Risk Theory and Related Topics

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Credit Risk: Reduced Form Approach

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Hedging of Defaultable Claims

The Model

Default Time

- The **default time** τ is a non-negative random variable on $(\Omega, \mathcal{G}, \mathbb{Q})$.
- Note that $\mathbb{Q}$ is the **statistical probability measure**.
- The filtration generated by the **default process** $H_t = \mathbb{1}_{\{\tau \le t\}}$ is denoted by $\mathbb{H}$.
- We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for every $t \in \mathbb{R}_+$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a **reference filtration.**
- We define the processes F_t and G_t as

$$F_t = \mathbb{Q}\{\tau \le t \,|\, \mathcal{F}_t\}$$

and

$$G_t = 1 - F_t = \mathbb{Q}\{\tau > t \,|\, \mathcal{F}_t\}.$$

Hazard Process

- The process Γ, given as

$$\Gamma_t = -\ln(1 - F_t) = -\ln G_t$$

  is the **$\mathbb{F}$-hazard process** under the statistical probability $\mathbb{Q}$.

- We shall assume that the $\mathbb{F}$-hazard process is absolutely continuous: $\Gamma_t = \int_0^t \gamma_u \, du$.

- Hence, the **compensated default process**

$$M_t = H_t - \int_0^{t \wedge \tau} \gamma_u \, du = H_t - \int_0^t U_u \, du,$$

  is a $\mathbb{G}$-martingale under $\mathbb{Q}$, where we denote $U_t = \gamma_t \mathbb{1}_{\{t<\tau\}}$.

Hypothesis (H): immersion property. We assume throughout that any $\mathbb{F}$-martingale under $\mathbb{Q}$ is also a $\mathbb{G}$-martingale under $\mathbb{Q}$.

- Hypothesis (H) is satisfied if a random time τ is defined through the canonical construction.
- If the representation theorem holds for the filtration $\mathbb{F}$ and a finite family $Z^i, i \leq n$, of $\mathbb{F}$-martingales then, under Hypothesis (H), it holds also for the filtration $\mathbb{G}$ and with respect to the $\mathbb{G}$-martingales $Z^i, i \leq n$ and M.

Remark. Hypothesis (H) is not invariant with respect to an equivalent change of a probability measure, in general.

The filtration $\mathbb{F}$ is said to be **immersed** in $\mathbb{G}$ if any (square-integrable) $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. It is also referred to as the $(\mathcal{H})$ hypothesis in Brémaud and Yor.

In our setting, hypothesis $(\mathcal{H})$ is equivalent to

$$\mathbb{P}(\tau \leq t|\mathcal{F}_t) = \mathbb{P}(\tau \leq t|\mathcal{F}_\infty)$$

In particular,

$$F_t := \mathbb{P}(\tau \leq t|\mathcal{F}_t)$$

is increasing.

Prices of Traded Assets

- Let Y^1, Y^2, Y^3 be semimartingales on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$. We interpret Y^i_t as the **cash price** at time t of the ith traded asset in the market model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$, where Φ stands for the class of all **self-financing trading strategies**.

- We postulate that the process Y^i is governed by the SDE

$$dY^i_t = Y^i_{t-}\big(\mu_i\, dt + \sigma_i\, dW_t + \kappa_i\, dM_t\big), \quad i = 1, 2, 3,$$

  with $Y^i_0 > 0$.

- Here W is a one-dimensional **Brownian motion** and M is the compensated martingale of the default process H.

Assumptions

- We assume that $\kappa_i \geq -1$ and $\kappa_1 > -1$ so that $Y^1_t > 0$ for every $t \in \mathbb{R}_+$. This assumption allows us to take the first asset as a **numeraire**.

- Note that the constant coefficient $\kappa_1 > -1$ corresponds to a **fractional recovery of market value** for the first asset.

- In general, we do not assume that a risk-free security exists. Hence we do not refer to the theory involving the risk-neutral probability associated with the choice of a **savings account** as a numeraire.

Let F be a $C^{1,2}$ function, and $dX_t = a_t dt + \sigma_t dW_t + \varphi_t dM_t$. Then, the jump of X is $\Delta X_s = \varphi_s \Delta H_s$ and

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \partial_s F(s, X_s)\, ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\ &+ \frac{1}{2} \int_0^t \partial_{xx} F(s, X_s) \sigma_s^2\, ds \\ &+ \sum_{s \le t} [F(s, X_s) - F(s, X_{s^-}) - \partial_x F(s, X_{s-}) \Delta X_s] \end{aligned}$$

Let F be a $C^{1,2}$ function, and $dX_t = a_t dt + \sigma_t dW_t + \varphi_t dM_t$. Then, the jump of X is $\Delta X_s = \varphi_s \Delta H_s$ and

$$\begin{aligned}
F(t, X_t) &= F(0, X_0) + \int_0^t \partial_s F(s, X_s)\, ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\
&+ \frac{1}{2} \int_0^t \partial_{xx} F(s, X_s) \sigma_s^2\, ds \\
&+ \sum_{s \le t} [F(s, X_s) - F(s, X_{s^-}) - \partial_x F(s, X_{s-}) \Delta X_s] \\
&= F(0, X_0) + \int_0^t \partial_s F(s, X_s)\, ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\
&+ \frac{1}{2} \int_0^t \partial_{xx} F(s, X_s) \sigma_s^2\, ds \\
&+ \int_0^t [F(s, X_{s^-} + \varphi_s) - F(s, X_{s^-}) - \varphi_s \partial_x F(s, X_{s-})] dH_s \ .
\end{aligned}$$

Change of Numéraire

- An **equivalent martingale measure** $\mathbb{Q}^*$ is characterized by the property that the relative prices $Y^i(Y^1)^{-1}$, $i = 1, 2, 3$, are $\mathbb{Q}^*$-martingales.
- We will derive the dynamics for the process $Y^{i,1} = Y^i(Y^1)^{-1}$ for $i = 2, 3$.
- From Itô's formula, we first obtain

$$d\left(\frac{1}{Y_t^1}\right) = \frac{1}{Y_{t-}^1}\left(-\mu_1 + \sigma_1^2 + U_t\left(\frac{1}{1+\kappa_1} - 1 + \kappa_1\right)\right)dt - \frac{1}{Y_{t-}^1}\left(\sigma_1 dW_t + \frac{\kappa_1}{1+\kappa_1}\, dM_t\right).$$

Dynamics of Relative Prices

Consequently, the Itô integration by parts formula yields the following dynamics for the processes $Y^{i,1}$

$$\begin{aligned} dY_t^{i,1} &= Y_{t-}^{i,1}\Bigg\{\left(\mu_i - \mu_1 - \sigma_1(\sigma_i - \sigma_1) - U_t(\kappa_i - \kappa_1)\frac{\kappa_1}{1+\kappa_1}\right)dt \\ &+ (\sigma_i - \sigma_1)\,dW_t + \frac{\kappa_i - \kappa_1}{1+\kappa_1}\,dM_t\Bigg\}. \end{aligned}$$

Equivalent Martingale Measure

- By assumption, $\mathbb{Q}^*$ is equivalent to the statistical probability $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ and such that $Y^{i,1}$, $i = 2, 3$ are $\mathbb{Q}^*$-martingales.
- Kusuoka (1999) showed that any probability equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ is defined by means of its Radon-Nikodým density process η satisfying the SDE

$$d\eta_t = \eta_{t-}\big(\theta_t\, dW_t + \zeta_t\, dM_t\big), \quad \eta_0 = 1,$$

  where θ and ζ are $\mathbb{G}$-predictable processes satisfying mild technical conditions (in particular, $\zeta_t > -1$ for $t \in [0, T]$).
- Since M is stopped at τ, we may and do assume that ζ is stopped at τ.

Radon-Nikodým Density

We define $\mathbb{Q}^*$ by setting

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \eta_T = \mathcal{E}_T(\theta W)\mathcal{E}_T(\zeta M), \quad \mathbb{Q}\text{-a.s.}$$

Then **the processes $\widehat{W}$ and $\widehat{M}$** given by, for $t \in [0, T]$,

$$\begin{aligned}
\widehat{W}_t &= W_t - \int_0^t \theta_u \, du, \\
\widehat{M}_t &= M_t - \int_0^t U_u \zeta_u \, du = H_t - \int_0^t U_u(1 + \zeta_u) \, du = H_t - \int_0^t \widehat{U}_u \, du,
\end{aligned}$$

where $\widehat{U}_u = U_u(1 + \zeta_u)$, are **$\mathbb{G}$-martingales under $\mathbb{Q}^*$.**

Martingale Condition

Proposition 1 *Processes $Y^{i,1}$, $i = 2,3$ are $\mathbb{Q}^*$-martingales if and only if drifts in their dynamics, when expressed in terms of $\widehat{W}$ and $\widehat{M}$, vanish.*

Hence the following equalities hold for $i = 2,3$ and every $t \in [0,T]$

$$Y^{i,1}_{t-}\left\{\mu_1 - \mu_i + (\sigma_1 - \sigma_i)(\theta_t - \sigma_1) + U_t(\kappa_1 - \kappa_i)\frac{\zeta_t - \kappa_1}{1+\kappa_1}\right\} = 0.$$

Equivalently, we have for $i = 2,3$, on the set $Y^{i,1}_{t^-} \neq 0$,

$$\mu_1 - \mu_i + (\sigma_1 - \sigma_i)(\theta_t - \sigma_1) + U_t(\kappa_1 - \kappa_i)\frac{\zeta_t - \kappa_1}{1+\kappa_1} = 0.$$

Case A: Strictly Positive Primary Assets

Case A: standing assumptions:

- We postulate that $\kappa_1 > -1$ so that $Y^1 > 0$.
- We assume, in addition, that $\kappa_i > -1$ for $i = 2, 3$, so that the price processes Y^2 and Y^3 are strictly positive as well.

Martingale Condition

- From the general theory of arbitrage pricing, it follows that the market model $\mathcal{M}$ is complete and arbitrage-free if there exists a unique solution (θ, ζ) such that the process $\zeta > -1$.
- Since $Y^{i,1} > 0$, we search for processes (θ, ζ) such that for $i = 2, 3$

$$\theta_t(\sigma_1 - \sigma_i) + \zeta_t U_t \frac{\kappa_1 - \kappa_i}{1 + \kappa_1} = \mu_i - \mu_1 + \sigma_1(\sigma_1 - \sigma_i) + U_t(\kappa_1 - \kappa_i)\frac{\kappa_1}{1 + \kappa_1}.$$

Martingale Condition

Since $U_t = \gamma \mathbb{1}_{\{t \leq \tau\}}$, we deal here with four linear equations.

- For $t \leq \tau$:

$$
\begin{aligned}
\theta_t(\sigma_1 - \sigma_2) + \zeta_t \gamma \frac{\kappa_1 - \kappa_2}{1 + \kappa_1} &= \mu_2 - \mu_1 + \sigma_1(\sigma_1 - \sigma_2) + \gamma \frac{(\kappa_1 - \kappa_2)\kappa_1}{1 + \kappa_1}, \\
\theta_t(\sigma_1 - \sigma_3) + \zeta_t \gamma \frac{\kappa_1 - \kappa_3}{1 + \kappa_1} &= \mu_3 - \mu_1 + \sigma_1(\sigma_1 - \sigma_3) + \gamma \frac{(\kappa_1 - \kappa_3)\kappa_1}{1 + \kappa_1}.
\end{aligned}
$$

- For $t > \tau$:

$$
\begin{aligned}
\theta_t(\sigma_1 - \sigma_2) &= \mu_2 - \mu_1 + \sigma_1(\sigma_1 - \sigma_2), \\
\theta_t(\sigma_1 - \sigma_3) &= \mu_3 - \mu_1 + \sigma_1(\sigma_1 - \sigma_3).
\end{aligned}
$$

- The first (the second, resp.) pair of equations is referred to as the pre-default (post-default, resp.) no-arbitrage condition.

Notation

To solve explicitly these equations, we find it convenient to write

$$a = \det A, \quad b = \det B, \quad c = \det C,$$

where A, B and C are the following matrices:

$$A = \begin{bmatrix} \sigma_1 - \sigma_2 & \kappa_1 - \kappa_2 \\ \sigma_1 - \sigma_3 & \kappa_1 - \kappa_3 \end{bmatrix}, \quad B = \begin{bmatrix} \sigma_1 - \sigma_2 & \mu_1 - \mu_2 \\ \sigma_1 - \sigma_3 & \mu_1 - \mu_3 \end{bmatrix},$$

$$C = \begin{bmatrix} \kappa_1 - \kappa_2 & \mu_1 - \mu_2 \\ \kappa_1 - \kappa_3 & \mu_1 - \mu_3 \end{bmatrix}.$$

Auxiliary Lemma

Lemma 1 *The pair* (θ, ζ) *satisfies the following equations*

$$\begin{aligned} \theta_t a &= \sigma_1 a + c, \\ \zeta_t U_t a &= \kappa_1 U_t a - (1+\kappa_1) b. \end{aligned}$$

In order to ensure the validity of the second equation after the default time τ (i.e., on the set $\{U_t = 0\}$), we need to impose an additional condition, $b = 0$, or more explicitly,

$$(\sigma_1 - \sigma_2)(\mu_1 - \mu_3) - (\sigma_1 - \sigma_3)(\mu_1 - \mu_2) = 0.$$

If this holds, then we obtain the following equations

$$\begin{aligned} \theta_t a &= \sigma_1 a + c, \\ \zeta_t U_t a &= \kappa_1 U_t a. \end{aligned}$$

Existence of a Martingale Measure

Proposition 2 (i) *If $a \neq 0$ and $b = 0$ then the unique martingale measure $\mathbb{Q}^*$ has the Radon-Nikodým density of the form*

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \mathcal{E}_T(\theta W)\mathcal{E}_T(\zeta M),$$

where the constants θ and ζ are given by

$$\theta = \sigma_1 + \frac{c}{a}, \quad \zeta = \kappa_1 > -1,$$

and where we write, for $t \in [0, T]$,

$$\mathcal{E}_t(\theta W) = \exp\Big(\theta W_t - \frac{1}{2}\theta^2 t\Big)$$

$$\mathcal{E}_t(\zeta M) = \big(1 + \mathbb{1}_{\{\tau \leq t\}}\zeta\big) \exp\big(- \zeta\gamma(t \wedge \tau)\big).$$

Existence of a Martingale Measure (Continued)

(ii) *If* $a \neq 0$ *and* $b = 0$ *then the model* $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ *is arbitrage-free and complete. Moreover, the process* (Y^1, Y^2, Y^3, H) *has the Markov property under* $\mathbb{Q}^*$*.*

(iii) *If* $a = 0$ *and* $b = 0$ *then a solution* (θ, ζ) *exists provided that* $c = 0$ *and the uniqueness of a martingale measure* $\mathbb{Q}^*$ *fails to hold. In this case, the model* $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ *is arbitrage-free, but it is not complete.*

(iv) *If* $b \neq 0$ *then a martingale measure fails to exist and consequently the model* $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ *is not arbitrage-free.*

Example A: Extension of the Black-Scholes Model

- Assume that the asset Y^1 is risk-free, the asset $Y^2 \neq Y^1$ is default-free, and Y^3 is a defaultable asset with non-zero recovery, so that

$$
\begin{aligned}
dY_t^1 &= rY_t^1\, dt, \\
dY_t^2 &= Y_t^2\big(\mu_2\, dt + \sigma_2\, dW_t\big), \\
dY_t^3 &= Y_{t-}^3\big(\mu_3\, dt + \sigma_3\, dW_t + \kappa_3\, dM_t\big).
\end{aligned}
$$

- We thus have $\sigma_1 = \kappa_1 = 0$, $\mu_1 = r$, $\sigma_2 \neq 0$, $\kappa_2 = 0$, and $\kappa_3 \neq 0$, $\kappa_3 > -1$.

- Therefore,

$$a = \sigma_2\kappa_3 \neq 0, \quad c = \kappa_3(r - \mu_2),$$

and the equality $b = 0$ holds if and only if

$$\sigma_2(r - \mu_3) = \sigma_3(r - \mu_2).$$

Example A (Continued)

- It is easy to check that

$$\theta = \frac{r - \mu_2}{\sigma_2}, \quad \zeta = 0,$$

and thus under the martingale measure $\mathbb{Q}^*$ we have (irrespective of whether $\sigma_3 > 0$ or $\sigma_3 = 0$)

$$\begin{aligned} dY_t^1 &= rY_t^1\, dt, \\ dY_t^2 &= Y_t^2\big(r\, dt + \sigma_2\, d\widehat{W}_t\big), \\ dY_t^3 &= Y_{t-}^3\big(r\, dt + \sigma_3\, d\widehat{W}_t + \kappa_3\, dM_t\big). \end{aligned}$$

- Since $\zeta = 0$ the risk-neutral default intensity $\widehat{\gamma}$ coincides here with the statistical default intensity γ. This implies the equality $\widehat{M} = M$.

Case B: Defaultable Asset with Zero Recovery

Case B: standing assumptions:

- We postulate that $\kappa_i > -1$ for $i = 1, 2$ and $\kappa_3 = -1$.
- This implies that the price of a defaultable asset Y^3 vanishes after τ, and thus the findings of the preceding section are no longer valid.

Martingale Condition

- Since Y^3 jumps to zero at τ, the first equality in the martingale condition

$$\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) + U_t(\kappa_2 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} = 0$$

  should still be satisfied for every $t \in [0, T]$.

- The second equality in the martingale condition

$$\mu_3 - \mu_1 + (\sigma_3 - \sigma_1)(\theta_t - \sigma_1) + U_t(\kappa_3 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} = 0$$

  is required to hold on the set $\{\tau > t\}$ only (i.e. when $U_t = \gamma$).

Martingale Condition

Lemma 2 *Under the present assumptions, the unknown processes θ and ζ in the Radon-Nikodým density of $\mathbb{Q}^*$ with respect to $\mathbb{Q}$ satisfy the following equations*

$$\begin{aligned}
\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) &= 0, \quad \textit{for } t > \tau, \\
\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) + \gamma(\kappa_2 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} &= 0, \quad \textit{for } t \leq \tau, \\
\mu_3 - \mu_1 + (\sigma_3 - \sigma_1)(\theta_t - \sigma_1) + \gamma(-1 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} &= 0, \quad \textit{for } t \leq \tau.
\end{aligned}$$

This leads to the following result.

Martingale Measure

Proposition 3 *The pair (θ, ζ) satisfies the following equations, for $t \leq \tau$,*

$$\theta_t a = \sigma_1 a + c, \quad \zeta_t \gamma a = \kappa_1 \gamma a - (1 + \kappa_1) b.$$

Moreover, for $t > \tau$,

$$\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) = 0.$$

Let $a \neq 0$, $\sigma_1 \neq \sigma_2$ and $\gamma > b/a$. Then the unique solution is

$$\theta_t = 1\!\!1_{\{t \leq \tau\}} \left(\sigma_1 + \frac{c}{a} \right) + 1\!\!1_{\{t > \tau\}} \left(\sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} \right), \ \zeta_t = \kappa_1 - \frac{(1 + \kappa_1) b}{\gamma a} > -1.$$

The model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is arbitrage-free, complete, and has the Markov property under the unique martingale measure $\mathbb{Q}^*$.

Example B : Extension of the Black-Scholes Model

- Assume that the asset Y^1 is risk-free, the asset $Y^2 \neq Y^1$ is default-free, and Y^3 is a defaultable asset with zero recovery, so that

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(\mu_2\,dt + \sigma_2\,dW_t\big), \\ dY_t^3 &= Y_{t-}^3\big(\mu_3\,dt + \sigma_3\,dW_t - dM_t\big). \end{aligned}$$

- This corresponds to the following conditions:

$$\sigma_1 = \kappa_1 = 0,\ \mu_1 = r,\ \sigma_2 \neq 0,\ \kappa_2 = 0,\ \kappa_3 = -1.$$

Hence $a = -\sigma_2 \neq 0$. Assume, in addition, that

$$\gamma > b/a = r - \mu_3 - \frac{\sigma_3}{\sigma_2}(r - \mu_2).$$

Example B (Continued)

- Then we obtain

$$\theta = \frac{r-\mu_2}{\sigma_2}, \quad \zeta = -\frac{b}{\gamma a} = \frac{1}{\gamma}\left(\mu_3 - r - \frac{\sigma_3}{\sigma_2}(\mu_2 - r)\right) > -1.$$

- Consequently, we have under the unique martingale measure $\mathbb{Q}^*$

$$\begin{aligned} dY_t^1 &= rY_t^1\, dt, \\ dY_t^2 &= Y_t^2\big(r\, dt + \sigma_2\, d\widehat{W}_t\big), \\ dY_t^3 &= Y_{t-}^3\big(r\, dt + \sigma_3\, d\widehat{W}_t - d\widehat{M}_t\big). \end{aligned}$$

- We do not assume here that $b = 0$; if this holds then $\zeta = 0$, as in Example A.

- In Case B, the risk-neutral default intensity $\widehat{\gamma}$ and the statistical default intensity γ are different, in general,

Stopped Trading

- Suppose that the recovery payoff at the time of default is exogenously specified in terms of some economic factors related to the prices of traded assets (e.g. credit spreads).
- The valuation problem for a defaultable claim is reduced to finding its pre-default value, and it is natural to search for a replicating strategy up to default time only.
- It thus suffices to examine the stopped model in which asset prices and all trading activities are stopped at time τ.
- In this case, we search for a pair (θ, ζ) of real numbers satisfying

$$\begin{aligned} \theta a &= \sigma_1 a + c, \\ \zeta \gamma a &= \kappa_1 \gamma a - (1 + \kappa_1) b. \end{aligned}$$

Case of Stopped Trading

- If $a \neq 0$ then the unique solution (θ, ζ) to the above pair of equations is

$$\theta = \sigma_1 + \frac{c}{a}, \quad \zeta = \kappa_1 - \frac{(1+\kappa_1)b}{\gamma a} > -1,$$

where the last inequality holds provided that $\gamma > b/a$.

- As expected, in the stopped model, we obtain the unique martingale measure $\mathbb{Q}^*$ for any choice of recovery coefficients κ_2 and κ_3.

- In the case of stopped trading, hedging of a contingent claim after the default time τ is not considered.

Case A: Pricing PDEs and Hedging

Pricing PDEs

Contingent Claim

Let us now discuss the PDE approach in a model in which the prices of all three primary assets are non-vanishing.

- It is natural to focus on the case when the market model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is complete and arbitrage-free.
- Therefore, we shall work under the assumptions of part (i) in the proposition on the existence of a martingale measure.
- We are interested in the valuation and hedging of a generic contingent claim with maturity T and the terminal payoff $Y = G(Y_T^1, Y_T^2, Y_T^3, H_T)$.
- The technique derived for this case can be easily applied to a defaultable claim that is subject to a fairly general recovery scheme.

Risk-Neutral Price

- Let $a \neq 0$ and $b = 0$, and let $\mathbb{Q}^*$ be the unique martingale measure associated with the numeraire Y^1. Then

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \mathcal{E}_T(\theta W)\mathcal{E}_T(\zeta M)$$

where θ and ζ are explicitly known.

- If $Y(Y_T^1)^{-1}$ is $\mathbb{Q}^*$-integrable then the risk-neutral price of Y equals, for every $t \in [0,T]$,

$$\begin{aligned} \pi_t(Y) &= Y_t^1 \, \mathbb{E}_{\mathbb{Q}^*}\big((Y_T^1)^{-1} Y \,\big|\, \mathcal{G}_t\big) \\ &= Y_t^1 \, \mathbb{E}_{\mathbb{Q}^*}\,\big((Y_T^1)^{-1} G(Y_T^1, Y_T^2, Y_T^3, H_T) \,\big|\, Y_t^1, Y_t^2, Y_t^3, H_t\big) \end{aligned}$$

where the second equality is a consequence of the Markov property of (Y^1, Y^2, Y^3, H) under $\mathbb{Q}^*$.

Pricing PDEs: Case A

Proposition 4 *Let the price processes* Y^i, $i = 1, 2, 3$ *satisfy*

$$dY_t^i = Y_{t-}^i \big(\mu_i \, dt + \sigma_i \, dW_t + \kappa_i \, dM_t \big)$$

with $\kappa_i > -1$ *for* $i = 1, 2, 3$. *Assume that* $a \neq 0$ *and* $b = 0$. *Then the risk-neutral price* $\pi_t(Y)$ *of the claim* Y *equals*

$$\pi_t(Y) = \mathbb{1}_{\{t<\tau\}} C(t, Y_t^1, Y_t^2, Y_t^3, 0) + \mathbb{1}_{\{t \geq \tau\}} C(t, Y_t^1, Y_t^2, Y_t^3, 1)$$

for some function

$$C : [0, T] \times \mathbb{R}_+^3 \times \{0, 1\} \to \mathbb{R}.$$

Assume that for $h = 0$ *and* $h = 1$ *the function* $C(\cdot, h) : [0, T] \times \mathbb{R}_+^3 \to \mathbb{R}$ *belongs to the class* $\mathrm{C}^{1,2}([0, T] \times \mathbb{R}_+^3, \mathbb{R})$.

Pricing PDEs: Case A

Then the functions $C(\cdot,0)$ *and* $C(\cdot,1)$ *solve the following PDEs:*

$$\partial_t C(\cdot,0) + \sum_{i=1}^{3} (\alpha - \gamma\kappa_i) y_i \partial_i C(\cdot,0) + \frac{1}{2}\sum_{i,j=1}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C(\cdot,0) - \alpha C(\cdot,0)$$
$$+ \gamma\big[C(t, y_1(1+\kappa_1), y_2(1+\kappa_2), y_3(1+\kappa_3), 1) - C(t, y_1, y_2, y_3, 0)\big] = 0$$

and

$$\partial_t C(\cdot,1) + \alpha \sum_{i=1}^{3} y_i \partial_i C(\cdot,1) + \frac{1}{2}\sum_{i,j=1}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C(\cdot,1) - \alpha C(\cdot,1) = 0$$

where $\alpha = \mu_i + \sigma_i \frac{c}{a}$, *subject to the terminal conditions*

$$C(T, y_1, y_2, y_3, 0) = G(y_1, y_2, y_3, 0),\ \ C(T, y_1, y_2, y_3, 1) = G(y_1, y_2, y_3, 1).$$

Comments

- The valuation problem splits into two pricing PDEs, which are solved recursively.
  - In the first step, we solve the PDE satisfied by the post-default pricing function $C(\cdot, 1)$.
  - Next, we substitute this function into the first PDE, and we solve it for the pre-default pricing function $C(\cdot, 0)$.
- The assumption that we deal with only three primary assets and the coefficients are constant can be easily relaxed, but a general result is too heavy to be stated here.
- Observe that the real-world default intensity γ under $\mathbb{Q}$, rather than the risk-neutral default intensity $\widehat{\gamma}$ under $\mathbb{Q}^*$, enters the valuation PDE.

Black and Scholes PDE

- We consider the set-up of Example A, with $a \neq 0$ and $b = 0$.
- Let $Y = G(Y_T^2)$ for some function $G : \mathbb{R} \to \mathbb{R}$ such that $Y(Y_T^1)^{-1}$ is $\mathbb{Q}^*$-integrable.
- It is possible to show that $\pi_t(Y) = C(t, Y_t^2)$.
- The two valuation PDEs of Proposition A2 reduce to a single PDE

$$\partial_t C + (\mu_2 - \sigma_2\theta) y_2 \partial_2 C + \frac{1}{2}\sigma_2^2 y_2^2 \partial_{22} C - (\mu_2 - \sigma_2\theta) C = 0$$

  with $\theta = (r - \mu_2)/\sigma_2$.
- After simplifications, we obtain the classic Black and Scholes PDE

$$\partial_t C + r y_2 \partial_2 C + \frac{1}{2}\sigma_2^2 y_2^2 \partial_{22} C - rC = 0.$$

Trading Strategies

- Recall that $\phi = (\phi^1, \phi^2, \phi^3)$ is a self-financing strategy if the processes ϕ^1, ϕ^2, ϕ^3 are $\mathbb{G}$-predictable and the wealth process

$$V_t(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3$$

satisfies

$$dV_t(\phi) = \phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2 + \phi_t^3 \, dY_t^3.$$

- We say that ϕ replicates a contingent claim Y if $V_T(\phi) = Y$. If ϕ is a replicating strategy for a claim Y then, for $t \in [0, T]$,

$$\pi_t(Y) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3.$$

- To find a replicating strategy, we combine the sensitivities of the valuation function C with respect to primary assets with the jump $\Delta C_t = C_t - C_{t-}$ associated with default event.

Hedging with Sensitivities and Jumps

Proposition 5 *Under the present the assumptions, the claim* $G(Y_T^1, Y_T^2, Y_T^3, H_T)$ *is replicated by* $\phi = (\phi^1, \phi^2, \phi^3)$*, where the components* ϕ^i*,* $i = 2, 3$*, are given in terms of the valuation functions* $C(\cdot, 0)$ *and* $C(\cdot, 1)$*:*

$$\phi_t^2 = \frac{1}{aY_{t-}^2}\left((\kappa_3 - \kappa_1)\Big(\sum_{i=1}^{3} \sigma_i Y_{t-}^i \partial_i C - \sigma_1 C\Big) - (\sigma_3 - \sigma_1)(\Delta C - \kappa_1 C)\right)$$

$$\phi_t^3 = \frac{1}{aY_{t-}^3}\left((\kappa_2 - \kappa_1)\Big(\sum_{i=1}^{3} \sigma_i Y_{t-}^i \partial_i C - \sigma_1 C\Big) - (\sigma_2 - \sigma_1)(\Delta C - \kappa_1 C)\right)$$

and ϕ^1 *equals*

$$\phi_t^1 = (Y_t^1)^{-1}\Big(C_t - \sum_{i=2}^{3} \phi_t^i Y_t^i\Big).$$

Example A: Extension of the Black-Scholes Model

- Assume that the asset Y^1 is risk-free, the asset $Y^2 \neq Y^1$ is default-free, and Y^3 is a defaultable asset with non-zero recovery, so that

$$\begin{aligned} dY_t^1 &= rY_t^1\, dt, \\ dY_t^2 &= Y_t^2 \big(\mu_2\, dt + \sigma_2\, dW_t \big), \\ dY_t^3 &= Y_{t-}^3 \big(\mu_3\, dt + \sigma_3\, dW_t + \kappa_3\, dM_t \big) \end{aligned}$$

  with $\sigma_2 \neq 0$ and $\kappa_3 \neq 0, \kappa_3 > -1$.

- We may assume, without loss of generality, that C does not depend explicitly on the variable y_1.

- Assume that $a = \sigma_2 \kappa_3 \neq 0$ and $\sigma_2(r - \mu_3) = \sigma_3(r - \mu_2)$. The following result combines and adapts previous results to the present situation.

Example A: Pricing PDEs

Corollary 1 *The arbitrage price of a claim* $Y = G(Y_T^2, Y_T^3, H_T)$ *can be represented as* $\pi_t(Y) = C(t, Y_t^2, Y_t^3, H_t)$, *where* $C(t, y_2, y_3, 0)$ *satisfies*

$$\partial_t C(\cdot,0) + ry_2\partial_2 C(\cdot,0) + y_3(r-\kappa_3\gamma)\partial_3 C(\cdot,0) - rC(\cdot,0)$$
$$+\frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij}C(\cdot,0) + \gamma\big(C(t,y_2,y_3(1+\kappa_3),1) - C(t,y_2,y_3,0)\big) = 0$$

with $C(T, y_2, y_3, 0) = G(y_2, y_3, 0)$, *and* $C(t, y_2, y_3, 1)$ *satisfies*

$$\partial_t C(t,y_2,y_3,1) + ry_2\partial_2 C(t,y_2,y_3,1) + ry_3\partial_3 C(t,y_2,y_3,1) - rC(t,y_2,y_3,1)$$
$$+\frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij}C(t,y_2,y_3,1) = 0$$

with $C(T, y_2, y_3, 1) = G(y_2, y_3, 1)$.

Example A: Hedging

Corollary 2 *The replicating strategy for Y equals $\phi = (\phi^1, \phi^2, \phi^3)$, where*

$$
\begin{aligned}
\phi_t^1 &= (Y_t^1)^{-1}\left(C_t - \sum_{i=2}^{3} \phi_t^i Y_t^i\right), \\
\phi_t^2 &= \frac{1}{\sigma_2 \kappa_3 Y_{t-}^2}\Bigg(\kappa_3 \sum_{i=2}^{3} \sigma_i y_i \partial_i C(t, Y_{t-}^2, Y_{t-}^3, H_{t-}) \\
&\quad - \sigma_3 \big(C(t, Y_{t-}^2, Y_{t-}^3(1+\kappa_3), 1) - C(t, Y_{t-}^2, Y_{t-}^3, 0)\big)\Bigg), \\
\phi_t^3 &= \frac{1}{\kappa_3 Y_{t-}^3}\big(C(t, Y_{t-}^2, Y_{t-}^3(1+\kappa_3), 1) - C(t, Y_{t-}^2, Y_{t-}^3, 0)\big).
\end{aligned}
$$

Example A: Survival Claim

- By a survival claim we mean a claim of the form $Y = \mathbb{1}_{\{\tau > T\}} X$, where an $\mathcal{F}_T$-measurable random variable X represents the promised payoff.

- In other words, a survival claim is a contract with zero recovery in the case of default prior to maturity T.

- We assume that the promised payoff has the form $X = G(Y_T^2, Y_T^3)$, where Y_T^i is the (pre-default) value of the ith asset at time T.

- It is obvious that the pricing function $C(\cdot, 1)$ is now equal to zero, and thus we are only interested in the pre-default pricing function $C(\cdot, 0)$.

Example A: Survival Claim

Corollary 3 *The pre-default pricing function $C(\cdot,0)$ of a survival claim of the form $Y = \mathbb{1}_{\{\tau > T\}} G(Y_T^2, Y_T^3)$ solves the PDE*

$$\partial_t C(\cdot,0) + r y_2 \partial_2 C(\cdot,0) + y_3(r - \kappa_3\gamma)\partial_3 C(\cdot,0) + \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,0) - (r+\gamma)C(\cdot,0) = 0$$

with $C(T, y_2, y_3, 0) = G(y_2, y_3)$. The components ϕ^2 and ϕ^3 of a replicating strategy ϕ are given by the following expressions

$$\phi_t^2 = \frac{1}{\kappa_3 \sigma_2 Y_{t-}^2}\Big(\kappa_3 \sum_{i=2}^{3} \sigma_i Y_{t-}^i \partial_i C(\cdot,0) - \sigma_3 C(\cdot,0)\Big), \quad \phi_t^3 = -\frac{C(\cdot,0)}{\kappa_3 Y_{t-}^3}.$$

Case B: Pricing PDEs and Hedging

Pricing PDEs

Case B: Defaultable Asset with Zero Recovery

Standing assumptions:

- We now assume that the prices Y^1 and Y^2 are strictly positive, but $\kappa_3 = -1$ so that Y^3 is a defaultable asset with zero recovery.
- Of course, the price Y_t^3 vanishes after default, that is, on the set $\{t \geq \tau\}$.
- We assume here that $a \neq 0$ and $\sigma_1 \neq \sigma_2$, but we no longer postulate that $b = 0$.
- We still assume that $\gamma > b/a$, however. Let us denote

$$\alpha_i = \mu_i + \sigma_i \frac{c}{a}, \quad \beta_i = \mu_i - \sigma_i \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}.$$

Valuation PDEs: Case B

Proposition 6 *Let the price processes* Y^i, $i = 1, 2, 3$, *satisfy*

$$dY_t^i = Y_{t-}^i \big(\mu_i dt + \sigma_i \, dW_t + \kappa_i \, dM_t \big)$$

with $\kappa_i > -1$ *for* $i = 1, 2$ *and* $\kappa_3 = -1$. *Assume that*

$$a \neq 0, \;\; \sigma_1 \neq \sigma_2, \;\; \gamma > b/a.$$

Consider a contingent claim Y *with maturity date* T *and the terminal payoff* $G(Y_T^1, Y_T^2, Y_T^3, H_T)$.

In addition, we postulate that the pricing functions $C(\cdot, 0)$ *and* $C(\cdot, 1)$ *belong to the class* $\mathrm{C}^{1,2}([0, T] \times \mathbb{R}_+^3, \mathbb{R})$.

Pricing PDEs: Case B

Proposition 7 *Then the pre-default pricing function* $C(t, y_1, y_2, y_3, 0)$ *satisfies the* pre-default PDE

$$\partial_t C(\cdot,0) + \sum_{i=1}^{3}(\alpha_i - \gamma\kappa_i)y_i\partial_i C(\cdot,0) + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij}C(\cdot,0)$$
$$+\left(\gamma - \frac{b}{a}\right)\left[C(t, y_1(1+\kappa_1), y_2(1+\kappa_2), 0, 1) - C(t, y_1, y_2, y_3, 0)\right]$$
$$-\left(\alpha_1 + \kappa_1\frac{b}{a}\right)C(\cdot,0) = 0$$

subject to the terminal condition

$$C(T, y_1, y_2, y_3, 0) = G(y_1, y_2, y_3, 0).$$

Pricing PDEs: Case B

Proposition 8 *The post-default pricing function $C(t, y_1, y_2, 1)$ solves the* post-default PDE

$$\partial_t C(\cdot, 1) + \sum_{i=1}^{2} \beta_i y_i \partial_i C(\cdot, 1) + \frac{1}{2} \sum_{i,j=1}^{2} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot, 1) - \beta_1 C(\cdot, 1) = 0$$

subject to the terminal condition

$$C(T, y_1, y_2, 1) = G(y_1, y_2, 0, 1).$$

The components of the replicating strategy ϕ are given by the general formulae.

Example B (Continued)

- We assume that the processes Y^1, Y^2, Y^3 satisfy

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(\mu_2\,dt + \sigma_2\,dW_t\big), \\ dY_t^3 &= Y_{t-}^3\big(\mu_3\,dt + \sigma_3\,dW_t - dM_t\big). \end{aligned}$$

- Let us write $\widehat{r} = r + \widehat{\gamma}$, where

$$\widehat{\gamma} = \gamma(1+\zeta) = \gamma - \frac{b}{a} = \gamma + \mu_3 - r + \frac{\sigma_3}{\sigma_2}(r - \mu_2) > 0$$

  stands for the default intensity under $\mathbb{Q}^*$.

- The quantity $\widehat{r}$ is interpreted as the credit-risk adjusted short-term rate.

- Straightforward calculations show that the following corollary is valid.

Example B: Pricing PDEs

Corollary 4 *Assume that* $\sigma_1 = \kappa_1 = \kappa_2 = 0$, $\kappa_3 = -1$ *and*

$$\gamma > b/a = r - \mu_3 - \frac{\sigma_3}{\sigma_2}(r - \mu_2).$$

Then $C(\cdot, 0)$ *satisfies the PDE*

$$\partial_t C(t, y_2, y_3, 0) + r y_2 \partial_2 C(t, y_2, y_3, 0) + \widehat{r} y_3 \partial_3 C(t, y_2, y_3, 0) - \widehat{r} C(t, y_2, y_3, 0)$$
$$+ \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3, 0) + \widehat{\gamma} C(t, y_2, 1) = 0,$$

with $C(T, y_2, y_3, 0) = G(y_2, y_3, 0)$, *and the function* $C(\cdot, 1)$ *solves*

$$\partial_t C(t, y_2, 1) + r y_2 \partial_2 C(t, y_2, 1) + \frac{1}{2} \sigma_2^2 y_2^2 \partial_{22} C(t, y_2, 1) - r C(t, y_2, 1) = 0,$$

with $C(T, y_2, 1) = G(y_2, 0, 1)$.

Example B: Survival Claim

For a survival claim, we have $C(\cdot, 1) = 0$, and thus we obtain following results.

Corollary 5 *The pre-default pricing function $C(\cdot, 0)$ of a survival claim $Y = \mathbb{1}_{\{\tau > T\}} G(Y_T^2, Y_T^3)$ solves the following PDE:*

$$\partial_t C(t, y_2, y_3, 0) + r y_2 \partial_2 C(t, y_2, y_3, 0) + \widehat{r} y_3 \partial_3 C(t, y_2, y_3, 0) + \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3, 0) - \widehat{r} C(t, y_2, y_3, 0) = 0$$

with the terminal condition $C(T, y_2, y_3, 0) = G(y_2, y_3)$.

Corollary B2 (Continued)

Corollary 6 *The components ϕ^2 and ϕ^3 of the replicating strategy are, for every $t < \tau$,*

$$\begin{aligned}
\phi_t^2 &= \frac{1}{\sigma_2 Y_{t-}^2} \Big(\sum_{i=2}^{3} \sigma_i Y_{t-}^i \partial_i C(t, Y_{t-}^2, Y_{t-}^3, 0) + \sigma_3 C(t, Y_{t-}^2, Y_{t-}^3, 0) \Big), \\
\phi_t^3 &= \frac{1}{Y_{t-}^3} C(t, Y_{t-}^2, Y_{t-}^3, 0).
\end{aligned}$$

- We have $\phi_t^3 Y_{t-}^3 = C(t, Y_{t-}^2, Y_{t-}^3, 0)$ for every $t \in [0, T]$. Hence the following relationships holds, for every $t < \tau$,

$$\phi_t^3 Y_t^3 = C(t, Y_t^2, Y_t^3, 0), \quad \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0.$$

- The last equality is a special case of a balance condition introduced in Bielecki et al. (2006) in a semimartingale set-up.

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