the first variable. To simplify the notation, we shall usually suppress the second variable, and we shall write $I_X(\cdot)$ in place of $I_X(\cdot,\omega)$.

The following lemma provides the form of the optimal solution for the problem $(\mathcal{P}^X_{\mathbf{F}})$,

Lemma 6.2.1 *The optimal terminal wealth for the problem $(\mathcal{P}^X_{\mathbf{F}})$ is given by $V_T^{v-p,*} = I_X(\lambda^*\zeta_T)$, $\mathbb{P}$-a.s., for some λ^* such that*

$$v - p = \mathbb{E}_{\mathbb{P}}\big(\zeta_T V_T^{v-p,*}\big). \tag{6.2}$$

Thus the optimal global wealth equals $V_T^{v-p,X,*} = V_T^{v-p,*} + X = I_X(\lambda^*\zeta_T) + X$ and the value function of the objective criterion for the problem $(\mathcal{P}^X_{\mathbf{F}})$ is

$$\mathcal{V}^{\mathbf{F}}_X(v-p) = \mathbb{E}_{\mathbb{P}}(u(V_T^{v-p,X,*})) = \mathbb{E}_{\mathbb{P}}(u(I_X(\lambda^*\zeta_T) + X)). \tag{6.3}$$

PROOF: It is well known (see, e.g., Karatzas and Shreve [130]) that, in order to find the optimal wealth it is enough to maximize $u(\Delta)$ over the set of square-integrable and $\mathcal{F}_T$-measurable random variables Δ, subject to the budget constraint, given by

$$\mathbb{E}_{\mathbb{P}}(\zeta_T\Delta) \le v - p.$$

The mapping $J_X(\cdot)$ is strictly concave (for all ω). Hence, for every pair of $\mathcal{F}_T$-measurable random variables (Δ,Δ^*) subject to the budget constraint, by tangent inequality, we have

$$\mathbb{E}_{\mathbb{P}}\big\{J_X(\Delta) - J_X(\Delta^*)\big\} \;\le\; \mathbb{E}_{\mathbb{P}}\big\{(\Delta-\Delta^*)J'_X(\Delta^*)\big\}.$$

For $\Delta^* = V_T^{v-p,*}$ given in the formulation of the Lemma we obtain

$$\mathbb{E}_{\mathbb{P}}\big\{J_X(\Delta) - J_X(V_T^{v-p,*})\big\} \le \lambda^*\mathbb{E}_{\mathbb{P}}\big\{\zeta_T(\Delta - V_T^{v-p,*})\big\} \le 0,$$

where the last inequality follows from the budget constraint and the choice of λ^*. Hence, for any $\phi \in \Phi(F)$,

$$\mathbb{E}_{\mathbb{P}}\big\{J_X(V_T^{v-p}(\phi)) - J_X(V_T^{v-p,*})\big\} \le 0\,.$$

To end the proof, it remains to observe that the first order conditions are also sufficient in the case of a concave criterion. Moreover, by virtue of strict concavity of the function J_X, the optimal strategy is unique. □

6.2.2 Exponential Utility: Explicit Computation of the Hodges Price

For the sake of simplicity, we assume here that $r = 0$.

Proposition 6.2.1 *Let $u(x) = 1 - \exp(-\varrho x)$ for some $\varrho > 0$. Assume that the random variables $\zeta_T e^{-\varrho X_i}$, $i = 1, 2$ are $\mathbb{P}$-integrable. Then the $\mathbf{F}$-Hodges buying price is given by*

$$p^*_{\mathbf{F}}(v) = -\frac{1}{\varrho}\,\mathbb{E}_{\mathbb{P}}\left(\zeta_T \ln\bigl((1-F_T)e^{-\varrho X_1} + F_T e^{-\varrho X_2}\bigr)\right) = \mathbb{E}_{\mathbb{P}}(\zeta_T \Psi),$$

where the $\mathcal{F}_T$-measurable random variable Ψ equals

$$\Psi = -\frac{1}{\varrho}\ln\bigl((1-F_T)e^{-\varrho X_1} + F_T e^{-\varrho X_2}\bigr). \tag{6.4}$$

Thus, the $\mathbf{F}$-Hodges buying price $p^*_{\mathbf{F}}(v)$ is the arbitrage price of the associated claim Ψ. In addition, the claim Ψ enjoys the following meaningful property

$$\mathbb{E}_{\mathbb{P}}\bigl\{u\bigl(X - \Psi\bigr)\,\big|\,\mathcal{F}_T\bigr\} = 0. \tag{6.5}$$

Proof: In view of the form of the solution to the problem $(\mathcal{P})$, we obtain

$$V^{v,*}_T = -\frac{1}{\varrho}\ln\left(\frac{\mu^*\zeta_T}{\varrho}\right).$$

The budget constraint $\mathbb{E}_{\mathbb{P}}(\zeta_T V^{v,*}_T) = v$ implies that the Lagrange multiplier μ^* satisfies

$$\frac{1}{\varrho}\ln\left(\frac{\mu^*}{\varrho}\right) = -\frac{1}{\varrho}\mathbb{E}_{\mathbb{P}}\bigl(\zeta_T \ln\zeta_T\bigr) - v. \tag{6.6}$$

The solution to the problem $(\mathcal{P}^X_{\mathbf{F}})$ is obtained in a general setting in Lemma 6.2.1. In the case of an exponential utility, we have (recall that the variable ω is suppressed)

$$J_X(y) = (1 - e^{-\varrho(y+X_1)})(1-F_T) + (1 - e^{-\varrho(y+X_2)})F_T,$$

so that

$$J'_X(y) = \varrho\, e^{-\varrho y}(e^{-\varrho X_1}(1-F_T) + e^{-\varrho X_2}F_T).$$

Thus, setting

$$A = e^{-\varrho X_1}(1-F_T) + e^{-\varrho X_2}F_T = e^{-\varrho\Psi},$$

we obtain

$$I_X(z) = -\frac{1}{\varrho}\ln\left(\frac{z}{A\varrho}\right) = -\frac{1}{\varrho}\ln\left(\frac{z}{\varrho}\right) - \Psi.$$

It follows that the optimal terminal wealth for the initial endowment $v - p$ is

$$V^{v-p,*}_T = -\frac{1}{\varrho}\ln\left(\frac{\lambda^*\zeta_T}{A\varrho}\right) = -\frac{1}{\varrho}\ln\left(\frac{\lambda^*}{\varrho}\right) - \frac{1}{\varrho}\ln\zeta_T - \Psi,$$

where the Lagrange multiplier λ^* is chosen to satisfy the budget constraint $\mathbb{E}_{\mathbb{P}}(\zeta_T V^{v-p,*}_T) = v - p$, that is,

$$\frac{1}{\varrho}\ln\left(\frac{\lambda^*}{\varrho}\right) = -\frac{1}{\varrho}\,\mathbb{E}_{\mathbb{P}}\bigl(\zeta_T\ln\zeta_T\bigr) - \mathbb{E}_{\mathbb{P}}\bigl(\zeta_T\Psi\bigr) - v + p. \tag{6.7}$$

¿From definition, the **F**-Hodges buying price is a real number $p^* = p^*_{\mathbf{F}}(v)$ such that

$$\mathbb{E}_{\mathbb{P}}\bigl(\exp(-\varrho V^{v,*}_T)\bigr) = \mathbb{E}_{\mathbb{P}}\bigl(\exp(-\varrho(V^{v-p^*,*}_T + X))\bigr),$$

where μ^* and λ^* are given by (6.6) and (6.7), respectively. After substitution and simplifications, we arrive at the following equality

$$\mathbb{E}_{\mathbb{P}}\Big\{ \exp\Big(-\varrho\big(\mathbb{E}_{\mathbb{P}}(\zeta_T\Psi) - p^* + X - \Psi\big)\Big)\Big\} = 1. \tag{6.8}$$

It is easy to check that

$$\mathbb{E}_{\mathbb{P}}\big(e^{-\varrho(X-\Psi)} \,\big|\, \mathcal{F}_T\big) = 1 \tag{6.9}$$

so that equality (6.5) holds, and $\mathbb{E}_{\mathbb{P}}\big(e^{-\varrho(X-\Psi)}\big) = 1$. Combining (6.8) and (6.9), we conclude that $p^*_{\mathbf{F}}(v) = \mathbb{E}_{\mathbb{P}}(\zeta_T\Psi)$. □

We briefly provide the analog of (6.4) for the **F**-Hodges selling price of X . We have $p_*^{\mathbf{F}}(v) = \mathbb{E}_{\mathbb{P}}(\zeta_T\widetilde{\Psi})$, where

$$\widetilde{\Psi} = \frac{1}{\varrho}\ln\big((1-F_T)e^{\varrho X_1} + F_T e^{\varrho X_2}\big). \tag{6.10}$$

Remark 6.2.1 It is important to notice that the **F**-Hodges prices $p^*_{\mathbf{F}}(v)$ and $p_*^{\mathbf{F}}(v)$ do not depend on the initial endowment v. This is an interesting property of the exponential utility function. In view of (6.5), the random variable Ψ will be called the *indifference conditional hedge.*

From concavity of the logarithm function we obtain

$$\ln((1-F_T)e^{-\varrho X_1} + F_T e^{-\varrho X_2}) \geq (1-F_T)(-\varrho X_1) + F_T(-\varrho X_2).$$

Hence, using that ζ_T is $\mathbf{F}_T$-measurable,

$$p^*_{\mathbf{F}}(v) \leq \mathbb{E}_{\mathbb{P}}(\zeta_T((1-F_T)X_1 + F_T X_2)) = \mathbb{E}_{\mathbb{Q}}(X).$$

Comparison with the Davis price. Let us present the results derived from the marginal utility pricing approach. The *Davis price* (see Davis [58]) is given by

$$d^*(v) = \frac{\mathbb{E}_{\mathbb{P}}\{u'\big(V_T^{v,*}\big)X\}}{\mathcal{V}'(v)}\,.$$

In our context, this yields

$$d^*(v) = \mathbb{E}_{\mathbb{P}}\big\{\zeta_T\big(X_1F_T + X_2(1-F_T)\big)\big\}.$$

In this case, the risk aversion ϱ has no influence on the pricing of the contingent claim. In particular, when F is deterministic, the Davis price reduces to the arbitrage price of each (default-free) financial asset X^i, $i = 1, 2$, weighted by the corresponding probabilities F_T and $1 - F_T$.

6.2.3 Risk-Neutral Spread Versus Hodges Spreads

In our setting the price process of the T-maturity unit discount Treasury (default-free) bond is $B(t,T) = e^{-r(T-t)}$. Let us consider the case of a defaultable bond with zero recovery, i.e., $X_1 = 1$ and $X_2 = 0$. It follows from (6.10) that the **F**-Hodges buying and selling prices of the bond are (it will be convenient here to indicate the dependence of the Hodges price on maturity T)

$$D^*_{\mathbf{F}}(0,T) = -\frac{1}{\varrho}\,\mathbb{E}_{\mathbb{P}}\big\{\zeta_T \ln(e^{-\varrho}(1-F_T) + F_T)\big\}$$

and

$$D_*^{\mathbf{F}}(0,T) = \frac{1}{\varrho}\,\mathbb{E}_{\mathbb{P}}\big\{\zeta_T \ln(e^{\varrho}(1-F_T) + F_T)\big\},$$

respectively.

Let $\widetilde{\mathbb{Q}}$ be a risk-neutral probability for the filtration $\mathbf{G}$, that is, for the enlarged market. The "market" price at time $t = 0$ of defaultable bond, denoted as $D^0(0,T)$, is thus equal to the expectation under $\widetilde{\mathbb{Q}}$ of its discounted pay-off, that is,

$$D^0(0,T) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\big(\mathbb{1}_{\{\tau>T\}} R_T\big) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\big((1-\widetilde{F}_T)R_T\big),$$

where $\widetilde{F}_t = \widetilde{\mathbb{Q}}\{\tau \le t \,|\, \mathcal{F}_t\}$ for every $t \in [0,T]$. Let us emphasize that the risk-neutral probability $\widetilde{\mathbb{Q}}$ is chosen by the market, via the price of the defaultable asset. The Hodges buying and selling spreads at time $t = 0$ are defined as

$$S^*(0,T) = -\frac{1}{T}\ln\frac{D^*_{\mathbf{F}}(0,T)}{B(0,T)}$$

and

$$S_*(0,T) = -\frac{1}{T}\ln\frac{D^{\mathbf{F}}_*(0,T)}{B(0,T)}\,,$$

respectively. Likewise, the *risk-neutral spread* at time $t = 0$ is given as

$$S^0(0,T) = -\frac{1}{T}\ln\frac{D^0(0,T)}{B(0,T)}\,.$$

Since $D^*_{\mathbf{F}}(0,0) = D^{\mathbf{F}}_*(0,0) = D^0(0,0) = 1$, the respective *backward short spreads* at time $t = 0$ are given by the following limits (provided the limits exist)

$$s^*(0) = \lim_{T\downarrow 0} S^*(0,T) = -\frac{d^+\ln D^*_{\mathbf{F}}(0,T)}{dT}\Big|_{T=0} - r$$

and

$$s_*(0) = \lim_{T\downarrow 0} S_*(0,T) = -\frac{d^+\ln D^{\mathbf{F}}_*(0,T)}{dT}\Big|_{T=0} - r,$$

respectively. We also set

$$s^0(0) = \lim_{T\downarrow 0} S^0(0,T) = -\frac{d^+\ln D^0(0,T)}{dT}\Big|_{T=0} - r.$$

Assuming, as we do, that the processes $\widetilde{F}_T$ and F_T are absolutely continuous with respect to the Lebesgue measure, and using the observation that the restriction of $\widetilde{\mathbb{Q}}$ to $\mathcal{F}_T$ is equal to $\mathbb{Q}$, we find out that

$$\begin{aligned}\frac{D^*_{\mathbf{F}}(0,T)}{B(0,T)} &= -\frac{1}{\varrho}\,\mathbb{E}_{\mathbb{Q}}\big\{\ln\big(e^{-\varrho}(1-F_T)+F_T\big)\big\}\\ &= -\frac{1}{\varrho}\,\mathbb{E}_{\mathbb{Q}}\Big\{\ln\Big(e^{-\varrho}\Big(1-\int_0^T f_t\,dt\Big)+\int_0^T f_t\,dt\Big)\Big\},\end{aligned}$$

and

$$\begin{aligned}\frac{D^{\mathbf{F}}_*(0,T)}{B(0,T)} &= \frac{1}{\varrho}\,\mathbb{E}_{\mathbb{Q}}\big\{\ln\big(e^{\varrho}(1-F_T)+F_T\big)\big\}\\ &= \frac{1}{\varrho}\,\mathbb{E}_{\mathbb{Q}}\Big\{\ln\Big(e^{\varrho}\Big(1-\int_0^T f_t\,dt\Big)+\int_0^T f_t\,dt\Big)\Big\}.\end{aligned}$$

Furthermore,

$$\frac{D^0(0,T)}{B(0,T)} = \mathbb{E}_{\mathbb{Q}}(1-\widetilde{F}_T) = \mathbb{E}_{\mathbb{Q}}\Big(1-\int_0^T \widetilde{f}_t\,dt\Big).$$

Consequently,

$$s^*(0) = \frac{1}{\varrho}\left(e^{\varrho} - 1\right) f_0, \quad s_*(0) = \frac{1}{\varrho}\left(1 - e^{-\varrho}\right) f_0,$$

and $s^0(0) = \widetilde{f}_0$. Now, if we postulate, for instance, that $s_*(0) = s^0(0)$ (it would be the case if the market price is the selling Hodges price), then we must have

$$\widetilde{f}_0 = \frac{1}{\varrho}\left(1 - e^{-\varrho}\right) f_0 = \frac{1}{\varrho}\left(1 - e^{-\varrho}\right) \gamma_0$$

so that $\widetilde{\gamma}_0 < \gamma_0$. Similar calculations can be made for any $t \in [0, T[$. It can be noticed that, if the market price is the selling Hodges price, $\widetilde{f}_0$ corresponds to the risk-neutral intensity at time 0 whereas γ_0 is the historical intensity. The reader may refer to Bernis and Jeanblanc [11] for other comments.

6.2.4 Recovery paid at time of default

Assume now that the recovery payment is made at time τ, if $\tau \leq T$. More precisely, let $(X_t^3, t \geq 0)$ be some $\mathbf{F}$-adapted process. If $\tau < T$, the payoff X_t^3 is paid at time $t = \tau$ and re-invested in the riskless asset. The terminal global wealth is now

$$(V_T^{v-p}(\pi) + X_1)1\!\!1_{T<\tau} + (V_T^{v-p}(\pi) + Z_\tau)1\!\!1_{\tau \leq T}$$

where $Z_t = X_t^3 e^{r(T-t)}$, and we are still interested in optimization of wealth at time T.

The corresponding optimization problem is

$$(\widehat{\mathcal{P}}_{\mathbf{F}}^Z) : \mathcal{V}(v-p) := \sup_{\phi \in \Phi(F)} \mathbb{E}_{\mathbb{P}} \left(U(V_T^{v-p}(\phi) + X_1)1\!\!1_{T<\tau} + U(V_T^{v-p}(\phi) + Z_\tau)1\!\!1_{\tau \leq T} \right).$$

The supremum part above can be written as

$$\sup_{\phi \in \Phi(F)} \mathbb{E}_{\mathbb{P}} \left\{ \widetilde{J}\left(V_T^{v-p}(\phi)\right) \right\},$$

where, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\widetilde{J}(y, \omega) = U(y + X_1(\omega))(1 - F_T(\omega)) + \int_0^T U(y + Z_t(\omega)) f_t dt.$$

Let us introduce the conditional indifference hedge:

$$\Phi := -\frac{1}{\varrho} \ln\Big(\int_0^T \exp(-\varrho Z_t) f_t dt + \exp(-\varrho X_1)(1 - F_T) \Big). \tag{6.11}$$

We have the following result,

Théorème 6.1 *Assume that* $\sup_{0 \leq t \leq T} \exp(-\varrho Z_t)$ *and* $\exp(-\varrho X^1)$ *are* $\mathbb{Q}$*-integrable. The Hodges price of* $(X^1, X^3_\cdot)$ *is the arbitrage price of the indifference conditional hedge*Φ*, the pay-off of which is given by (6.11).*

PROOF: Observe first that problem $(\widehat{\mathcal{P}}_{\mathbf{F}}^Z)$ can be written as

$$\mathcal{V}(x - p) = \sup_{\phi \in \Phi(F)} \mathbb{E}_{\mathbb{P}} \left\{ \exp\left(-\varrho \left[V_T^{v-p}(\phi) + \Phi\right]\right) \right\}.$$

Thus, problem $(\widehat{\mathcal{P}}_{\mathbf{F}}^Z)$ is the same as problem $(\mathcal{P}_{\mathbf{F}}^X)$ with $X = \Phi$, so that finding the Hodges price of $(X^1, X^3_\cdot)$ amounts to finding the Hodges price of Φ. But now, the claim Φ is a $\mathcal{F}_T$-measurable random variable. Thus, its Hodges price must coincide with its arbitrage price.

6.3 Optimization Problems and BSDEs

We now consider strategies ϕ that are predictable with respect to the full filtration $\mathbf{G}$. The dynamics of the risky asset $(S_t, t \geq 0)$ are

$$dS_t = S_t(\nu dt + \sigma dW_t). \tag{6.12}$$

In order to simplify notation, we denote by $(\xi_t, t \geq 0)$ the $\mathbf{G}$-predictable process such that $dM_t = dH_t - \xi_t\, dt$ is a $\mathbf{G}$-martingale, i.e., $\xi_t = \gamma_t(1 - H_{t-})$.)

We assume for simplicity that $r = 0$, so that now $\theta = \nu/\sigma$, and we change the definition of admissible portfolios to one that will be more suitable for problems considered here: instead of using the number of shares ϕ as before, we set $\pi = \phi S$, so that π represents the value invested in the risky asset. In addition, we adopt here the following relaxed definition of admissibility of trading strategies.

Definition 6.3.1 *The class $\Pi(\mathbf{F})$ ($\Pi(\mathbf{G})$, respectively) of $\mathbf{F}$-admissible ($\mathbf{G}$-admissible, respectively) trading strategies is the set of all $\mathbf{F}$-adapted ($\mathbf{G}$-predictable, respectively) processes π such that $\int_0^T \pi_t^2\, dt < \infty$, $\mathbb{P}$-a.s.*

The wealth process of a strategy π satisfies

$$dV_t(\pi) = \pi_t\big(\nu dt + \sigma dW_t\big). \tag{6.13}$$

Let X be a given contingent claim, represented by a $\mathcal{G}_T$-measurable random variable. We shall study the following problem:

$$\sup_{\pi \in \Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^v(\pi) + X\big)\big\}.$$

in the case of the exponential utility. In a last step, for the determination of Hodges' price, we shall change v into $v - p$.

6.3.1 Optimization Problem

Our first goal is to solve an optimization problem for an agent who sells a claim X. To this end, it suffices to find a strategy $\pi \in \Pi(\mathbf{G})$ that maximizes $\mathbb{E}_{\mathbb{P}}(u(V_T^v(\pi) + X))$, where the wealth process $(V_t = V_t^v(\pi), t \geq 0)$ (for simplicity, we shall frequently skip v and π from the notation) satisfies

$$dV_t = \phi_t\, dS_t = \pi_t(\nu dt + \sigma dW_t),\ V_0 = v.$$

We consider the exponential utility function $u(x) = 1 - e^{-\varrho x}$, with $\varrho > 0$. Therefore,

$$\sup_{\pi \in \Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}}\big\{u(V_T^v(\pi) + X)\big\} = 1 - \inf_{\pi \in \Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi)} e^{-\varrho X}\big).$$

We shall give three different methods to solve $\inf_{\pi \in \Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi)} e^{-\varrho X}\big)$.

Direct method

We describe the idea of a solution; the idea follows the dynamic programming principle.

Suppose that we can find a $\mathbf{G}$-adapted process $(Z_t, t \geq 0)$ with $Z_T = e^{-\varrho X}$, which depends only on the claim X and parameters ϱ, σ, ν, and such that the process $(e^{-\varrho V_t^v(\pi)} Z_t, t \geq 0)$ is a $(\mathbb{P}, \mathbf{G})$-submartingale for any admissible strategy π, and is a martingale under $\mathbb{P}$ for some admissible strategy $\pi^* \in \Pi(\mathbf{G})$. Then, we would have

$$\mathbb{E}_{\mathbb{P}}(e^{-\varrho V_T^v(\pi)} Z_T) \geq e^{-\varrho V_0^v(\pi)} Z_0 = e^{-\varrho v} Z_0$$

for any $\pi \in \Pi(\mathbf{G})$, with equality for some strategy $\pi^* \in \Pi(\mathbf{G})$. Consequently, we would obtain

$$\inf_{\pi\in\Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi)}e^{-\varrho X}\big) = \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi^*)}e^{-\varrho X}\big) = e^{-\varrho v} Z_0, \tag{6.14}$$

and thus we would be in the position to conclude that π^* is an optimal strategy. In fact, it will turn out that in order to implement the above idea we shall need to restrict further the class of **G**-admissible trading strategies to such strategies that the "martingale part" in (6.16) determines a true martingale rather than a local-martingale.

In what follows, we shall use the BSDE framework. We refer the reader to the chapter by ElKaroui and Hamadéne in the volume on Indifference prices and to the papers of Barles (1997), Rong [165] and the thesis of Royer [166] for BSDE with jumps.

We shall search the process Z in the class of all processes satisfying the following BSDE

$$dZ_t = z_t\,dt + \widehat{z}_t\,dW_t + \widetilde{z}_t\,dM_t,\ t\in[0,T[,\ Z_T = e^{-\varrho X}, \tag{6.15}$$

where the process $z = (z_t, t \geq 0)$ will be determined later (see equation (6.18) below). By applying Itô's formula, we obtain

$$d(e^{-\varrho V_t}) = e^{-\varrho V_t}\left(\left(\frac{1}{2}\,\varrho^2\pi_t^2\sigma^2 - \varrho\pi_t\nu\right)dt - \varrho\pi_t\sigma\,dW_t\right),$$

so that

$$\begin{aligned} d(e^{-\varrho V_t}Z_t) &= e^{-\varrho V_t}\big(z_t + Z_t(\frac{1}{2}\varrho^2\pi_t^2\sigma^2 - \varrho\pi_t\nu) - \varrho\pi_t\sigma\widehat{z}_t\big)dt \\ &\quad + e^{-\varrho V_t}\big((\widehat{z}_t - \varrho\pi_t\sigma Z_t)\,dW_t + \widetilde{z}_t\,dM_t\big). \end{aligned} \tag{6.16}$$

Let us choose $\pi^* = (\pi_t^*, t\geq 0)$ such that it minimizes, for every t, the following expression

$$Z_t\big(\frac{1}{2}\,\varrho^2\pi_t^2\sigma^2 - \varrho\pi_t\nu\big) - \varrho\pi_t\sigma\widehat{z}_t = -\varrho\pi_t(\nu Z_t + \sigma\widehat{z}_t) + \frac{1}{2}\,\varrho^2\pi_t^2\sigma^2 Z_t.$$

It is easily seen that, assuming that the process Z is strictly positive, we have

$$\pi_t^* = \frac{\nu Z_t + \sigma\widehat{z}_t}{\varrho\sigma^2 Z_t} = \frac{1}{\varrho\sigma}\left(\theta + \frac{\widehat{z}_t}{Z_t}\right). \tag{6.17}$$

Now, let us choose the process z as follows

$$\begin{aligned} z_t &= Z_t\left(\varrho\pi_t^*\nu - \frac{1}{2}\,\varrho^2(\pi_t^*)^2\sigma^2\right) + \varrho\pi_t^*\sigma\widehat{z}_t \\ &= \varrho\pi_t^*(Z_t\nu + \sigma\widehat{z}_t) - \frac{1}{2}\,\varrho^2(\pi_t^*)^2\sigma^2 Z_t = \frac{(\nu Z_t + \sigma\widehat{z}_t)^2}{2\sigma^2 Z_t} \\ &= \frac{1}{2}\theta^2 Z_t + \theta\widehat{z}_t + \frac{1}{2Z_t}\widehat{z}_t^2\,. \end{aligned} \tag{6.18}$$

Note that with the above choice of the process z the drift term in (6.16) is positive for any admissible strategy π, and it is zero for $\pi = \pi^*$.

Given the above, it appears that we have reduced our problem to the problem of solving the BSDE (6.15) with the process z given by (6.18), i.e.,

$$\begin{cases} dZ_t = (\frac{1}{2}\theta^2 Z_t + \theta\widehat{z}_t + \frac{1}{2Z_t}\widehat{z}_t^2)dt + \widehat{z}_t dW_t + \widetilde{z}_t dM_t,\ t\in[0,T), \\ Z_T = e^{-\varrho X}\,. \end{cases} \tag{6.19}$$

In fact, assuming that (6.19) admits a solution $(Z, \widehat{z}, \widetilde{z})$, so that with $\pi = \pi^*$ the "martingale part" in (6.16) is a true martingale part rather than a local-martingale part, then the process

$$\pi_t^* = \frac{1}{\varrho\sigma}\left(\theta + \frac{\widehat{z}_t}{Z_t}\right),$$

will be an optimal portfolio, i.e.,

$$\inf_{\pi\in\Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi)}e^{-\varrho X}\big) = \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi^*)}e^{-\varrho X}\big)\,.$$

However, this BSDE is not of standard. This is a BSDE with jumps, and existence theorems and comparison theorems are known only if the driver is Lipschitz. Hence, we shall establish the existence using another approach, an approach due to Mania and Tevzadze.

Mania and Tevzadze approach

In a very general setting, when the underlying asset is of the form

$$dS_t = d\mu_t + \lambda_t d\langle\mu\rangle_t$$

where μ is a continuous local martingale, Mania and Tevzadze [156, 155] study the family of processes

$$\mathcal{V}_t(v) = \max_{\phi} \mathbb{E}_{\mathbb{P}}(U(v + \int_t^T \phi_s dS_s)|\mathcal{G}_t)$$

where v is a real-valued deterministic parameter. They establish that the process $(\mathcal{V}(t,v) = \mathcal{V}_t(v), t \geq 0)$ (which depends on the parameter v) is solution of a BSDE

$$\begin{aligned} d\mathcal{V}(t,v) &= \frac{1}{2}\frac{1}{\mathcal{V}_{vv}(t,v)}(\varphi_v(t,v) + \lambda_t\mathcal{V}_v(t,v))^2 d\langle\mu\rangle_t + \varphi(t,v)d\mu_t + dN_t(v), \\ \mathcal{V}(T,v) &= U(v), \end{aligned} \tag{6.20}$$

where N is a martingale orthogonal to μ, and the optimal portfolio is proved to be

$$\phi_t^* = -S_t\frac{\varphi_v(t,V_t^*) - \lambda_t\mathcal{V}_v(t,V_t^*)}{\mathcal{V}_{vv}(t,V_t^*)}.$$

Analysis of the proof of the equation (1.4) in Mania and Tevzadze [156] reveals that their results carry to the case when

$$\mathcal{V}_t(v) = \max_{\phi} E(U(v + \int_t^T \phi_s dS_s + X)|\mathcal{G}_t)$$

for a claim X satisfying appropriate integrability conditions, in which case the process $(\mathcal{V}_t(v), t \geq 0)$ satisfies the BSDE (6.20) with terminal condition $\mathcal{V}(T,v) = U(v+X)$. We note however that there are several technical conditions postulated in Mania and Tevzadze [156] that need to be verified before their results can be adopted.

In the particular case when the dynamics of the underlying asset follows

$$dS_t = S_t(\nu dt + \sigma dW_t)$$

we have $d\mu_t = S_t\sigma dW_t$ and $\lambda_t = \nu/(S_t\sigma^2)$, and the BSDE (6.20) reads

$$\begin{aligned} d\mathcal{V}(t,v) &= \frac{S_t^2\sigma^2}{2\mathcal{V}_{vv}(t,v)}(\varphi(t,v) + \frac{\nu}{\sigma^2 S_t}\mathcal{V}_v(t,v))^2 dt + \varphi(t,v)S_t\sigma dW_t + dN_t \\ &= \frac{1}{2\sigma^2\mathcal{V}_{vv}(t,v)}(\varphi(t,v)\sigma^2 S_t + \nu\mathcal{V}_v(t,v))^2 dt + \varphi(t,v)S_t\sigma dW_t + dN_t \end{aligned}$$

where N is a martingale orthogonal to W (hence, in our setting a martingale of the form $\int_0^t \psi_s dM_s$). The terminal condition is

$$\mathcal{V}(T,v) = U(v+X)\,.$$

and the optimal portfolio is

$$\phi_t^* = -S_t \frac{\varphi_v + \mathcal{V}_v \nu/(\sigma^2 S_t)}{\mathcal{V}_{vv}}.$$

Here, U is an exponential function. Thus, it is convenient to factorize process $\mathcal{V}$ as $\mathcal{V}(t,v) = e^{-\varrho v} Z_t$, and to factorize process φ as $\varphi(t,v) = \widehat{\varphi}(t) e^{-\varrho v}$. It follows that Z satisfies

$$dZ_t = \frac{(\widehat{\varphi}(t) + \frac{\nu}{\sigma^2 S_t} Z_t)^2}{2Z_t} S_t^2 \sigma^2 dt + \widehat{\varphi}(t) S_t \sigma dW_t + dN_t, \quad Z_T = e^{-\varrho X}.$$

Setting $\widehat{z}_t = \widehat{\varphi}(t)\sigma S_t$, we get

$$dZ_t = \frac{1}{2Z_t}(\widehat{z}_t + \frac{\nu}{\sigma} Z_t)^2 dt + \widehat{z}_t dW_t + dN_t, \quad Z_T = e^{-\varrho X},$$

which is exactly equation (6.18), where N is a stochastic integral w.r.t. the martingale M, orthogonal to W. Thus, it appears that a solution to equation (6.18) is given as

$$Z_t = e^{\varrho v}\mathcal{V}(t,v), \quad \widehat{z}_t = \widehat{\varphi}(t)\sigma S_t, \quad \text{and} \quad \widetilde{z}_t = \frac{dN_t}{dM_t}.$$

The optimal portfolio is

$$\frac{\sigma \widehat{z}_t + Z_t \nu}{\varrho \sigma^2 Z_t}$$

which is exactly (6.17).

Remark 6.3.1 Analogous results follow from by Mania and Tevzadze [156] where a more general case of utility function is studied.

Duality Approach

We present now the duality approach (See for example Delbaen et al. [61], or Mania and Tevzadze [155]). In the case $dS_t = S_t(\nu dt + \sigma dW_t)$, the set of equivalent martingale measure (emm) is the set of probability measures $\mathbb{Q}^\psi$ defined as

$$d\mathbb{Q}^\psi|_{\mathcal{G}_t} = L_t d\mathbb{P}|_{\mathcal{G}_t}$$

where

$$dL_t = L_{t-}(-\theta dW_t + \psi_t dM_t)$$

where ψ is a **G**-predictable process, with $\psi > -1$ and θ is the risk premium $\theta = \nu/\sigma$. Indeed, using Kusuoka representation theorem [140], we know that any strictly positive martingale can be written of the form

$$dL_t = L_{t-}(\ell_t dW_t + \psi_t dM_t)\,.$$

The discounted price of the default-free asset is a martingale under the change of probability, hence, it is easy to check that $\ell_t = -\theta$. (We have already noticed that the restriction of any emm to the filtration $\mathbf{F}$ is equal to $\mathbb{Q}$.) Let us denote by $W_t^{\mathbb{Q}} = W_t + \theta t$ and $\widehat{M}_t = M_t - \int_0^t \psi_s \xi_s ds$. The processes $W^{\mathbb{Q}}$ and $\widehat{M}$ are $\mathbb{Q}^\psi$ martingales. Then,

$$\begin{aligned} L_t &= \exp\left(-\theta W_t - \frac{1}{2}\theta^2 t + \int_0^t \ln(1+\psi_s)dH_s - \int_0^t \psi_s \xi_s ds\right) \\ &= \exp\left(-\theta W_t^{\mathbb{Q}} + \frac{\theta^2 t}{2} + \int_0^t \ln(1+\psi_s)d\widehat{M}_s + \int_0^t [(1+\psi_s)\ln(1+\psi_s) - \psi_s]\xi_s ds\right) \end{aligned}$$

Hence, the relative entropy of $\mathbb{Q}^\psi$ with respect to $\mathbb{P}$ is

$$H(\mathbb{Q}^\psi|\mathbb{P}) = E_{\mathbb{Q}^\psi}(\ln L_T) = E_{\mathbb{Q}^\psi}\left(\frac{1}{2}\theta^2 T + \int_0^T [(1+\psi_s)\ln(1+\psi_s) - \psi_s]\xi_s ds\right).$$

From duality theory, the optimization problem

$$\inf_{\pi\in\Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}}\left(e^{-\varrho V_T^v(\pi)}e^{-\varrho X}\right)$$

reduces to maximization over ψ of

$$E_{\mathbb{Q}^\psi}(X - \frac{1}{\varrho}H(\mathbb{Q}^\psi|\mathbb{P})),$$

that is, maximization over ψ of

$$E_{\mathbb{Q}^\psi}\left(X - \frac{1}{2\varrho}\theta^2 T - \frac{1}{\varrho}\int_0^T [(1+\psi_s)\ln(1+\psi_s) - \psi_s]\xi_s ds\right).$$

We solve this latter problem by operating

$$\begin{aligned} dU_t &= \left(\frac{1}{\varrho}[(1+\psi_t)\ln(1+\psi_t) - \psi_t]\xi_t\right)dt + \widehat{u}_t dW_t^{\mathbb{Q}} + \widetilde{u}_t d\widehat{M}_t, \\ U_T &= X - \frac{1}{2\varrho}\,\theta^2 T. \end{aligned}$$

Setting $Y_t = \varrho U_t$ we obtain

$$\begin{aligned} dY_t &= ([(1+\psi_t)\ln(1+\psi_t) - \psi_t]\xi_t)\,dt + \widehat{y}_t dW_t^{\mathbb{Q}} + \widetilde{y}_t d\widehat{M}_t, \\ Y_T &= \varrho X - \frac{1}{2}\theta^2 T. \end{aligned}$$

In terms of the martingale M, we get

$$dY_t = ([(1+\psi_t)\ln(1+\psi_t) - \psi_t(1+\widetilde{y}_t)]\xi_t)\,dt + \widehat{y}_t dW_t^{\mathbb{Q}} + \widetilde{y}_t dM_t,$$

The solution is obtained by maximization of the drift in the above equation w.r.t. ψ, which leads to $1+\psi_s = \widetilde{y}_s$. Consequently, the BSDE reads

$$dY_t = -\left(e^{\widetilde{y}_t} - 1 - \widetilde{y}_t\right)\xi_t dt + \widehat{y}_t dW_t^{\mathbb{Q}} + \widetilde{y}_t dM_t, \quad Y_T = \varrho X - \frac{1}{2}\theta^2 T,$$

and setting $Z_t^* = \exp(-Y_t)$ we conclude that

$$dZ_t^* = \frac{1}{2}Z_t^*\widehat{y}_t^2 dt - Z_t^*\widehat{y}_t dW_t^{\mathbb{Q}} + Z_{t-}^*(e^{\widehat{y}_t} - 1)dM_t, \quad Z_T^* = \exp(-\varrho X + \frac{1}{2}\theta^2 T),$$

or, denoting $\widehat{z}_t = -Z_t^*\widehat{y}_t, \widetilde{z}_t = Z_{t-}^*(e^{\widehat{y}_t} - 1)$

$$dZ_t^* = \frac{1}{2Z_t^*}\widehat{z}_t^2 dt + \widehat{z}_t dW_t^{\mathbb{Q}} + \widehat{z}_t dM_t, \quad Z_T^* = \exp(-\varrho X + \frac{1}{2}\theta^2 T),$$

which is equivalent to (6.19). (Note that $Z_t = Z_t^* e^{-\frac{1}{2}\theta^2(T-t)}$.)

6.3.2 Hodges Buying and Selling Prices

Particular case: attainable claims

Assume, as before, that $r = 0$ and let us check that the Hodges buying price is the hedging price in case of attainable claims. Assume that a claim X is $\mathcal{F}_T$-measurable. By virtue of the predictable representation theorem, there exists a pair $(x, \widehat{x})$, where x is a constant and $\widehat{x}_t$ is an $\mathbf{F}$-adapted process, such that $X = x + \int_0^T \widehat{x}_u \, dW_u^{\mathbb{Q}}$, where $W_t^{\mathbb{Q}} = W_t + \theta t$. Here $x = \mathbb{E}_{\mathbb{Q}} X$ is the arbitrage price of X and the replicating portfolio is obtained through $\widehat{x}$. Hence, the time t value of X is $X_t = x + \int_0^t \widehat{x}_u \, dW_u^{\mathbb{Q}}$. Then $dX_t = \widehat{x}_t \, dW_t^{\mathbb{Q}}$ and the process

$$Z_t = e^{-\theta^2 (T-t)/2} e^{-\varrho X_t}$$

satisfies

$$\begin{aligned} dZ_t &= Z_t \Big(\big(\frac{1}{2}\theta^2 + \frac{1}{2}\varrho^2 \widehat{x}_t^2 \big) \, dt + \varrho \widehat{x}_t \, dW_t^{\mathbb{Q}} \Big) \\ &= \frac{1}{2\sigma^2 Z_t} (\nu Z_t + \sigma \varrho Z_t \widehat{x}_t)^2 \, dt + \varrho Z_t \widehat{x}_t \, dW_t, \\ Z_T &= e^{-\varrho X}. \end{aligned}$$

Hence $(Z_t, \varrho Z_t \widehat{x}_t, 0)$ is the solution of (6.19) with the terminal condition $e^{-\varrho X}$, and

$$Z_0 = e^{-\theta^2 T/2} e^{-\varrho x} .$$

Note that, for $X = 0$, we get $Z_0 = e^{-\theta^2 T/2}$, therefore

$$\inf_{\pi \in \Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}} (e^{-\varrho V_T^v(\pi)}) = e^{-\varrho v} e^{-\theta^2 T/2}.$$

The $\mathbf{G}$-Hodges buying price of X is the value of p such that

$$\inf_{\pi \in \Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}} \big(e^{-\varrho V_T^v(\pi)} \big) = \inf_{\pi \in \Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}} \big(e^{-\varrho (V_T^{v-p}(\pi) + X)} \big),$$

that is,

$$e^{-\varrho v} e^{-\theta^2 T/2} = e^{-\varrho (v - p + \mathbb{E}_{\mathbb{Q}} X)} e^{-\theta^2 T/2}.$$

We conclude easily that $p_*^{\mathbf{G}}(X) = \mathbb{E}_{\mathbb{Q}} X$. Similar arguments show that $p^*_{\mathbf{G}}(X) = \mathbb{E}_{\mathbb{Q}} X$.

General case

Assume now that a claim X is $\mathcal{G}_T$-measurable. Assuming that the process Z introduced in (6.19) is strictly positive, we can use its logarithm. Let us denote $\widehat{\psi}_t = Z_t / \widehat{z}_t =$, $\widetilde{\psi}_t = Z_t / \widetilde{z}_t =$ and

$$\kappa_t = \frac{\widetilde{\psi}_t}{\ln(1 + \widetilde{\psi}_t)} \geq 0.$$

Then we get

$$d(\ln Z_t) = \tfrac{1}{2} \theta^2 dt + \widehat{\psi}_t \, dW_t^{\mathbb{Q}} + \ln(1 + \widetilde{\psi}_t) \big(dM_t + \xi_t (1 - \kappa_t) \, dt \big),$$

and thus

$$d(\ln Z_t) = \tfrac{1}{2} \theta^2 dt + \widehat{\psi}_t \, dW_t^{\mathbb{Q}} + \ln(1 + \widetilde{\psi}_t) \, d\widehat{M}_t,$$

where

$$d\widehat{M}_t = dM_t + \xi_t (1 - \kappa_t) \, dt = dH_t - \xi_t \kappa_t \, dt.$$

The process $\widehat{M}$ is a martingale under the probability measure $\widehat{\mathbb{Q}}$ defined as $d\widehat{\mathbb{Q}}|_{\mathcal{G}_t} = \widehat{\eta}_t \, d\mathbb{P}|_{\mathcal{G}_t}$, where $\widehat{\eta}$ satisfies

$$d\widehat{\eta}_t = -\widehat{\eta}_{t-} \big(\theta \, dW_t + \xi_t (1 - \kappa_t) \, dM_t \big)$$

with $\widehat{\eta}_0 = 1$.

Proposition 6.3.1 *The* $\mathbf{G}$*-Hodges buying price of* X *with respect to the exponential utility is the real number* p *such that* $e^{-\varrho(v-p)}Z_0^X = e^{-\varrho v}Z_0^0$*, that is,* $p^*_{\mathbf{G}}(X) = \varrho^{-1}\ln(Z_0^0/Z_0^X)$ *or, equivalently,* $p^*_{\mathbf{G}}(X) = \mathbf{E}_{\widehat{\mathbb{Q}}}X$*.*

Our previous study establishes that the dynamic hedging price of a claim X is the process $X_t = \mathbf{E}_{\widehat{\mathbb{Q}}}(X \,|\, \mathcal{G}_t)$. This price is the expectation of the payoff, under some martingale measure, as is any price in the range of no-arbitrage prices.

Remark All the results presented in this section remain valid if ν and σ are adapted processes.

6.4 Quadratic Hedging

We work under the same hypothesis as before; in particular, the wealth process follows

$$dV_t^v(\pi) = \pi_t(\nu dt + \sigma dW_t), \quad V_0^v(\pi) = v\,.$$

In the last part of this section we shall study a more general case.

The objective of this section is to examine the issue of quadratic pricing and hedging. Specifically, for a given $\mathbb{P}$-square-integrable claim $X \in \mathcal{G}_T$, we study the following problems:

• For a given initial endowment v, solve the minimization problem:

$$\min_{\pi} \mathbb{E}_{\mathbb{P}}((V_T^v(\pi) - X)^2)\,.$$

A solution to this problem provides the portfolio which, among the portfolios with a *given initial wealth*, has the closest terminal wealth to a given claim X, in the sense of L^2-norm under the historical probability $\mathbb{P}$. The solution of this problem exists, since the set of stochastic integrals of the form $\int_0^T \phi_s dS_s$ is closed in L^2.

• Solve the minimization problem:

$$\min_{\pi,v} \mathbb{E}_{\mathbb{P}}((V_T^v(\pi) - X)^2)\,.$$

The optimal value of v is called the *quadratic hedging price* and the optimal π the *quadratic hedging strategy.*

The quadratic hedging problem was examined in a fairly general framework of incomplete markets by means of BSDEs in several papers; see, for example, Mania [154], Mania and Tevzadze [156], Bobrovnytska and Schweizer [29], Hu and Zhou [103] or Lim [151]. Since this list is by no means exhaustive, the interested reader is referred to the references quoted in the above-mentioned papers. The reader may refer to Bielecki et al. [13] for a study of the same problem under a constraint on the expectation. Also, some additional references can be found in that paper.

6.4.1 Quadratic Hedging with F-Adapted Strategies

We shall first solve, for a given initial endowment v, the following minimization problem

$$\min_{\pi \in \Pi(\mathbf{F})} \mathbb{E}_{\mathbb{P}}((V_T^v(\pi) - X)^2),$$

where X is given as

$$X = X_1 \mathbb{1}_{\{\tau > T\}} + X_2 \mathbb{1}_{\{\tau \le T\}}$$

for some $\mathcal{F}_T$-measurable, $\mathbb{P}$-square-integrable random variables X_1 and X_2. Using the same approach as in Section 6.2.1, we define

$$J_X(y) = (y - X_1)^2(1 - F_T) + (y - X_2)^2 F_T$$

and its derivative

$$J'_X(y) = 2\left[(y - X_1)(1 - F_T) + (y - X_2)F_T\right] = 2\left[y - X_1(1 - F_T) - X_2 F_T\right].$$

Hence, the inverse of $J'_X(y)$ is

$$I_X(z) = \frac{1}{2}z + X_1(1 - F_T) + X_2 F_T$$

and thus the optimal terminal wealth equals

$$V_T^{v,*} = \frac{1}{2}\,\lambda^* \zeta_T + X_1(1 - F_T) + X_2 F_T,$$

where λ^* is specified through the budget constraint:

$$\mathbb{E}_{\mathbb{P}}(\zeta_T V_T^{v,*}) = \frac{1}{2}\lambda^*\,\mathbb{E}_{\mathbb{P}}(\zeta_T^2) + \mathbb{E}_{\mathbb{P}}(\zeta_T X_1(1 - F_T)) + \mathbb{E}_{\mathbb{P}}(\zeta_T X_2 F_T) = v\,.$$

The optimal strategy is the one, which hedges the $\mathcal{F}_T$-measurable contingent claim

$$\lambda^*\zeta_T + X_1(1 - F_T) + X_2 F_T = 2e^{-\theta_2 T}(v - \mathbb{E}_{\mathbb{Q}}(X))\zeta_T + X_1(1 - F_T) + X_2 F_T\,.$$

We deduce that

$$\begin{aligned}
&\min_{\pi} \mathbb{E}_{\mathbb{P}}((V_T^v - X)^2)\\
&\quad= \mathbb{E}_{\mathbb{P}}\left[\left(\frac{1}{2}\lambda^*\zeta_T + X_1(1 - F_T) + X_2F_T - X_1\right)^2 (1 - F_T)\right]\\
&\qquad+\mathbb{E}_{\mathbb{P}}\left[\left(\frac{1}{2}\lambda^*\zeta_T + X_1(1 - F_T) + X_2F_T) - X_2\right)^2 F_T\right]\\
&\quad= \tfrac{1}{4}(\lambda^*)^2\,\mathbb{E}_{\mathbb{P}}(\zeta_T^2) + \mathbb{E}_{\mathbb{P}}\left((X_1 - X_2)^2 F_T(1 - F_T)\right)\\
&\quad= \frac{1}{2\mathbb{E}_{\mathbb{P}}(\zeta_T^2)}\Big(v - \mathbb{E}_{\mathbb{P}}(\zeta_T(X_1 + F_T(X_2 - X_1))\Big)^2\\
&\qquad+\mathbb{E}_{\mathbb{P}}((X_1 - X_2)^2 F_T(1 - F_T)).
\end{aligned}$$

It remains to minimize over v the right-hand side, which is now simple. Therefore, we obtain the following result.

Proposition 6.4.1 *If we restrict our attention to* $\mathbf{F}$*-adapted strategies, the quadratic hedging price of the claim* $X = X_1 1\!\!1_{\{\tau > T\}} + X_2 1\!\!1_{\{\tau \le T\}}$ *equals*

$$\mathbb{E}_{\mathbb{P}}(\zeta_T(X_1 + F_T(X_2 - X_1)) = \mathbb{E}_{\mathbb{Q}}(X_1(1 - F_T) + F_T X_2)\,.$$

The optimal quadratic hedging of X *is the strategy which replicates the* $\mathcal{F}_T$*-measurable contingent claim* $X_1(1 - F_T) + F_T X_2$.

Let us now examine the case of a generic $\mathcal{G}_T$-measurable random variable X. Here, we shall only examine the solution of the second problem introduced above, that is,

$$\min_{v,\pi} \mathbb{E}_{\mathbb{P}}((V_T^v(\pi) - X)^2)\,.$$

As explained in Bielecki et al. [176], this problem is essentially equivalent to a problem where we restrict our attention to the terminal wealth so that we may reduce the problem to $\min_{V\in\mathcal{F}_T} \mathbb{E}_{\mathbb{P}}((V - X)^2)$. From the properties of conditional expectations, we have

$$\min_{V\in\mathcal{F}_T} \mathbb{E}_{\mathbb{P}}((V - X)^2) = \mathbb{E}_{\mathbb{P}}((\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_T) - X)^2)$$

and the initial value of the strategy with terminal value $\mathbb{E}_{\mathbb{P}}(X \mid \mathcal{F}_T)$ is

$$\mathbb{E}_{\mathbb{P}}(\zeta_T \mathbb{E}_{\mathbb{P}}(X \mid \mathcal{F}_T)) = \mathbb{E}_{\mathbb{P}}(\zeta_T X).$$

In essence, the latter statement is a consequence of the completeness of the default-free market model. Indeed, the fact that the conditional expectation $\mathbb{E}_{\mathbb{P}}(X \mid \mathcal{F}_T)$ can be written as a stochastic integral w.r.t. S follows directly from the completeness of the default-free market. In conclusion, the quadratic hedging price equals $\mathbb{E}_{\mathbb{P}}(\zeta_T X) = \mathbb{E}_{\mathbb{Q}} X$ and the quadratic hedging strategy is the replicating strategy of the attainable claim $\mathbb{E}_{\mathbb{P}}(X \mid \mathcal{F}_T)$ associated with X.

6.4.2 Quadratic Hedging with G-Adapted Strategies

Similarly as in the previous subsection we assume here that the price process of the underlying asset obeys

$$dS_t = S_t(\nu dt + \sigma dW_t).$$

The wealth process follows

$$dV_t^v(\pi) = \pi_t(\nu dt + \sigma dW_t), \quad V_0^v(\pi) = v\,.$$

We shall first solve, for a given initial endowment v, the following minimization problem

$$\min_{\pi \in \Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}}((V_T^v(\pi) - X)^2).$$

As discussed in Bielecki et al.[176] one way of solving this problem is to project the random variable X on the closed set of stochastic integrals of the form $\int_0^T \varphi_s dS_s$. Here, we present an alternative approach. We are looking for $\mathbf{G}$-adapted processes X, Θ and Ψ such that the process

$$J_t(\pi, v) = \big(V_t^v(\pi) - X_t\big)^2 \Theta_t + \Psi_t, \quad \forall\, t \in [0, T], \tag{6.21}$$

is a $\mathbf{G}$-submartingale for any $\mathbf{G}$-adapted trading strategy π and a $\mathbf{G}$-martingale for some strategy π^*. In addition, we require that $X_T = X$, $\Theta_T = 1$, $\Phi_T = 0$ so that $J_T(\pi, v) = (V_T^v(\pi) - X)^2$. Let us assume that the dynamics of these processes are of the form

$$\begin{aligned}
dX_t &= x_t\, dt + \widehat{x}_t\, dW_t + \widetilde{x}_t\, dM_t, && (6.22)\\
d\Theta_t &= \Theta_{t-}\big(\vartheta_t\, dt + \widehat{\vartheta}_t\, dW_t + \widetilde{\vartheta}_t\, dM_t\big), && (6.23)\\
d\Psi_t &= \psi_t\, dt + \widehat{\psi}_t\, dW_t + \widetilde{\psi}_t\, dM_t, && (6.24)
\end{aligned}$$

where the drifts x_t, ϑ_t and ψ_t are yet to be determined. From Itô's formula, we obtain (recall that $\xi_t = \gamma_t \mathbb{1}_{\{\tau > t\}}$)

$$\begin{aligned}
d(V_t - X_t)^2 = {} & 2(V_t - X_t)(\pi_t\sigma - \widehat{x}_t)\, dW_t - 2(V_t - X_{t-})\widetilde{x}_t\, dM_t \\
& + \big[(V_t - X_{t-} - \widetilde{x}_t)^2 - (V_t - X_{t-})^2\big] dM_t \\
& + \Big(2(V_t - X_t)(\pi_t\nu - x_t) + (\pi_t\sigma - \widehat{x}_t)^2 \\
& + \xi_t\big[(V_t - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2\big]\Big) dt,
\end{aligned}$$

where we denote $V_t = V_t^v(\pi)$. Then, using integration by parts formula, we obtain by straightforward calculations

$$J_t(\pi) = k(t, \pi_t, \vartheta_t, x_t, \psi_t) dt + martingale$$

where

$$\begin{aligned}
k(t, \pi_t, \vartheta_t, x_t, \psi_t) = {} & \psi_t + \Theta_t\Big[\vartheta_t(V_t - X_t)^2 \\
& + 2(V_t - X_t)\big[(\pi_t\nu - x_t) + \widehat{\vartheta}_t(\pi_t\sigma - \widehat{x}_t) + \xi_t\widetilde{x}_t\big] \\
& + (\pi_t\sigma - \widehat{x}_t)^2 + \xi_t(\widetilde{\vartheta}_t + 1)\big[(V_t - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2\big]\Big]\,.
\end{aligned}$$

The process $J(\pi)$ is a (local) martingale if and only if its drift term $k(t,\pi_t,x_t,\vartheta_t,\psi_t)$ equals 0 for every $t\in[0,T]$.

In the first step, for any $t\in[0,T]$ we shall find π_t^* such that the minimum of $k(t,\pi_t,x_t,\vartheta_t,\psi_t)$ is attained. Subsequently, we shall choose the processes $x=x^*,\vartheta=\vartheta^*$ and $\psi=\psi^*$ in such a way that $k(t,\pi_t^*,x_t^*,\vartheta_t^*,\psi_t^*)=0$. This choice will imply that $k(t,\pi_t,x_t^*,\vartheta_t^*,\psi_t^*)\geq 0$ for any trading strategy π and any $t\in[0,T]$.

The strategy π^* which minimizes $k(t,\pi_t,x_t,\vartheta_t,\psi_t)$ is the solution of the following equation:

$$(V_t^v(\pi)-X_t)(\nu+\widehat{\vartheta}_t\sigma)+\sigma(\pi_t\sigma-\widehat{x}_t)=0,\quad \forall\, t\in[0,T].$$

Hence, the strategy π^* is implicitly given by

$$\pi_t^*=\sigma^{-1}\widehat{x}_t-\sigma^{-2}(\nu+\widehat{\vartheta}_t\sigma)(V_t^v(\pi^*)-X_t)=A_t-B_t(V_t^v(\pi^*)-X_t),$$

where we denote

$$A_t=\sigma^{-1}\widehat{x}_t,\quad B_t=\sigma^{-2}(\nu+\widehat{\vartheta}_t\sigma).$$

After some computations, we see that the drift term of the process $J(\pi^*)$ admits the following representation:

$$\begin{aligned}k(t,\pi_t,\vartheta_t,x_t,\psi_t)&=\psi_t+\Theta_t(V_t-X_t)^2(\vartheta_t-\sigma^2B_t^2)\\&+2\Theta_t(V_t-X_t)\big(\sigma^2A_tB_t-\widehat{\vartheta}_t\widehat{x}_t-\widetilde{\vartheta}_t\widetilde{x}_t\xi_t-x_t\big)+\Theta_t\xi_t(\widetilde{\vartheta}_t+1)\widetilde{x}_t^2.\end{aligned}$$

From now on, we shall assume that the auxiliary processes ϑ,x and ψ are chosen as follows:

$$\begin{aligned}\vartheta_t&=\vartheta_t^*=\sigma^2B_t^2,\\x_t&=x_t^*=\sigma^2A_tB_t-\widehat{\vartheta}_t\widehat{x}_t-\widetilde{\vartheta}_t\widetilde{x}_t\xi_t,\\\psi_t&=\psi_t^*=-\Theta_t\xi_t(\widetilde{\vartheta}_t+1)\widetilde{x}_t^2.\end{aligned}$$

Straightforward computation verifies that if the drift coefficients ϑ,x,ψ in (6.22)-(6.24) are chosen as above, then the drift term in dynamics of J is always non-negative, and it is equal to 0 for $\pi_t^*=A_t-B_t(V_t^v(\pi^*)-X_t)$.

Our next goal is to solve equations (6.22)-(6.24). Since $\vartheta_t=\sigma^2B_t^2$, the three-dimensional process $(\Theta,\widehat{\vartheta},\widetilde{\vartheta})$ is the unique solution to the linear BSDE (6.23)

$$d\Theta_t=\Theta_t\big(\sigma^{-2}(\nu+\widehat{\vartheta}_t\sigma)^2\,dt+\widehat{\vartheta}_t\,dW_t+\widetilde{\vartheta}_t\,dM_t\big),\ \Theta_T=1.$$

It is obvious that a solution is

$$\widehat{\vartheta}_t=0,\quad \widetilde{\vartheta}_t=0,\quad \Theta_t=\exp(-\theta^2(T-t)),\quad \forall\, t\in[0,T]. \tag{6.25}$$

The three-dimensional process $(X,\widehat{x},\widetilde{x})$ solves equation (6.22) with $x_t=x_t^*=\sigma^2A_t(\nu/\sigma^2)=\theta\widehat{x}_t$. This means that $(X,\widehat{x},\widetilde{x})$ is the unique solution to the linear BSDE

$$dX_t=\theta\widehat{x}_t\,dt+\widehat{x}_t\,dW_t+\widetilde{x}_t\,dM_t,\ X_T=X.$$

The unique solution to the last equation is $X_t=\mathbb{E}_{\mathbb{Q}}(X\,|\,\mathcal{G}_t)$. The components $\widehat{x}$ and $\widetilde{x}$ are given by the integral representation of the $\mathbf{G}$-martingale $(X_t,t\geq 0)$ with respect to $W^{\mathbb{Q}}$ and M, where $W_t^{\mathbb{Q}}=W_t+\theta t$. Notice also that since $\widehat{\vartheta}=0$, the optimal portfolio π^* is given by the feedback formula

$$\pi_t^*=\sigma^{-1}\big(\widehat{x}_t-\theta(V_t^v(\pi^*)-X_t)\big).$$

Finally, since $\widetilde{\vartheta}=0$, we have $\psi_t=-\xi_t\widetilde{x}_t^2\Theta_t$. Therefore, we can solve explicitly the BSDE (6.24) for the process Ψ. Indeed, we are now looking for a three-dimensional process $(\Psi,\widehat{\psi},\widetilde{\psi})$, which is the unique solution of the BSDE

$$d\Psi_t=-\Theta_t\xi_t\widetilde{x}_t^2\,dt+\widehat{\psi}_t\,dW_t+\widetilde{\psi}_t\,dM_t,\ \Psi_T=0.$$

Noting that the process

$$\Psi_t + \int_0^t \Theta_s \xi_s \widetilde{x}_s^2 \, ds$$

is a **G**-martingale under $\mathbb{P}$ with terminal value $\int_0^T \Theta_s \xi_s \widetilde{x}_s^2 \, ds$, we obtain the value of Ψ in a closed form:

$$\begin{aligned} \Psi_t &= \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \Theta_s \xi_s \widetilde{x}_s^2 \, ds \,\Big|\, \mathcal{G}_t \Big) \\ &= \int_t^T e^{-\theta^2 (T-s)} \, \mathbb{E}_{\mathbb{P}}\big(\gamma_s \widetilde{x}_s^2 1\!\!1_{\{\tau > s\}} \,\big|\, \mathcal{G}_t \big) \, ds \\ &= \int_t^T e^{-\theta^2 (T-s)} \, \mathbb{E}_{\mathbb{P}}\big(\gamma_s \widetilde{x}_s^2 e^{\Gamma_t - \Gamma_s} \,\big|\, \mathcal{F}_t \big) \, ds \end{aligned} \tag{6.26}$$

where we have identified the process $\widetilde{x}$ with its **F**-adapted version (recall that any **G**-predictable process is equal, prior to default, to an **F**-predictable process).

Substituting (6.25) and (6.26) in (6.21), we conclude that for a fixed v the value function for our problem is $J_t^*(v) = J_t(\pi^*, v)$, where in turn

$$J_t(\pi^*, v) = (V_t^v(\pi^*) - X_t)^2 e^{-\theta^2 (T-t)} + 1\!\!1_{\{\tau > t\}} \int_t^T e^{-\theta^2 (T-s)} \, \mathbb{E}_{\mathbb{P}}\big(\gamma_s \widetilde{x}_s^2 e^{\Gamma_t - \Gamma_s} \,\big|\, \mathcal{F}_t \big) \, ds \, .$$

In particular,

$$J_0^*(v) = e^{-\theta^2 T} \Big((v - X_0)^2 + \mathbb{E}_{\mathbb{P}}\Big(\int_0^T e^{\theta^2 s} \gamma_s \widetilde{x}_s^2 e^{-\Gamma_s} \, ds \Big) \Big).$$

The quadratic hedging price, say v^*, is obtained by minimizing $J_0^*(v)$ with respect to v. From the last formula, it is obvious that the quadratic hedging price is $v^* = X_0 = \mathbb{E}_{\mathbb{Q}} X$. We are in the position to formulate the main result of this section. A corresponding theorem for a default-free financial model was established by Kohlmann and Zhou [138].

Proposition 6.4.2 *Let a claim X be $\mathcal{G}_T$-measurable and square-integrable under $\mathbb{P}$. The optimal trading strategy π^*, which solves the quadratic problem*

$$\min_{\pi \in \Pi(\mathbf{G})} \mathbb{E}_{\mathbb{P}}((V_T^v(\pi) - X)^2),$$

is given by the feedback formula

$$\pi_t^* = \sigma^{-1} \big(\widehat{x}_t - \theta (V_t^v(\pi^*) - X_t) \big),$$

where $X_t = \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}_t)$ for every $t \in [0, T]$, and the process $\widehat{x}_t$ is specified by

$$dX_t = \widehat{x}_t \, dW_t^{\mathbb{Q}} + \widetilde{x}_t \, dM_t.$$

The quadratic hedging price of X is $\mathbb{E}_{\mathbb{Q}} X$.

Example: Survival Claim

Let us consider a simple survival claim $X = 1\!\!1_{\{\tau > T\}}$, and let us assume that Γ is deterministic, specifically, $\Gamma(t) = \int_0^t \gamma(s) \, ds$. In that case, from the representation theorem (see Bielecki and Rutkowski (2002), Page 159), we have $dX_t = \widetilde{x}_t \, dM_t$ with $\widetilde{x}_t = -e^{\Gamma(t) - \Gamma(T)}$. Hence

$$\Psi_t = \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \Theta_s \xi_s \widetilde{x}_s^2 \, ds \,\Big|\, \mathcal{G}_t \Big)$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \Theta_s \gamma(s) 1\!\!1_{\{\tau > s\}} e^{2\Gamma(s)-2\Gamma(T)}\, ds \,\Big|\, \mathcal{G}_t \Big) \\
&= 1\!\!1_{\{\tau > t\}}\, e^{\Gamma(t)-2\Gamma(T)}\, \mathbb{E}_{\mathbb{P}}\Big(\int_t^T e^{-\theta^2(T-s)} \gamma(s) e^{\Gamma(s)}\, ds \,\Big|\, \mathcal{F}_t \Big) \\
&= 1\!\!1_{\{\tau > t\}}\, e^{\Gamma(t)-2\Gamma(T)} \int_t^T e^{-\theta^2(T-s)} \gamma(s) e^{\Gamma(s)}\, ds.
\end{aligned}
$$

One can check that, at time 0, the value function is indeed smaller that the one obtained with **F**-adapted portfolios.

Case of an Attainable Claim

Assume now that a claim X is $\mathcal{F}_T$-measurable. Then $X_t = \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}_t)$ is the price of X, and it satisfies $dX_t = \widehat{x}_t\, dW_t^{\mathbb{Q}}$. The optimal strategy is, in a feedback form,

$$\pi_t^* = \sigma^{-1}\big(\widehat{x}_t - \theta(V_t - X_t)\big)$$

and the associated wealth process satisfies

$$dV_t = \pi_t^*(\nu dt + \sigma dW_t) = \pi_t^* \sigma\, dW_t^{\mathbb{Q}} = \sigma^{-1}\big(\sigma \widehat{x}_t - \nu(V_t - X_t)\big)\, dW_t^{\mathbb{Q}}.$$

Therefore,

$$d(V_t - X_t) = -\theta(V_t - X_t)\, dW_t^{\mathbb{Q}}.$$

Hence, if we start with an initial wealth equal to the arbitrage price $\mathbb{E}_{\mathbb{Q}} X$ of X, then we that $V_t = X_t$ for every $t \in [0, T]$, as expected.

Hodges Price

Let us emphasize that the Hodges price has no real meaning here, since the problem $\min \mathbb{E}_{\mathbb{P}}((V_T^v)^2)$ has no financial interpretation. We have studied in Bielecki et al. [176] a more pertinent problem, with a constraint on the expected value of V_T^v under $\mathbb{P}$. Nevertheless, from a mathematical point of view, the Hodges price would be the value of p such that

$$(v^2 - (v-p)^2) = \int_0^T e^{\theta^2 s} \mathbb{E}_{\mathbb{P}}(\gamma_s \widetilde{x}_s^2 e^{-\Gamma_s}) 1\!\!1_{\{\tau > t\}} ds$$

In the case of the example studied in Section 6.4.2, the Hodges price would be the non-negative value of p such that

$$2vp - p^2 = e^{-2\Gamma_T} \int_0^T e^{\theta^2 s} \gamma_s e^{\Gamma_s}\, ds.$$

Let us also mention that our results are different from results of Lim [151]. Indeed, Lim studies a model with Poisson component, and thus in his approach the intensity of this process does not vanish after the first jump.

6.4.3 Jump-Dynamics of Price

We assume here that the price process follows

$$dS_t = S_{t-}(\nu dt + \sigma dW_t + \varphi dM_t), \quad S_0 > 0$$

where the constant φ satisfy $\varphi > -1$ so that the price S_t is strictly positive. Hence, the primary market, where the savings account and the asset S are traded is arbitrage free, but incomplete (in general). It follows that the wealth process follows

$$dV_t^v(\pi) = \pi_t(\nu dt + \sigma dW_t + \varphi dM_t), \quad V_0^v(\pi) = v.$$

As in the previous subsection, our aim is, for a given initial endowment v, solve the minimization problem:

$$\min_{\pi} \mathbb{E}_{\mathbb{P}}((V_T^v(\pi) - X)^2).$$

In order to characterize the value function we proceed analogously as before. That is, we are looking for processes X, Θ and Ψ such that the process (for simplicity we write V_t in place of $V_t^v(\pi)$)

$$J(t, V_t) = (V_t - X_t)^2 \Theta_t + \Psi_t$$

is a submartingale for any π and a martingale for some π^*, and such that $\Psi_T = 0$, $X_T = X$, $\Theta_T = 1$. (Note that Mania and Tevzadze[156] did a similar approach for continuous processes, with a value function of the form $J_t = \Phi_0(t) + \Phi_1(t)V_t + \Phi_2(t)V_t^2$.) Let us assume that the dynamics of these processes are of the form

$$\begin{aligned} dX_t &= f_t dt + \widehat{x}_t dW_t + \widetilde{x}_t dM_t, &(6.27)\\ d\Theta_t &= \Theta_t(\vartheta_t dt + \widehat{\vartheta}_t dW_t + \widetilde{\vartheta}_t dM_t) &(6.28)\\ d\Psi_t &= \psi_t dt + \widehat{\psi}_t dW_t + \widetilde{\psi}_t dM_t &(6.29) \end{aligned}$$

where the drifts f_t, ϑ_t and ψ_t have to be determined.

From Itô's formula we obtain

$$\begin{aligned} d(V_t - X_t)^2 &= 2(V_t - X_t)(\pi_t\sigma - \widehat{x}_t)dW_t \\ &+ \left[(V_t + \pi_t\varphi - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2\right] dM_t \\ &+ \big(2(V_t - X_t)(\pi_t\mu - f_t) + (\pi_t\sigma - \widehat{x}_t)^2 \\ &+ \xi_t \left[(V_t + \pi_t\varphi - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2 - 2(V_t - X_t)(\pi_t\varphi - \widetilde{x}_t)\right]\big)\, dt. \end{aligned}$$

Process $\Theta_t(V_t - X_t)^2 + \Psi_t$ is a (local) martingale iff $k(\pi_t, f_t, \vartheta_t, \psi_t) = 0$ for all t, where

$$\begin{aligned} k(\pi, \vartheta, f, \psi) &= \psi + \Theta_t \big[\vartheta_t(V_t - X_t)^2 \\ &+ 2(V_t - X_t)\Big((\pi\mu - f) + \widehat{\vartheta}_t(\pi\sigma - \widehat{x}_t) - \xi_t(\pi\varphi - \widetilde{x}_t)\Big) \\ &+ (\pi\sigma - \widehat{x}_t)^2 \\ &+ \xi_t(\widetilde{\vartheta}_t + 1)\left((V_t + \pi\varphi - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2\right)\Big]\,. \end{aligned}$$

In the first step, we find $\pi^\sharp$ such that the maximum of $k(\pi)$ is obtained. Then, one defines $(f^*, \vartheta^*, \psi^*)$ such that $k(\pi^\sharp, f^*, \vartheta^*, \psi^*) = 0$. This implies that, for any π, $k(\pi, f^*, \vartheta^*, \psi^*) \leq 0$, and that $k(\pi^\sharp, f^*, \vartheta^*, \psi^*) = 0$.

The optimal $\pi^\sharp$ is the solution of

$$\begin{aligned} &(V_t - X_t)(\mu - \xi_t\varphi + \widehat{\vartheta}_t\sigma) + \sigma(\pi\sigma - \widehat{x}_t) \\ &+ \xi_t(\widetilde{\vartheta}_t + 1)\varphi(V_t + \pi\varphi - X_t - \widetilde{x}_t) = 0\,. \end{aligned}$$

hence

$$\begin{aligned} \pi_t^\sharp &= \frac{1}{\sigma^2 + \varphi^2\xi_t(\widetilde{\vartheta}_t + 1)} \Big((\sigma\widehat{x}_t + \xi_t\varphi(\widetilde{\vartheta}_t + 1)\widetilde{x}_t) - (\mu + \widehat{\vartheta}_t\sigma + \xi_t\varphi\widetilde{\vartheta}_t)(V_t - X_t)\Big) \\ &= A_t - B_t(V_t - X_t) \end{aligned}$$

with

$$\begin{aligned} A_t &= \Big(\sigma\widehat{x}_t + \xi_t\varphi(\widetilde{\vartheta}_t + 1)\widetilde{x}_t\Big)\,\Delta_t^{-1} \\ B_t &= \Big(\mu + \widehat{\vartheta}_t\sigma + \xi_t\varphi\widetilde{\vartheta}_t\Big)\,\Delta_t^{-1} \\ \Delta_t &= \sigma^2 + \varphi^2\xi_t(\widetilde{\vartheta}_t + 1)\,. \end{aligned}$$

After some computations the drift term of $\Theta_t(V_t - X_t) + \Psi_t$ is found to be

$$\begin{aligned}&\Theta_t(V_t - X_t)^2(\vartheta_t - B_t^2\Delta_t) + 2\Theta_t(V_t - X_t)\left(A_tB_t\Delta_t - \widehat{\vartheta}_t\widehat{x}_t - \xi_t\widetilde{\vartheta}_t\widetilde{x}_t - f_t\right)\\ &+ \quad \Theta_t\xi_t(\widetilde{\vartheta}_t + 1)(A_t\varphi - \widetilde{x}_t)^2 + \Theta_t(A_t\sigma - \widehat{x}_t)^2 + \psi_t\,.\end{aligned}$$

Then, we choose

$$\begin{aligned}\vartheta_t^* &= B_t^2\Delta_t\\ f_t^* &= A_tB_t\Delta_t - \widehat{\varphi}_t\widehat{x}_t - \xi_t\widetilde{\vartheta}_t\widetilde{x}_t\\ \psi_t^* &= -\Theta_t\xi_t(\widetilde{\vartheta}_t + 1)(A_t\varphi - \widetilde{x}_t)^2 - \Theta_t(A_t\sigma - \widehat{x}_t)^2.\end{aligned}$$

Let us suppose that with this choice of drifts equations (6.28)–(6.29) admit solutions (we shall discuss this issue below). Next, let us denote these solutions as $(\Theta^*, \widehat{\vartheta}^*, \widetilde{\vartheta}^*)$, $(X^*, \widehat{x}^*, \widetilde{x}^*)$ and $(\Psi^*, \widehat{\psi}^*, \widetilde{\psi}^*)$; the corresponding processes A, B and Δ will be denoted as A^*, B^* and Δ^*. Consequently, the drift term of $\Theta_t^*(V_t^*(\pi) - X_t^*) + \Psi_t^*$ is non-positive for any admissible π and it is equal to 0 for $\pi^* = A_t^* - B_t^*(V_t^{v,*}(\pi^*) - X_t^*)$.

The three dimensional process $(\Theta^*, \widehat{\vartheta}^*, \widetilde{\vartheta}^*)$ is supposed to satisfy the BSDE

$$\begin{aligned}d\Theta_t &= \Theta_t\left(\frac{(\mu + \widehat{\vartheta}_t\sigma + \xi_t\varphi\widetilde{\vartheta}_t)^2}{\sigma^2 + \varphi^2\xi_t(\widetilde{\vartheta}_t + 1)}dt + \widehat{\vartheta}_t dW_t + \widetilde{\vartheta}_t dM_t\right)\\ \Theta_T &= 1.\end{aligned} \tag{6.30}$$

We shall discuss this equation later.

The three dimensional process $(X^*, \widehat{x}^*, \widetilde{x}^*)$ is a solution of the **linear BSDE**

$$\begin{aligned}dX_t &= \frac{1}{\Delta_t}\left(\kappa_{1,t}\widehat{x}_t + \kappa_{2,t}\widetilde{x}_t\right)dt + \widehat{x}_t dW_t + \widetilde{x}_t dM_t\\ X_T &= X\end{aligned}$$

where

$$\kappa_{1,t} = \sigma\mu + \sigma\varphi\xi_t\widetilde{\vartheta}_t - \varphi^2\widehat{\vartheta}_t\xi_t(1 + \widetilde{\vartheta}_t),\ \kappa_{2,t} = \varphi\xi_t(1 + \widetilde{\vartheta}_t)(\mu + \sigma\widehat{\vartheta}_t) - \sigma^2\xi_t\widetilde{\vartheta}_t\,.$$

Thus,

$$X_t^* = \mathbf{E}_{\mathbb{Q}^\kappa}(X|\mathcal{G}_t),$$

where $d\mathbb{Q}^\kappa|_{\mathcal{G}_t} = L_t^{(\kappa)}d\mathbb{P}|_{\mathcal{G}_t}$ and

$$dL_t^{(\kappa)} = -L_{t-}^{(\kappa)}\left(\frac{\kappa_{1,t}}{\Delta_t}dW_t + \frac{\kappa_{2,t}}{\xi\Delta_t}dM_t\right).$$

The three dimensional process $(\Psi^*, \widehat{\psi}^*, \widetilde{\psi}^*)$ is solution of

$$\begin{aligned}d\Psi_t &= -\Theta_t\left(\xi_t(\widetilde{\vartheta}_t + 1)(A_t\varphi - \widetilde{x}_t)^2 + (A_t\sigma - \widehat{x}_t)^2\right)dt + \widehat{\psi}_t dW_t + \widetilde{\psi}_t dM_t\\ \Psi_T &= 0.\end{aligned}$$

Thus, noting that

$$\Psi_t^* + \int_0^t \Theta_s\left(\xi_s(\widetilde{\vartheta}_s + 1)(A_s\varphi - \widetilde{x}_s)^2 + (A_s\sigma - \widehat{x}_s)^2\right)ds$$

is a $\mathbf{G}$-martingale, we obtain that

$$\Psi_t^* = E\left(\int_t^T \Theta_s\left(\xi_s(\widetilde{\vartheta}_s + 1)(A_s\varphi - \widetilde{x}_s)^2 + (A_s\sigma - \widehat{x}_s)^2\right)ds|\mathcal{G}_t\right). \tag{6.31}$$

Discussion of equation (6.30): Duality approach

Our aim is here to prove that the BSDE (6.30) has a solution. We take the opportunuity to correct a mistake in Bielecki et al [176] where we claim that, in the particular case where the intensity γ_t is constant, we get a solution of the form $\widetilde{\theta}_t$ constant. The solution that appear in Bielecki et al. is valid only in the case $P(\tau < T) = 1$. We proceed using duality approach.

The set of equivalent martingale measure is determined by the set of densities. From Kusuoka [140] representation theorem, it follows that any strictly positive martingale in the filtration $\mathbf{G}$ can be written as

$$dL_t = L_{t-}(\ell_t dW_t + \chi_t dM_t) \tag{6.32}$$

for a $\mathbf{G}$-predictable process χ satisfying $\chi_t > -1$. In order that L corresponds to the Radon-Nikodym density of an emm, a relation between ℓ and χ has to be satisfied in order to imply that process $L_t S_t$ is a $\mathbb{P}$ (local) martingale. (Recall that $r = 0$.) Straightforward application of integration by parts formula proves that the drift term of LS vanishes iff

$$\varphi \chi_t \xi_t + \sigma \ell_t + \nu = 0$$

Recall that by definition the variance optimal measure for L is a probability measure $\mathbb{Q}^*$ such that it minimizes $\mathbf{E}_{\mathbb{Q}^*}(L_T^2)$. At this moment we are unable to verify existence/uniqueness of such a measure in the context of our model. We thus assume that the measure exists,

Hypothesis: We assume that the variance optimal measure exists.

In what follows we shall use the same argument as in Bobrovnytska and Schweizer [29]. Towards this end we denote by L^* the Radon-Nikodym density of the variance optimal martingale measure. Let Z be the martingale $Z_t = \mathbf{E}_{\mathbb{Q}^*}(L_T^*|\mathcal{G}_t)$ and $U = L^*/Z$. It is proved in Delbaen and Schachermayer [62] (Lemma 2.2) that, if the variance optimal martingale measure exists, then there exists a predictable process $\widehat{z}$ such that

$$dZ_t/Z_{t-} = \widehat{z}_t dS_t = z_t(\sigma dW_t + \varphi dM_t + \nu dt)$$

where $z_t = \widehat{z}_t S_{t-}$ (in the proof of lemma 2.2, the hypothesis of continuity of the asset is not required). The process L^* is a $(\mathbb{P}, \mathbf{G})$ martingale, hence there exist ℓ and χ such that

$$dL_t^* = L_{t-}^*(\ell_t dW_t + \chi_t dM_t)$$

From Itô's calculus, setting $U = L^*/Z$, we obtain

$$dU_t = U_{t-}\left(A_t dt + (\ell_t - z_t\sigma)dW_t + \left(\frac{1}{1+z_t\varphi}(\chi_t + 1) - 1\right)dM_t\right), \quad U_T = 1,$$

where

$$\begin{aligned} A_t &= z_t^2\sigma^2 + \xi_t(1+\chi_t)(z_t\varphi + \frac{1}{1+z_t\varphi} - 1) \\ &= z_t^2\sigma^2 + \xi_t(1+\chi_t)\frac{z_t^2\varphi^2}{1+z_t\varphi} \\ &= z_t^2\left(\sigma^2 + \xi_t(1+\chi_t)\frac{\varphi^2}{1+z_t\varphi}\right). \end{aligned}$$

We recall that $\varphi\chi_t\xi_t + \sigma\ell_t + \nu = 0$. Hence, letting

$$\begin{aligned} \widehat{u}_t &= \ell_t - z_t\sigma \\ \widetilde{u}_t &= \frac{1}{1+z_t\varphi}(\chi_t + 1) - 1\,, \end{aligned}$$

we get

$$z_t = -\frac{\nu + \sigma\widehat{u}_t + \varphi\xi_t\widetilde{u}_t}{\sigma^2 + \varphi^2\xi_t(1+\widetilde{u}_t)}\,.$$

It follows that

$$\begin{aligned}
A_t &= z_t^2\left(\sigma^2 + \xi_t(1+\chi_t)\frac{\varphi^2}{1+z_t\varphi}\right)\\
&= z_t^2\left(\sigma^2 + \xi_t(1+z_t\varphi)(1+\widetilde{u}_t)\frac{\varphi^2}{1+z_t\varphi}\right)\\
&= z_t^2\left(\sigma^2 + \xi_t(1+\widetilde{u}_t)\varphi^2\right)\\
&= \frac{(\nu + \sigma\widehat{u}_t + \varphi\xi_t\widetilde{u}_t)^2}{\sigma^2 + \varphi^2\xi_t(1+\widetilde{u}_t)}
\end{aligned}$$

so that process U is a solution of

$$dU_t = U_{t-}\left(\frac{(\nu + \sigma\widehat{u}_t + \varphi\xi_t\widetilde{u}_t)^2}{\sigma^2 + \varphi^2\xi_t(1+\widetilde{u}_t)}dt + \widehat{u}_t dW_t + \widetilde{u}_t dM_t\right), \quad U_T = 1,$$

which establishes that the BSDE (6.30) has a solution as long as the variance optimal martingale measure exists in our set-up.

6.5 MeanVariance Hedging

TO BE WRITTEN

6.6 Quantile Hedging

TO BE WRITTEN

Chapter 7

Dependent Defaults and Credit Migrations

Arguably, dependent defaults is the most important and the most difficult research area with regard to credit risk and credit derivatives. We describe the case of conditionally independent default time, the copula-based approach, as well as the Jarrow and Yu [117] approach to the modeling of dependent stochastic intensities. We conclude by summarizing one of the approaches that were recently developed for the purpose of modeling term structure of corporate interest rates.

Let us start by providing a tentative classification of issues and techniques related to dependent defaults and credit ratings.

Valuation of basket credit derivatives covers, in particular:

- Default swaps of type F (Duffie [70], Kijima and Muromachi [133]) – they provide a protection against the first default in a basket of defaultable claims.
- Default swaps of type D (Kijima and Muromachi [133]) – a protection against the first two defaults in a basket of defaultable claims.
- The i^{th}-to-default claims (Bielecki and Rutkowski [21]) – a protection against the first i defaults in a basket of defaultable claims.

Technical issues arising in the context of dependent defaults include:

- Conditional independence of default times (Kijima and Muromachi [133]).
- Simulation of correlated defaults (Duffie and Singleton [73]).
- Modeling of infectious defaults (Davis and Lo [59]).
- Asymmetric default intensities (Jarrow and Yu [117]).
- Copulas (Schönbucher and Schubert[169], Laurent and Gregory [144], Frey and McNeil [88]).
- Dependent credit ratings (Lando [142], Bielecki and Rutkowski [24]).
- Simulation of dependent credit migrations (Kijima et al.[132], Bielecki [12]).
- Simulation of correlated defaults via Marshall Olkin copula, Elouerkhaoui [82]

7.1 Basket Credit Derivatives

Basket credit derivatives are credit derivatives deriving their cash flows values (and thus their values) from credit risks of several reference entities (or prespecified credit events).

Standing assumptions. We assume that:

- We are given a collection of default times $\tau_1, \ldots, \tau_n$ defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{Q})$.
- $\mathbb{Q}\{\tau_i = 0\} = 0$ and $\mathbb{Q}\{\tau_i > t\} > 0$ for every i and t.
- $\mathbb{Q}\{\tau_i = \tau_j\} = 0$ for arbitrary $i \neq j$ (in a continuous time setup).

We associate with the collection $\tau_1, \ldots, \tau_n$ of default times the ordered sequence $\tau_{(1)} < \tau_{(2)} < \cdots < \tau_{(n)}$, where $\tau_{(i)}$ stands for the random time of the i^{th} default. Formally,

$$\tau_{(1)} = \min\{\tau_1, \tau_2, \ldots, \tau_n\}$$

and for $i = 2, \ldots, n$

$$\tau_{(i)} = \min\big\{\tau_k : k = 1, \ldots, n,\ \tau_k > \tau_{(i-1)}\big\}.$$

In particular,

$$\tau_{(n)} = \max\{\tau_1, \tau_2, \ldots, \tau_n\}.$$

7.1.1 Different Filtrations

As we shall see, one has to make precise the choice of filtrations, especially while working with intensities.

We set $H_t^i = \mathbb{1}_{\{\tau_i \le t\}}$ and we denote by $\mathbf{H}^i$ the filtration generated by the process H^i, that is, by the observations of the default time τ_i. In addition, we are given a reference filtration $\mathbf{F}$ on the space $(\Omega, \mathcal{G}, \mathbb{Q})$. The filtration $\mathbf{F}$ is related to some other market risks, for instance, to the interest rate risk. Finally, we introduce the enlarged filtration $\mathbf{G}$ by setting

$$\mathbf{G} = \mathbf{F} \vee \mathbf{H}^1 \vee \mathbf{H}^2 \vee \ldots \vee \mathbf{H}^n.$$

The σ-field $\mathcal{G}_t$ models the information available at time t. In this chapter, the trivial filtration will be denoted by $\mathbf{T}$

Definition 7.1.1 *A process λ^i is said to be an $\mathbf{F}$-intensity of τ^i is*

- *the process λ^i is $\mathbf{F}$-adapted,*
- *the process*

$$H_t^i - \int_0^{t \wedge \tau_i} \lambda_s^i ds$$

is a $\mathbf{F} \vee \mathbf{H}^i$ martingale.

Hence, we can associate to a random time τ_i, its $\mathbf{T}$ intensity as well as $\mathbf{F} \vee \mathbf{H}^j$ intensity, where $j \neq i$.

7.1.2 The i^{th}-to-Default Contingent Claims

A general i^{th}-to-default contingent claim which matures at time T is specified by the following covenants:

- If $\tau_{(i)} = \tau_k \le T$ for some $k = 1, \ldots, n$ it pays at time $\tau_{(i)}$ the amount $Z^k_{\tau_{(i)}}$ where Z^k is an $\mathbf{F}$-predictable recovery process.
- If $\tau_{(i)} > T$ it pays at time T an $\mathcal{F}_T$-measurable promised amount X.

7.1.3 Case of Two Entities

For the sake of notational simplicity, we shall frequently consider the case of two reference credit risks.

Cash flows of the first-to-default contract (FDC):

- If $\tau_{(1)} = \min\{\tau_1, \tau_2\} = \tau_i \le T$ for $i = 1, 2$, the claim pays at time τ_i the amount $Z^i_{\tau_i}$.
- If $\min\{\tau_1, \tau_2\} > T$, it pays at time T the amount X.

Cash flows of the last-to-default contract (LDC):

- If $\tau_{(2)} = \max\{\tau_1, \tau_2\} = \tau_i \le T$ for $i = 1, 2$, the claim pays at time τ_i the amount $Z^i_{\tau_i}$.
- If $\max\{\tau_1, \tau_2\} > T$, it pays at time T the amount X.

We recall that throughout these lectures the savings account B equals

$$B_t = \exp\Big(\int_0^t r_u\,du\Big),$$

and $\mathbb{Q}$ stands for the martingale measure for our model of the financial market (including defaultable securities, such as: corporate bonds and credit derivatives). Consequently, the price $B(t,T)$ of a zero-coupon default-free bond equals

$$B(t,T) = B_t\,\mathbb{E}_{\mathbb{Q}}\big(B_T^{-1}\,|\,\mathcal{G}_t\big)$$

Values of FDC and LDC

In general, the value at time t of a defaultable claim (X, Z, τ) is given by the *risk-neutral valuation formula*

$$S_t = B_t\,\mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1}\,dD_u\,\Big|\,\mathcal{G}_t\Big)$$

where D is the *dividend process,* which describes all the cash flows of the claim. Consequently, the value at time t of the FDC equals:

$$\begin{aligned} S_t^{(1)} &= B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_{\tau_1}^{-1} Z^1_{\tau_1} \mathbb{1}_{\{\tau_1<\tau_2,\, t<\tau_1\le T\}}\,\Big|\,\mathcal{G}_t\Big) \\ &+ B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_{\tau_2}^{-1} Z^2_{\tau_2} \mathbb{1}_{\{\tau_2<\tau_1,\, t<\tau_2\le T\}}\,\Big|\,\mathcal{G}_t\Big) + B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} X \mathbb{1}_{\{T<\tau_{(1)}\}}\,\Big|\,\mathcal{G}_t\Big). \end{aligned}$$

The value at time t of the LDC equals:

$$\begin{aligned} S_t^{(2)} &= B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_{\tau_1}^{-1} Z^1_{\tau_1} \mathbb{1}_{\{\tau_2<\tau_1,\, t<\tau_1\le T\}}\,\Big|\,\mathcal{G}_t\Big) \\ &+ B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_{\tau_2}^{-1} Z^2_{\tau_2} \mathbb{1}_{\{\tau_1<\tau_2,\, t<\tau_2\le T\}}\,\Big|\,\mathcal{G}_t\Big) + B_t\,\mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} X \mathbb{1}_{\{T<\tau_{(2)}\}}\,\Big|\,\mathcal{G}_t\Big). \end{aligned}$$

Both expressions above are merely special cases of a general formula. The goal is to derive more explicit representations under various assumptions about τ_1 and τ_2, or to provide ways of efficient calculation of involved expected values by means of simulation (using perhaps another probability measure).

7.1.4 Role of $(\mathcal{H})$ hypothesis

If one assumes that $(\mathcal{H})$ hypothesis holds between the filtration $\mathbf{F}$ and $\mathbf{G}$ under $\mathbb{Q}$, then, it holds between $\mathbf{F}$ and $\mathbf{F} \vee \mathbf{H}^{i_1} \vee \cdots \vee \mathbf{H}^{i_k}$ for any $i_1, \cdots i_k$. However, there is no reason for hypothesis $(\mathcal{H})$ hold between $\mathbf{F} \vee \mathbf{H}^{i_1}$ and $\mathbf{G}$. Note that, if $(\mathcal{H})$ hypothesis holds, for $t_1, \cdots t_n \le T$, one has

$$\mathbb{Q}(\tau_1 > t_1, \cdots, \tau_n > t_n|\mathcal{F}_T) = \mathbb{Q}(\tau_1 > t_1, \cdots, \tau_n > t_n|\mathcal{F}_\infty)$$

7.2 Conditionally Independent Defaults

Definition 7.2.1 *The random times τ_i, $i = 1, \dots, n$ are said to be conditionally independent with respect to* $\mathbf{F}$ *under* $\mathbb{Q}$ *if for any $T > 0$ and any $t_1, \dots, t_n \in [0, T]$ we have:*

$$\mathbb{Q}\{\tau_1 > t_1, \dots, \tau_n > t_n \,|\, \mathcal{F}_T\} = \prod_{i=1}^{n} \mathbb{Q}\{\tau_i > t_i \,|\, \mathcal{F}_T\}.$$

Let us comment briefly on Definition 7.2.1.

- Conditional independence has the following intuitive interpretation: the reference credits (credit names) are subject to common risk factors that may trigger credit (default) events. In addition, each credit name is subject to idiosyncratic risks that are specific for this name.
- Conditional independence of default times means that once the common risk factors are fixed then the idiosyncratic risk factors are independent of each other.
- The property of conditional independence is not invariant with respect to an equivalent change of a probability measure.
- Conditional independence fits into static and dynamic theories of default times.
- A stronger condition would be a full conditionally independence, i.e., for any $T > 0$ and any intervals $I_1, \dots, I_n$ we have:

$$\mathbb{Q}\{\tau_1 \in I_1, \dots, \tau_n \in I_n \,|\, \mathcal{F}_T\} = \prod_{i=1}^{n} \mathbb{Q}\{\tau_i \in I_i \,|\, \mathcal{F}_T\}$$

.

7.2.1 Independent Default Times

We shall first examine the case of default times $\tau_1, \dots, \tau_n$ that are mutually independent under $\mathbb{Q}$. Suppose that for every $k = 1, \dots, n$ we know the cumulative distribution function $F_k(t) = \mathbb{Q}\{\tau_k \le t\}$ of the default time of the k^{th} reference entity. The cumulative distribution functions of $\tau_{(1)}$ and $\tau_{(n)}$ are:

$$F_{(1)}(t) = \mathbb{Q}\{\tau_{(1)} \le t\} = 1 - \prod_{k=1}^{n} (1 - F_k(t))$$

and

$$F_{(n)}(t) = \mathbb{Q}\{\tau_{(n)} \le t\} = \prod_{k=1}^{n} F_k(t).$$

More generally, for any $i = 1, \dots, n$ we have

$$F_{(i)}(t) = \mathbb{Q}\{\tau_{(i)} \le t\} = \sum_{m=i}^{n} \sum_{\pi \in \Pi^m} \prod_{j \in \pi} F_{k_j}(t) \prod_{l \notin \pi} (1 - F_{k_l}(t))$$

where Π^m denote the family of all subsets of $\{1, \dots, n\}$ consisting of m elements.

Suppose, in addition, that the default times $\tau_1, \dots, \tau_n$ admit deterministic intensity functions $\gamma_1(t), \dots, \gamma_n(t)$, such that

$$H_t^i - \int_0^{t \wedge \tau_i} \gamma_i(s)\,ds$$

are $\mathbf{H}^i$-martingales. Recall that $\mathbb{Q}\{\tau_i > t\} = e^{-\int_0^t \gamma_i(v)\,dv}$. It is easily seen that, for any $t \in I\!\!R_+$,

$$\mathbb{Q}\{\tau_{(1)} > t\} = \prod \mathbb{Q}\{\tau_i > t\} = e^{-\int_0^t \gamma_{(1)}(v)\,dv}.$$

where

$$\gamma_{(1)}(t) = \gamma_1(t) + \ldots + \gamma_n(t)$$

hence

$$H_t^{(1)} - \int_0^{t\wedge\tau_{(1)}} \gamma_{(1)}(t)dt$$

is a $\mathbf{H}^{(1)}$-martingale, where $\mathcal{H}_t^{(1)} = \sigma(\tau_{(1)} \wedge t)$.

Example 7.2.1 We shall consider a digital default put of basket type. To be more specific, we postulate that a contract pays a fixed amount (e.g., one unit of cash) at the i^{th} default time $\tau_{(i)}$ provided that $\tau_{(i)} \le T$. Assume that the interest rates is deterministic. Then the value at time 0 of the contract equals

$$S_0 = \mathbb{E}_{\mathbb{Q}}\big(B^{-1}_{\tau_{(i)}} 1\!\!1_{\{\tau_{(i)}\le T\}}\big) = \int_{]0,T]} B_u^{-1}\, dF_{(i)}(u).$$

If $\tau_1, \ldots, \tau_n$ admit intensities then

$$S_0 = \int_0^T B_u^{-1}\, dF_{(i)}(u)$$

where $F_{(i)}(t) = \mathbb{Q}(\tau_{(i)} \le t)$.

7.2.2 Signed Intensities

Some authors (e.g., Kijima and Muromachi [133]) examine credit risk models in which the negative values of "intensities" are not precluded. In that case, the process chosen as the "intensity" does not play the role of a real intensity, in particular, it is not true that $H_t - \int_0^{t\wedge\tau} \gamma_t dt$ is a martingale and negative values of the "intensity" process clearly contradict the interpretation of the intensity as the conditional probability of survival over an infinitesimal time interval. More precisely, for a given collection Γ^i, $i = 1, \ldots, n$ of $\mathbf{F}$-adapted continuous stochastic processes, one can define τ_i, $i = 1, \ldots, n$,

$$\tau_i = \inf\{\, t \in I\!R_+ : \Gamma_t^i \ge -\ln \xi_i \,\}$$

where the r.v. ξ_i are iid, uniformly distributed and independent of $\mathcal{F}_\infty$. Let us denote $\widehat{\Gamma}_t^i = \max_{u\le t} \Gamma_u^i$.

The following result examines the case of signed intensities.

Lemma 7.2.1 *Random times τ_i, $i = 1, \ldots, n$ are conditionally independent with respect to $\mathbf{F}$ under $\mathbb{Q}$. For every $t_1, \ldots, t_n \le T$,*

$$\mathbb{Q}\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = \prod_{i=1}^n e^{-\hat{\Gamma}_{t_i}^i} = e^{-\sum_{i=1}^n \hat{\Gamma}_{t_i}^i}.$$

7.2.3 Valuation of FDC and LDC

Valuation of the first-to-default or last-to-default contingent claim in relatively straightforward under the assumption of conditional independence of default times. We have the following result in which, for notational simplicity, we consider only the case of two entities. As usual, we do not state explicitly integrability conditions that should be imposed on recovery processes Z and the terminal payoff X.

Proposition 7.2.1 *Let the default times τ_j, $j = 1, 2$ be $\mathbf{F}$-conditionally independent with $\mathbf{F}$-intensities γ^j (i.e. $H_t^i - \int_0^{t\wedge\tau_i} \gamma_s^i ds$ are $\mathbf{G}^i$ martingales and γ^i is $\mathbf{F}$ adapted) Assume that the recovery Z is a $\mathbf{F}$-predictable process, and that the terminal payoff X is $\mathcal{F}_T$-measurable. (We assume that the payoffs associated to τ_1 and τ_2 are equal: the recovery is Z_t if the first default occurs at time t). We*

also assume that (H) hypothesis holds between $\mathbf{F}$ *and* $\mathbf{G}$.
The price at time $t=0$ *of the first-to-default claim equals*

$$S_0^{(1)} = \sum_{i,j=1,\, i\neq j}^{2} \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T B_u^{-1} Z_u^j\, e^{-\Gamma_u^i} \gamma_u^j e^{-\Gamma_u^j}\, du\Big) + \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} XG\big),$$

where we denote

$$G = e^{-(\Gamma_T^1+\Gamma_T^2)} = \mathbb{Q}\{\tau_1 > T,\, \tau_2 > T \,|\, \mathcal{F}_T\}.$$

PROOF: We have to compute $\mathbb{E}(Z_\tau \mathbb{1}_{\tau<T})$ for $\tau = \tau_1 \wedge \tau_2$. We know that, if Z is $\mathbf{F}$-predictable $\mathbb{E}(Z_\tau \mathbb{1}_{\tau<T}) = \mathbb{E}\int_0^T Z_u dF_u$ where $F_u = \mathbb{Q}(\tau \leq u|\mathcal{F}_u)$.
Let F^i be the increasing process defined as $1 - F_u^i = \mathbb{Q}(\tau_i > u|\mathcal{F}_u) = e^{-FGamma_u^i}$. The conditional independence assumption yields

$$1 - F_u = \mathbb{Q}(\tau_1 > u, \tau_2 > u|\mathcal{F}_u) = \mathbb{Q}(\tau_1 > u|\mathcal{F}_u)\,\mathbb{Q}(\tau_2 > u|\mathcal{F}_u) = (1-F_u^1)(1-F_u^2)\,,$$

Since $(\mathcal{H})$ hypothesis holds between $\mathbf{F}$ and $\mathbf{G}^i$, for $i=1,2$, the processes F^i are increasing hence

$$dF_u = e^{-\Gamma_u^1} dF_u^2 + e^{-\Gamma_u^2} dF_u^1 = e^{-\Gamma_u^1} e^{-\Gamma_u^2}(\gamma_u^1+\gamma_u^2)du$$

It follows that

$$\mathbb{E}(Z_{\tau_1\wedge\tau_2}\mathbb{1}_{\tau_1\wedge\tau_2<T}) = \mathbb{E}\int_0^T Z_u e^{-\Gamma_u^1} e^{-\Gamma_u^2}(\gamma_u^1+\gamma_u^2)du = \mathbb{E}\sum_i \int_0^T Z_u e^{-\Gamma_u^1-\Gamma_u^2}\gamma_u^i du\,.$$

□

Comments 7.2.1 (i) It is also possible to prove, under the hypothesis of the Proposition that, for $t<\tau$

$$S_t^{(1)} = e^{\Gamma_t^1+\Gamma_t^2}\mathbb{E}(\int_t^T Z_u e^{-(\Gamma_u^1+\Gamma_u^2)}(\gamma_u^1+\gamma_u^2)du|\mathcal{F}_t)$$

(ii) In the general case, setting $F_t^i = P(\tau_i \leq t|\mathcal{F}_t) = Z_t^i + A_t^i$ where Z is an $\mathbf{F}$ martingale,

$$S_0^{(1)} = \mathbb{E}\int_0^T Z_u(e^{-(\Gamma_u^1+\Gamma_u^2)}(\gamma_u^1+\gamma_u^2)du + d<Z_1,Z_2>_u) + \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1}XG\big)$$

Indeed, the Doob-Meyer decomposition of F_i is $F_i = Z_i + A_i$ and

$$H_t^i - \int_0^{t\wedge\tau_i} \gamma_s^i ds$$

is a $\mathcal{G}^i$-martingale where $\gamma_s^i = \frac{a_s^i}{1-F_s^i}$

$$dF_u = e^{-\Gamma_u^1} dF_u^2 + e^{-\Gamma_u^2} dF_u^1 + d<Z_1,Z_2>_u$$

It follows that

$$\begin{aligned}\mathbb{E}(Z_{\tau_1\wedge\tau_2}\mathbb{1}_{\tau_1\wedge\tau_2<T}) &= \mathbb{E}\int_0^T Z_u(e^{-\Gamma_u^1}dA_u^2 + e^{-\Gamma_u^2}dA_u^1 + d<Z_1,Z_2>_u)\\ &= \mathbb{E}\int_0^T Z_u(e^{-(\Gamma_u^1+\Gamma_u^2)}(\gamma_u^1+\gamma_u^2) + d<Z_1,Z_2>_u)\end{aligned}$$

The bracket must be related with some correlation of default times.

7.3 Copula-Based Approaches

7.3.1 Direct Application

In a direct application, we first postulate a (univariate marginal) probability distribution for each random variable τ_i. Let us denote it by F_i for $i = 1, 2, \ldots, n$. Then, a suitable copula function C is chosen in order to introduce an appropriate dependence structure of the random vector $(\tau_1, \tau_2, \ldots, \tau_n)$. Finally, the joint distribution of the random vector $(\tau_1, \tau_2, \ldots, \tau_n)$ is derived, specifically,

$$\mathbb{Q}\{\tau_i \le t_i,\, i = 1, 2, \ldots, n\} = C\big(F_1(t_1), \ldots, F_n(t_n)\big).$$

In the finance industry, the most commonly used are elliptical copulas (such as the Gaussian copula and the t-copula). The direct approach has an apparent drawback. It is essentially a static approach; it makes no account of changes in credit ratings, and no conditioning on the flow of information is present. Let us mention, however, an interesting theoretical issue, namely, the study of the effect of a change of probability measures on the copula structure.

See the Appendix for more information on copula.

7.3.2 Indirect Application

A less straightforward application of copulas is based on an extension of the canonical construction of conditionally independent default times. This can be considered as the first step towards a dynamic theory, since the techniques of copulas is merged with the flow of available information, in particular, the information regarding the observations of defaults.

Let Γ^i, $i = 1, \ldots, n$ be a given family of $\mathbf{F}$-adapted, increasing, continuous processes, defined on a probability space $(\tilde{\Omega}, \mathbf{F}, \mathbb{Q})$. We assume that $\Gamma^i_0 = 0$ and $\Gamma^i_\infty = \infty$. We assume that there exists a sequence ξ_i, $i = 1, \ldots, n$ of mutually independent random variables uniformly distributed on $[0, 1]$. We set

$$\tau_i = \inf\{\, t \in I\!R_+ \,:\, \Gamma^i_t \ge -\ln \xi_i \,\}\,.$$

We endow the space $(\Omega, \mathcal{G}, \mathbb{Q})$ with the filtration $\mathbf{G} = \mathbf{F} \vee \mathbf{H}^1 \vee \cdots \vee \mathbf{H}^n$. Then,

$$\mathbb{Q}\{\tau_i > s \,|\, \mathcal{F}_t \vee \mathcal{H}^i_t\} = 1\!\!1_{\{\tau_i > t\}}\, \mathbb{E}_{\mathbb{Q}}\big(e^{\Gamma^i_t - \Gamma^i_s} \,|\, \mathcal{F}_t\big).$$

Proposition 7.3.1 *If the r.v's ξ_k are iid, we have $\mathbb{Q}\{\tau_i = \tau_j\} = 0$ for every $i \neq j$. Moreover, default times $\tau_1, \ldots, \tau_n$ are conditionally independent with respect to $\mathbf{F}$ under $\mathbb{Q}$.*

PROOF: It suffices to note that, for $t_i < T$,

$$\begin{aligned}
\mathbb{Q}(\tau_1 > t_1, \ldots, \tau_n > t_n |\, \mathcal{F}_T) &= \mathbb{Q}(\Gamma^1_{t_1} \ge -\ln \xi_1, \ldots, \Gamma^n_{t_n} \ge -\ln \xi_n |\, \mathcal{F}_T) \\
&= \prod_{i=1}^n e^{\Gamma^i_{t_i}}
\end{aligned}$$

□

Recall that if $\Gamma^i_t = \int_0^t \gamma^i_u\, du$ then γ^i is the $\mathbf{F}$-intensity of τ_i. Intuitively

$$\mathbb{Q}\{\tau_i \in [t, t + dt] \,|\, \mathcal{F}_t \vee \mathcal{H}^i_t\} \approx 1\!\!1_{\{\tau_i > t\}} \gamma^i_t\, dt.$$

In the more general case where the r.v's ξ_i are correlated, we introduce their cumulative distribution function

$$C(u_1, \cdots, u_n) = \mathbb{Q}(\xi_1 > u_1, \cdots, \xi_n > u_n)$$

Assume that the cumulative distribution function of $(\xi_1, \ldots, \xi_n)$ in the canonical construction (cf. Section 7.3.2) is given by an n-dimensional copula C, and that the univariate marginal laws are uniform on $[0,1]$. Similarly as in Section 7.3.2, we postulate that $(\xi_1,\dots,\xi_n)$ are independent of $\mathbf{F}$, and we set

$$\tau_i(\tilde\omega,\hat\omega) = \inf\{\, t\in I\!R_+ \,:\, \Gamma^i_t(\tilde\omega) \ge -\ln \xi_i(\hat\omega)\,\}.$$

Then, $\{\tau_i > t_i\} = \{e^{-\Gamma^i_{t_i}} > \xi_i\}$.
However, we do not assume that the ξ_k are iid, and we denote by C their copula. Then:

- The case of default times conditionally independent with respect to $\mathbf{F}$ corresponds to the choice of the product copula Π. In this case, for $t_1,\dots,t_n \le T$ we have
$$\mathbb{Q}\{\tau_1 > t_1,\dots,\tau_n > t_n \,|\, \mathcal{F}_T\} = \Pi(Z^1_{t_1},\dots,Z^n_{t_n}),$$
where we set $Z^i_t = e^{-\Gamma^i_t}$.
- In general, for $t_1,\dots,t_n \le T$ we obtain
$$\mathbb{Q}\{\tau_1 > t_1,\dots,\tau_n > t_n \,|\, \mathcal{F}_T\} = C(Z^1_{t_1},\dots,Z^n_{t_n}),$$
where C is the copula used in the construction of $\xi_1,\dots,\xi_n$.

Survival Intensities

Proposition 7.3.2 *For arbitrary $s\le t$ on the set $\{\tau_1 > s,\dots,\tau_n > s\}$ we have*

$$\mathbb{Q}\{\tau_i > t \,|\, \mathcal{G}_s\} = \mathbb{E}_{\mathbb{Q}}\left(\frac{C(Z^1_s,\dots,Z^i_t,\dots,Z^n_s)}{C(Z^1_s,\dots,Z^n_s)}\,\Big|\, \mathcal{F}_s\right).$$

PROOF: The proof is straightforward, and follows from the key lemma

$$\mathbb{Q}\{\tau_i > t \,|\, \mathcal{G}_s\}1\!\!1_{\{\tau_1>s,\dots,\tau_n>s\}} = 1\!\!1_{\{\tau_1>s,\dots,\tau_n>s\}}\frac{\mathbb{Q}(\tau_1 > s,\dots,\tau_i>t,\dots,\tau_n>s\,|\,\mathcal{F}_s)}{\mathbb{Q}(\tau_1 > s,\dots,\tau_i>s,\dots,\tau_n>s\,|\,\mathcal{F}_s)}$$

□

Consequently, assuming that the derivatives $\gamma^i_t = \frac{d\Gamma^i_t}{dt}$ exist, the i^{th} intensity of survival equals, on the set $\{\tau_1 > t,\dots,\tau_n > t\}$,

$$\lambda^i_t = \gamma^i_t\, Z^i_t\, \frac{\frac{\partial}{\partial v_i} C(Z^1_t,\dots,Z^n_t)}{C(Z^1_t,\dots,Z^n_t)} = \gamma^i_t\, Z^i_t\, \frac{\partial}{\partial v_i}\ln C(Z^1_t,\dots,Z^n_t),$$

where λ^i_t is understood as the limit:

$$\lambda^i_t = \lim_{h\downarrow 0} h^{-1}\,\mathbb{Q}\{t<\tau_i\le t+h\,|\,\mathcal{F}_t,\tau_1>t,\dots,\tau_n>t\}.$$

It appears that, in general, the i^{th} intensity of survival jumps at time t, if the j^{th} entity defaults at time t for some $j\ne i$. In fact, it holds that

$$\lambda^{i,j}_t = \gamma^i_t\, Z^i_t\, \frac{\frac{\partial^2}{\partial v_i \partial v_j} C(Z^1_t,\dots,Z^n_t)}{\frac{\partial}{\partial v_j} C(Z^1_t,\dots,Z^n_t)},$$

where

$$\lambda^{i,j}_t = \lim_{h\downarrow 0} h^{-1}\,\mathbb{Q}\{t<\tau_i\le t+h\,|\,\mathcal{F}_t,\tau_k>t,k\ne j,\tau_j=t\}.$$

Schönbucher and Schubert [169] also examine the intensities of survival after the default times of some entities. Let us fix s, and let $t_i\le s$ for $i=1,2,\dots,k<n$, and $T_i\ge s$ for $i=k+1,k+2,\dots,n$. Then,

$$\begin{aligned}\mathbb{Q}\big\{&\tau_i > T_i,\, i=k+1,k+2,\dots,n\,|\,\mathcal{F}_s,\,\tau_j=t_j,\,j=1,2,\dots,k,\\ &\tau_i>s,\, i=k+1,k+2,\dots,n\big\}\\ &= \frac{\mathbb{E}_{\mathbb{Q}}\Big(\frac{\partial^k}{\partial v_1\dots\partial v_k} C(Z^1_{t_1},\dots,Z^k_{t_k},Z^{k+1}_{T_{k+1}},\dots,Z^n_{T_n})\,\Big|\,\mathcal{F}_s\Big)}{\frac{\partial^k}{\partial v_1\dots\partial v_k} C(Z^1_{t_1},\dots,Z^k_{t_k},Z^{k+1}_s,\dots,Z^n_s)}.\end{aligned} \tag{7.1}$$

Remark 7.3.1 Jumps of intensities cannot be efficiently controlled, except for the choice of C. In the approach described above, the dependence between the default times is implicitly introduced through Γ^is, and explicitly introduced by the choice of a copula C.

See Schönbucher and Schubert [169].

7.3.3 Laurent and Gregory's model

Laurent and Gregory [144] examine a simplified version of the framework of Schönbucher and Schubert [169]. Namely, they assume that the reference filtration is trivial – that is, $\mathcal{F}_t = \{\Omega, \emptyset\}$ for every $t \in I\!R_+$. This implies, in particular, that the default intensities γ^i are deterministic functions, and

$$\mathbb{Q}\{\tau_i > t\} = 1 - F_i(t) = e^{-\int_0^t \gamma^i(u)\,du}.$$

They obtain closed-form expressions for certain conditional intensities of default.
Example: This example describes the use of one-factor Gaussian copula (Bank of International Settlements (BIS) standard). Let

$$X_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i,$$

where V, $\bar{V}_i,\ i = 1,2,\ldots,n$, are independent, standard Gaussian variables under the probability measure $\mathbb{Q}$,. Define

$$\tau_i = \inf\{t\,:\,\int_0^t \gamma_u^i du > -\ln U_i\} = \inf\{t\,:\,1 - F_i(t) < U_i\}$$

where the random barriers are defined as $U_i = 1 - \mathcal{N}(X_i)$ where as usual $\mathcal{N}$ is the cumulative distribution function of a Gaussian r.v..
Then,

$$\{\tau_i \le t\} = \{U_i \ge 1 - F_i(t)\} = \{X_i \le \frac{\mathcal{N}^{-1}(F_i(t)) - \rho_i V}{\sqrt{1-\rho_i^2}}\}$$

Define $q_t^{i|V} = \mathbb{Q}(\tau_i > t|V)$ and $p_t^{i|V} = 1 - q_t^{i|V}$. Then,

$$\mathbb{Q}(\tau_i \le t_i, \forall i \le n) = \int \prod_i p_{t_i}^{i|v} f(v)dv$$

where f is the density of V.
It is easy to check that

$$p_t^{i|V} = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(F_i(t)) - \rho_i V}{\sqrt{1-\rho_i^2}}\right)$$

and

$$\mathbb{Q}(\tau_i \le t_i, \forall i \le n) = \int \prod_i \mathcal{N}\left(\frac{\mathcal{N}^{-1}(F_i(t_i)) - \rho_i V}{\sqrt{1-\rho_i^2}}\right) f(v)dv\,.$$

7.4 Two defaults, trivial reference filtration

We present general results on the case of two default times, as presented in Section 2.5.1. We use the same notation. We assume in particular that the reference filtration $\mathbf{F}$ is the trivial filtration $\mathbf{T}$. We denote by $\mathbf{G}$ the filtration $\mathbf{H}^1 \vee \mathbf{H}^2$

Martingales

we present the computation of the martingales associated to the times τ_i in different filtrations. In particular, we shall obtain the computation of the intensities in various filtrations.

• **Filtration $\mathbf{H}^i$** We study the decomposition of the semi-martingales H^i in the filtration $\mathbf{H}^i$. From our study presented Proposition 2.2.1, for any $i = 1, 2$, the process

$$M_t^i = H_t^i - \int_0^{t\wedge\tau_i} \frac{f_i(s)}{1 - F_i(s)} ds \tag{7.2}$$

where $F_i(s) = \mathbb{P}(\tau_i \le s) = \int_0^s f_i(u)du$ is a $\mathbf{H}^i$-martingales. In other terms, the process $\frac{f_i(t)}{1-F_i(t)}$ is the $\mathbf{T}$-intensity of τ^i

• **Filtration G** From the results obtained in Proposition 4.1.3, the process

$$H_t^1 - \int_0^{t\wedge\tau_1} \frac{a_s^{(1)}}{1 - F_{s-}^{1|2}} ds$$

is a $\mathbf{G}$-martingale where $F^{1|2}$ is the $\mathbf{H}^2$-submartingale $F_t^{1|2} = \mathbb{P}(\tau_1 \le t|\mathcal{H}_t^2)$ with decomposition $F_t^{1|2} = Z_t^{1|2} + \int_0^t a_s^{(1)} ds$ where $Z^{1|2}$ is a $\mathbf{H}^2$-martingale (it suffices to apply the general result established in Proposition 4.1.3 in the case $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ to the case $\mathbf{F} = \mathbf{H}^2$ and $\mathbf{H} = \mathbf{H}^1$). The process $A_t^{(1)} = \int_0^{t\wedge\tau_1} \frac{a_s^{(1)}}{1-F_{s-}^{1|2}} ds$ is the $\mathbf{H}^2$-adapted compensator of H^1. The same methodology can be applied for the compensator of H^2.

We now compute in an explicit form the compensator of H^1 in order to establish the proposition

Proposition 7.4.1 *The process*

$$H_t^1 - \int_0^{t\wedge\tau_1} \frac{a^{(1)}(s)}{1 - F^{1|2}(s)} ds$$

where $a^{(1)}(t) = H_t^2 \partial_1 h^{(1)}(t, \tau_2) - (1 - H_t^2) \frac{\partial_1 G(t,t)}{G(0,t)}$ *and*

$$h^{(1)}(t, s) = 1 - \frac{\partial_2 G(t, s)}{\partial_2 G(0, s)}.$$

is a $\mathbf{G}$*-martingale.*
The process

$$H_t^2 - \int_0^{t\wedge\tau_2} \frac{a^{(2)}(s)}{1 - F^{2*}(s)} ds$$

where $a^{(2)}(t) = H_t^2 \partial_2 h^{(2)}(\tau_1, 1) - (1 - H_t^1) \frac{\partial_2 G(t,t)}{G(t,0)}$ *and*

$$h^{(2)}(t, s) = 1 - \frac{\partial_1 G(t, s)}{\partial_1 G(t, 0)}.$$

is a $\mathbf{G}$*-martingale.*

Remark 7.4.1 Note that

$$\begin{aligned}
H_t^1 - \int_0^{t\wedge\tau_1} \frac{a_s^{(1)}}{1 - F^{1|2}(s)} ds &= H_t^1 - \int_0^{t\wedge\tau_1} \frac{H_s^2 \partial_1 h^{(1)}(s, \tau_2) - (1 - H_s^2)\partial_1 G(s, s)/G(0, s)}{1 - H_s^2 h^{(1)}(s, \tau_2) - (1 - H_s^2)\psi(s)} ds \\
&= H_t^1 - \int_0^{t\wedge\tau_1} H_s^2 \frac{\partial_1 h^{(1)}(s, \tau_2)}{1 - h^{(1)}(s, \tau_2)} - (1 - H_s^2) \frac{\partial_1 G(s, s)/G(0, s)}{1 - \psi(s)} ds \\
&= H_t^1 - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \left(\frac{\partial_1 h^{(1)}(s, \tau_2)}{1 - h^{(1)}(s, \tau_2)} ds - \int_0^{t\wedge\tau_1\wedge\tau_2} \frac{\partial_1 G(s, s)}{G(s, s)} \right) ds \\
&= H_t^1 - \ln \frac{1 - h^{(1)}(t \wedge \tau_1 \wedge \tau_2, \tau_2)}{1 - h^{(1)}(t \wedge \tau_1, \tau_2)} - \int_0^{t\wedge\tau_1\wedge\tau_2} \frac{\partial_1 G(s, s)}{G(s, s)} ds
\end{aligned}$$

It follows that the intensity of τ_1 in the **G**-filtration is $\frac{\partial_1 G(s,s)}{G(s,s)}$ on the set $\{t < \tau_2 \vee \tau_1\}$ and $\frac{\partial_1 h^{(1)}(s,\tau_2)}{1-h^{(1)}(s,\tau_2)}$ on the set $\{\tau_2 < t < \tau_1\}$. It can be proved that the intensity of $\tau_1 \wedge \tau_2$ is

$$\frac{\partial_1 G(s,s)}{G(s,s)} + \frac{\partial_2 G(s,s)}{G(s,s)} = \frac{g(t)}{G(t,t)}$$

where $g(t) = \frac{d}{dt} G(t,t)$

Proof: Some easy computation enables us to write

$$\begin{aligned} F_t^{1|2} &= H_t^2 \mathbb{P}(\tau_1 \le t | \tau_2) + (1 - H_t^2) \frac{\mathbb{P}(\tau_1 \le t < \tau_2)}{\mathbb{P}(\tau_2 > t)} \\ &= H_t^2 h^{(1)}(t, \tau_2) + (1 - H_t^2) \frac{G(0,t) - G(t,t)}{G(0,t)} \end{aligned} \tag{7.3}$$

where

$$h^{(1)}(t,v) = 1 - \frac{\partial_2 G(t,v)}{\partial_2 G(0,v)} .$$

Introducing the deterministic function $\psi(t) = 1 - G(t,t)/G(0,t)$, the submartingale $F_t^{1|2}$ writes (we delete the superscript (1) for h in what follows)

$$F_t^{1|2} = H_t^2 h(t,\tau_2) + (1 - H_t^2)\psi(t) \tag{7.4}$$

Function $t \to \psi(t)$ and process $t \to h(t,\tau_2)$ are continuous and of finite variation, hence Ito's rule leads to

$$\begin{aligned} dF_t^{1|2} &= h(t,\tau_2) dH_t^2 + H_t^2 \partial_1 h(t,\tau_2) dt + (1 - H_t^2)\psi'(t) dt - \psi(t) dH_t^2 \\ &= (h(t,\tau_2) - \psi(t))\, dH_t^2 + \left(H_t^2 \partial_1 h(t,\tau_2) + (1 - H_t^2)\psi'(t)\right) dt \\ &= \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,\tau_2)}{\partial_2 G(0,\tau_2)}\right) dH_t^2 + \left(H_t^2 \partial_1 h(t,\tau_2) + (1 - H_t^2)\psi'(t)\right) dt \end{aligned}$$

From the computation of the Stieljes integral, we can rewrite it as

$$\begin{aligned} \int_0^T \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,\tau_2)}{\partial_2 G(0,\tau_2)}\right) dH_t^2 &= \left(\frac{G(\tau_2,\tau_2)}{G(0,\tau_2)} - \frac{\partial_2 G(\tau_2,\tau_2)}{\partial_2 G(0,\tau_2)}\right) 1_{\{\tau_2 \le t\}} \\ &= \int_0^T \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right) dH_t^2 \end{aligned}$$

and substitute it in the expression of dF^* :

$$dF_t^{1|2} = \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right) dH_t^2 + \left(H_t^2 \partial_1 h(t,\tau_2) + (1 - H_t^2)\psi'(t)\right) dt$$

From

$$dH_t^2 = dM_t^2 - \left(1 - H_t^2\right) \frac{\partial_2 G(0,t)}{G(0,t)} dt$$

with M^2 $\mathbb{H}^2$-martingale, we get the $\mathbb{H}^2-$ semimartingale decomposition of $F^{1|2}$:

$$\begin{aligned} dF_t^{1|2} = & \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right) dM_t^2 - \left(1 - H_t^2\right) \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right) \frac{\partial_2 G(0,t)}{G(0,t)} dt \\ & + \left(H_t^2 \partial_1 h^{(1)}(t,\tau_2) + (1 - H_t^2)\psi'(t)\right) dt \end{aligned}$$

and from

$$\psi'(t) = \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right) \frac{\partial_2 G(0,t)}{G(0,t)} - \frac{\partial_1 G(t,t)}{G(0,t)}$$

we conclude

$$dF_t^{1|2} = \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right) dM_t^2 + \left(H_t^2 \partial_1 h^{(1)}(t,\tau_2) - (1-H_t^2)\frac{\partial_1 G(t,t)}{G(0,t)}\right) dt$$

We can also check that this is the dynamics of $F^{1|2}$ From (7.3), the process $F^{1|2}$ has a single jump of size $\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}$. From (7.3),

$$F^{1|2} = \frac{G(0,t) - G(t,t)}{G(0,t)} = \Psi(t)$$

on the set $\tau_2 > t$, and its bounded variation part is $\Psi'(t)$. On can check that

$$\left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right)\frac{\partial_2 G(0,t)}{G(0,t)} - \frac{\partial_1 G(t,t)}{G(0,t)} = -\frac{d}{dt}\frac{G(t,t)}{G(0,t)} = \Psi'(t)$$

The hazard process has a non null martingale part, except if $\frac{G(t,t)}{G(0,t)} = \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}$ (this is the case if the default are independent). Hence, (H) hypothesis is not satisfied in a general setting between $\mathbf{H}^i$ and $\mathbf{G}$.

• **Filtration H** We reproduce now the result of Chou and Meyer [45], in order to obtain the martingales in the filtration **H**, in case of two default times. Here, we denote by **H** the filtration generated by the process $H_t = H_t^1 + H_t^2$. This filtration is smaller than the filtration **H**. We dnote by $T_1 = \tau_1 \wedge \tau_2$ the infimum of the two default times and by $T_2 = \tau_1 \vee \tau_2$ the supremum. The filtration **H** is the filtration generated by $\sigma(T_1 \wedge t) \vee \sigma - t_2 \wedge t)$, up to completion with negligeable sets.

Let us denote by $G_1(t)$ the survival distribution function of T_1, i.e., $G_1(t) = \mathbb{P}(\tau_1 > t, \tau_2 > t) = G(t,t)$ and by $G_2(t;u)$ the survival conditional distribution function of T_2 with respect to T_1, i.e., for $t > u$,

$$G_2(u;t) = \mathbb{P}(T_2 > t | T_1 = u) = \frac{1}{g(u)}\left(\partial_1 G(u,t) + \partial_2 G(t,u)\right),$$

where $g(t) = \frac{d}{dt}G(t,t) = \frac{1}{dt}\mathbb{P}(T_1 \in dt)$. We shall also note

$$K(u;t) = \mathbb{P}(T_2 - T_1 > t | T_1 = u) = G_2(u; t+u)$$

The process $M_t \stackrel{def}{=} H_t - \Lambda_t$ is a **H**-martingale, where

$$\Lambda_t = \Lambda_1(t) 1\!\!1_{t<T_1} + \left[\Lambda_1(T_1) + \Lambda_2(T_1, t-T_1)\right] 1\!\!1_{T_1 \le t < T_2}$$

with

$$\Lambda_1(t) = -\int_0^t \frac{dG_1(s)}{G_1(s)} = \int_0^t \frac{g(s)}{G(s,s)} ds = -\ln\frac{G(t,t)}{G(0,0)} = -\ln G(t,t)$$

and

$$\Lambda_2(s;t) = -\int_0^t \frac{d_u K(s;u)}{K(s,u)} = -\ln\frac{K(s;t)}{K(s;0)}$$

hence

$$\begin{aligned}\Lambda_2(T_1, t-T_1) &= -\ln\frac{K(T_1;t-T_1)}{K(T_1;0)} = -\ln\frac{G_2(T_1;t)}{G_2(T_1;T_1)} \\ &= -\ln\frac{\partial_1 G(T_1,t) + \partial_2 G(t,T_1)}{\partial_1 G(T_1,T_1) + \partial_2 G(T_1,T_1)}\end{aligned}$$

It is proved in Chou-Meyer [45] that any **H**-martingale is a stochastic integral with respect to M. This result admits an immediate extension to the case of n successive defaults.

This representation theorem has an interesting consequence: a single asset is enough to get a complete market. This asset with price M, and final payoff $H_T - \Lambda_T$. It corresponds to a swap with cumulative premium leg Λ_t

7.4.1 Application of Norros lemma for two defaults

Norros's lemma

Proposition 7.4.2 *Let $\tau_i, i = 1, \cdots, n$ be n finite-valued random times and $\mathcal{G}_t = \mathcal{H}^1_t \vee \cdots \vee \mathcal{H}^n_t$. Assume that*

$P(\tau_i = \tau_j) = 0, \forall i \neq j$

there exists continuous processes A^i such that $M^i_t = H^i_t - A^i_{t\wedge\tau_i}$ are $\mathbf{G}$-martingales

then, the r.v's $A^i_{\tau_i}$ are independent with exponential law.

Proof. For any $\mu_i > -1$ the processes $L^i_t = (1+\mu_i)^{H^i_t} e^{-\mu_i A^i_t}$, solution of

$$dL^i_t = L^i_{t-}\mu_i dM^i_t$$

are uniformly integrable martingales. Moreover, these martingales have no commun jumps, and are orthogonal. Hence $E(\prod_i (1+\mu_i) e^{-\mu_i A^i_\infty}) = 1$, which implies

$$E(\prod_i e^{-\mu_i A^i_\infty}) = \prod_i (1+\mu_i)^{-1}$$

hence the independence property. □

Application

In case of two defaults, this implies that U_1 and U_2 are independent, where

$$U_i = \int_0^{\tau_i} \frac{a_i(s)}{1 - F^*_i(s)} ds$$

and

$$a_1(t) = -(1 - H^2_t)\frac{\partial_1 G(t,t)}{G(0,t)} + H^2_t \partial_1 h^{(1)}(t, \tau_2), \quad F^*_1(t) = H^2_t h^{(1)}(t, \tau_2) + (1 - H^2_t)\frac{G(0,t) - G(t,t)}{G(0,t)},$$

$$a_2(t) = -(1 - H^1_t)\frac{\partial_2 G(t,t)}{G(t,0)} + H^1_t \partial_2 h^{(2)}(\tau_1, t), \quad F^*_2(t) = H^1_t h^{(2)}(\tau_1, t) + (1 - H^1_t)\frac{G(t,0) - G(t,t)}{G(t,0)}$$

are independent. In a more explicit form,

$$\int_0^{\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds + \ln \frac{1 - h^{(1)}(\tau_1, \tau_2)}{1 - h^{(1)}(\tau_1 \wedge \tau_2, \tau_2)} = \int_0^{\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds + \ln \frac{\partial_2 G(\tau_1, \tau_2)}{\partial_2 G(\tau_1 \wedge \tau_2, \tau_2)}$$

is independent from

$$\int_0^{\tau_1\wedge\tau_2} \frac{\partial_2 G(s,s)}{G(s,s)} ds + \ln \frac{1 - h^{(2)}(\tau_1, \tau_2)}{1 - h^{(2)}(\tau_1, \tau_1 \wedge \tau_2)} = \int_0^{\tau_1\wedge\tau_2} \frac{\partial_2 G(s,s)}{G(s,s)} ds + \ln \frac{\partial_1 G(\tau_1, \tau_2)}{\partial_1 G(\tau_1, \tau_1 \wedge \tau_2)}$$

Example of Poisson process

In the case where τ_1 and τ_2 are the two first jumps of a Poisson process, we have

$$G(t,s) = \begin{cases} e^{-\lambda t} \text{ for } s < t \\ e^{-\lambda s}(1 + \lambda(s-t)) \text{ for } s > t \end{cases}$$

with partial derivatives

$$\partial_1 G(t,s) = \begin{cases} -\lambda e^{-\lambda t} \text{ for } t > s \\ -\lambda e^{-\lambda s} \text{ for } s > t \end{cases}, \quad \partial_2 G(t,s) = \begin{cases} 0 \text{ for } t > s \\ -\lambda^2 e^{-\lambda s}(s-t) \text{ for } s > t \end{cases}$$

and

$$h(t,s) = \begin{cases} 1 \text{ for } t > s \\ \frac{t}{s} \text{ for } s > t \end{cases}, \quad \partial_1 h(t,s) = \begin{cases} 0 \text{ for } t > s \\ \frac{1}{s} \text{ for } s > t \end{cases}$$

$$k(t,s) = \begin{cases} 0 \text{ for } t > s \\ 1 - e^{-\lambda(s-t)} \text{ for } s > t \end{cases}, \quad \partial_2 k(t,s) = \begin{cases} 0 \text{ for } t > s \\ \lambda e^{-\lambda(s-t)} \text{ for } s > t \end{cases}$$

Then, one obtains $U_1 = \tau_1$ et $U_2 = \tau_2 - \tau_1$

7.5 Jarrow and Yu Model

Jarrow and Yu [117] approach can be considered as another step towards a dynamic theory of dependence between default times. For a given finite family of reference credit names, Jarrow and Yu [117] propose to make a distinction between the *primary firms* and the *secondary firms.*

At the intuitive level:

- The class of primary firms encompasses these entities whose probabilities of default are influenced by macroeconomic conditions, but not by the credit risk of counterparties. The pricing of bonds issued by primary firms can be done through the standard intensity-based methodology.
- It suffices to focus on securities issued by secondary firms, that is, firms for which the intensity of default depends on the status of some other firms.

Formally, the construction is based on the assumption of asymmetric information. Unilateral dependence is not possible in the case of complete (i.e., symmetric) information.

7.5.1 Construction and Properties of the Model

Let $\{1, \ldots, n\}$ represent the set of all firms, and let $\mathbf{F}$ be the reference filtration. We postulate that:

- For any firm from the set $\{1, \ldots, k\}$ of primary firms, the 'default intensity' depends only on $\mathbf{F}$.
- The 'default intensity' of each firm belonging to the set $\{k+1, \ldots, n\}$ of secondary firms may depend not only on the filtration $\mathbf{F}$, but also on the status (default or no-default) of the primary firms.

Construction of Default Times $\tau_1, \ldots, \tau_n$

First step. We first model default times of primary firms. To this end, we assume that we are given a family of $\mathbf{F}$-adapted 'intensity processes' $\lambda^1, \ldots, \lambda^k$ and we produce a collection $\tau_1, \ldots, \tau_k$ of $\mathbf{F}$-conditionally independent random times through the canonical method:

$$\tau_i = \inf \left\{ t \in I\!R_+ \, : \, \int_0^t \lambda_u^i \, du \geq -\ln \xi_i \right\}$$

where ξ_i, $i = 1, \ldots, k$ are mutually independent identically distributed random variables with uniform law on $[0,1]$ under the martingale measure $\mathbb{Q}$.
Second step. We now construct default times of secondary firms. We assume that:

- The probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ is large enough to support a family ξ_i, $i = k+1, \ldots, n$ of mutually independent random variables, with uniform law on $[0,1]$.

- These random variables are independent not only of the filtration $\mathbf{F}$, but also of the already constructed in the first step default times $\tau_1, \ldots, \tau_k$ of primary firms.

The default times τ_i, $i = k+1, \ldots, n$ are also defined by means of the standard formula:

$$\tau_i = \inf \left\{ t \in I\!R_+ \,:\, \int_0^t \lambda_u^i \, du \geq -\ln \xi_i \right\}.$$

However, the 'intensity processes' λ^i for $i = k+1, \ldots, n$ are now given by the following expression:

$$\lambda_t^i = \mu_t^i + \sum_{l=1}^k \nu_t^{i,l} \mathbb{1}_{\{\tau_l \leq t\}},$$

where μ^i and $\nu^{i,l}$ are $\mathbf{F}$-adapted stochastic processes. If the default of the j^{th} primary firm does not affect the default intensity of the i^{th} secondary firm, we set $\nu^{i,j} \equiv 0$.

Main Features

Let $\mathbf{G} = \mathbf{F} \vee \mathbf{H}^1 \vee \ldots \vee \mathbf{H}^n$ stand for the enlarged filtration and let $\widehat{\mathbf{F}} = \mathbf{F} \vee \mathbf{H}^{k+1} \vee \ldots \vee \mathbf{H}^n$ be the filtration generated by the reference filtration $\mathbf{F}$ and the observations of defaults of secondary firms. Then:

- The default times $\tau_1, \ldots, \tau_k$ of primary firms are conditionally independent with respect to $\mathbf{F}$.
- The default times $\tau_1, \ldots, \tau_k$ of primary firms are no longer conditionally independent when we replace the filtration $\mathbf{F}$ by $\widehat{\mathbf{F}}$.
- In general, the default intensity of a primary firm with respect to the filtration $\widehat{\mathbf{F}}$ differs from the intensity λ^i with respect to $\mathbf{F}$.

We conclude that defaults of primary firms are also 'dependent' of defaults of secondary firms.

Case of Two Firms

To illustrate the present model, we now consider only two firms, A and B say, and we postulate that A is a primary firm, and B is a secondary firm. We restrict our attention to the case where $\mathbf{F}$ is the trivial filtration $\mathbf{T}$. Let the constant process $\lambda_t^1 \equiv \lambda_1$ represent the $\mathbf{T}$-intensity of default for firm A, so that

$$\tau_1 = \inf \left\{ t \in I\!R_+ \,:\, \int_0^t \lambda_u^1 \, du = \lambda_1 t \geq -\ln \xi_1 \right\},$$

where ξ_1 is a random variable independent of $\mathbf{F}$, with the uniform law on $[0,1]$. For the second firm, the 'intensity' of default is assumed to satisfy

$$\lambda_t^2 = \lambda_2 \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \leq t\}} = \lambda_2 + (\alpha_2 - \lambda_2) \mathbb{1}_{\{\tau_1 \leq t\}}$$

for some positive constants λ_2 and α_2, and thus

$$\tau_2 = \inf \left\{ t \in I\!R_+ \,:\, \int_0^t \lambda_u^2 \, du \geq -\ln \xi_2 \right\}$$

where ξ_2 is a random variable with the uniform law, independent of $\mathbf{F}$, and such that ξ_1 and ξ_2 are mutually independent. Then the following properties hold:

- λ^1 is the intensity of τ_1 with respect to $\mathbf{T}$,
- λ^2 is the intensity of τ_2 with respect to $\mathbf{T} \vee \mathbf{H}^1$,

- λ^1 is not the intensity of τ_1 with respect to $\mathbf{T} \vee \mathbf{H}^2$.
- λ^2 is not the intensity of τ_2 with respect to $\mathbf{T}$.

Assume for simplicity that $r = 0$ and compute the value of defaultable zero-coupon bonds with default time τ_i, with a rebate δ_i, paid at maturity:

$$D_i^{\delta_i,T}(t,T) = \mathbb{E}(1\!\!1_{\{\tau_i>T\}} + \delta_i 1\!\!1_{\{\tau_i<T\}}|\mathcal{G}_t), \text{ for } \mathcal{G}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2 .$$

As seen in section 2.5.1, we need to compute the joint law of the pair (τ_1, τ_2). Let

$$G(s,t) = \mathbb{P}(\tau_1 > s, \tau_2 > t) .$$

Wet set $\Theta_i = -\ln \xi_i$.

• **Case** $t \leq s$

For $t < s$, one has $\lambda_2(t) = \lambda_2 t$ on the set $s < \tau_1$. Hence, the following equality

$$\{\tau_1 > s\} \cap \{\tau_2 > t\} = \{\tau_1 > s\} \cap \{\Lambda_2(t) < \Theta_2\} = \{\tau_1 > s\} \cap \{\lambda_2 t < \Theta_2\} = \{\lambda_1 s < \Theta_1\} \cap \{\lambda_2 t < \Theta_2\}$$

leads to

$$\text{for } \ t < s, \ \mathbb{P}(\tau_1 > s, \tau_2 > t) = e^{-\lambda_1 s} e^{-\lambda_2 t} .$$

• **Case** $t > s$

$$\begin{aligned}
\{\tau_1 > s\} \cap \{\tau_2 > t\} &= \{\{t > \tau_1 > s\} \cap \{\tau_2 > t\}\} \cup \{\{\tau_1 > t\} \cap \{\tau_2 > t\}\} \\
\{t > \tau_1 > s\} \cap \{\tau_2 > t\} &= \{t > \tau_1 > s\} \cap \{\Lambda_2(t) < \Theta_2\} \\
&= \{t > \tau_1 > s\} \cap \{\lambda_2 \tau_1 + \alpha_2(t - \tau_1) < \Theta_2\}
\end{aligned}$$

The independence between Θ_1 and Θ_2 implies that the r.v. τ_1 is independent from Θ_2 (use that $\tau_1 = \Theta_1(\lambda_1)^{-1}$), hence

$$\begin{aligned}
\mathbb{P}(t > \tau_1 > s, \tau_2 > t) &= \mathbb{E}\left(1\!\!1_{\{t>\tau_1>s\}} e^{-(\lambda_2\tau_1 + \alpha_2(t-\tau_1))}\right) \\
&= \int du\, 1\!\!1_{\{t>u>s\}} e^{-(\lambda_2 u + \alpha_2(t-u))} \lambda_1 e^{-\lambda_1 u} \\
&= \frac{1}{\lambda_1 + \lambda_2 - \alpha_2} \lambda_1 e^{-\alpha_2 t} \left(e^{-s(\lambda_1+\lambda_2-\alpha_2)} - e^{-t(\lambda_1+\lambda_2-\alpha_2)}\right) .
\end{aligned}$$

Setting $\Delta = \lambda_1 + \lambda_2 - \alpha_2$, it follows that

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \frac{1}{\Delta} \lambda_1 e^{-\alpha_2 t} \left(e^{-s\Delta} - e^{-t\Delta}\right) + e^{-\lambda_1 t} e^{-\lambda_2 t} . \tag{7.5}$$

In particular, for $s = 0$,

$$\mathbb{P}(\tau_2 > t) = \frac{1}{\Delta} \left(\lambda_1 \left(e^{-\alpha_2 t} - e^{-(\lambda_1+\lambda_2)t}\right) + \Delta e^{-(\lambda_1+\lambda_2)t}\right)$$

• The computation of $D_1^{\delta_1,T}$ reduces to that of

$$\mathbb{P}(\tau_1 > T|\mathcal{G}_t) = \mathbb{P}(\tau_1 > T|\mathcal{F}_t \vee \mathcal{H}_t^1)$$

where $\mathcal{F}_t = \mathcal{H}_t^2$. From the results obtained in Section 2.5.1,

$$\mathbb{P}(\tau_1 > T|\mathcal{G}_t) = 1 - D^1(t,T) \quad = \quad 1\!\!1_{\{\tau_1>t\}} \left(1\!\!1_{\{\tau_2\leq t\}} \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + 1\!\!1_{\{\tau_2>t\}} \frac{G(T,t)}{G(t,t)}\right)$$

Therefore,

$$D_1^{\delta_1,T}(t,T) = \delta_1 + \mathbb{1}_{\{\tau_1>t\}}(1-\delta_1)e^{-\lambda_1(T-t)}\,.$$

One can also use

• The computation of $D_2^{\delta_2,T}$ follows from the computation of

$$\mathbb{P}(\tau_2 > T|\mathcal{G}_t) = \mathbb{1}_{\{t<\tau_2\}}\frac{\mathbb{P}(\tau_2 > T|\mathcal{H}_t^1)}{\mathbb{P}(\tau_2 > t|\mathcal{H}_t^1)} + \mathbb{1}_{\{\tau_2<t\}}\mathbb{P}(\tau_2 > T|\tau_2)$$

$$\begin{aligned} D_2^{\delta_2,T}(t,T) &= \delta_2 + (1-\delta_2)\mathbb{1}_{\{\tau_2>t\}}\Big(\mathbb{1}_{\{\tau_1\le t\}}e^{-\alpha_2(T-t)} \\ &\quad +\mathbb{1}_{\{\tau_1>t\}}\frac{1}{\Delta}(\lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2-\alpha_2)e^{-(\lambda_1+\lambda_2)(T-t)})\Big) \end{aligned}$$

Special Case: Zero Recovery

Assume that $\lambda_1 + \lambda_2 - \alpha_2 \neq 0$ and the bond is subject to the zero recovery scheme. For the sake of brevity, we set $r = 0$ so that $B(t,T) = 1$ for $t \le T$. Under the present assumptions:

$$D_2(t,T) = \mathbb{Q}\{\tau_2 > T \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2\}$$

and the general formula yields

$$D_2(t,T) = \mathbb{1}_{\{\tau_2>t\}}\,\frac{\mathbb{Q}\{\tau_2 > T \,|\, \mathcal{H}_t^1\}}{\mathbb{Q}\{\tau_2 > t \,|\, \mathcal{H}_t^1\}}\,.$$

If we set $\Lambda_t^2 = \int_0^t \lambda_u^2\,du$ then

$$D_2(t,T) = \mathbb{1}_{\{\tau_2>t\}}\,\mathbb{E}_{\mathbb{Q}}(e^{\Lambda_t^2-\Lambda_T^2} \,|\, \mathcal{H}_t^1).$$

Finally, we have the following explicit result.

Corollary 7.5.1 *If $\delta_2 = 0$ then $D_2(t,T) = 0$ on $\{\tau_2 \le t\}$. On the set $\{\tau_2 > t\}$ we have*

$$\begin{aligned} D_2(t,T) &= \mathbb{1}_{\{\tau_1\le t\}}\,e^{-\alpha_2(T-t)} \\ &\quad +\mathbb{1}_{\{\tau_1>t\}}\,\frac{1}{\lambda-\alpha_2}\Big(\lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2-\alpha_2)e^{-\lambda(T-t)}\Big). \end{aligned}$$

Exercise 7.5.1 Compute the $\mathbf{T}$ intensity of τ_2 and the $\mathbf{T} \vee \mathbf{H}^2$-intensity of τ_1.

7.6 Extension of Jarrow and Yu Model

We shall now argue that the assumption that some firms are primary while other firms are secondary is not relevant. For simplicity of presentation, we assume that:

- We have $n = 2$, that is, we consider two firms only.
- The interest rate r is zero, so that $B(t,T) = 1$ for every $t \le T$.
- The reference filtration $\mathbf{F}$ is trivial and denoted by $\mathbf{T}$.
- Corporate bonds are subject to the zero recovery scheme.

Since the situation is symmetric, it suffices to analyze a bond issued by the first firm. By definition, the price of this bond equals

$$D_1(t,T) = \mathbb{Q}\{\tau_1 > T \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2\}.$$

For the sake of comparison, we shall also evaluate the following values, which are based on partial observations,

$$\tilde{D}_1(t,T) = \mathbb{Q}\{\tau_1 > T \,|\, \mathcal{H}_t^2\}$$

and

$$\widehat{D}_1(t,T) = \mathbb{Q}\{\tau_1 > T \,|\, \mathcal{H}_t^1\}.$$

7.6.1 Kusuoka's Construction

We follow here Kusuoka [140]. Under the original probability measure $\mathbb{P}$ the random times τ_i, $i = 1,2$ are assumed to be mutually independent random variables with exponential laws with parameters λ_1 and λ_2, respectively.
For any i, the processes

$$M_t^i = H_t^i - \int_0^{t\wedge\tau_i} \lambda_i \, du$$

are $\mathbf{H}^i$-martingales, and λ_i is the $\mathbf{T}$ intensity of τ_i.

Girsanov's theorem. For a fixed $T > 0$, we define a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$ by setting

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \eta_T, \quad \mathbb{P}\text{-a.s.}$$

where the Radon-Nikodým density process η_t, $t \in [0,T]$, satisfies

$$\eta_t = 1 + \sum_{i=1}^{2} \int_{]0,t]} \eta_{u-} \kappa_u^i \, dM_u^i$$

Here processes κ^1 and κ^2 are given by

$$\kappa_t^1 = \mathbb{1}_{\{\tau_2 < t\}} \Big(\frac{\alpha_1}{\lambda_1} - 1 \Big), \quad \kappa_t^2 = \mathbb{1}_{\{\tau_1 < t\}} \Big(\frac{\alpha_2}{\lambda_2} - 1 \Big).$$

We set

$$\begin{aligned} \lambda_t^1 &= \lambda_1 \mathbb{1}_{\{\tau_2 > t\}} + \alpha_1 \mathbb{1}_{\{\tau_2 \le t\}} \\ \lambda_t^2 &= \lambda_2 \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \le t\}} \,. \end{aligned}$$

Using Girsanov's theorem, under $\mathbb{Q}$, the processes

$$H_t^i - \int_0^{t\wedge\tau_i} \lambda_u^i du$$

are $\mathbf{G} = \mathbf{H}^1 \vee \mathbf{H}^2$ martingales, hence the $\mathbf{H}^2$-intensity of τ_1 is λ^1 (and the $\mathbf{H}^1$-intensity of τ_2 is λ^1).

Main features. We focus on τ_1 and we denote $\Lambda_t^1 = \int_0^t \lambda_u^1 \, du$. In general, we have

$$\mathbb{Q}\{\tau_1 > s \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2\} \neq \mathbb{1}_{\{\tau_1 > t\}} \, \mathbb{E}_{\mathbb{Q}}\big(e^{\Lambda_t^1 - \Lambda_s^1} \,|\, \mathcal{H}_t^2\big),$$

since $(\mathcal{H})$ hypothesis does not hold between $\mathbf{H}^1$ and $\mathbf{H}^1 \vee \mathbf{H}^2$.

7.6.2 Interpretation of Intensities

Recall that the processes λ^1 and λ^2 have jumps if $\alpha_i \neq \lambda_i$.

Proposition 7.6.1 *For $i = 1, 2$ and every $t \in I\!R_+$ we have*

$$\lambda_i = \lim_{h \downarrow 0} h^{-1}\, \mathbb{Q}\{t < \tau_i \leq t + h \,|\, \tau_1 > t,\, \tau_2 > t\}. \tag{7.6}$$

Moreover:

$$\alpha_1 = \lim_{h \downarrow 0} h^{-1}\, \mathbb{Q}\{t < \tau_1 \leq t + h \,|\, \tau_1 > t,\, \tau_2 \leq t\}.$$

and

$$\alpha_2 = \lim_{h \downarrow 0} h^{-1}\, \mathbb{Q}\{t < \tau_2 \leq t + h \,|\, \tau_2 > t,\, \tau_1 \leq t\}.$$

7.6.3 Bond Valuation

Proposition 7.6.2 *The price $D_1^{\delta_1,T}(t,T)$ on $\{\tau_1 > t\}$ equals*

$$\begin{aligned} D_1^{\delta_1,T}(t,T) = {} & 1\!\!1_{\{\tau_2 \leq t\}}\, e^{-\alpha_1(T-t)} \\ & + 1\!\!1_{\{\tau_2 > t\}}\, \frac{1}{\lambda - \alpha_1} \left(\lambda_2 e^{-\alpha_1(T-t)} + (\lambda_1 - \alpha_1) e^{-\lambda(T-t)} \right). \end{aligned}$$

Furthermore

$$\begin{aligned} \widetilde{D}_1^{\delta_1,T} = {} & 1\!\!1_{\{\tau_2 \leq t\}}\, \frac{(\lambda - \alpha_2)\lambda_2 e^{-\alpha_1(T-\tau_2)}}{\lambda_1 \alpha_2 e^{(\lambda-\alpha_2)\tau_2} + \lambda(\lambda_2 - \alpha_2)} \\ & + 1\!\!1_{\{\tau_2 > t\}}\, \frac{\lambda - \alpha_2}{\lambda - \alpha_1}\, \frac{(\lambda_1 - \alpha_1) e^{-\lambda(T-t)} + \lambda_2 e^{-\alpha_1(T-t)}}{\lambda_1 e^{-(\lambda-\alpha_2)t} + \lambda_2 - \alpha_2} \end{aligned}$$

and

$$\widehat{D}_1^{\delta_1,T}(t,T) = 1\!\!1_{\{\tau_1 > t\}}\, \frac{\lambda_2 e^{-\alpha_1 T} + (\lambda_1 - \alpha_1) e^{-\lambda T}}{\lambda_2 e^{-\alpha_1 t} + (\lambda_1 - \alpha_1) e^{-\lambda t}}\,.$$

Observe that:

- Formula for $D_1^{\delta_1,T}(t,T)$ coincides with the Jarrow and Yu formula for the bond issued by a secondary firm.
- Processes $D_1^{\delta_1,T}(t,T)$ and $\widehat{D}_1^{\delta_1,T}(t,T)$ represent ex-dividend values of the bond, and thus they vanish after default time τ_1.
- The latter remark does not apply to the process $\widetilde{D}_1^{\delta_1,T}(t,T)$.

7.7 Defaultable Term Structure

It this section, we shall summarize the model of defaultable term structure of interest rates developed by Bielecki and Rutkowski [22] and Schönbucher[173], and then further generalized by Eberlein and Özkan [78]. Essentially, the model extends the Heath-Jarrow-Morton (HJM) model of term structure of default-free interest rates to the case of defaultable bonds.

7.7.1 Standing Assumptions

Standard intensity-based models (as, for instance, in Jarrow and Turnbull [116] or Jarrow et al.[115]) rely on the following assumptions:

- Existence of the martingale measure $\mathbb{Q}$ is postulated.
- Relationship between the statistical probability $\mathbb{P}$ and the risk-neutral probability $\mathbb{Q}$ is derived via calibration.
- Credit migrations process is modeled as a Markov chain.
- Market and credit risks are separated (independent).

The HJM-type model of defaultable term structure with multiple ratings was proposed by Bielecki and Rutkowski [22]. The main features of this approach are:

- The model formulates sufficient consistency conditions that tie together credit spreads and recovery rates in order to construct a risk-neutral probability $\mathbb{Q}$ and the corresponding risk-neutral intensities of credit events.
- Statistical probability $\mathbb{P}$ and the risk-neutral probability $\mathbb{Q}$ are connected via the market price of interest rate risk and the market price of credit risk.
- Market and credit risks are combined in a flexible way.

Term Structure of Credit Spreads

Suppose that we are given a filtered probability space $(\Omega, \mathbf{F}, \mathbb{P})$ endowed with a d-dimensional standard Brownian motion W. We assume that the reference filtration satisfies $\mathbf{F} = \mathbf{F}^W$. For any fixed maturity $0 < T \leq T^*$, the price of a zero-coupon Treasury bond equals

$$B(t,T) = \exp\Big(- \int_t^T f(t,u)\,du \Big),$$

where the default-free instantaneous forward rate $f(t,T)$ process is subject to the standard (HJM) assumption.

(HJM) Dynamics of the instantaneous forward rate $f(t,T)$ are given by the expression

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\,dW_u$$

for some function $f(0,\cdot) : [0,T^*] \to I\!R$, where, for any T, the processes $\alpha(t,T), t \in [0,T]$ and $\sigma(t,T), t \in [0,T]$ are $\mathbf{F}$-adapted.

Credit Classes

Suppose there are $K \geq 2$ credit rating classes, where the K^{th} class corresponds to the default-free bond. Essentially, credit rating classes are distinguished by the yields on the corresponding bonds. In other words, for any fixed maturity $0 < T \leq T^*$, the *defaultable instantaneous forward rate* $g_i(t,T)$ corresponds to the rating class $i = 1, \ldots, K-1$. We assume that:

(HJM^i) Dynamics of the instantaneous defaultable forward rates $g_i(t,T)$ are given by

$$g_i(t,T) = g_i(0,T) + \int_0^t \alpha_i(u,T)\,du + \int_0^t \sigma_i(u,T)\,dW_u$$

for some deterministic functions $g_i(0,\cdot) : [0,T^*] \to I\!R$, , where, for any T, the processes $\alpha_i(t,T), t \in [0,T]$ and $\sigma_i(t,T), t \in [0,T]$ are $\mathbf{F}$-adapted.

Credit Spreads

It is natural (although not necessary for further developments) to assume that

$$g_{K-1}(t,T) > g_{K-2}(t,T) > \ldots > g_1(t,T) > f(t,T)$$

for every $t \leq T$.

Definition 7.7.1 *For every* $i = 1, 2, \ldots, K-1$, *the* i^{th} *forward credit spread equals* $s_i(\cdot, T) = g_i(\cdot, T) - f(\cdot, T)$.

Martingale Measure $\mathbb{Q}$

It is known from the HJM theory that the following condition (M) is sufficient to exclude arbitrage across default-free bonds for all maturities $T \leq T^*$ and the savings account.

Condition (M) There exists an $\mathbf{F}$-adapted $I\!R^d$-valued process β such that

$$\mathbb{E}_{\mathbb{P}}\Big\{ \exp\Big(\int_0^{T^*} \theta_u \, dW_u - \frac{1}{2} \int_0^{T^*} |\theta_u|^2 \, du \Big) \Big\} = 1$$

and, for any maturity $T \leq T^*$, we have

$$\alpha^*(t,T) = \frac{1}{2} |\sigma^*(t,T)|^2 - \sigma^*(t,T)\theta_t$$

where

$$\alpha^*(t,T) = \int_t^T \alpha(t,u) \, du$$

$$\sigma^*(t,T) = \int_t^T \sigma(t,u) \, du.$$

Let γ be some process satisfying Condition (M). Then the probability measure $\mathbb{Q}$, given by the formula

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big(\int_0^{T^*} \theta_u \, dW_u - \frac{1}{2} \int_0^{T^*} |\theta_u|^2 \, du \Big), \quad \mathbb{P}\text{-a.s.},$$

is a *martingale measure* for the default-free term structure. We will see that for any T the prices $B(t,T)$ is a martingale under the measure $\mathbb{Q}$, when discounted with the savings account B_t.

Zero-Coupon Bonds

As we have seen before, the price of the T-maturity default-free zero-coupon bond is given by the equality

$$B(t,T) = \exp\Big(- \int_t^T f(t,u) \, du \Big).$$

Formally, such Treasury bond corresponds to credit class K. Similarly, the 'conditional value' of T-maturity defaultable zero-coupon bond belonging at time t to the credit class $i = 1, 2, \ldots, K-1$, equals

$$D_i(t,T) = \exp\Big(- \int_t^T g_i(t,u) \, du \Big).$$

We consider discounted price processes

$$Z(t,T) = B_t^{-1} B(t,T), \; Z_i(t,T) = B_t^{-1} D_i(t,T),$$

where B is the savings account

$$B_t = \exp\Big(\int_0^t f(u,u)\,du\Big).$$

Let us define a Brownian motion W^* under $\mathbb{Q}$ by setting

$$W_t^* = W_t - \int_0^t \theta_u\,du, \quad \forall\, t \in [0, T^*].$$

Conditional Dynamics of the Bond Price

Lemma 7.7.1 *Under the martingale measure $\mathbb{Q}$, for any fixed $T \le T^*$, the discounted price processes $Z(t,T)$ and $Z_i(t,T)$ satisfy*

$$dZ(t,T) = Z(t,T)b(t,T)\,dW_t^*,$$

where $b(t,T) = -\sigma^*(t,T)$, and

$$dZ_i(t,T) = Z_i(t,T)\big(\lambda_i(t)\,dt + b_i(t,T)\,dW_t^*\big)$$

where

$$\lambda_i(t) = a_i(t,T) - f(t,t) + b_i(t,T)\theta_t$$

and

$$a_i(t,T) = g_i(t,t) - \alpha_i^*(t,T) + \frac{1}{2}\,|\sigma_i^*(t,T)|^2$$
$$b_i(t,T) = -\sigma_i^*(t,T).$$

Observe that usually the process $Z_i(t,T)$ is not a martingale under the martingale measure $\mathbb{Q}$. This feature is related to the fact that it does not represent the (discounted) price of a tradeable security.

7.7.2 Credit Migration Process

Recall that we assumed that the set of rating classes is $\mathcal{K} = \{1, \dots, K\}$, where the class K corresponds to default. The *migration process* C is constructed in Bielecki and Rutkowski [22] as a (nonhomogeneous) conditionally Markov process on $\mathcal{K}$, with the state K as the unique *absorbing* state for this process. The process C is constructed on some enlarged probability space $(\Omega^*, \mathbf{G}, \mathbb{Q})$, where the probability measure $\mathbb{Q}$ is the extended martingale measure. The reference filtration $\mathbf{F}$ is contained in the extended filtration $\mathbf{G}$. For simplicity of presentation, we summarize the results for the case $K = 3$.

Given some non-negative and $\mathbf{F}$-adapted processes $\lambda_{1,2}(t)$, $\lambda_{1,3}(t)$, $\lambda_{2,1}(t)$ and $\lambda_{2,3}(t)$, a migration process C is constructed as a conditional Markov process with the conditional intensity matrix (infinitesimal generator)

$$\Lambda(t) = \begin{pmatrix} \lambda_{1,1}(t) & \lambda_{1,2}(t) & \lambda_{1,3}(t) \\ \lambda_{2,1}(t) & \lambda_{2,2}(t) & \lambda_{2,3}(t) \\ 0 & 0 & 0 \end{pmatrix}$$

where $\lambda_{i,i}(t) = -\sum_{j \ne i} \lambda_{i,j}(t)$ for $i = 1, 2$.

The conditional Markov property (with respect to the reference filtration $\mathbf{F}$) means that if we denote by $\mathcal{F}_t^C$ the σ-field generated by C up to time t then for arbitrary $s \ge t$ and $i, j \in \mathcal{K}$ we have

$$\mathbb{Q}\big\{C_{t+s} = i \,|\, \mathcal{F}_t \vee \mathcal{F}_t^C\big\} = \mathbb{Q}\big\{C_{t+s} = i \,|\, \mathcal{F}_t \vee \{C_t = j\}\big\}.$$

The formula above provides the risk-neutral conditional probability that the defaultable bond is in class i at time $t + s$, given that it was in the credit class C_t at time t. For any date t, we denote by $\hat{C}_t$ the previous bond's rating; we shall need this notation later.

Finally, the default time τ is introduced by setting

$$\tau = \inf\{t \in I\!R_+ \,:\, C_t = 3\}.$$

Let $H_i(t) = 1\!\!1_{\{C_t=i\}}$ for $i = 1, 2$, and let $H_{i,j}(t)$ represent the number of transitions from i to j by C over the time interval $(0, t]$. It can be shown that the process

$$M_{i,j}(t) = H_{i,j}(t) - \int_0^t \lambda_{i,j}(s) H_i(s)\, ds, \quad \forall\, t \in [0, T],$$

for $i = 1, 2$ and $j \neq i$, is a martingale on the enlarged probability space $(\Omega^*, \mathbf{G}, \mathbb{Q})$. Let us emphasize that due to the judicious construction of the migration process C, appropriate version of the hypotheses (H.1)-(H.3) remain valid here.

7.7.3 Defaultable Term Structure

We maintain the simplified framework with $K = 3$. We assume the fractional recovery of Treasury value scheme. To be more specific, to each credit rating $i = 1, \dots, K-1$, we associate the recovery rate $\delta_i \in [0, 1)$, where δ_i is the fraction of par paid at bond's maturity, if a bond belonging to the i^{th} class defaults prior to its maturity. Thus, the cash flow at maturity is

$$X = 1\!\!1_{\{\tau > T\}} + \delta_{\hat{C}_\tau} 1\!\!1_{\{\tau \le T\}}.$$

In order to provide the model with arbitrage free properties, Bielecki and Rutkowski [22] postulate that the risk-neutral intensities of credit migrations $\lambda_{1,2}(t)$, $\lambda_{1,3}(t)$, $\lambda_{2,1}(t)$ and $\lambda_{2,3}(t)$ are specified by the *no-arbitrage condition* (also termed the *consistency condition*):

$$\begin{aligned}
\lambda_{1,2}(t)\big(Z_2(t,T) - Z_1(t,T)\big) &+ \lambda_{1,3}(t)\big(\delta_1 Z(t,T) - Z_1(t,T)\big) \\
&+ \lambda_1(t) Z_1(t,T) = 0, \\
\lambda_{2,1}(t)\big(Z_1(t,T) - \hat{Z}_2(t,T)\big) &+ \lambda_{2,3}(t)\big(\delta_2 Z(t,T) - Z_2(t,T)\big) \\
&+ \lambda_2(t) Z_2(t,T) = 0.
\end{aligned}$$

Martingale Dynamics of a Defaultable Bond

First, we introduce the process $\widehat{Z}(t,T)$ as a solution to the following SDE

$$\begin{aligned}
d\widehat{Z}(t,T) = &\big(Z_2(t,T) - Z_1(t,T)\big)\, dM_{1,2}(t) + \big(Z_1(t,T) - Z_2(t,T)\big)\, dM_{2,1}(t) \\
&+ \big(\delta_1 Z(t,T) - Z_1(t,T)\big)\, dM_{1,3}(t) + \big(\delta_2 Z(t,T) - Z_2(t,T)\big)\, dM_{2,3}(t) \\
&+ H_1(t) Z_1(t,T) b_1(t,T)\, dW_t^* + H_2(t) Z_2(t,T) b_2(t,T)\, dW_t^* \\
&+ \big(\delta_1 H_{1,3}(t) + \delta_2 H_{2,3}(t)\big) Z(t,T) b(t,T)\, dW_t^*,
\end{aligned}$$

with the initial condition $\widehat{Z}(0,T) = H_1(0) Z_1(0,T) + H_2(0) Z_2(0,T)$.

It appears that the process $\widehat{Z}(t,T)$ follows a martingale on $(\Omega^*, \mathbf{G}, \mathbb{Q})$, so that it is justified to refer to $\mathbb{Q}$ as the *extended martingale measure*). The proof of the next result employs the no-arbitrage condition.

Lemma 7.7.2 *For any maturity $T \le T^*$ and for every $t \in [0, T]$ we have*

$$\widehat{Z}(t,T) = 1\!\!1_{\{C_t \neq 3\}}\, Z_{C_t}(t,T) + 1\!\!1_{\{C_t = 3\}}\, \delta_{\hat{C}_t} Z(t,T)$$

Next, we define the price process of a T-maturity defaultable zero-coupon bond by setting

$$D_C(t,T) = B_t \widehat{Z}(t,T)$$

for any $t \in [0,T]$. In view of Lemma 7.7.2, we have that

$$D_C(t,T) = \mathbb{1}_{\{C_t \neq 3\}} D_{C_t}(t,T) + \mathbb{1}_{\{C_t = 3\}} \delta_{\hat{C}_t} B(t,T).$$

The defaultable bond price $D_C(t,T)$ satisfies the following properties:

- The process $D_C(t,T)$ is a **G**-martingale under $\mathbb{Q}$, when discounted by the savings account.
- In contrast to the 'conditional price' $D_i(t,T)$, the process $D_C(t,T)$ admits discontinuities. Jumps are directly associated with changes in credit quality (ratings migrations).
- The process $D_C(t,T)$ represents the price of a tradeable security: the corporate zero-coupon bond of maturity T.

Risk-Neutral Representations

Recall that $\delta_i \in [0,1)$ is the recovery rate for a bond which was in the i^{th} rating class just prior to default.

Proposition 7.7.1 *The price process $D_C(t,T)$ of a T-maturity defaultable zero-coupon bond equals*

$$\begin{aligned} D_C(t,T) &= \mathbb{1}_{\{C_t \neq 3\}} B(t,T) \exp\Big(-\int_t^T s_{C_t}(t,u)\,du\Big) \\ &\quad + \mathbb{1}_{\{C_t = 3\}} \delta_{\hat{C}_t} B(t,T) \end{aligned}$$

where $s_i(t,u) = g_i(t,u) - f(t,u)$ is the i^{th} credit spread.

Proposition 7.7.2 *The price process $D_C(t,T)$ satisfies the risk-neutral valuation formula*

$$D_C(t,T) = B_t\,\mathbb{E}_{\mathbb{Q}}\big(\delta_{\hat{C}_T} B_T^{-1} \mathbb{1}_{\{\tau \le T\}} + B_T^{-1} \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t\big).$$

It is also clear that

$$D_C(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\big(\delta_{\hat{C}_T} \mathbb{1}_{\{\tau \le T\}} + \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t\big),$$

where $\mathbb{Q}_T$ stands for the T-forward measure associated with the extended martingale measure $\mathbb{Q}$.

Let us end this section by mentioning that Eberlein and Özkan [78] have generalized the model presented above to the case of term structures driven by Lévy processes.

7.7.4 Premia for Interest Rate and Credit Event Risks

We shall now change, using a suitable version of Girsanov's theorem, the measure $\mathbb{Q}$ to the equivalent probability measure $\mathbb{Q}$. In the financial interpretation, the probability measure $\mathbb{Q}$ will play the role of the statistical probability (i.e., the real-world probability). It is thus natural to postulate that the restriction of the probability measure $\mathbb{Q}$ to the original probability space Ω necessarily coincides with the statistical probability $\mathbb{P}$ for the default-free market. From now on, we shall assume that the following condition holds.

Condition (P) We postulate that

$$\frac{d\mathbb{Q}}{d\mathbb{Q}} = \hat{\eta}_{T^*}, \quad \mathbb{Q}\text{-a.s.},$$

where the positive $\mathbb{Q}$-martingale $\hat{\eta}$ is given by the formula

$$d\hat{\eta}_t = -\hat{\eta}_t \theta_t \, dW_t^* + \hat{\eta}_{t-}\, dM_t, \quad \eta_0 = 1,$$

for some $I\!R^d$-valued **F**-predictable process θ, where the $\mathbb{Q}$-local martingale M equals

$$\begin{aligned} dM_t &= \sum_{i\neq j} \kappa_{i,j}(t)\, dM_{i,j}(t) \\ &= \sum_{i\neq j} \kappa_{i,j}(t) \big(dH_{i,j}(t) - \lambda_{i,j}(t) H_i(t)\, dt \big) \end{aligned}$$

for some **F**-predictable processes $\kappa_{i,j} > -1$.

Assume that for any $i \neq j$

$$\int_0^{T^*} \big(\kappa_{i,j}(t) + 1\big) \lambda_{i,j}(t)\, dt < \infty, \quad \mathbb{Q}\text{-a.s.}$$

In addition, we postulate that $\mathbb{E}_{\mathbb{Q}}(\hat{\eta}_{T^*}) = 1$, so that the probability measure $\mathbb{Q}$ is indeed well defined on $(\Omega^*, \mathcal{G}_{T^*})$. The financial interpretation of processes θ and κ is

- The vector-valued process θ corresponds to the *premium for the interest rate risk.*
- The matrix-valued process κ represents the *premium for the credit event risk.*

Statistical Default Intensities

We define processes $\lambda^{\mathbb{Q}}_{i,j}$ by setting, for $i \neq j$,

$$\lambda^{\mathbb{Q}}_{i,j}(t) = (\kappa_{i,j}(t) + 1)\lambda_{i,j}(t), \quad \lambda^{\mathbb{Q}}_{i,i}(t) = -\sum_{j\neq i} \lambda^{\mathbb{Q}}_{i,j}(t).$$

Proposition 7.7.3 *Under an equivalent probability $\mathbb{Q}$ given by condition (P), the process C is a conditionally Markov process. The matrix of conditional intensities of C under $\mathbb{Q}$ equals*

$$\Lambda^{\mathbb{Q}}_t = \begin{pmatrix} l^{\mathbb{Q}}_{1,1}(t) & \ldots & l^{\mathbb{Q}}_{1,K}(t) \\ \cdot & \ldots & \cdot \\ l^{\mathbb{Q}}_{K-1,1}(t) & \ldots & l^{\mathbb{Q}}_{K-1,K}(t) \\ 0 & \ldots & 0 \end{pmatrix}.$$

If the market price for credit risk depends only on the current rating i (and not on the rating j after jump), so that $\kappa_{i,j} = \kappa_{i,i}$ for every $j \neq i$. Then $\Lambda^{\mathbb{Q}}_t = \Phi_t \Lambda_t$, where $\Phi_t = \operatorname{diag}[\phi_i(t)]$ with $\phi_i(t) = \kappa_{i,i}(t) + 1$ is the diagonal matrix (this case was examined, e.g., by Jarrow et al. [115]).

7.7.5 Defaultable Coupon Bond

Consider a defaultable coupon bond with the face value L that matures at time T and promises to pay coupons c_i at times $T_1 < \ldots < T_n < T$. The coupon payments are only made prior to default, and the recovery payment is made at maturity T, and is proportional to the bond's face value. Notice that the migration process C introduced in Section 7.7.2 may depend on both the maturity T and on recovery rates. Therefore, it is more appropriate to write $C_t = C_t(\delta, T)$, where $\delta = (\delta_1, \ldots, \delta_K)$. Similarly, we denote the price of a defaultable zero-coupon bond $D_{C(\delta,T)}(t, T)$, rather than $D_C(t, T)$.

A defaultable coupon bond can be treated as a portfolio consisting of:

- Defaultable coupons – that is, defaultable zero-coupon bonds with maturities $T_1, \ldots, T_n$, which are subject to zero recovery.
- Defaultable face value – that is, a T-maturity defaultable zero-coupon bond with a constant recovery rate δ.

We conclude that the arbitrage price of a defaultable coupon bond equals

$$D_c(t,T) = \sum_{i=1}^{n} c_i D_{C(0,T_i)}(t,T_i) + L D_{C(\delta,T)}(t,T),$$

where, by convention, we set $D_{C(0,T_i)}(t,T_i) = 0$ for $t > T_i$.

7.7.6 Examples of Credit Derivatives

Credit Default Swap

Consider first a basic credit default swap, as described, e.g., in Section 1.3.1 of Bielecki and Rutkowski [23]. In the present setup, the contingent payment is triggered by the event $\{C_t = K\}$. The contract is settled at time $\tau = \inf\{t < T : C_t = K\}$, and the payoff equals

$$Z_\tau = \big(1 - \delta_{\hat{C}_T} B(\tau,T)\big).$$

Notice the dependence of Z_τ on the initial rating C_0 through the default time τ and the recovery rate $\delta_{\hat{C}_T}$. The following two market conventions are common in practice:

- The buyer pays a lump sum at contract's inception (*default option*).
- The buyer pays annuities up to default time (*default swap*).

In the first case, the value at time 0 of a default option equals

$$S_0 = \mathbb{E}_{\mathbb{Q}}\Big(B_\tau^{-1}\big(1 - \delta_{\hat{C}_T} B(\tau,T)\big)\mathbb{1}_{\{\tau \le T\}}\Big).$$

In the second case, the annuity κ can be found from the equation

$$S_0 = \kappa\, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{i=1}^{T} B_{t_i}^{-1}\, \mathbb{1}_{\{t_i < \tau\}}\Big).$$

Notice that both the price S_0 and the annuity κ depend on the initial bond's rating C_0.

Total Rate of Return Swap

As a reference asset we take the coupon bond with the promised cash flows c_i at times T_i. Suppose the contract maturity is $\hat{T} \le T$. In addition, suppose that the *reference rate payments* (the *annuity payments*) are made by the investor at fixed scheduled times $t_i \le \hat{T}$, $i = 1, 2, \ldots, m$. The owner of a total rate of return swap is entitled not only to all coupon payments during the life of the contract, but also to the change in the value of the underlying bond. By convention, we assume that the default event occurs when $C_t(\delta,T) = K$. According to this convention, the reference rate κ to be paid by the investor satisfies

$$\mathbb{E}_{\mathbb{Q}}\Big(\sum_{i=1}^{n} c_i B_{T_i}^{-1}\mathbb{1}_{\{T_i \le \hat{T}\}}\Big) + \mathbb{E}_{\mathbb{Q}}\Big(B_\tau^{-1}\big(D_c(\tau,T) - D_c(0,T)\big)\Big)$$

$$= \kappa\, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{i=1}^{m} B_{t_i}^{-1}\, \mathbb{1}_{\{C_{t_i}(\delta,T) \ne K\}}\Big),$$

where $\tau = \inf\{t \ge 0 : C_t(\delta,T) = K\} \wedge \hat{T}$.

7.8 Markovian Market Model

In this section we give a brief description of a Markovian market model that can be efficiently used for evaluating and hedging basket credit instruments. This framework, is a special case of a more general model introduced in Bielecki et al (2006), which allows to incorporate information relative to the dynamic evolution of credit ratings and credit migration processes in the pricing of basket instruments. Empirical study of the model is carried in Bielecki, Vidozzi and Vidozzi (2006).

We start with some notation. Let the underlying probability space be denoted by $(\Omega, \mathcal{G}, \mathbf{G}, \mathbb{P})$, where $\mathbb{P}$ is a risk neutral measure inferred from the market (we shall discuss this in further detail when addressing the issue of model calibration), $\mathbf{G} = \mathbf{H} \vee \mathbf{F}$ is a filtration containing all information available to market agents. The filtration $\mathbf{H}$ carries information about evolution of credit events, such as changes in credit ratings or defaults of respective credit names. The filtration $\mathbf{F}$ is a reference filtration containing information pertaining to the evolution of relevant macroeconomic variables.

We consider L obligors (or credit names) and we assume that the current credit quality of each reference entity can be classified into $\mathcal{K} := \{1, 2, \dots, K\}$ rating categories. By convention, the category K corresponds to default. Let X^ℓ, $\ell = 1, 2, \dots, L$ be some processes on $(\Omega, \mathcal{G}, \mathbb{P})$ taking values in the finite state space $\mathcal{K}$. The processes X^ℓ represent the evolution of credit ratings of the ℓ^{th} reference entity. We define the *default time* τ_l of the ℓ^{th} reference entity by setting

$$\tau_l = \inf\{\, t > 0 \,:\, X_t^\ell = K\} \tag{7.7}$$

We assume that the default state K is absorbing, so that for each name the default event can only occur once.

We denote by $X = (X^1, X^2, \dots, X^L)$ the joint credit rating process of the portfolio of L credit names. The state space of X is $\mathcal{X} := \mathcal{K}^L$ and the elements of $\mathcal{X}$ will be denoted by x. We postulate that the filtration $\mathbf{H}$ is the natural filtration of the process X and that the filtration $\mathbf{F}$ is generated by a $I\!R^n$ valued factor process, Y, representing the evolution of relevant economic variables, like short rate or equity price processes.

We assume that the process $\mp = (X, Y)$ is jointly Markov under $\mathbb{P}$, so that we have, for every $0 \le t \le s$, $x \in \mathcal{X}$, and any set $\mathcal{Y}$ from the state space of Y,

$$\mathbb{P}(X_s = x, Y_s \in \mathcal{Y} \,|\, \mathcal{H}_t \vee \mathcal{F}_t^Y) = \mathbb{P}(X_s = x, Y_s \in \mathcal{Y} \,|\, X_t, Y_t). \tag{7.8}$$

We construct the process $\mp$ as a Markov chain modulated by a Lévy process, and vice versa. We shall refer to X (Y, respectively) as the *Markov chain component* of $\mp$ (the *Lévy component* of $\mp$, respectively). We provide the following structure to the generator of the process $\mp$.

$$\begin{aligned}
\mathbf{A}f(x,y) &= (1/2)\sum_{i,j=1}^{n} a_{ij}(y)\partial_i\partial_j f(x,y) + \sum_{i=1}^{n} b_i(y)\partial_i f(x,y) \\
&\quad + \int_{I\!R^n} \big(f(x, y + g(y,y')) - f(x,y)\big)\nu(dy') \\
&\quad + \sum_{\ell=1}^{L} \sum_{x^{\ell'} \in \mathcal{K}} \lambda^\ell(x, x'_\ell; y) f(x'_\ell, y),
\end{aligned} \tag{7.9}$$

where we write $x'_\ell = (x^1, x^2, \dots, x^{\ell-1}, x'^{\ell}, x^{\ell+1}, \dots, x^L)$. At any time t, the intensity matrix of the Markov chain component is given as $\Lambda_t = [\lambda(x, x'; Y_t)]_{x, x' \in \mathcal{X}}$. The Lévy component satisfies the SDE:

$$dY_t = b(Y_t)\, dt + \sigma(Y_t)\, dW_t + \int_{I\!R^n} g(Y_{t-}, y')\, N(dy', dt),$$

where, for a fixed $y \in I\!R^n$, $N(dy', dt)$ is a counting process with Lévy measure $\nu(dy')dt$, and $\sigma(y)$ satisfies the equality $\sigma(y)\sigma(y)^{\mathsf{T}} = a(y)$.

Note that the model specified by (7.9) does not allow for simultaneous jumps of the components X^ℓ and $X^{\ell'}$ for $\ell \neq \ell'$. In other words, the ratings of different credit names may not change simultaneously. Nevertheless, this is not a serious lack of generality, as the ratings of both credit names may still change in an arbitrarily small time interval. The advantage is that, for the purpose of simulation of paths of process X, rather than dealing with $\mathcal{X} \times \mathcal{X}$ intensity matrix $[\lambda(x, x'; y)]$, we shall deal with L intensity matrices $[\lambda^\ell(x, x'_l; y)]$, each of dimension $\mathcal{K} \times \mathcal{K}$ (for any fixed y). We stress that within the present set-up the current credit rating of the credit name ℓ directly impacts the intensity of transition of the rating of the credit name ℓ', and vice versa. This property, known as *frailty*, may contribute to default contagion.

7.8.1 Description of some credit basket products

In this section, we describe the cash-flows associated to the main-stream basket credit products, focusing in particular on the recently developed standardized instruments like the Dow Jones Credit Default Swap indices (iTraxx and CDX), and the relative derivative contracts (Collateralized Debt Obligations and First to Default Swaps).

CDS indices

CDS indices are static portfolios of L equally weighted credit default swaps (CDSs) with standard maturities, typically five or ten years. Typically, the index matures few months before the underlying CDSs. For instance, the five years iTraxx S3 (series three) and its underlying CDSs mature on June 2010 and December 2010 respectively. The debt obligations underlying the CDSs in the pool are selected from among those with highest CDS trading volume in the respective industry sector. We shall refer to the underlying debt obligations as reference entities. We shall denote by $T > 0$ the maturity of any given CDS index.

CDS indices are typically issued by a pool of licensed financial institutions, which we shall call the market maker. At time of issuance of a CDS index, say at time $t = 0$, the market maker determines an annual rate known as *index spread*, to be paid out to investors on a periodic basis. We shall denote this rate by η_0.

In what follows, we shall assume that, at some time $t \in [0, T]$, an investor purchases one unit of CDS index issued at time zero. By purchasing the index, he/she enters into a binding contract whose main provisions are summarized below,

(i) The time of issuance of the contract 0. The inception time of the contract is time t; the maturity time of the contract is T.

(ii) By purchasing the index, the investor sells protection to the market makers. Thus, the investor assumes the role of a protection seller and the market makers assume the role of protection buyers. In practice, the investors agrees to absorb all losses due to defaults in the reference portfolio, occurring between the time of inception t and the maturity T. In case of default of a reference entity, the protection seller pays to the market makers the protection payment in the amount of $(1-\delta)$, where $\delta \in [0, 1]$ is the agreed recovery rate (typically 40%). (We assume that the face value of each reference entity is one. Thus the total notional of the index is L.) The notional on which the market maker pays the spread, henceforth referred to as *residual protection* is then reduced by such amount. For instance, after the first default, the residual protection is updated as follows (recall that, at inception the notional is L):

$$L \to L - (1 - \delta)$$

(iii) In exchange, the protection seller receives from the market maker a periodic fixed premium on the residual protection at the annual rate of η_t, that represents the fair index spread. (Whenever a reference entity defaults, its weight in the index is set to zero. By purchasing one unit of index the protection seller owes protection only on those names that have not yet defaulted at time of inception.) If, at inception of the contract, the market index spread is different from the issuance spread, i.e. $\eta_t \neq \eta_0$, the present value of the difference is settled through an upfront payment.

We denote by τ_i the random default time of the i^{th} name in the index and by H_t^i the right continuous process defined as $H_t^i = 1\!\!1_{\{\tau_i \leq t\}}$, $i = 1, 2, ..., L$. Also, let $\{t_j, j = 0, 1, ..., J\}$ with $t = t_0$ and $t_J \leq T$ denote the tenor of the premium leg payments dates. The discounted cumulative cash flows associated with a CDS index are as follows:

$$\textit{Premium Leg} = \sum_{j=0}^{J} \frac{B_t}{B_{t_j}} \Big(\sum_{i=1}^{L} 1 - H_{t_j}^i (1-\delta) \Big) \eta_t$$

$$\textit{Protection Leg} = \sum_{i=1}^{L} \frac{B_t}{B_{\tau_i}} \Big((1-\delta)(H_T^i - H_t^i) \Big)$$

where

$B_t = \exp(\int_0^t r_t dt)$ is the discount factor.

Collateralized Debt Obligations

Collateralized Debt Obligations (CDO) are credit derivatives backed by portfolios of assets. If the underlying portfolio is made up of bonds, loans or other securitized receivables, such products are known as *cash* CDOs. Alternatively, the underlying portfolio may consist of credit derivatives referencing a pool of debt obligations. In the latter case, CDOs are said to be *synthetic*. Because of their recently acquired popularity, we focus our discussion on standardized (synthetic) CDO contracts backed by CDS indices.

We begin with an overview of the product.

(i) The time of issuance of the contract is 0. The time of inception of the contract is $t \geq 0$, the maturity is T. The notional of the CDO contract is the residual protection of the underlying CDS index at the time of inception.

(ii) The credit risk (the potential loss due to credit events) borne by the reference pool is layered into different risk levels. The range in between two adjacent risk levels is called a *tranche*. The lower bound of a tranche is usually referred to as *attachment point* and the upper bound as *detachment point*. The credit risk is sold in these tranches to protection sellers. For instance, in a typical CDO contract on iTraxx, the credit risk is split into equity, mezzanine, and senior tranches corresponding to $0-3\%, 3-6\%, 6-9\%, 9-12\%$, and $12-22\%$ of the losses, respectively. At inception, the notional value of each tranche is the CDO residual notional weighted by the respective *tranche width*.

(iii) The tranche buyer sells partial protection to the pool owner, by agreeing to absorb the pool's losses comprised in between the tranche attachment and detachment point. This is better understood by an example. Assume that, at time t, the protection seller purchases one currency unit worth of the $6-9\%$ tranche. One year later, consequently to a default event, the cumulative loss breaks through the attachment point, reaching 8%. The protection seller then fulfills his obligation by disbursing two thirds ($= \frac{8\%-6\%}{9\%-6\%}$) of a currency unit. The tranche notional is then reduced to one third of its pre-default event value. We refer to the remaining tranche notional as *residual tranche protection.*

(iv) In exchange, as of time t and up to time T, the CDO issuer (protection buyer) makes periodic payments to the tranche buyer according to a predetermined rate (termed tranche spread) on the residual tranche protection. We denote the time t spread of the l^{th} tranche by κ_t^l. Returning to our example, after the loss reaches 8%, premium payments are made on $\frac{1}{3}$ $(=\frac{9\%-8\%}{9\%-6\%})$ of the tranche notional, until the next credit event occurs or the contract matures.

We denote by L_l and U_l the lower and upper attachment points for the l^{th} tranche, κ_t^l its time t spread. It is also convenient to introduce the percentage loss process,

$$\Gamma_s^t = \frac{\sum_{i=1}^{L}(H_s^i - H_t^i)(1-\delta)}{\sum_{i=1}^{L}(1-H_t^i)}$$

where L is the number of reference names in the basket. (Note that the loss is calculated only on the names which are not defaulted at the time of inception t.) Finally define by $C^l = U_l - L_l$ the portion of credit risk assigned to the l^{th} tranche.

Purchasing one unit of the l^{th} tranche at time t generates the following discounted cash flows:

$$\textit{Premium Leg} = \sum_{j=0}^{J}\frac{B_t}{B_{t_j}}\kappa_t^l\sum_{i=1}^{L}(1-H_t^i)\Big(C^l - \min(C^l, \max(\Gamma_{t_j}^t - L_l, 0))\Big)$$

$$\textit{Protection Leg} = \sum_{i=1}^{L}\frac{B_t}{B_{t_j}}(H_T^i - H_t^i)(1-\delta)\mathbb{1}_{\{L_k \le \Gamma_{\tau_i}^t \le U_k\}}$$

We remark here that the equity tranche of the CDO on iTraxx or CDX is quoted as an upfront rate, say κ_t^0, on the total tranche notional, in addition to 500 basis points (5% rate) paid annually on the residual tranche protection. The premium leg payment, in this case, is as follows:

$$\kappa_t^0 C^0\sum_{i=1}^{L}(1-H_t^i) + \sum_{j=0}^{J}\frac{B_t}{B_{t_j}}(.05)\sum_{i=1}^{L}(1-H_t^i)\Big(C^0 - \min(C^0, \max(\Gamma_{t_j}^t - L_0, 0))\Big)$$

N$^{\text{th}}$-to-default Swaps

N$^{\text{th}}$-to-default swaps (NTDS) are basket credit instruments backed by portfolios of single name CDSs. Since the growth in popularity of CDS indices and their associated derivatives, NTDS have become rather illiquid. Currently, such products are typically customized bank to client contracts, and hence relatively bespoke to the client's credit portfolio. For this reason, we focus our attention on First to Default Swap contracts issued on the iTraxx index, which are the only ones with a certain degree of liquidity. Standardized FTDS are now issued on each of the iTraxx sector sub-indices. Each FTDS is backed by an equally weighted portfolio of five single name CDSs in the relative sub-index, chosen according to some liquidity criteria. The main provisions contained in a FTDS contract are the following:

(i) The time of issuance of the contract is 0. The time of inception of the contract is t, the maturity is T.

(ii) By investing in a FTDS, the protection seller agrees to absorb the loss produced by the first default in the reference portfolio

(iii) In exchange, the protection seller is paid a periodic premium, known as FTDS spread, computed on the residual protection. We denote the time-t spread by φ_t.

Recall that $\{t_j, j = 0, 1, ..., J\}$ with $t = t_0$ and $t_J \le T$ denotes the tenor of the premium leg payments dates. Also, denote by $\tau^{(1)}$ the (random) time of the first default in the pool. The discounted cumulative cash flows associated with a FTDS on an iTraxx sub-index containing N names are as follows (again we assume that each name in the basket has notional equal to one):

$$\textit{Premium Leg} = \sum_{j=0}^{J}\varphi_t\frac{B_t}{B_{t_j}}(\mathbb{1}_{\{\tau^{(1)} \ge t_j\}})$$

$$\textit{Protection Leg} = \frac{B_t}{B_{\tau^{(1)}}}(1-\delta)(\mathbb{1}_{\{\tau^{(1)}\leq T\}})$$

Step-up corporate bonds.

As of now, these products are not traded in baskets, however they are of interest because they offer protection against credit events other than defaults. In particular, step up bonds are corporate coupon issues for which the coupon payment depends on the issuer's credit quality: the coupon payment increases when the credit quality of the issuer declines. In practice, for such bonds, credit quality is reflected in credit ratings assigned to the issuer by at least one credit ratings agency (Moody's-KMV or Standard&Poor's). The provisions linking the cash flows of the step-up bonds to the credit rating of the issuer have different step amounts and different rating event triggers. In some cases, a step-up of the coupon requires a downgrade to the trigger level by both rating agencies. In other cases, there are step-up triggers for actions of each rating agency. Here, a downgrade by one agency will trigger an increase in the coupon regardless of the rating from the other agency. Provisions also vary with respect to step-down features which, as the name suggests, trigger a lowering of the coupon if the company regains its original rating after a downgrade. In general, there is no step-down below the initial coupon for ratings exceeding the initial rating.

Let X_t stand for some indicator of credit quality at time t. Assume that t_i, $i = 1, 2, \ldots, n$ are coupon payment dates and let $c_n = c(X_{t_{n-1}})$ be the coupons ($t_0 = 0$). The time t cumulative cash flow process associated to the step-up bond equals

$$D_t = (1-H_T)\frac{B_t}{B_T} + \int_{(t,T]}(1-H_u)\frac{B_t}{B_u}\,dC_u + \text{ possible recovery payment}$$

where $C_t = \sum_{t_i \leq t} c_i$.

7.8.2 Valuation of Basket Credit Derivatives in the Markovian Framework

We now discuss the pricing of the basket instruments introduced in previous sub-section. In particular, computing the fair spreads of such products involves evaluating the conditional expectation under the risk neutral measure $\mathbb{P}$ of some quantities related to the cash flows associated to each instrument. In the case of CDS indexes, CDOs and FTDS, the fair spread is such that, at inception, the value of the contract is exactly zero, i.e the risk neutral expectations of the fixed leg and protection leg payments are identical. The following expressions can be easily derived from the discounted cumulative cash flows given in the previous sub-section.

- the time t fair spread of a single name CDS (we shall need this during the calibration phase):

$$\eta_t^\ell = \frac{\mathbf{E}_{\mathbf{P}}^{X_t,Y_t}\Big(\frac{B_t}{B_{\tau^\ell}}H_T^\ell\Big)(1-\delta)}{\mathbf{E}_{\mathbf{P}}^{X_t,Y_t}\Big(\sum_{j=0}^{J}\frac{B_t}{B_{t_j}}(1-H_{t_j}^\ell)\Big)}$$

- the time t fair spread of a CDS index is:

$$\eta_t = \frac{\mathbf{E}_{\mathbf{P}}^{X_t,Y_t}\Big(\sum_{i=1}^{L}\frac{B_t}{B_{\tau_i}}(1-\delta)(H_T^i - H_t^i)\Big)}{\mathbf{E}_{\mathbf{P}}^{X_t,Y_t}\Big(\sum_{j=0}^{J}\frac{B_t}{B_{t_j}}\Big(\sum_{i=1}^{L}1-H_{t_j}^i(1-\delta)\Big)\Big)}$$

- the time t fair spread of the CDO equity tranche is:

$$\kappa_t^0 = \frac{1}{C^0\sum_{i=1}^{L}(1-H_t^i)}\Big(\mathbf{E}_{\mathbf{P}}^{X_t,Y_t}\sum_{i=1}^{L}\frac{B_t}{B_{\tau_i}}(H_T^i - H_t^i)(1-\delta)\mathbb{1}_{\{L_0\leq\Gamma_{\tau_i}^t\leq U_0\}}$$

$$-\mathbf{E}_{\mathbf{P}}^{X_t,Y_t} \sum_{j=0}^{J} \frac{B_t}{B_{t_j}} (.05) \sum_{i=1}^{L} (1 - H_t^i) \Big(C^0 - \min(C^0, \max(\Gamma_{t_j}^t - L_0, 0)) \Big) \Big)$$

- the time t fair spread of the ℓ^{th} CDO tranche is:

$$\kappa_t^\ell = \frac{\mathbf{E}_{\mathbf{P}}^{X_t,Y_t} \Big(\sum_{i=1}^{L} \frac{B_t}{B_{\tau_i}} (H_T^i - H_t^i)(1-\delta) 1\!\!1_{\{L_\ell \le \Gamma_{\tau_i}^t \le U_\ell\}} \Big)}{\mathbf{E}_{\mathbf{P}}^{X_t,Y_t} \Big(\sum_{j=0}^{J} \frac{B_t}{B_{t_j}} \sum_{i=1}^{L} (1 - H_t^i) \Big(C^l - \min(C^l, \max(\Gamma_{t_j}^t - L_l, 0)) \Big) \Big)}$$

- the time t fair spread of a First To Default Swap is:

$$\varphi_t = \frac{\frac{B_t}{B_{\tau^{(1)}}} (1-\delta)(1\!\!1_{\{\tau^{(1)} \le T\}})}{\sum_{j=0}^{J} \frac{B_t}{B_{t_j}} (1\!\!1_{\{\tau^{(1)} \ge t_j\}})}$$

- the time t fair value of the step up bond is:

$$B^{su} = \mathbf{E}_{\mathbf{P}}^{X_t,Y_t} \Big((1 - H_T) \frac{B_t}{B_T} + \int_{(t,T]} (1 - H_u) \frac{B_t}{B_u} \, dC_u + \text{ possible recovery payment} \Big)$$

Depending on the dimensionality of the problem, the above conditional expectations will be evaluated either by means of Monte Carlo simulation, or by means of some other numerical method and in the low dimensional cases even analytically .

Chapter 8

Appendix

The main part of the appendix comes from the forthcoming book of Jeanblanc et al. [123].

8.1 Hitting times

In this chapter, a Brownian motion $(W_t, t \geq 0)$ starting from 0 is given on a probability space $(\Omega, \mathcal{F}, P)$, and $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$ is its natural filtration. The function $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$ is the cumulative function of the standard Gaussian law.

8.1.1 Hitting times of a level and law of the maximum for Brownian motion

Let us study the law of the pair of random variables (W_t, M_t) where M is the maximum process of the Brownian motion, i.e., $M_t \stackrel{def}{=} \sup_{s \leq t} W_s$. The law of hitting times of a given level by the Brownian motion will be obtained.

Law of the pair of the random variables (W_t, M_t)

Let us remark that the process M is an increasing process, with non negative values.

Théorème 8.1 *Let W be a Brownian motion starting from 0 and $M_t = \sup(W_s, 0 \leq s \leq t)$.*

$$\begin{cases} \textit{for} \quad y \geq 0, x \leq y & P(W_t \leq x, M_t \leq y) = \mathcal{N}(\frac{x}{\sqrt{t}}) - \mathcal{N}(\frac{x-2y}{\sqrt{t}}), \\ \textit{for} \quad y \geq 0, x \geq y & P(W_t \leq x, M_t \leq y) = P(M_t \leq y) = \mathcal{N}(\frac{y}{\sqrt{t}}) - \mathcal{N}(\frac{-y}{\sqrt{t}}), \\ \textit{for} \quad y \leq 0 & P(W_t \leq x, M_t \leq y) = 0\,. \end{cases} \tag{8.1}$$

$$P(W_t \in dx, M_t \in dy) = \mathbb{1}_{y \geq 0} \mathbb{1}_{x \leq y} \sqrt{\frac{2}{\pi t^3}} (2y - x) \exp\left(-\frac{(x-2y)^2}{2t}\right). \tag{8.2}$$

Law of the supremum

Proposition 8.1.1 *The random variable M_t has the same law as $|W_t|$.*

Law of the hitting time

For $x > 0$, the law of $T_x = \inf\{s : W_s \geq x\}$ can be now easily deduced from

$$P(T_x \leq t) = P(x \leq M_t) = P(x \leq |W_t|) = P(x \leq |G|\sqrt{t}) = P(\frac{x^2}{G^2} \leq t)\,, \tag{8.3}$$

where, as usual G stands for a Gaussian random variable, with zero expectation and unit variance. Hence, $T_x \stackrel{law}{=} \frac{x^2}{G^2}$ and the density of T_x follows:

$$P(T_x \in dt) = \frac{x}{\sqrt{2\pi t^3}} \exp(-\frac{x^2}{2t}) \mathbb{1}_{t\geq 0}\, dt\,.$$

For $x < 0$, we have, using the symmetry of the BM

$$T_x = \inf\{t \,:\, W_t \leq x\} = \inf\{t \,:\, W_t = x\} \stackrel{law}{=} T_{-x}$$

and

$$P(T_x \in dt) = \frac{|x|}{\sqrt{2\pi t^3}} \exp(-\frac{x^2}{2t}) \mathbb{1}_{t\geq 0}\, dt\,. \tag{8.4}$$

Law of the infimum

The law of the infimum of a Brownian motion is obtained by relying on the same procedure. It can also be deduced by observing that

$$m_t \stackrel{def}{=} \inf_{s\leq t} W_s = -\sup_{s\leq t}(-W_s) = -\sup_{s\leq t}(B_s)$$

where $B = -W$ is a Brownian motion. Hence

$$\begin{array}{lll} \text{for} \quad y \leq 0, x \geq y & P(W_t \geq x, m_t \geq y) & = \mathcal{N}(\frac{-x}{\sqrt{t}}) - \mathcal{N}(\frac{2y-x}{\sqrt{t}}), \\ \text{for} \quad y \leq 0,\, x \leq y & P(W_t \geq x, m_t \geq y) & = \mathcal{N}(\frac{-y}{\sqrt{t}}) - \mathcal{N}(\frac{y}{\sqrt{t}})\,, \\ \text{for} \quad y \geq 0 & P(W_t \geq x, m_t \geq y) & = 0\,. \end{array} \tag{8.5}$$

In particular, $P(m_t \geq y) = \mathcal{N}(\frac{-y}{\sqrt{t}}) - \mathcal{N}(\frac{y}{\sqrt{t}})$. As an immediate consequence, we obtain that, for $x > 0$ and $y > 0$,

$$\begin{aligned} P_x(W_t \in dy, T_0 > t) &= P_0(W_t + x \in dy, T_{-x} > t) = P_0(W_t + x \in dy, m_t > -x) \\ &= \frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{(x-y)^2}{2t}\right) - \exp\left(-\frac{(x+y)^2}{2t}\right)\right] dx\, dy\,. \end{aligned} \tag{8.6}$$

Laplace transform of the hitting time

We have recalled that, for any $\lambda > 0$ the process $(\exp(\lambda W_t - \frac{\lambda^2}{2}t),\, t \geq 0)$ is a martingale. Let $y \geq 0$, $\lambda \geq 0$ and T_y be the hitting time of y. The martingale

$$(\exp(\lambda W_{t\wedge T_y} - \frac{\lambda^2}{2}(t \wedge T_y)),\ t \geq 0)$$

is bounded by $e^{\lambda y}$. Doob's optional sampling theorem yields

$$E[\exp(\lambda W_{T_y} - \frac{\lambda^2}{2}T_y)] = 1\,.$$

The case where $y < 0$ is obtained by studying the Brownian motion $-W$.

Warning 1 In order to apply Doob's optional sampling theorem, we have to check carefully that the martingale $\exp(\lambda W_{t\wedge T_y} - \frac{\lambda^2}{2}(t \wedge T_y))$ is uniformely integrable. In the case $\lambda > 0$ and $y < 0$, a wrong use of this theorem would lead to the equality between 1 and

$$E[\exp(\lambda W_{T_y} - \frac{\lambda^2}{2}T_y)] = e^{\lambda y}E[\exp(-\frac{\lambda^2}{2}T_y)]$$

that is between $E[\exp(-\frac{\lambda^2}{2}T_y)]$ and $\exp(-y\lambda)$ This is obvioulsly wrong since the quantity $E[\exp(-\frac{\lambda^2}{2}T_y)]$ is smaller than 1 whereas $\exp(-y\lambda)$ is strictly greater than 1.

Proposition 8.1.2 *Let T_y be the hitting time of $y \in I\!R$ for a standard Brownian motion. Then, for $\lambda > 0$*

$$E[\exp(-\frac{\lambda^2}{2}T_y)] = \exp(-|y|\lambda)\,.$$

8.1.2 Hitting times for a Drifted Brownian motion

We study now the case where $X_t = \nu t + W_t$, where W is a Brownian motion. Let $M_t^X = \sup(X_s, s \le t)$, $m_t^X = \inf(X_s, s \le t)$ and $T_y(X) = \inf\{t \ge 0 \,|\, X_t = y\}$.

Laws of the pairs M, X and m, X at time t

Proposition 8.1.3 *For $y \ge 0, y \ge x$*

$$P(X_t \le x, M_t^X \le y) = \mathcal{N}(\frac{x - \nu t}{\sqrt{t}}) - e^{2\nu y}\mathcal{N}(\frac{x - 2y - \nu t}{\sqrt{t}})$$

and for $y \le 0, y \le x$

$$P(X_t \ge x, m_t^X \ge y) = \mathcal{N}(\frac{-x + \nu t}{\sqrt{t}}) - e^{2\nu y}\mathcal{N}(\frac{-x + 2y + \nu t}{\sqrt{t}})\,.$$

Laws of maximum, minimum and hitting times

In particular, the laws of the maximum and of the minimum are deduced :

$$\begin{aligned} P(M_t^X \le y) &= \mathcal{N}(\frac{y - \nu t}{\sqrt{t}}) - e^{2\nu y}\mathcal{N}(\frac{-y - \nu t}{\sqrt{t}}), \quad y \ge 0 \\ P(M_t^X \ge y) &= \mathcal{N}(\frac{-y + \nu t}{\sqrt{t}}) + e^{2\nu y}\mathcal{N}(\frac{-y - \nu t}{\sqrt{t}}), \quad y \ge 0 \end{aligned}$$

and for $y > 0$

$$P(T_y(X) \ge t) = P(M_t^X \le y)\,.$$

The law of the variable $T_y(X)$ has density

$$P(T_y(X) \in dt) = \frac{dt}{\sqrt{2\pi t^3}}\, y e^{\nu y} \exp\left(-\frac{1}{2}\left(\frac{y^2}{t} + \nu^2 t\right)\right) = \frac{dt}{\sqrt{2\pi t^3}}\, y \exp\left(-\frac{1}{2t}(y - \nu t)^2\right),$$

named inverse Gaussian law with parameter (y, ν). In particular, when $t \to \infty$ in

$$P(T_y \ge t) = \mathcal{N}(\frac{y - \nu t}{\sqrt{t}}) - e^{2\nu y}\mathcal{N}(\frac{-y - \nu t}{\sqrt{t}})\,,$$

we obtain $P(T_y = \infty) = 1 - e^{2\nu y}$, for $\nu \le 0$ and $y > 0$.

$$P(m_t^X \geq y) = \mathcal{N}(\frac{-y+\nu t}{\sqrt{t}}) - e^{2\nu y}\mathcal{N}(\frac{y+\nu t}{\sqrt{t}}), \quad y \leq 0 \tag{8.7}$$

$$P(m_t^X \leq y) = \mathcal{N}(\frac{y-\nu t}{\sqrt{t}}) + e^{2\nu y}\mathcal{N}(\frac{y+\nu t}{\sqrt{t}}), \quad y \leq 0\,. \tag{8.8}$$

Laplace transforms

From Cameron-Martin's theorem

$$E(\exp -\frac{\lambda^2}{2}T_y(X)) = E\left(\exp(\nu W_{T_y} - \frac{\nu^2+\lambda^2}{2}T_y(W))\right)\,.$$

From Proposition 8.1.2, the right hand side equals

$$e^{\nu y}E[\exp(-\frac{1}{2}(\nu^2+\lambda^2)T_y(W))] = e^{\nu y}\exp[-|y|\sqrt{\nu^2+\lambda^2}]\,.$$

Therefore

$$E(\exp -\frac{\lambda^2}{2}T_y(X)) = e^{\nu y}\exp[-|y|\sqrt{\nu^2+\lambda^2}]\,. \tag{8.9}$$

8.1.3 Hitting Times for Geometric Brownian Motion

Let us assume that the dynamics of the risky asset are, under the risk neutral probability Q, given by

$$dS_t = S_t(\mu dt + \sigma dW_t)\,, S_0 = x \tag{8.10}$$

with $\sigma > 0$, i.e.,

$$S_t = x\exp\left((\mu - \sigma^2/2)t + \sigma W_t\right) = xe^{\sigma X_t}\,,$$

where $\mu = r - \delta$, $X_t = \nu t + W_t$, $\nu = \frac{\mu}{\sigma} - \frac{\sigma}{2}$. We denote the first hitting time of a by

$$T_a(S) = \inf\{t \geq 0 \,:\, S_t = a\} = \inf\{t \geq 0 \,:\, X_t = \frac{1}{\sigma}\ln(a/x)\}$$

Then $T_a(S) = T_\alpha(X)$ where $\alpha = \frac{1}{\sigma}\ln(a/x)$. When a level b is used for the geometric Brownian motion S, we shall denote $\beta = \frac{1}{\sigma}\ln(b/x)$.

Law of the pair (maximum, minimum)

We deduce from Proposition 8.1.3 that for $b > a, b > x$

$$P(S_t \leq a, M_t^S \leq b) = P(X_t \leq \alpha, M_t^X \leq \beta) = \mathcal{N}(\frac{\alpha - \nu t}{\sqrt{t}}) - e^{2\nu\beta}\mathcal{N}(\frac{\alpha - 2\beta - \nu t}{\sqrt{t}})$$

whereas, for $a > b, b < x$

$$P(S_t \geq a, m_t^S \geq b) = P(X_t \geq \alpha, m_t^X \geq \beta) = \mathcal{N}(\frac{-\alpha+\nu t}{\sqrt{t}}) - e^{2\nu\beta}\mathcal{N}(\frac{-\alpha+2\beta+\nu t}{\sqrt{t}}) \tag{8.11}$$

It follows that, for $a > x$ (or $\alpha > 0$)

$$\begin{aligned}
P(T_a(S) < t) &= P(T_\alpha(X) < t) = 1 - P(M_t^X \leq \alpha) \\
&= 1 - P(X_t \leq \alpha, M_t^X \leq \alpha) \\
&= 1 - \mathcal{N}(\frac{\alpha - \nu t}{\sqrt{t}}) + e^{2\nu\alpha}\mathcal{N}(\frac{-\nu t - \alpha}{\sqrt{t}}) \\
&= \mathcal{N}(\frac{-\alpha + \nu t}{\sqrt{t}}) + e^{2\nu\alpha}\mathcal{N}(\frac{-\nu t - \alpha}{\sqrt{t}})
\end{aligned}$$

and, for $a < x$ (or $\alpha < 0$)

$$\begin{aligned} P(T_a(S) < t) &= P(T_\alpha(X) < t) = 1 - P(m_t^X \geq \alpha) \\ &= \mathcal{N}(\frac{\alpha - \nu t}{\sqrt{t}}) + e^{2\nu\alpha}\mathcal{N}(\frac{\nu t + \alpha}{\sqrt{t}}) . \end{aligned}$$

It follows, from Markov property that

$$P(T_a(S) > T|\mathcal{F}_t) = \mathcal{N}(h_1(S_t, T-t)) - \left(\frac{a}{S_t}\right)^{\sigma^{-2}(r-\sigma^2/2)} \mathcal{N}(h_2(S_t, T-t)) \tag{8.12}$$

with

$$h_1(x,u) = \frac{1}{\sigma\sqrt{u}}\left(\ln\frac{x}{a} + (r - \frac{1}{2}\sigma^2)(u)\right), \; h_2(x,u) = \frac{1}{\sigma\sqrt{u}}\left(\ln\frac{a}{x} + (r - \frac{1}{2}\sigma^2)(u)\right)$$

Laplace transforms

From the previous remarks

$$E(\exp -\frac{\lambda^2}{2}T_a(S)) = E^{(\nu)}(\exp -\frac{\lambda^2}{2}T_\alpha(X)) .$$

Therefore, from (8.9)

$$E(\exp -\frac{\lambda^2}{2}T_a(S)) = \exp\left(\nu\alpha - |\alpha|\sqrt{\nu^2 + \lambda^2}\right) . \tag{8.13}$$

8.1.4 Other processes

OU Process

Let $(r_t, t \geq 0)$ be defined as

$$dr_t = k(\theta - r_t)\,dt + \sigma\,dW_t, \quad r_0 = 0,$$

and $\tau_\rho = \inf\{t \geq 0 \,:\, r_t \geq \rho\}$. For any $\rho > r_0 = 0$, the density function of τ_ρ equals

$$f(t) = \frac{\rho}{\sigma\sqrt{2\pi}}\left(\frac{k}{\sinh kt}\right)^{3/2} e^{kt/2} \exp\Big[-\frac{k}{2\sigma 2}\Big(\big(\rho - \theta\big)^2 - \theta^2 + \rho^2 \coth kt\Big)\Big].$$

For the derivation of the last formula, the reader is referred to Göing and Yor [93]. The formula in Leblanc and Scaillet [146] is only valid for $r_0 = 0$. The Laplace transform of the stopping time τ_ρ is known (see Borodin and Salminen [30]):

$$E_r\left(\exp\left(-\delta\tau_\rho\right)\right) = \frac{\Upsilon(r)}{\Upsilon(\rho)}$$

where

$$\Upsilon(r) = \exp\left(\frac{(r-\theta)^2}{4\sigma^2}\right) D_{-k}\left(-\frac{r-k}{\sigma}\right)$$

where D is the parabolic cylinder function :

$$\begin{aligned} D_{-\nu}(z) \;&=\; \exp\left(-\frac{z^2}{4}\right) 2^{-\nu/2}\sqrt{\pi} \\ &\left\{\frac{1}{\Gamma\left((\nu+1)/2\right)}\left(1 + \sum_{k=1}^{+\infty} \frac{\nu\,(\nu+2)\ldots(\nu+2k-2)}{3.5\ldots(2k-1)\,k!}\left(\frac{z^2}{2}\right)^k\right)\right. \\ &\left. -\frac{z\sqrt{2}}{\Gamma\left(\nu/2\right)}\left(1 + \sum_{k=1}^{+\infty} \frac{(\nu+1)\,(\nu+3)\ldots(\nu+2k-1)}{3.5\ldots(2k+1)\,k!}\left(\frac{z^2}{2}\right)^k\right)\right\} . \end{aligned}$$

CEV Process

The **Constant Elasticity of Variance (CEV) process** has dynamics

$$dZ_t = Z_t(\mu dt + \sigma Z_t^\beta dW_t)\,. \tag{8.14}$$

Lemma 8.1.1 *For $\beta > 0$, or $\beta < -\frac{1}{2}$ a CEV process is a deterministic time changed process of a power of a BESQ process:*

$$\left(S_t = e^{\mu t}\left(\rho_{c(t)}\right)^{-1/(2\beta)}, t \geq 0\right)$$

where ρ is a BESQ *with dimension $\delta = 2 + \dfrac{1}{\beta}$ and $c(t) = \dfrac{\beta\sigma^2}{2\mu}(e^{2\mu\beta t} - 1)$.*
If $0 > \beta > -\frac{1}{2}$

$$\left(S_t = e^{\mu t}\left(\rho_{c(t)}\right)^{-1/(2\beta)}, t \leq T_0\right)$$

where T_0 is the first hitting time of 0 for the BESQ *ρ.*
For any β and $y > 0$, one has

$$\begin{aligned} P_x(S_t \in dy) &= \frac{|\beta|}{c(t)} e^{\mu(2\beta+1/2)t} x^{1/2} y^{-2\beta-3/2} \exp\left(-\frac{1}{2c(t)}\left(x^{-2\beta} + y^{-2\beta}e^{2\mu\beta t}\right)\right) \\ &\quad \times I_{1/(2\beta)}\left(\frac{1}{\gamma(t)} x^{-\beta} y^{-\beta} e^{\mu\beta t}\right) dy\,. \end{aligned}$$

Let X be a Bessel process with dimension $\delta < 2$, starting at $x > 0$ and $T_0 = \inf\{t : X_t = 0\}$. Using time reversed process, Göing-Jaeschke and Yor [93] proved that the density of T_0 is

$$\frac{1}{t\Gamma(\alpha)}\left(\frac{x^2}{2t}\right)^\alpha e^{-x^2/(2t)}$$

where $\alpha = (4 - \delta)/2 - 1$.

8.1.5 Non-constant Barrier

The case of non-constant barrier would be of great interest. For example, the process X is a geometric Brownian motion with deterministic volatility

$$dS_t = S_t(rdt + \sigma(t)dB_t), S_0 = x$$

and

$$T_a(S) = \inf\{t : S_t = a\} = \inf\{t : rt - \frac{1}{2}[\int_0^t \sigma(s)]^2 ds + \int_0^t \sigma_s dB_s = \alpha\}\,,$$

where $\alpha = \ln(a/x)$ As we shall see below, the process $U_t = \displaystyle\int_0^t \sigma_s dB_s$ is a changed time Brownian motion and can be written as $Z_{A(t)}$ where Z is a brownian motion and $A(t) = \displaystyle\int_0^t [\sigma(s)]^2 ds$. Hence, introducing the inverse C of the function A

$$T_a(S) = \inf\{t : rt - \frac{1}{2}A(t) + Z_{A(t)} = \alpha\} = \inf\{C(u) : rC(u) - \frac{1}{2}u + Z_u = \alpha\}\,,$$

and we are reduced to the study of the hitting time of the non-constant boundary $C(u)$ by the drifted Brownian motion $Z_t - \frac{1}{2}t$.

Bibliography

More generally, let $\tau_f(V) = \inf\{t \geq 0 : V_t = f(t)\}$, where f is a deterministic function and V a diffusion process. There are only few cases for which the law of $\tau_f(V)$ is explicitly known; for instance, the previous case when V is a Brownian motion and f is an affine function.

This problem is studied in a general framework in Alili's thesis [1], Barndorff-Nielsen et al. [9], Daniels [55], Durbin [77], Ferebee [85], Hobson et al. [102], Jennen and Lerche [124][125], Lerche [149], Salminen [168] and Siegmund and Yuh [175].
Breiman [31] studies the case of a square root boundary, i.e. $T = \inf\{t : x + B_t = \alpha\sqrt{t}\}$.

Groeneboom [95] studies the case $T = \inf\{t : x + B_t = \alpha t^2\}$. For any $x > 0$ and $\alpha < 0$,

$$P_x(T \in dt) = 2(\alpha c)^2 \Sigma_{n=0}^{\infty} \exp(-\mu_n - \frac{2}{3}\alpha^2 t^3) \frac{\mathrm{Ai}(\lambda_n - 2\alpha c x)}{\mathrm{Ai}'(\lambda_n)}$$

here λ_n are the zeros on the negative half-line of the Airy function Ai, the unique bounded solution of $u'' - xu = 0$, $u(0) = 1$, and $\mu_n = -\lambda_n/c$. This last expression was obtained by Salminen [168]. The Airy function is defined as

$$(\mathrm{Ai})(x) \stackrel{def}{=} \frac{1}{\pi}\left(\frac{x}{3}\right)^{1/2} K_{1/3}\left(\frac{2}{3}x^{3/2}\right) .$$

8.1.6 Fokker-Planck equation

Let

$$dX_t = b(t, x_t)dt + \sigma(t, X - t)dW_t$$

be a diffusion.

Proposition 8.1.4 *Let h be a deterministic function, $\tau = \inf\{t \geq 0 : X_t \leq h(t)\}$ and*

$$g(t,x)dx = P(X_t \in dx, \tau > t)\,.$$

The measure $g(t,x)dx$ satisfies

$$\frac{d}{dt}g(t,x) = -\frac{\partial}{\partial x}(b(t,x)g(t,x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma^2(t,x)g(t,x)\,; x > h(t)$$

and the boundary conditions

$$\begin{aligned} g(t,x)dx|_{t=0} &= \delta(x - X_0) \\ g(t,x)|_{x=h(t)} &= 0 \end{aligned}$$

Using Fokker-Planck equation, He et al. [99] and Iyengar established the following result (See also Patras [163] for a different approach)

Proposition 8.1.5 *Let $X_i(t) = \alpha_i t + \sigma_i W_i(t)$ where W_1, W_2 are two correlated Brownian motion, with correlation ρ, and M_i, m_i the running maximum and minimum. The probability density*

$$\begin{aligned} &P(X_1(t) \in dx_1, X_2(t) \in dx_2, m_1(t) \in dm_1, m_2(t) \in dm_2) \\ &\quad = p(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho) \end{aligned}$$

is

$$\frac{e^{a_1 x_1 + a_2 x_2 + bt}}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} h(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho)$$

with

$$h(x_1,x_2,t;m_1,m_2,\alpha_1,\alpha_2,\sigma_1,\sigma_2,\rho)=\frac{2}{\beta t}\sum_{n=1}^{\infty}e^{-(r^2+r_0^2)/(2t)}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\sin\left(\frac{n\pi\theta}{\beta}\right)I_{(n\pi)/\beta}\left(\frac{rr_0}{t}\right)$$

and

$$\begin{aligned}
a_1 &= \frac{\alpha_1\sigma_2-\rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2} \qquad a_2=\frac{\alpha_2\sigma_1-\rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2}\\
b &= -\alpha_1a_1-\alpha_2a_2+\frac{1}{2}\left(\sigma_1^2a_1^2+\sigma_2^2a_2^2\right)+\rho\sigma_1\sigma_2a_1a_2\\
\tan\beta &= -\frac{\sqrt{1-\rho^2}}{\rho},\\
z_1 &= \frac{1}{\sqrt{1-\rho^2}}\left[\left(\frac{x_1-m_1}{\sigma_1}\right)-\rho\left(\frac{x_2-m_2}{\sigma_2}\right)\right], \quad z_2=\frac{x_2-m_2}{\sigma_2}\\
z_{10} &= \frac{1}{\sqrt{1-\rho^2}}\left[-\frac{m_1}{\sigma_1}\right)+\rho\frac{m_2}{\sigma_2}, \quad z_{20}=-\frac{m_2}{\sigma_2}\\
r &= \sqrt{z_1^2+z_2^2}, \quad \tan\theta=\frac{z_2}{z_1}, \quad \theta\in[0,\beta]\\
r_0 &= \sqrt{z_{10}^2+z_{20}^2}, \quad \tan\theta_0=\frac{z_{20}}{z_{10}}, \quad \theta_0\in[0,\beta]
\end{aligned}$$

$$\begin{aligned}
&P(X_1(t)\in dx_1,X_2(t)\in dx_2,m_1(t)\geq m_1,M_2(t)\leq M_2)\\
&=p(x_1,-x_2,t;m_1,-M_2,\alpha_1,-\alpha_2,\sigma_1,\sigma_2,-\rho)dx_1dx_2
\end{aligned}$$

8.2 Copulas

The concept of a *copula function* allows to produce various multidimensional probability distributions with prespecified univariate marginal laws.

Definition 8.2.1 *A function $C:[0,1]^n\to[0,1]$ is called a copula if the following conditions are satisfied:*
(i) $C(1,\dots,1,v_i,1,\dots,1)=v_i$ for any i and any $v_i\in[0,1]$,
(ii) C is an n-dimensional cumulative distribution function (c.d.f.).

Let us give few examples of copulas:

- Product copula: $\Pi(v_1,\dots,v_n)=\Pi_{i=1}^n v_i$,
- Gumbel copula: for $\theta\in[1,\infty)$ we set
$$C(v_1,\dots,v_n)=\exp\left(-\left[\sum_{i=1}^n(-\ln v_i)^\theta\right]^{1/\theta}\right),$$
- Gaussian copula:
$$C(v_1,\dots,v_n)=N_\Sigma^n\left(N^{-1}(v_1),\dots,N^{-1}(v_n)\right),$$
where N_Σ^n is the c.d.f for the n-variate central normal distribution with the linear correlation matrix Σ, and N^{-1} is the inverse of the c.d.f. for the univariate standard normal distribution.
- t-copula:
$$C(v_1,\dots,v_n)=\Theta_{\nu,\Sigma}^n\left(t_\nu^{-1}(v_1),\dots,t_\nu^{-1}(v_n)\right),$$
where $\Theta_{\nu,\Sigma}^n$ is the c.d.f for the n-variate t-distribution with ν degrees of freedom and with the linear correlation matrix Σ, and t_ν^{-1} is the inverse of the c.d.f. for the univariate t-distribution with ν degrees of freedom.

The following theorem is the fundamental result underpinning the theory of copulas.

Théorème 8.2 (Sklar) *For any cumulative distribution function F on $I\!R^n$ there exists a copula function C such that*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

where F_i is the i^{th} marginal cumulative distribution function. If, in addition, F is continuous then C is unique.

To be completed

8.3 Poisson processes

We give some results on Poisson processes and martingales with jumps. For more details, see [123]

8.3.1 Standard Poisson process

Definition

The **standard Poisson process** is a counting process such that the random variables $(T_{n+1} - T_n,\, n \geq 0)$ are independent and identically distributed with exponential law of parameter λ with $\lambda > 0$. Hence, the explosion time is infinite and

$$P(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

The standard Poisson process can be redefined as follows (See e.g., Cinlar [47]): it is a counting process without explosion (i.e., $T = \infty$) such that
- for every s, t, $N_{t+s} - N_t$ is independent of $\mathcal{F}_t^N$,
- for every s, t, the r.v. $N_{t+s} - N_t$ has the same law as N_s.
or, in an equivalent way, its increments are independent and stationary.

Definition 8.3.1 *Let* $\mathbf{F}$ *be a given filtration and λ a positive constant. The process N is an* $\mathbf{F}$*-Poisson process with intensity λ if N is an* $\mathbf{F}$*-adapted process, such that for any (t, s), the random variable $N_{t+s} - N_t$ is independent of $\mathcal{F}_t$ and follows the Poisson law with parameter λs.*

Martingale Properties

From the independence of the increments of the Poisson process, we derive the following martingale properties:

Proposition 8.3.1 *Let N be an* $\mathbf{F}$*-Poisson process. For each $\alpha \in I\!R$, for each bounded Borel function h, for any $\beta > -1$, and any bounded Borel function φ valued in $]-1, \infty[$, the processes the following processes are* $\mathbf{F}$*-martingales:*

$$M_t = N_t - \lambda t, \qquad M_t^2 - \lambda t = (N_t - \lambda t)^2 - \lambda t,$$

$$\exp(\alpha N_t - \lambda t(e^\alpha - 1)), \qquad \exp\Big[\int_0^t h(s) dN_s - \lambda \int_0^t (e^{h(s)} - 1) ds\Big]$$

$$\exp[\ln(1+\beta) N_t - \lambda \beta t] = (1+\beta)^{N_t} e^{-\lambda \beta t},$$

$$\exp\Big[\int_0^t h(s) dN_s + \lambda \int_0^t (1 - e^{h(s)}) ds\Big]$$

$$= \exp\Big[\int_0^t h(s)dM_s + \lambda \int_0^t (1 + h(s) - e^{h(s)})ds\Big] \,,$$

$$\exp\Big[\int_0^t \ln(1+\varphi(s))dN_s - \lambda \int_0^t \varphi(s)ds\Big]$$
$$= \exp\Big[\int_0^t \ln(1+\varphi(s))dM_s + \lambda \int_0^t (\ln(1+\varphi(s)) - \varphi(s))ds\Big] \,,$$

Definition 8.3.2 *The martingale $(M_t = N_t - \lambda t,\, t \geq 0)$ is called the* ***compensated process*** *of N, and λ is the* ***intensity*** *of the process N.*

Proposition 8.3.2 *Let N be an $\mathbf{F}$-Poisson process and H be an $\mathbf{F}$-predictable bounded process, then the following processes are martingales*

$$\begin{aligned} &(H \star M)_t = \int_0^t H_s dM_s = \int_0^t H_s dN_s - \lambda \int_0^t H_s ds \\ &((H \star M)_t)^2 - \lambda \int_0^t H_s^2 ds \\ &\exp\left(\int_0^t H_s dN_s + \lambda \int_0^t (1 - e^{H_s})ds\right) \end{aligned} \tag{8.15}$$

Watanabe's Characterization of the Poisson Process

Let N be a counting process and assume that there exists a constant $\lambda > 0$ such that $M_t = N_t - \lambda t$ is a martingale. Then N is a Poisson process with intensity λ.

Change of Probability

Proposition 8.3.3 *Let Π^λ be the probability on the canonical space such that the canonical process is a Poisson process with intensity λ. Then, the following absolute continuity relationship holds*

$$\boxed{\Pi^{(1+\beta)\lambda}|_{\mathcal{F}_t} = \left((1+\beta)^{N_t} e^{-\lambda\beta t}\right)\, \Pi^\lambda|_{\mathcal{F}_t} \,.}$$

8.3.2 Inhomogeneous Poisson Processes

Definition

Instead of considering a constant intensity λ as before, now $(\lambda(t), t \geq 0)$ is an $\mathbb{R}^+$-valued function satisfying $\int_0^t \lambda(u)du < \infty, \forall t$. An **inhomogeneous Poisson process** N with intensity λ is a counting process with independent increments which satisfies for $t > s$

$$P(N_t - N_s = n) = e^{-\Lambda(s,t)} \frac{(\Lambda(s,t))^n}{n!} \tag{8.16}$$

where $\Lambda(s,t) = \Lambda(t) - \Lambda(s) = \int_s^t \lambda(u)du$, and $\Lambda(t) = \int_0^t \lambda(u)du$.
If $(T_n, n \geq 1)$ is the sequence of successive jump times associated with N, the law of T_n is:

$$P(T_n \leq t) = \frac{1}{n!} \int_0^t \exp(-\Lambda(s))\, (\Lambda(s))^{n-1}\, d\Lambda(s) \,.$$

It can easily be shown that an inhomogeneous Poisson process with deterministic intensity is an inhomogeneous Markov process. Moreover, $E(N_t) = \Lambda(t), \mathrm{Var}(N_t) = \Lambda(t)$. An inhomogeneous Poisson process can be constructed as a deterministic changed time Poisson process.

Martingale Properties

Proposition 8.3.4 *Let N be an inhomogeneous Poisson process with deterministic intensity λ and $\mathbf{F}^N$ its natural filtration. The process*

$$(M_t = N_t - \int_0^t \lambda(s)ds,\, t \geq 0)$$

is an $\mathbf{F}^N$-martingale, and the increasing function $\Lambda(t) = \int_0^t \lambda(s)ds$ is called the (deterministic) **compensator** *of N.*

Let ϕ be an $\mathbf{F}^N$-predictable process such that $E(\int_0^t |\phi_s|\lambda(s)ds) < \infty$ for every t. Then, the process $(\int_0^t \phi_s dM_s, t \geq 0)$ is an $\mathbf{F}^N$-martingale. In particular,

$$E\left(\int_0^t \phi_s \, dN_s\right) = E\left(\int_0^t \phi_s \lambda(s)ds\right) . \tag{8.17}$$

As in the constant intensity case, for any bounded predictable process H, the following processes are martingales

$$\begin{array}{ll} a) & (H \star M)_t = \int_0^t H_s dM_s = \int_0^t H_s dN_s - \int_0^t \lambda(s) H_s ds \\ b) & ((H \star M)_t)^2 - \int_0^t \lambda(s) H_s^2 ds \\ c) & \exp\left(\int_0^t H_s dN_s - \int_0^t \lambda(s)(e^{H_s} - 1)ds\right) . \end{array}$$

Stochastic Calculus

In this section, M is the compensated martingale of an inhomogeneous Poisson process N with deterministic intensity $(\lambda(s), s \geq 0)$. From now on, we restrict our attention to integrals of predictable processes, even if the stochastic integral is defined in a more general setting.

Integration by parts formula

Let g and $\tilde{g}$ be two predictable processes and define two processes X and Y as $X_t = x + \int_0^t g_s dN_s$ and $Y_t = y + \int_0^t \widetilde{g}_s dN_s$. The jumps of X (resp. of Y) occur at the same times as the jumps of N and $\Delta X_s = g_s \Delta N_s, \Delta Y_s = \tilde{g}_s \Delta N_s$. The processes X and Y are of finite variation and are constant between two jumps. Then

$$\begin{aligned} X_t Y_t &= xy + \sum_{s \leq t} \Delta(XY)_s = xy + \sum_{s \leq t} X_{s-} \Delta Y_s + \sum_{s \leq t} Y_{s-} \Delta X_s + \sum_{s \leq t} \Delta X_s \, \Delta Y_s \\ &= xy + \int_0^t Y_{s-} dX_s + \int_0^t X_{s-} dY_s + [X, Y]_t \end{aligned}$$

where (note that $(\Delta N_t)^2 = \Delta N_t$)

$$[X, Y]_t = \sum_{s \leq t} \Delta X_s \, \Delta Y_s = \sum_{s \leq t} \widetilde{g}_s g_s \Delta N_s = \int_0^t \widetilde{g}_s g_s dN_s \, .$$

More generally, if $dX_t = h_t dt + g_t dN_t$ with $X_0 = x$ and $dY_t = \widetilde{h}_t dt + \widetilde{g}_t dN_t$ with $Y_0 = y$, one gets

$$X_t Y_t = xy + \int_0^t Y_{s-} dX_s + \int_0^t X_{s-} dY_s + [X, Y]_t$$

where

$$[X,Y]_t = \int_0^t \widetilde{g}_s g_s dN_s \,.$$

In particular, if $dX_t = g_t dM_t$ and $dY_t = \widetilde{g}_t dM_t$, the process $X_tY_t - [X,Y]_t$ is a martingale.

Itô's Formula

For Poisson processes, Itô's formula is obvious as we now explain.
Let N be a Poisson process and f a bounded Borel function. The decomposition

$$f(N_t) = f(N_0) + \sum_{0<s\leq t} [f(N_s) - f(N_{s^-})] \tag{8.18}$$

is trivial and is the main step to obtain Itô's formula for a Poisson process.
We can write the right-hand side of (8.18)as a stochastic integral:

$$\begin{aligned} \sum_{0<s\leq t} [f(N_s) - f(N_{s^-})] &= \sum_{0<s\leq t} [f(N_{s^-}+1) - f(N_{s^-})]\Delta N_s \\ &= \int_0^t [f(N_{s^-}+1) - f(N_{s^-})] dN_s \,, \end{aligned}$$

hence, the canonical decomposition of $f(N_t)$ as the sum of a martingale and an absolute continuous adapted process is

$$f(N_t) = f(N_0) + \int_0^t [f(N_{s^-}+1) - f(N_{s^-})] dM_s + \int_0^t [f(N_{s^-}+1) - f(N_{s^-})]\lambda ds \,.$$

It is straightforward to generalize this result. Let

$$X_t = x + \int_0^t g_s dN_s = x + \sum_{T_n \leq t} g_{T_n} \,,$$

with g a predictable process. The process $(X_t, t \geq 0)$ jumps at time T_n, the size of the jump is g_{T_n}, the process is constant between two jumps. The obvious identity

$$F(X_t) = F(X_0) + \sum_{s\leq t} \left(F(X_s) - F(X_{s-})\right) \,,$$

holds for any bounded function F. The number of jumps before t is a.s. finite, and the sum is well defined. This formula can be written in an equivalent form:

$$\begin{aligned} F(X_t) - F(X_0) &= \sum_{s\leq t} \left(F(X_s) - F(X_{s-})\right) \Delta N_s \\ &= \int_0^t \left(F(X_s) - F(X_{s-})\right) dN_s = \int_0^t \left(F(X_{s-} + g_s) - F(X_{s-})\right) dN_s \,, \end{aligned}$$

where the integral on the right-hand side is a Stieltjes integral. More generally again, let

$$dX_t = h_t dt + g_t dM_t = (h_t - g_t\lambda(t))dt + g_t dN_t$$

and $F \in C^{1,1}(\mathbb{R}^+ \times \mathbb{R})$. Then

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_{s-})(h_s - g_s\lambda(s)) ds \\ &\quad + \sum_{s\leq t} F(s, X_s) - F(s, X_{s-}) \end{aligned} \tag{8.19}$$

$$= F(0,X_0) + \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_{s-})dX_s$$
$$+\sum_{s\le t} [F(s,X_s) - F(s,X_{s-}) - \partial_x F(s,X_{s-})g_s \Delta N_s] \,.$$

Indeed, between two jumps, $dX_t = (h_t - \lambda(t)g_t)dt$, and for $T_n < s < t < T_{n+1}$,

$$F(t,X_t) = F(s,X_s) + \int_s^t \partial_t F(u,X_u)du + \int_s^t \partial_x F(u,X_u)(h_u - g_u\lambda(u))du \,.$$

At jump times, $F(T_n, X_{T_n}) = F(T_n, X_{T_n-}) + \Delta F(\cdot, X)_{T_n}$.
The formula (8.19) can be written as

$$\begin{aligned} F(t,X_t) \quad - \quad F(0,X_0) &= \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_s)(h_s - g_s\lambda(s))ds \\ &\quad + \int_0^t [F(s,X_s) - F(s,X_{s-})]dN_s \qquad (8.20) \\ &= \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_{s-})dX_s \\ &\quad + \int_0^t [F(s,X_s) - F(s,X_{s-}) - \partial_x F(s,X_{s-})g_s]dN_s \\ &= \int_0^t \partial_t F(s,X_s)ds + \int_0^t \partial_x F(s,X_{s-})dX_s \\ &\quad + \int_0^t [F(s,X_{s-}+g_s) - F(s,X_{s-}) - \partial_x F(s,X_{s-})g_s]dN_s \,, \end{aligned}$$

$$= \int_0^t \left(\partial_t F(s,X_s) + [F(s,X_{s-}+g_s) - F(s,X_{s-}) - \partial_x F(s,X_{s-})g_s]\lambda\right) ds$$
$$+ \int_0^t [F(s,X_{s-}+g_s) - F(s,X_{s-})]dM_s \,. \qquad (8.21)$$

Remark that, in the "ds" integrals, we can write X_{s-} or X_s, since, for any bounded Borel function f,

$$\int_0^t f(X_{s-})ds = \int_0^t f(X_s)ds \,.$$

Note that since dN_s a.s. $N_s = N_{s-} + 1$, one has

$$\int_0^t f(N_{s-})dN_s = \int_0^t f(N_s+1)dN_s \,.$$

We shall use systematically use the form $\int_0^t f(N_{s-})N_s$, even if the $\int_0^t f(N_s+1)dN_s$ has a meaning. The reason is that $\int_0^t f(N_{s-})dM_s = \int_0^t f(N_{s-})dN_s + \lambda \int_0^t f(N_{s-})ds$ is a martingale, whereas $\int_0^t f(N_s+1)dM_s$ is not.

Predictable Representation Property

Proposition 8.3.5 *Let $\mathbf{F}^N$ be the completion of the canonical filtration of the Poisson process N and $H \in L^2(\mathcal{F}^N_\infty)$, a square integrable random variable. Then, there exists a unique predictable process $(h_t, t \ge 0)$ such that*

$$H = E(H) + \int_0^\infty h_s dM_s$$

and $E(\int_0^\infty h_s^2 ds) < \infty$.

Comments 8.3.1 This result goes back to Brémaud and Jacod [33], Chou and Meyer [45], Davis [60].

8.4 General theory

8.4.1 Semimartingales

A semi martingale is a càdlàg process X with decomposition $X_t = M_t + A_t$ where M is a martingale and A a bounded variation process. If A is predictable, the decomposition with predictable bounded variation process is unique and the semi martingale is said to be special.
The martingales M^1 and M^2 are orthogonal if the product $M^1 M^2$ is a martingale.
If X is a submartingale, then the process A in its decomposition $X = M + A$ is increasing (Doob-Meyer decomposition)
A semimartingale can be written as $X_t = M_t^d + M_t^c + A_t$ where $M_t = M_t^c + M_t^d$ is the decomposition of the martingale M into a continuous martingale M^c and a discontinuous martingale M^d (orthogonal to any continuous martingale)

8.4.2 Integration by parts formula for finite variation processes

If U and V are two finite variation processes, Stieltjes' integration by parts formula can be written as follows

$$\begin{aligned} U(t)V(t) &= U(0)V(0) + \int_{]0,t]} V(s)dU(s) + \int_{]0,t]} U(s-)dV(s) \qquad (8.22) \\ &= U(0)V(0) + \int_{]0,t]} V(s-)dU(s) + \int_{]0,t]} U(s-)dV(s) \\ &\qquad + \sum_{s\le t} \Delta U(s)\,\Delta V(s)\,. \end{aligned}$$

We shall often write $\int_0^t V(s)dU(s)$ for $\int_{]0,t]} V(s)dU(s)$.

8.4.3 Integration by parts formula for mixed processes

Let $dX_t = \mu_t dt + \sigma_t dW_t + \varphi_t dM_t$ where M is a compensated martingale of a compound Poisson process. Then

For

8.4.4 Doléans-Dade exponential

If X is a semimartingale, then the process $Z = \mathcal{E}(X)$ is the unique solution to the SDE (called the Doléans Dade exponential)

$$Z_t = 1 + \int_{]0,t]} Z_{u-}\, dX_u.$$

It is known that

$$\mathcal{E}_t(X) = \exp\big(X_t - X_0 - \frac{1}{2}\langle X^c\rangle_t\big) \prod_{u\le t}(1 + \Delta X_u)e^{-\Delta X_u},$$

where X^c is the continuous martingale component of X.

8.4.5 Itô's formula

8.4.6 Stopping times

Definition 8.4.1 *A stopping time T is* ***predictable*** *if there exists an increasing sequence (T_n) of stopping times such that almost surely*

i) $\lim_n T_n = T$

ii) $T_n < T$ *for every n on the set* $\{T > 0\}$.

A stopping time T is ***totally inaccessible*** *if $P(T = S < \infty) = 0$ for any predictable stopping time S. An equivalent definition is: for any increasing sequence of stopping times T_n, $P(\{\lim T_n = T\} \cap A) = 0$ where $A = \cap\{T_n < T\}$.*

8.5 Enlargements of Filtrations

In general, if **G** is a filtration larger than **F**, it is not true that an **F**-martingale remains a martingale in the filtration **G**. From the end of the 1970's the French school of probability studied the problem of enlargement of filtration, and obtained results on the decomposition of the **F**-martingales in the filtration **G**. The main papers and books are Brémaud and Yor [34], Jacod [111, 110], Jeulin [126], Jeulin and Yor [127, 128] and Protter [164]. See also Barlow [8] and Dellacherie and Meyer [65]. The book of Yor [179] contains a concise introduction to enlargement of filtrations, and the book of Mansuy and Yor [157] presents this theory in details. These results are extensively used in finance to study the problem of insider trading, an incomplete list of authors is: Amendinger [2], Amendinger et al. [3], Ankrichner et al. [4], Corcuera et al. [50], Eyraud-Loisel [83], Gasbarra et al. [90] Grorud and Pontier [96], Hiliaret [101], Imkeller [108], Imkeller et al. [109], Karatzas and Pikovsky [129], Kohatsu-Higa [135, 136], Kohatsu-Higa and Oksendal [137]. They are also used to study asymmetric information, see e. g. Föllmer et al. [87] and for the study of default in the reduced form approach by Bielecki et al. [19, 16, 14], Elliott et al.[81] and Kusuoka [140].

8.5.1 Progressive Enlargement

We consider the case where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$ when τ is a finite random time, i.e., a finite non-negative random variable. For any **G**-predictable process H, there exists an **F**-predictable process h such that $H_t 1\!\!1_{t\leq\tau} = h_t 1\!\!1_{t\leq\tau}$. Under the condition $\forall t, P(\tau \leq t|\mathcal{F}_t) < 1$, the process $(h_t, t \geq 0)$ is unique (See [64] page 186).

Let us first investigate the case where the (H) hypothesis holds

Lemma 8.5.1 *In the progressive enlargement setting,* (H) *is equivalent to one of the following equivalent conditions*

$$\begin{array}{llrcl} (i) & \forall\, s \leq t, & P(\tau \leq s|\mathcal{F}_\infty) & = & P(\tau \leq s|\mathcal{F}_t), \\ (ii) & \forall t, & P(\tau \leq t|\mathcal{F}_\infty) & = & P(\tau \leq t|\mathcal{F}_t). \end{array} \tag{8.23}$$

In particular, if (H) holds, then the process $(P(\tau \leq t|\mathcal{F}_t), t \geq 0)$ is increasing.

The decomposition of the **F**-martingales in the filtration **G** are known up to time τ.

Proposition 8.5.1 *Every* **F***-martingale M stopped at time τ is a* **G***-semi-martingale with canonical decomposition*

$$M_{t\wedge\tau} = \widetilde{M}_t + \int_0^{t\wedge\tau} \frac{d\langle M, \mu^\tau\rangle_s}{Z^\tau_{s-}}\,,$$

where $\widetilde{M}$ is a **G***-local martingale. The process*

$$1\!\!1_{\tau \le t} - \int_0^{t\wedge\tau} \frac{1}{Z^\tau_{s-}} dA^\tau_s$$

is a **G***-martingale.*

As we have mentioned, if (H) holds, the process $(Z^\tau_t, t \ge 0)$ is an increasing process. For a general random time τ, it is not true that **F**-martingales are **G** semi-martingales.

Definition 8.5.1 *Let $\tau \in \mathcal{F}_\infty$. The random time τ is* ***honest*** *if τ is equal, on $\{\tau < t\}$ to an $\mathcal{F}_t$-measurable random variable.*

It is proved in Jeulin [126] that a random time is honest if and only if it is the end of a predictable set, i.e., $\tau = \sup\{t : (t,\omega) \in \Gamma\}$, where Γ is an **F**-predictable set.

Proposition 8.5.2 *Let τ be honest. Then, if X is an* **F***-martingale, there exists a* **G***-martingale $\widetilde{X}$ such that*

$$X_t = \widetilde{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X, Z^\tau\rangle_s}{Z^\tau_{s-}} - \int_\tau^{\tau\vee t} \frac{d\langle X, Z^\tau\rangle_s}{1 - Z^\tau_{s-}}$$

8.6 Markov Chains

Let X be a right-continuous process with values in a finite set E. Le $\mathcal{G}$ be some filtration larger than the natural filtration of X.

Definition 8.6.1 *A process X is a continuous time* **G***-Markov-chain if for any function $h : E \to I\!R$ and any t, t*

$$E(h(X_{t+s})|\mathcal{G}_t) = E(h(X_{t+s})|X_t) = \Psi(t, X_t, t+s)$$

A continuous time **G***-Markov-chain is time homogeneous if $\Psi(t,x,t+s) = \Psi(t+u,x,t+s+u)$*

Definition 8.6.2 *A family $p_{i,j}(t,s)$ is called a transition probability matrix if*

$$P(X_s = j|X_t = i) = p_{i,j}(t,s)$$

In the case of time-homogeneous Markov chain, the one-parameter family $p_{i,j}(t)$ is the family of transition probability if

$$P(X_{s+t} = j|X_t = i) = p_{i,j}(s)$$

Observe that

for all i, $p_{i,j} \ge 0$

for all i, $\sum_{j\in E} p_{i,j} = 1$

Then

$$P(X_{s+t} \in A|X_t = i) = \sum_{j\in A} p_{i,j}(s)$$

The Chapman-Kolmogorov equation

$$p_{i,j}(t+s) = \sum_{k\in E} p_{i,k}(t)p_{k,j}(s) = \sum_{k\in E} p_{i,k}(s)p_{k,j}(s)$$

is satisfied and can be written in a matrix form

$$P(t+s) = P(t)P(s) = P(s)P(t)$$

The following limit

$$\lambda_{i,j} = \lim \frac{p_{i,j}(t) - p_{i,j}(0)}{t} = \lim \frac{p_{i,j}(t) - \delta_{i,j}}{t}$$

exists. Observe that

for $i \neq j$, $\lambda_{i,j} \geq 0$

$\lambda_{i,i} = -\sum_{j \neq i} \lambda i, j$

The matrix $\Lambda = (\lambda_{i,j})$ is called the infinitesimal generator matrix. The backward Kologorov equation is

$$\frac{dP(t)}{dt} = \Lambda P(t),\ P(0) = Id$$

The forward Kolmogorov equation is

$$\frac{dP(t)}{dt} = P(t)\Lambda,\ P(0) = Id$$

These equation have the unique solution

$$P(t) = e^{t\Lambda}$$

Note that, for any function h, the process

$$h(X_t) - \int_0^t (\Lambda h)(X_u) du$$

is a martingale.

`Elementary case` Let us study a continuous time Markov chain with two states 0 and 1. T If

$$P_{0,0}(t) = P(\tau > t) = e^{-\lambda t},\ P_{0,1}(t) = 1 - e^{-\lambda t},\ P_{1,0}(t) = 0,\ P_{1,1}(t) = 1$$

The transition matrix is

$$P(t) = \begin{bmatrix} e^{-\lambda t} & 1 - e^{-\lambda t} \\ 0 & 1 \end{bmatrix}$$

and can be written in the form

$$P(t) = e^{\Lambda t} = \sum_n \frac{(t\Lambda)^n}{n!}$$

with $\Lambda = \begin{bmatrix} -\lambda & \lambda \\ 0 & 0 \end{bmatrix}$. The matrix Λ is called the generator of the Markov chain. The probability for going from state 0 to state 1 between the date t and $t + dt$ is λdt. (See Karlin and Taylor [131])

8.7 Dividend paying assets

We also consider an underlying market, which is composed of the savings account and of d risky assets. We postulate that savings account, say $(B_t)_{t \in \mathbb{R}_+}$, is a **G**-predictable, positive, with bounded variation and $B_0 = 1$; we define the *discount factor* process, say $(\beta_t)_{t \in \mathbb{R}_+}$, as the inverse of the savings account; that is, $\beta_t = B_t^{-1}$.

8.7.1 Discounted Cum-dividend Prices

We denote by $\mathbf{X} = (X^1, \ldots, X^d)$ an $I\!R^d$-valued process representing ex-dividend prices of all the risky assets in our underlying market. It is assumed that $\mathbf{X}$ is $\mathbf{G}$-semimartingale under P. Also, for the sake of concreteness, we take the $X^1 = S$. Note that, in general, we do not assume that components of $\mathbf{X}$ take non-negative values, although we take it that process S is positive. An example of a risky asset whose ex-dividend price process is not necessarily non-negative would be a swap contract, such as Credit Default Swap (or CDS) contract.

Any risky asset X^i may pay a dividend. If this is the case, then we denote by $X^{i,cum}$ the cum-dividend price of the asset. Accordingly, we denote by $\mathbf{X}^{cum}$ the vector of cum-dividend prices; of course, it may happen that some i-th asset does not pay any dividend, in which case $X^i = X^{i,cum}$.

Let now X denote the ex-dividend price process of a generic asset. For a dividend paying asset we have that $X^{cum} = X + D^X$, where D^X is the (cumulative) dividend process, which is assumed to be $\mathbf{G}$–adapted and of finite variation. Thus, X^{cum} is also a $\mathbf{G}$-semimartingale under P. The central role in financial applications is played by discounted price processes. The *discounted cum-dividend price* process will be denoted by ${}^{\beta}X^{cum}$ and defined as

$$\boxed{{}^{\beta}X^{cum}_t = \beta_t X_t + {}^{\beta}D^X_t ,} \tag{8.24}$$

where (it is additionally assumed that β is integrable w.r.t. D^X)

$${}^{\beta}D^X_t := \int_{(0,t]} \beta_u \, dD^X_u .$$

Now, we assume that the underlying market model is free of arbitrage opportunities (though presumably incomplete) and we denote by $\mathcal{M}$ the set of equivalent martingale measures on the underlying market, namely the set of probability measures $\mathbb{Q} \sim \mathbb{P}$ for which ${}^{\beta}\mathbf{X}^{cum}$ is a $I\!R^d$-valued $\mathbf{G}$-martingale under Q.

To simplify the notation we shall write ${}^{\beta}X$ instead of ${}^{\beta}X^{cum}$. Thus, ${}^{\beta}X$ will denote discounted cumulative price process, whereas βX will denote discounted ex–dividend price process. Of course, in case of absence of dividends we have ${}^{\beta}X = \beta X$.

8.8 Ornstein-Uhlenbeck processes

8.8.1 Vacisek model

Proposition 8.8.1 *Let k, θ and σ be bounded Borel functions, and W a Brownian motion. The solution of*

$$dr_t = k(t)(\theta(t) - r_t)dt + \sigma(t)dW_t \tag{8.25}$$

is

$$r_t = e^{-K(t)} \left(r_0 + \int_0^t e^{K(s)} k(s)\theta(s)ds + \int_0^t e^{K(s)} \sigma(s) dW_s \right)$$

where $K(t) = \displaystyle\int_0^t k(s)ds$. The process $(r_t, t \geq 0)$ is a Gaussian process with mean

$$E(r_t) = e^{-K(t)} \left(r_0 + \int_0^t e^{K(s)} k(s)\theta(s)ds \right)$$

and covariance

$$e^{-K(t)-K(s)} \int_0^{t \wedge s} e^{2K(u)} \sigma^2(u) du \, .$$

The **Hull and White model** correspond to the dynamics (8.25) where k is a positive function.

In the particular case where θ and k are constant, we obtain

Corollary 8.8.1 *The solution of*

$$\boxed{dr_t = k(\theta - r_t)dt + \sigma dW_t} \tag{8.26}$$

is

$$r_t = (r_0 - \theta)e^{-kt} + \theta + \sigma \int_0^t e^{-k(t-u)} dW_u .$$

The process $(r_t, t \geq 0)$ *is a Gaussian process with mean* $(r_0 - \theta)e^{-kt} + \theta$ *and covariance*

$$Cov(r_s, r_t) = \frac{\sigma^2}{2k} e^{-k(s+t)}(e^{2ks} - 1) = \frac{\sigma^2}{k} e^{-kt} \sinh(ks)$$

for $s \leq t$.

In finance, the solution of (8.26) is called a **Vasicek** process. In general, k is chosen to be positive, so that $E(r_t) \to \theta$ as $t \to \infty$. The process (8.25) is called a Generalized Vasicek process (GV). Since r is a Gaussian process, it can take negative values. This is one of the reasons why this process is no longer used for modelling interest rates. When $\theta = 0$, the process r is called an **Ornstein-Uhlenbeck** (OU) process. Consequently, for a general θ, the process $(r_t - \theta, t \geq 0)$ is a OU process with parameter k. More formally, here is a

Definition 8.8.1 *An Ornstein-Uhlenbeck (OU) process driven by a BM follows the dynamics* $dr_t = -kr_t dt + \sigma dW_t$.

An OU process can be constructed in terms of time-changed BM:

Proposition 8.8.2 *i) If* W *is a BM starting from* x *and* $A(t) = \sigma^2 \frac{e^{2kt} - 1}{2k}$*, the process* $Z_t = e^{-kt} W_{A(t)}$ *is an OU process starting from* x.
ii) Conversely, if U *is an OU process starting from* x*, then there exists a BM* W *starting from* x *such that* $U_t = e^{-kt} W_{A(t)}$.

Proof: Indeed, the process Z is a Gaussian process, with mean xe^{-kt} and covariance $e^{-k(t+s)}(A(t) \wedge A(s))$. □

From the Markov property of the process r it follows, in the case of constant coefficients:

Proposition 8.8.3 *Let* r *be the solution of (8.26) and* $\mathbf{F}$ *the natural filtration of the Brownian motion* W*. For* $s < t$*, the conditional expectation and the conditional variance of* r_t *with respect to* $\mathcal{F}_s$ *are given by*

$$\begin{aligned} E(r_t | r_s) &= (r_s - \theta)e^{-k(t-s)} + \theta \\ Var_s\ (r_t) &= \frac{\sigma^2}{2k}(1 - e^{-2k(t-s)}) . \end{aligned}$$

Note that the filtration generated by the process r is equal to the Brownian filtration. Due to the Gaussian property of the process r, the integrated process $\int_0^t r_s ds$ can be characterized as follows:

Proposition 8.8.4 *Let* r *be a solution of (8.26). The process* $\left(\int_0^t r_s ds,\ t \geq 0\right)$ *is Gaussian with mean* $E(\int_0^t r_s ds) = \theta t + (r_0 - \theta)\frac{1 - e^{-kt}}{k}$*, variance*

$$Var(\int_0^t r_s ds) = -\frac{\sigma^2}{2k^3}(1 - e^{-kt})^2 + \frac{\sigma^2}{k^2}(t - \frac{1 - e^{-kt}}{k})$$

and covariance (for $s < t$)

$$\frac{\sigma^2}{k^2}\left(s - e^{-kt}\frac{e^{ks}-1}{k} - \frac{1-e^{-ks}}{k} + e^{-k(t+s)}\frac{e^{2ks}-1}{2k}\right).$$

Proof: From the definition, $r_t = r_0 + k\theta t - k\int_0^t r_s ds + \sigma W_t$, hence

$$\begin{aligned}\int_0^t r_s ds &= \frac{1}{k}[-r_t + r_0 + k\theta t + \sigma W_t] \\ &= \frac{1}{k}[k\theta t + (r_0-\theta)(1-e^{-kt}) - \sigma\int_0^t e^{-k(t-u)}dW_u + \sigma W_t].\end{aligned}$$

Obviously, from the properties of the Wiener integral, the right-hand side defines a Gaussian process. It remains to compute the expectation and the variance of the Gaussian variable on the right-hand side. □

Zero-coupon Bond

Suppose that the dynamics of the interest rate under the risk-neutral probability are given by (8.26). The value $P(t,T)$ of a zero-coupon bond maturing at date T is given as the conditional expectation of the discounted payoff. Using the Laplace transform of a Gaussian law, and using Proposition 8.8.4, we obtain

$$P(t,T) = E\left(\exp\left(-\int_t^T r_u\,du\right)|\mathcal{F}_t\right) = \exp(-M(t,T) + \frac{1}{2}V(t,T)),$$

i.e.,

Proposition 8.8.5 *In a Vasicek model, the price of a zero-coupon with maturity T is*

$$\begin{aligned}P(t,T) &= \exp\left[-\theta(T-t) - (r_t-\theta)\frac{1-e^{-k(T-t)}}{k} - \frac{\sigma^2}{4k^3}(1-e^{-k(T-t)})^2 \right.\\ &\left. \qquad +\frac{\sigma^2}{2k^2}\left(T-t-\frac{1-e^{-k(T-t)}}{k}\right)\right] \\ &= \exp(a(t,T) - b(t,T)r_t),\end{aligned}$$

with $b(t,T) = \frac{1-e^{-k(T-t)}}{k}$.

It is not difficult to check that the risk-neutral dynamics of the zero-coupon bond is

$$dP(t,T) = P(t,T)(r_t dt - b(t,T)dW_t).$$

Note that we know in advance, without any computation that

$$dP(t,T) = P(t,T)(r_t dt - \sigma_t dW_t),$$

since the discounted value of the zero-coupon bond is a martingale. It suffices to identify the volatility term.

8.9 Cox-Ingersoll-Ross Processes

8.9.1 CIR Processes and BESQ

From general Theorem on the existence of solutions to one dimensional SDE, the equation

$$dr_t = k(\theta - r_t)\,dt + \sigma\sqrt{|r_t|}dW_t, \tag{8.27}$$

admits a unique solution which is strong. For $\theta = 0$ and $r_0 = 0$, the solution is $r_t = 0$, and from the comparison Theorem, we deduce that, in the case $k\theta > 0$, $r_t \geq 0$ for $r_0 \geq 0$. In that case, we omit the absolute value and consider the positive solution of

$$dr_t = k(\theta - r_t)\,dt + \sigma\sqrt{r_t}dW_t\,. \tag{8.28}$$

This solution is called a **Cox-Ingersoll-Ross (CIR)** process or a square-root process (See Feller [84]). For $\sigma = 2$, this process is the square of the norm of a δ-dimensional OU process, with parameter $k\theta$.

We shall denote by ${}^kQ^{k\theta,\sigma}$ the law of the CIR process solution of the equation (8.27). In the case $\sigma = 2$, we simply note ${}^kQ^{k\theta,2} = {}^kQ^{k\theta}$. The elementary change of time $A(t) = 4t/\sigma^2$ reduces the study of the solution of (8.28) to the case $\sigma = 2$: indeed, if $Z_t = r(4t/\sigma^2)$, then

$$dZ_t = k'(\theta - Z_t)\,dt + 2\sqrt{Z_t}dB_t$$

with $k' = 4k/\sigma^2$ and B a Brownian motion.

Many authors prefer to write the dynamics of a square root process as

$$dr_t = (\nu - \lambda r_t)\,dt + \sigma\sqrt{|r_t|}dW_t \tag{8.29}$$

allowing to consider the interesting case $\nu = 0$. In the case $\nu = 0$, when a CIR process hits 0, it remains at 0.

Proposition 8.9.1 *The CIR process (8.28) is a space-time changed BESQ process: more precisely,*

$$\boxed{r_t = e^{-kt}\rho\left(\frac{\sigma^2}{4k}(e^{kt} - 1)\right)}$$

where $(\rho(s), s \geq 0)$ is a BESQ$^\delta$*process, with dimension* $\delta = \dfrac{4k\theta}{\sigma^2}$.

PROOF: See [123]. □

It follows that for $2k\theta \geq \sigma^2$, a CIR process starting from a positive initial point stays always positive. For $0 \leq 2k\theta < \sigma^2$, a CIR process starting from a positive initial point hits 0 with probability $p \in]0,1[$ if $k < 0$ ($P(T_0^x < \infty) = p$) and almost surely if $k \geq 0$ ($P(T_0^x < \infty) = 1$). In the case $0 < 2k\theta$, the boundary 0 is instantaneously reflecting, whereas in the case $2k\theta < 0$, the process r starting from a positive initial point remains positive until $T_0 = \inf\{t : r_t = 0\}$. Setting $Z_t = -r_{T_0+t}$, we obtain that

$$dZ_t = (-\delta + \lambda Z_t)dt + \sigma\sqrt{|Z_t|}dB_t$$

where B is a BM. We know that $Z_t \geq 0$, thus r_{T_0+t} takes values in $I\!R_-$.

Absolute Continuity Relationship

A routine application of Girsanov's theorem leads to

$${}^kQ_x^{k\theta}|_{\mathcal{F}_t} = \exp\left(\frac{k}{4}\left[x + k\theta t - \rho_t\right] - \frac{k^2}{8}\int_0^t \rho_s\,ds\right) Q_x^{k\theta}|_{\mathcal{F}_t} \tag{8.30}$$

Comments 8.9.1 From an elementary point of view, if the process r reaches 0 at time t, the formal equality between dr_t and $k\theta dt$ explains that the increment of r_t is positive if $k\theta > 0$. Again formally, for $k > 0$, if at time t, the inequality $r_t > \theta$ holds (resp. $r_t < \theta$), then the drift $k(\theta - r_t)$ is negative

(resp. positive) and, at least in mean, r is decreasing (resp. increasing).
Here we have used the notation r for the CIR process. As shown above, this process is close to a BESQ ρ (and not to a BES R).
Dufresne [76] has obtained explicit formulae for the moments of the r.v. r_t. The process $(\int_0^t r_s ds, t \geq 0)$ is studied by Dufresne [76]; Dassios and Nagaradjasarma [57] present an explicit computation for the joint moments of r_t and $I_t = \int_0^t r_s ds$, and, in the case $\theta = 0$, the joint density of the pair (r_t, I_t).

8.9.2 Transition Probabilities for a CIR Process

From the expression of a CIR process as a squared Bessel process time-changed, using the transition density of the squared Bessel process given in ([123]), we obtain its transition density.

Proposition 8.9.2 *Let r be a CIR process following (8.28). The transition density ${}^kQ^{k\theta,\sigma}(r_{t+s} \in dy | r_s = x) = f_t(x,y)dy$ is given by*

$$f_t(x,y) = \frac{e^{kt}}{2c(t)} \left(\frac{ye^{kt}}{x}\right)^{\nu/2} \exp\left(-\frac{x + ye^{kt}}{2c(t)}\right) I_\nu\left(\frac{1}{c(t)}\sqrt{xye^{kt}}\right) 1\!\!1_{\{y \geq 0\}}\,,$$

where $c(t) = \dfrac{\sigma^2}{4k}(e^{kt} - 1)$ and $\nu = \dfrac{2k\theta}{\sigma^2} - 1$. The cumulative distribution function is

$${}^kQ_x^{k\theta,\sigma}(r_t < y) = \chi^2\left(\frac{4k\theta}{\sigma^2}, \frac{x}{c(t)}; \frac{ye^{kt}}{c(t)}\right),$$

where the function $\chi^2(\delta, \alpha)$ is T is a non-central chi-square with $\delta = 2(\nu + 1)$ degrees of freedom, and α the parameter of non-centrality

8.9.3 CIR Processes as Spot Rate Models

The Cox-Ingersoll-Ross model for the short interest rate is the object of many studies since the seminal paper of Cox et al. [53] where the authors assume that the riskless rate r follows a square root process under the historical probability given by

$$dr_t = \tilde{k}(\tilde{\theta} - r_t)\,dt + \sigma\sqrt{r_t}d\tilde{W}_t\,.$$

Here $\tilde{k}(\tilde{\theta} - r)$ defines a mean reverting drift pulling the interest rate towards its long term value $\tilde{\theta}$ with a speed of adjustment equal to $\tilde{k}$. In the risk adjusted economy, the dynamics are supposed to be given by:

$$dr_t = (\tilde{k}(\tilde{\theta} - r_t) - \lambda r_t)dt + \sigma\sqrt{r_t}dW_t$$

where $(W_t = \widetilde{W}_t + \int_0^t \frac{\lambda}{\sigma}\sqrt{r_s}ds, t \geq 0)$ is a Brownian motion under the risk adjusted probability Q where λ denotes the market price of risk. Setting $k = \tilde{k} + \lambda$, $\theta = \tilde{k}(\tilde{\theta}/k)$, the Q -dynamics of r are

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t\,.$$

Therefore, we shall establish formulae under general dynamics of the form (8.28).
Even if no closed-form expression as a functional of W can be written for r_t, it is remarkable that the Laplace transform of the process, i.e.

$${}^kQ_x^{k\theta,\sigma}\left[\exp\left(-\int_0^t du\,\phi(u)r_u\right)\right]$$

is known..

Théorème 8.3 *Let r be a CIR process, the solution of*

$$\boxed{dr_t \;=\; k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t.} \tag{8.31}$$

The conditional expectation and the conditional variance of the r.v. r_t are given by, for $s < t$,

$$\begin{aligned} {}^kQ_x^{k\theta,\sigma}(r_t\,|\mathcal{F}_s) &= r_s e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}), \\ Var(r_t\,|\mathcal{F}_s) &= r_s\,\frac{\sigma^2(e^{-k(t-s)} - e^{-2k(t-s)})}{k} + \frac{\theta\sigma^2(1 - e^{-k(t-s)})^2}{2k}. \end{aligned}$$

Note that, if $k > 0$, $E(r_t) \to \theta$ as t goes to infinity.

Comments 8.9.2 Using an induction procedure, or using computations done for squared Bessel processes, all the moments of r_t can be computed. See Dufresne [75].

Exercise 8.9.1 If r is a CIR process and $Z = r^\alpha$, prove that

$$dZ_t = \left(\alpha Z_t^{1-1/\alpha}(k\theta + (\alpha-1)\sigma^2/2) - Z_t\alpha k\right)dt + \alpha Z_t^{1-1/(2\alpha)}\sigma dW_t\,.$$

In particular, for $\alpha = -1$, $dZ_t = Z_t(k - Z_t(k\theta - \sigma^2))dt - Z_t^{3/2}\sigma dW_t$ is the so-called 3/2 model (see Section on CEV processes in [123] and Lewis [150]).

8.9.4 Zero-coupon Bond

We now address the problem of the valuation of a zero-coupon bond, i.e., we assume that the dynamics of the interest rate are given by a CIR process under the risk neutral probability and we compute $E\left(\exp\left(-\int_t^T r_u\,du\right)|\mathcal{F}_t\right)$.

Proposition 8.9.3 *Let r be a CIR process defined as in (8.28) by*

$$dr_t = k(\theta - r_t)\,dt + \sigma\sqrt{r_t}dW_t\,,$$

and ${}^kQ^{k\theta,\sigma}$ its law. Then, for any pair (λ,μ) of positive numbers

$${}^kQ_x^{k\theta,\sigma}\left(\exp\left(-\lambda r_T - \mu\int_0^T r_u\,du\right)\right) = \exp[-A_{\lambda,\mu}(T) - xG_{\lambda,\mu}(T)]$$

with

$$\begin{aligned} G_{\lambda,\mu}(s) &= \frac{\lambda(\gamma + k + e^{\gamma s}(\gamma - k)) + 2\mu(e^{\gamma s} - 1)}{\sigma^2\lambda(e^{\gamma s} - 1) + \gamma(e^{\gamma s} + 1) + k(e^{\gamma s} - 1)} \\ A_{\lambda,\mu}(s) &= -\frac{2k\theta}{\sigma^2}\ln\left(\frac{2\gamma e^{(\gamma+k)s/2}}{\sigma^2\lambda(e^{\gamma s} - 1) + \gamma(e^{\gamma s} + 1) + k(e^{\gamma s} - 1)}\right) \end{aligned}$$

where $\gamma = \sqrt{k^2 + 2\sigma^2\mu}\,$.

Corollary 8.9.1 *Let r be a CIR process defined as in (8.28) under the risk-neutral probability. Then, the t-time price of a zero-coupon bond maturing at T is*

$${}^kQ_x^{k\theta,\sigma}\left(\exp\left(-\int_t^T r_u\,du\right)\Big|\mathcal{F}_t\right) = \exp[-A(T-t) - r_tG(T-t)] = B(r_t, T-t)$$

with

$$B(r,s) = \exp(-A(s) - rG(s))$$

and

$$\begin{aligned}
G(s) &= \frac{2(e^{\gamma s}-1)}{(\gamma+k)(e^{\gamma s}-1)+2\gamma} = \frac{2}{k+\gamma\coth(\gamma s/2)} \\
A(s) &= -\frac{2k\theta}{\sigma^2}\ln\left(\frac{2\gamma e^{(\gamma+k)s/2}}{(\gamma+k)(e^{\gamma s}-1)+2\gamma}\right) \\
&= -\frac{2k\theta}{\sigma^2}\left[\frac{ks}{2} + \ln\left(\cosh\frac{\gamma s}{2} + \frac{k}{\gamma}\sinh\frac{\gamma s}{2}\right)^{-1}\right],
\end{aligned}$$

where $\gamma = \sqrt{k^2+2\sigma^2}$.
The dynamics of the zero-coupon bond $P(t,T) = B(r_t, T-t)$ *are, under the risk neutral probability*

$$dP(t,T) = P(t,T)\left(r_t dt + \sigma(T-t, r_t) dW_t\right)$$

with $\sigma(s,r) = -\sigma G(s)\sqrt{r}$.

Corollary 8.9.2 *The Laplace transform of the r.v.* r_T *is*

$${}^k Q_x^{k\theta,\sigma}(e^{-\lambda r_T}) = \left(\frac{1}{2\lambda\tilde{c}+1}\right)^{2k\theta/\sigma^2} \exp\left(-\frac{\lambda\tilde{c}\tilde{x}}{2\lambda\tilde{c}+1}\right)$$

with $\tilde{c} = c(T)e^{-kT}$ *and* $\tilde{x} = x/c(T)$, $c(T) = \dfrac{\sigma^2}{4k}(e^{kT}-1)$.

8.10 Parisian Options

In this section, our aim is to price an exotic option that we describe below, in a Black and Scholes framework: the underlying asset satisfies the stochastic differential equation

$$dS_t = S_t((r-\delta)\,dt + \sigma\,dW_t)$$

where W is a Brownian motion under the risk-neutral probability Q, and w.l.g. $\sigma > 0$. In a closed form,

$$S_t = xe^{\sigma X_t}$$

where $X_t = W_t + \nu t$ and $\nu = \frac{r-\delta}{\sigma} - \frac{\sigma}{2}$. The owner of an `Up-and-Out Parisian` option loses its value if the stock price reaches a level H and remains constantly above this level for a time interval longer than D (the window). A **Down-and-in** Parisian option is activated if the stock price reaches a level L and remains constantly below this level for a time interval longer than D. For a window length equal to zero, the Parisian option reduces to a standard barrier option. For a continuous process X and a given $t > 0$, we introduce $g_t^b(X)$, the last time before t at which the process X was at level b, i.e.,

$$g_t^b(X) = \sup\{s \le t : X_s = b\}.$$

For an Up-and-Out Parisian option we need to consider the first time at which the underlying asset S is above H for a period greater than D, i.e.,

$$G_D^{+,H}(S) = \inf\{t > 0 : (t - g_t^H(S))\mathbb{1}_{\{S_t > H\}} \ge D\}$$

or, written in terms of X

$$G_D^{+,h}(X) = \inf\{t > 0 : (t - g_t^h(X))\mathbb{1}_{\{X_t > h\}} \ge D\}$$

where $h = \ln(H/x)/\sigma$. If this stopping time occurs before the maturity then the Up-and-Out Parisian option is worthless. The price of an Up-and-Out Parisian call option is

$$\begin{aligned} \mathrm{PUO}(x,H,D;T) &= E_Q\left(e^{-rT}(S_T-K)^+ \mathbb{1}_{G_D^{+,H}(S)>T}\right) \\ &= E_Q\left(e^{-rT}(xe^{\sigma X_T}-K)^+ \mathbb{1}_{G_D^{+,h}(X)>T}\right) \end{aligned}$$

or, using a change of probability

$$\mathrm{PUO}(x,H,D;T) = e^{-(r+\nu^2/2)T} E\left(e^{\nu W_T}(xe^{\sigma W_T}-K)^+ \mathbb{1}_{G_D^{+,h}(W)>T}\right).$$

The sum of the price of an up-and-in and an up-and-out Parisian option is obviously the price of a plain-vanilla European call.

In the same way, the value of a Down-and-in Parisian option with level L is defined using

$$G_D^{-,L}(S) = \inf\{t>0 : (t-g_t^L(S))\mathbb{1}_{\{S_t<L\}} \geq D\}$$

which equals, in terms of X

$$G_D^{-,\ell}(X) = \inf\{t>0 : (t-g_t^\ell(X))\mathbb{1}_{\{X_t<\ell\}} \geq D\}$$

with $\ell = \dfrac{1}{\sigma}\ln(L/x)$ and is equal to

$$\begin{aligned} \mathrm{PDI}(x,L,D;T) &= E_Q\left(e^{-rT}(S_T-K)^+ \mathbb{1}_{G_D^{-,L}(S)<T}\right) \\ &= e^{-(r+\nu^2/2)T} E\left(e^{\nu W_T}(xe^{\sigma W_T}-K)^+ \mathbb{1}_{G_D^{-,\ell}(W)<T}\right) \\ &\stackrel{def}{=} e^{-(r+\nu^2/2)T}\, {}^\star\mathrm{PDI}(x,L,D;T)\,. \end{aligned}$$

8.10.1 The Law of $(G_D^{-,\ell}(W)\,,\, W_{G_D^{-,\ell}})$

Proposition 8.10.1 *Let W be a Brownian motion and $G_D^- = G_D^{-,0}(W)$. The random variables G_D^- and $W_{G_D^-}$ are independent and*

$$P(W_{G_D^-}\in dx) = \frac{-x}{D}\exp(-\frac{x^2}{2D})\,\mathbb{1}_{\{x<0\}}\,dx, \tag{8.32}$$

$$E(\exp(-\frac{\lambda^2}{2}G_D^-)) = \frac{1}{\Psi(\lambda\sqrt{D}\,)} \tag{8.33}$$

where

$$\Psi(z) = \int_0^\infty x\exp(zx-\frac{x^2}{2})dx = 1-2e^{z^2/2}\mathcal{N}(z)\,.$$

We can easily deduce from the above Proposition the law of the pair $(G_D^{-,\ell}, W_{G_D^{-,\ell}})$ in the case $\ell<0$, as we present now.

Corollary 8.10.1 *Let $\ell<0$. The random variables $G_D^{-,\ell}$ and $W_{G_D^{-,\ell}}$ are independent and their laws are given by*

$$P(W_{G_D^{-,\ell}}\in dx) = \frac{dx}{D}\,\mathbb{1}_{\{x<\ell\}}\,(\ell-x)\exp\left(-\frac{(x-\ell)^2}{2D}\right) \tag{8.34}$$

$$E\left(\exp(-\frac{\lambda^2}{2}G_D^{-,\ell})\right) = \frac{\exp(\ell\lambda)}{\Psi(\lambda\sqrt{D})}\,. \tag{8.35}$$

Proposition 8.10.2 *In the case $\ell > 0$, the random variables $G_D^{-,\ell}$ and $W_{G_D^{-,\ell}}$ are independent. Their laws are given by*

$$E(\exp(-\lambda G_D^{-,\ell})) = e^{-\lambda D}\,(1 - F_\ell(D)) + \frac{1}{\Psi(\sqrt{2\lambda D})} H(\sqrt{2\lambda}, \ell, D)\,.$$

where the function H is defined by

$$H(a,y,t) \stackrel{def}{=} e^{-ay}\mathcal{N}\Big(\frac{at-y}{\sqrt{t}}\Big) + e^{ay}\mathcal{N}\Big(\frac{-at-y}{\sqrt{t}}\Big)\,, \tag{8.36}$$

we get

$$\mathbb{E}(e^{-\lambda T_y} \mathbb{1}_{\{T_y<t\}}) = e^{\nu y} H(\gamma, |y|, t)\,,$$

and

$$\begin{aligned} P(W_{\widehat{G}_D^{-,\ell}} \in dx) &= \mathbb{1}_{\{x\le \ell\}} dx \left[e^{-(x-\ell)^2/(2D)} P(T_\ell < D)\,\frac{\ell - x}{D} \right. \\ &\quad \left. + \frac{1}{\sqrt{2\pi D}}\left(e^{-x^2/(2D)} - e^{-(x-2\ell)^2/(2D)}\right)\right] \end{aligned}$$

Proof: See [123] or [81] □

8.10.2 Valuation of a Down and In Parisian Option

Théorème 8.4 *In the case $x > L$ (i.e., $\ell < 0$) the function $t \to h_\ell(t,y)$ is characterized by its Laplace transform: for $\lambda > 0$,*

$$\widehat{h}_\ell(\lambda, y) = \frac{e^{\ell\sqrt{2\lambda}}}{D\sqrt{2\lambda}\,\Psi(\sqrt{2\lambda D})} \int_0^\infty dz\, z \exp\left(-\frac{z^2}{2D} - |y+z-\ell|\sqrt{2\lambda}\right)$$

where $\Psi(z)$ is defined in (8.34). If $y > \ell$, then

$$\widehat{h}_\ell(\lambda, y) = \frac{\Psi(-\sqrt{2\lambda D})}{\Psi(\sqrt{2\lambda D})}\,\frac{e^{(2\ell-y)\sqrt{2\lambda}}}{\sqrt{2\lambda}}\,.$$

Comments 8.10.1 Parisian options were studied in Avellaneda and Wu [5], Chesney et al. [44], Cornwall et al. [51], Gauthier [91], Haber et al. [98]. Numerical analysis is done in Bernard et al. [10] and Schröder [174]. The "Parisian" time models a default time in Çetin et al. [42]. Cumulative Parisian options are developed in Hugonnier [106] and Moraux [160]. A PDE approach of valuation of Parisian option is presented in Haber et al. [98] and in Wilmott [177].

Index

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