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Lévy processes

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Lévy Processes

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Some books

Barndorff-Nielsen, O.E. and Shephard, N., Financial volatility and Lévy based models, Forthcoming Cambridge University Press (CUP), 2008

Bertoin, J., Lévy Processes, CUP, 1996

Cont, R. and Tankov, P., Financial modeling with jump processes, Chapman & Hall/CRC, 2004

Jeanblanc, M. Yor, M. and Chesney, M., Mathematical Methods for Financial Markets, Forthcoming 2008, Springer

Kyprianou, A.E., Introductory Lectures on Fluctuations of Lévy Processes with Applications, Springer, 2006

Overhaus, M. et al. (Deutsche Bank team) Equity derivatives, Theory and applications, Wiley, 2002

Sato, K., Lévy processes and infinitely divisible distributions, CUP, 1999

Schoutens, W., Lévy Processes in Finance, Pricing Financial Derivatives, Wiley, 2003

Give us the tools, and we will finish the work.
Winston Churchill, February 9, 1941.

Definition and Main Properties of Lévy Processes

Definition

An \mathbb{R}^d -valued process X such that $X_0 = 0$ is a **Lévy process** if

- a) for every s, t , $X_{t+s} - X_t$ is independent of \mathcal{F}_t^X
- b) for every s, t the r.v.'s $X_{t+s} - X_t$ and X_s have the same law.
- c) X is continuous in probability, i.e., $\mathbb{P}(|X_t - X_s| > \epsilon) \rightarrow 0$ when $s \rightarrow t$ for every $\epsilon > 0$.

The sum of two independent Lévy processes is a Lévy process.

Property c) implies that a Lévy process has no jumps at fixed time.

A Lévy process admits a càdlàg modification

(A process Y is said to be a modification of X is $\mathbb{P}(X_t = Y_t) = 1, \forall t$)

Let $T > 0$ be fixed. For any $\epsilon > 0$, the set $\{t \in [0, T] : |\Delta X_t| > \epsilon\}$ is finite

The set $\{t \in [0, T] : |\Delta X_t| > 0\}$ is countable

Examples

- **Brownian motion**

The standard Brownian motion is a process W with **continuous paths** such that

- for every s, t , $W_{t+s} - W_t$ is independent of \mathcal{F}_t^W ,
- for every s, t , the r.v. $W_{t+s} - W_t$ has the same law as W_s .

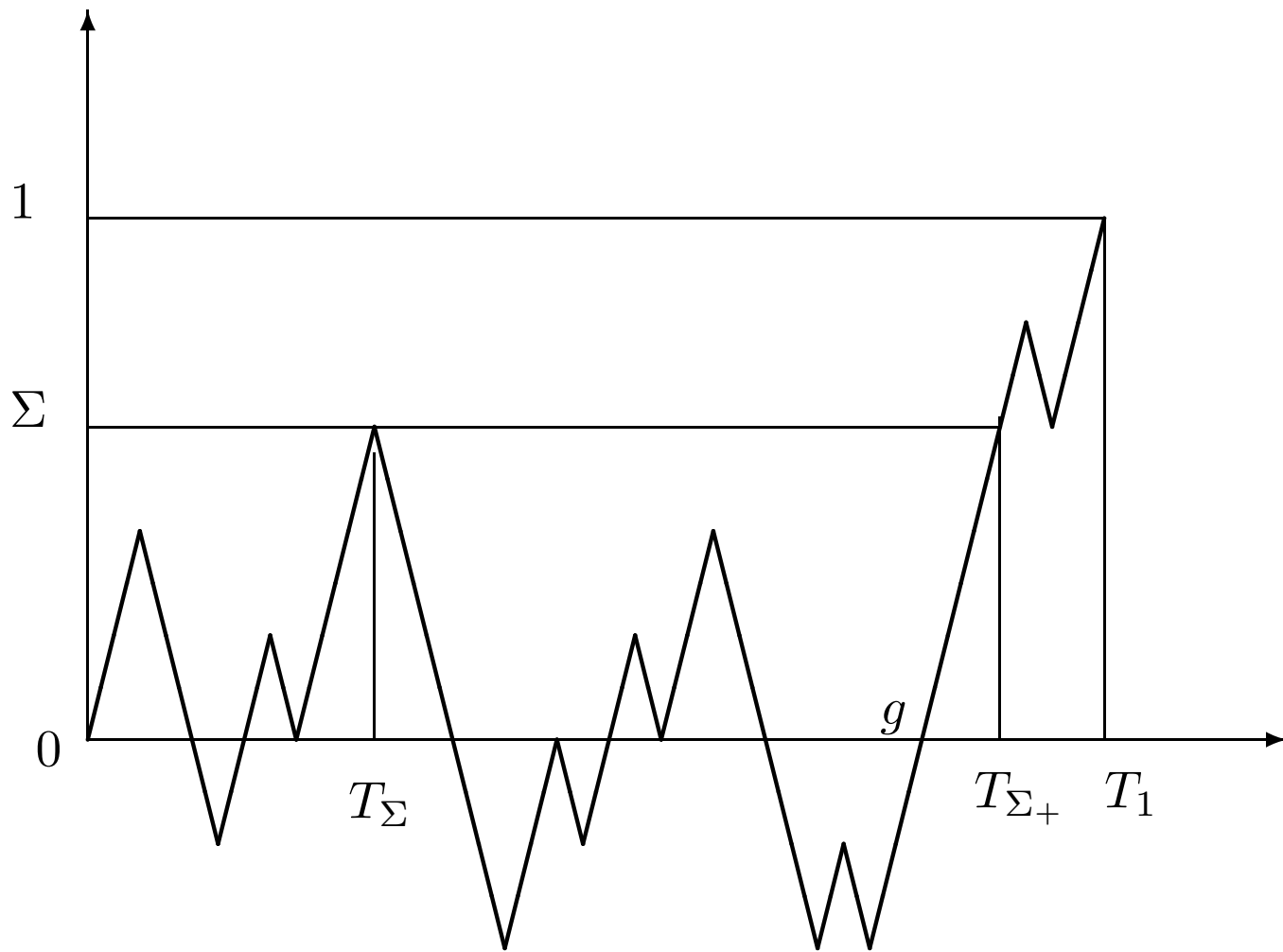
The law of W_s is $\mathcal{N}(0, s)$.

- **Brownian hitting time process**

Let W be a standard BM and for $a > 0$, define

$$T_a := \inf\{t : W_t = a\}$$

The process $(T_a, a \geq 0)$ is a Lévy process.



Non continuity of T_a

• Poisson process

The standard Poisson process is a **counting process** such that

- for every s, t , $N_{t+s} - N_t$ is independent of \mathcal{F}_t^N ,
- for every s, t , the r.v. $N_{t+s} - N_t$ has the same law as N_s .

Then, the r.v. N_t has a Poisson law with parameter λt

$$\mathbb{P}(N_t = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

If X has a Poisson law with parameter $\theta > 0$, then

(i) for any $s \in \mathbb{R}$, $\mathbb{E}[s^X] = e^{\theta(s-1)}$.

(ii) $\mathbb{E}[X] = \theta$, $\text{Var}(X) = \theta$.

(iii) for any $u \in \mathbb{R}$, $\mathbb{E}(e^{iuX}) = \exp(\theta(e^{iu} - 1))$

(iv) for any $\alpha \in \mathbb{R}$, $\mathbb{E}(e^{\alpha X}) = \exp(\theta(e^\alpha - 1))$

• Compound Poisson Process

Let λ be a positive number and $F(dy)$ be a probability law on \mathbb{R} (we assume that $\mathbb{P}(Y_1 = 0) = 0$). A (λ, F) -**compound Poisson process** is a process $X = (X_t, t \geq 0)$ of the form

$$X_t = \sum_{k=1}^{N_t} Y_k$$

where N is a Poisson process with intensity $\lambda > 0$ and the $(Y_k, k \in \mathbb{N})$ are i.i.d. random variables, independent of N , with law $F(dy) = \mathbb{P}(Y_1 \in dy)$.

If $\mathbb{E}(|Y_1|) < \infty$, for any t , $\mathbb{E}(X_t) = \lambda t \mathbb{E}(Y_1)$.

The characteristic function of the r.v. X_t is

$$\mathbb{E}[e^{iuX_t}] = e^{\lambda t(\mathbb{E}[e^{iuY_1}] - 1)} = \exp\left(\lambda t \int_{\mathbb{R}} (e^{iuy} - 1)F(dy)\right).$$

Assume that $\mathbb{E}[e^{\alpha Y_1}] < \infty$. Then, the Laplace transform of the r.v. X_t is

$$\mathbb{E}[e^{\alpha X_t}] = e^{\lambda t(\mathbb{E}[e^{\alpha Y_1}] - 1)} = \exp\left(\lambda t \int_{\mathbb{R}} (e^{\alpha y} - 1)F(dy)\right).$$

We shall note $\nu(dy) = \lambda F(dy)$ and say that X is a ν -compound Poisson process.

Martingales

Let X be a Lévy process.

- If $\mathbb{E}(|X_t|) < \infty$, the process $X_t - \mathbb{E}(X_t)$ is a martingale.
- For any u , the process $Z_t(u) := \frac{e^{iuX_t}}{\mathbb{E}(e^{iuX_t})}$ is a martingale.
- If $\mathbb{E}(e^{\lambda X_t}) < \infty$, the process $\frac{e^{\lambda X_t}}{\mathbb{E}(e^{\lambda X_t})}$ is a martingale

Examples

- **Brownian motion**

The standard Brownian motion is a martingale, the process $Y = \mathcal{E}(\lambda W)$ defined by $Y_t = e^{\lambda W_t - \frac{1}{2}\lambda^2 t}$ is a martingale.

The Doléans-Dade exponential $Y = \mathcal{E}(\lambda W)$ satisfies

$$dY_t = Y_t \lambda dW_t$$

- **Poisson process**

The process $M_t = N_t - \lambda t$ is a martingale.

For any α , the process

$$\exp(\alpha N_t - \lambda t(e^\alpha - 1)) = \exp(\alpha M_t - \lambda t(e^\alpha - 1 - \alpha)) = \mathcal{E}(\alpha M)_t$$

is a martingale.

For any β , the process $(1 + \beta)^{N_t} e^{-\lambda \beta t}$ is a martingale

If X is a counting process and if, for some λ the process $M_t = N_t - \lambda t$ is a martingale, then X is a Poisson process

If X is a counting process with stationary and independent increments, then X is a Poisson process.

• Compound Poisson Processes

Assume that $\mathbb{E}(|Y_1|) < \infty$. Then, the process

$(Z_t := X_t - t\lambda\mathbb{E}(Y_1), t \geq 0)$ is a martingale and in particular,

$$\mathbb{E}(X_t) = \lambda t \mathbb{E}(Y_1) = \lambda t \int_{-\infty}^{\infty} y F(dy) = t \int_{-\infty}^{\infty} y \nu(dy)$$

For any $\alpha \in \mathbb{R}$ such that $\int_{-\infty}^{\infty} |e^{\alpha x} - 1| F(dx) < \infty$, the process

$$\exp\left(\alpha X_t - t\lambda \int_{-\infty}^{\infty} (e^{\alpha x} - 1) F(dx)\right) = \exp\left(\alpha X_t - t \int_{-\infty}^{\infty} (e^{\alpha x} - 1) \nu(dx)\right)$$

is a martingale.

Random Measures

- **Counting process:** Let (T_n) be a sequence of random times, with

$$0 < T_1 < \cdots < T_n \dots$$

and $N_t = \sum_{n \geq 1} \mathbb{1}_{T_n \leq t}$. Let A be a Borel set in \mathbb{R}^+ and

$$\mathbf{N}(\omega; A) := \text{Card} \{n \geq 1 : T_n(\omega) \in A\}$$

The measure \mathbf{N} is a random measure and $N_t(\omega) = \mathbf{N}(\omega,]0, t])$.

For a Poisson process

$$\mathbb{E}(\mathbf{N}(A)) = \lambda \text{Leb}(A)$$

Let ν be a radon measure on E . A **random Poisson measure** \mathbf{N} on E with intensity ν is a measure such that

- $\mathbf{N}(A)$ is an integer valued random measure,
- $\mathbf{N}(A) < \infty$ for A bounded Borel set,
- for disjoint sets A_i , the r.v.'s $\mathbf{N}(A_i)$ are independent
- the r.v. $\mathbf{N}(A)$ is Poisson distributed with parameter $\nu(A)$

• **Compound Poisson process:** Define $\mathbf{N} = \sum_n \delta_{T_n, Y_n}$ on $\mathbb{R}^+ \times \mathbb{R}$,
i.e

$$\mathbf{N}(\omega, [0, t] \times A) = \sum_{n=1}^{N_t(\omega)} \mathbb{1}_{Y_n(\omega) \in A}.$$

We shall also write $\mathbf{N}_t(dx) = \mathbf{N}([0, t], dx)$. The measure \mathbf{N} is a random Poisson measure on $\mathbb{R}^+ \times \mathbb{R}$ with intensity $\lambda dt F(dx)$

We denote by $(f * \mathbf{N})_t$ the integral

$$\int_0^t \int_{\mathbb{R}} f(x) \mathbf{N}(ds, dx) = \int_{\mathbb{R}} f(x) \mathbf{N}_t(dx) = \sum_{k=1}^{N_t} f(Y_k) = \sum_{s \leq t} f(\Delta X_s) \mathbb{1}_{\Delta X_s \neq 0}.$$

In particular

$$X_t = \sum_{k=1}^{N_t} Y_k = \sum_{s \leq t} \Delta X_s = \int_0^t \int_{\mathbb{R}} x \mathbf{N}(ds, dx)$$

If $\nu(|f|) < \infty$, the process

$$\begin{aligned} M_t^f &: = (f * \mathbf{N})_t - t\nu(f) = \int_0^t \int_{\mathbb{R}} f(x)(\mathbf{N}(ds, dx) - ds\nu(dx)) \\ &= \sum_{s \leq t} f(\Delta X_s) \mathbb{1}_{\Delta X_s \neq 0} - t\nu(f) \end{aligned}$$

is a martingale.

PROOF: Indeed, the process $Z_t = \sum_{k=1}^{N_t} f(Y_k)$ is a $\hat{\nu}$ compound Poisson process, where $\hat{\nu}$, defined as

$$\hat{\nu}(A) = \lambda \mathbb{P}(f(Y_n) \in A)$$

is the image of ν by f . Hence, if $\mathbb{E}(|f(Y_1)|) < \infty$, the process $Z_t - t\lambda\mathbb{E}(f(Y_1)) = Z_t - t \int f(x)\nu(dx)$ is a martingale.

Using again that Z is a compound Poisson process, it follows that the process

$$\begin{aligned} & \exp \left(\sum_{k=1}^{N_t} f(Y_k) - t \int_{-\infty}^{\infty} (e^{f(x)} - 1) \nu(dx) \right) \\ &= \exp \left(\int_0^t \int_{\mathbb{R}} f(x) \mathbf{N}(ds, dx) - t \int_{-\infty}^{\infty} (e^{f(x)} - 1) \nu(dx) \right) \end{aligned}$$

is a martingale

If X is a pure jump process, if there exists λ and a probability measure σ such that $\sum_{s \leq t} f(\Delta X_s) \mathbb{1}_{\Delta X_s \neq 0} - t\lambda\sigma(f)$ is a martingale, then X is a compound Poisson process.

• Lévy Processes

The random variable $\mathbf{N}([s, t] \times A) = \sum_{s < u \leq t} \mathbb{1}_A(\Delta X_u)$ represents the number of jumps in the time interval $]s, t]$ with jump size in A .

We define ν by

$$\nu(A) = \mathbb{E}(\mathbf{N}([0, 1] \times A))$$

For A compact set such that $0 \notin A$, $\nu(A) < \infty$

The process

$$\mathbf{N}_t^A = \sum_{0 < s \leq t} \mathbb{1}_A(\Delta X_s) = \mathbf{N}([0, t] \times A)$$

is a Poisson process with intensity $\nu(A)$.

The processes \mathbf{N}^A and \mathbf{N}^C are independent if $\nu(A \cap C) = 0$, in particular if A and C are disjoint.

Let A be a Borel set of \mathbb{R}^d with $0 \notin \bar{A}$, and f a Borel function defined on A . We have

$$\int_A f(x) \mathbf{N}_t(\omega, dx) = \int_0^t \int_A f(x) \mathbf{N}(\omega, ds, dx) = \sum_{0 < s \leq t} f(\Delta X_s(\omega)) \mathbb{1}_A(\Delta X_s(\omega)).$$

The process

$$\int_A f(x) \mathbf{N}_t(\omega, dx)$$

is a Lévy process; if $\int_A |f(x)| \nu(dx) < \infty$, then

$$M_t^f = \int_A f(x) \mathbf{N}_t(\omega, dx) - t \int_A f(x) \nu(dx) = \int_0^t \int_A f(x) (\mathbf{N}(ds, dx) - \nu(dx) ds)$$

is a martingale.

If f is bounded and vanishes in a neighborhood of 0,

$$\mathbb{E} \left(\sum_{0 < s \leq t} f(\Delta X_s) \right) = t \int_{\mathbb{R}^d} f(x) \nu(dx).$$

The measure ν satisfies

$$\int (1 \wedge |x|^2) \nu(dx) < \infty$$

i.e. $\int_{|x| \geq 1} \nu(dx) < \infty$ and $\int_{|x| > 1} |x|^2 \nu(dx) < \infty$

Infinitely Divisible Random Variables

Definition

A random variable X taking values in \mathbb{R}^d is **infinitely divisible** if its characteristic function satisfies

$$\hat{\mu}(u) = \mathbb{E}(e^{i(u \cdot X)}) = (\hat{\mu}_n(u))^n$$

where $\hat{\mu}_n$ is a characteristic function.

Examples: The **Gaussian law** $\mathcal{N}(m, \sigma^2)$ has the characteristic function $\exp(ium - u^2\sigma^2/2)$.

Cauchy laws. The standard Cauchy law has the characteristic function $\exp(-c|u|)$.

The **hitting time of the level a for a Brownian motion** has Laplace transform

$$\mathbb{E}[\exp(-\frac{\lambda^2}{2}T_a)] = \exp(-|a|\lambda)$$

Poisson laws. The Poisson law with parameter λ has characteristic function

$$\exp(c(e^{iu} - 1))$$

Poisson Random Sum. Let X_i i.i.d. r.v.'s with characteristic function φ and N a r.v. independent of the X_i 's with a Poisson law. Let

$$X = X_1 + X_2 + \cdots + X_N$$

The characteristic function of X is

$$\exp(-\lambda(1 - \varphi(u)))$$

Gamma laws. The Gamma law $\Gamma(a, \nu)$ has density

$$\frac{\nu^a}{\Gamma(a)} x^{a-1} e^{-\nu x} \mathbb{1}_{x>0}$$

and characteristic function

$$(1 - iu/\nu)^{-a}$$

A **Lévy measure** ν is a positive measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{\mathbb{R}^d \setminus \{0\}} \min(1, \|x\|^2) \nu(dx) < \infty.$$

Lévy-Khintchine representation.

If X is an infinitely divisible random variable, there exists a triple (m, A, ν) where $m \in \mathbb{R}^d$, A is a non-negative quadratic form and ν is a Lévy measure such that

$$\hat{\mu}(u) = \exp \left(i(u \cdot m) - \frac{1}{2}(u \cdot Au) + \int_{\mathbb{R}^d} (e^{i(u \cdot x)} - 1 - i(u \cdot x) \mathbb{1}_{\{|x| \leq 1\}}) \nu(dx) \right).$$

The triple (m, A, ν) is called the *characteristic triple*.

If $\int |x| \mathbb{1}_{\{|x| \leq 1\}} \nu(dx) < \infty$, one writes the LK representation in a *reduced form*

$$\hat{\mu}(u) = \exp \left(i(u \cdot m_0) - \frac{1}{2}(u \cdot Au) + \int_{\mathbb{R}^d} (e^{i(u \cdot x)} - 1) \nu(dx) \right).$$

A first step is to prove that any function φ such that

$$\varphi(u) = \exp \left(i(u \cdot m) - \frac{1}{2}(u \cdot Au) + \int_{\mathbb{R}^d} (e^{i(u \cdot x)} - 1 - i(u \cdot x) \mathbb{1}_{\{|x| \leq 1\}}) \nu(dx) \right) \quad (*)$$

is a characteristic function (hence, i.d.).

If φ satisfies (*), one proves that

- it is continuous at 0
- it is the limit of characteristic functions.

The result follows.

Continuity: show that

$$\psi(u) = \int (e^{iux} - 1 - iux \mathbb{1}_{\{|x| \leq 1\}}) \nu(dx)$$

is continuous.

$$\psi(u) = \int_{|x| \leq 1} (e^{iux} - 1 - iux) \nu(dx) + \int_{|x| > 1} (e^{iux} - 1) \nu(dx)$$

Then, using the fact that

$$|e^{iux} - 1 - iux| \leq \frac{1}{2} u^2 x^2$$

the result is obtained

Limit:

$$\int_{|x| \leq 1} (e^{iux} - 1 - iux)\nu(dx) = \lim \int_{|x| \geq 1/n} (e^{iux} - 1 - iux)\nu(dx)$$

The right-hand side corresponds to compound Poisson process

If $\hat{\mu}$ is i.d., then it satisfies LK.

If $\hat{\mu}$ is i.d., then $\hat{\mu}(u)$ does not vanish.

Then,

$$\hat{\mu}(u) = (\hat{\mu}_n(u))^n$$

implies that

$$\Phi_n(u) := \exp(n(\hat{\mu}_n(u) - 1)) = \exp(n(e^{\frac{1}{n} \ln \hat{\mu}(u)} - 1))$$

converges to $\hat{\mu}(u)$.

$$\Phi_n(u) = \exp\left(n \int (e^{iux} - 1) \mu_n(dx)\right)$$

is associated with a compound Poisson process.

Examples:

Gaussian laws. The Gaussian law $\mathcal{N}(m, \sigma^2)$ has the characteristic function $\exp(ium - u^2\sigma^2/2)$. Its characteristic triple is $(m, \sigma, 0)$.

Cauchy laws. The standard Cauchy law has the characteristic function

$$\exp(-c|u|) = \exp\left(\frac{c}{\pi} \int_{-\infty}^{\infty} (e^{iux} - 1)x^{-2} dx\right).$$

Its reduced form characteristic triple is $(0, 0, \pi^{-1}x^{-2}dx)$.

Gamma laws. The Gamma law $\Gamma(a, \nu)$ has characteristic function

$$(1 - iu/\nu)^{-a} = \exp \left(a \int_0^{\infty} (e^{iux} - 1) e^{-\nu x} \frac{dx}{x} \right).$$

Its reduced form characteristic triple is $(0, 0, \mathbb{1}_{\{x>0\}} ax^{-1} e^{-\nu x} dx)$.

Brownian hitting times. The first hitting time of $a > 0$ for a Brownian motion has characteristic triple (in reduced form)

$$(0, 0, \frac{a}{\sqrt{2\pi}} x^{-3/2} \mathbb{1}_{\{x>0\}} dx).$$

Indeed $\mathbb{E}(e^{-\lambda T_a}) = e^{-a\sqrt{2\lambda}}$. Moreover

$$\sqrt{2\lambda} = \frac{1}{\sqrt{2}\Gamma(1/2)} \int_0^\infty (1 - e^{-\lambda x}) x^{-3/2} dx,$$

hence, using that $\Gamma(1/2) = \sqrt{\pi}$

$$\mathbb{E}(e^{-\lambda T_a}) = \exp\left(-\frac{a}{\sqrt{2\pi}} \int_0^\infty (1 - e^{-\lambda x}) x^{-3/2} dx\right).$$

Inverse Gaussian laws. The Inverse Gaussian law has density

$$\frac{a}{\sqrt{2\pi}} e^{a\nu} x^{-3/2} \exp\left(-\frac{1}{2}(a^2 x^{-1} + \nu^2 x)\right) \mathbb{1}_{\{x>0\}}$$

This is the law of the first hitting time of a for a Brownian motion with drift ν . The Inverse Gaussian law has characteristic triple (in reduced form)

$$\left(0, 0, \frac{a}{\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2}\nu^2 x\right) \mathbb{1}_{\{x>0\}} dx\right).$$

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$$\left(0, 0, \frac{a}{\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2}\nu^2 x\right) \mathbb{1}_{\{x>0\}} dx\right).$$

Indeed

$$\begin{aligned} & \exp\left(-\frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{dx}{x^{3/2}} (1 - e^{-\lambda x}) e^{-\nu^2 x/2}\right) \\ &= \exp\left(-\frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{dx}{x^{3/2}} \left((e^{-\nu^2 x/2} - 1) + (1 - e^{-(\lambda + \nu^2/2)x})\right)\right) \\ &= \exp(-a(-\nu + \sqrt{\nu^2 + 2\lambda})) \end{aligned}$$

is the Laplace transform of the first hitting time of a for a BM with

drift ν .

Stable Random Variables

A random variable is stable if for any $a > 0$, there exist $b > 0$ and $c \in \mathbb{R}$ such that $[\hat{\mu}(u)]^a = \hat{\mu}(bu) e^{icu}$.

X is stable if

$$\forall n, \exists(\beta_n, \gamma_n), \text{ such that } X_1^{(n)} + \dots + X_n^{(n)} \stackrel{\text{law}}{=} \beta_n X + \gamma_n$$

where $(X_i^{(n)}, i \leq n)$ are i.i.d. random variables with the same law as X .

A stable law is infinitely divisible.

The characteristic function of a stable law can be written

$$\hat{\mu}(u) = \begin{cases} \exp(i\beta u - \frac{1}{2}\sigma^2 u^2), & \text{for } \alpha = 2 \\ \exp(-\gamma|u|^\alpha[1 - i\beta \operatorname{sgn}(u) \tan(\pi\alpha/2)]), & \text{for } \alpha \neq 1, \neq 2 \\ \exp(\gamma|u|(1 - i\beta \operatorname{sgn}(u) \ln|u|)), & \alpha = 1 \end{cases},$$

where $\beta \in [-1, 1]$. For $\alpha \neq 2$, the Lévy measure of a stable law is absolutely continuous with respect to the Lebesgue measure, with density

$$\nu(dx) = \begin{cases} c^+ x^{-\alpha-1} dx & \text{if } x > 0 \\ c^- |x|^{-\alpha-1} dx & \text{if } x < 0. \end{cases}$$

Examples: A Gaussian variable is stable with $\alpha = 2$. The Cauchy law is stable with $\alpha = 1$.

Lévy-Khintchine Representation

Let X be a Lévy process. Then, X_1 is i.d.

There exists $m \in \mathbb{R}^d$, a non-negative semi-definite quadratic form A , a Lévy measure ν such that for $u \in \mathbb{R}^d$

$$\mathbb{E}(\exp(i(u \cdot X_1))) = \exp \left(i(u \cdot m) - \frac{1}{2}(u \cdot Au) + \int_{\mathbb{R}^d} (e^{i(u \cdot x)} - 1 - i(u \cdot x) \mathbb{1}_{|x| \leq 1}) \nu(dx) \right)$$

where ν is the Lévy measure.

- If $\nu(\mathbb{R} \setminus \{0\}) < \infty$, the process X has a finite number of jumps in any finite time interval. In finance, one refers to **finite activity**.

- If $\nu(\mathbb{R} \setminus \{0\}) < \infty$, the process X has a finite number of jumps in any finite time interval. In finance, one refers to **finite activity**.
- If $\nu(\mathbb{R} \setminus \{0\}) = \infty$, the process corresponds to **infinite activity**.

The complex valued continuous function Φ such that

$$\mathbb{E} [\exp(iuX_1)] = \exp(-\Phi(u))$$

is called the **characteristic exponent** (sometimes the Lévy exponent) of the Lévy process X .

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If $\mathbb{E} [e^{\lambda X_1}] < \infty$ for any $\lambda > 0$, the function Ψ defined on $[0, \infty[$, such that

$$\mathbb{E} [\exp(\lambda X_1)] = \exp(\Psi(\lambda))$$

is called the **Laplace exponent** of the Lévy process X .

The complex valued continuous function Φ such that

$$\mathbb{E} [\exp(iuX_1)] = \exp(-\Phi(u))$$

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If $\mathbb{E} [e^{\lambda X_1}] < \infty$ for any $\lambda > 0$, the function Ψ defined on $[0, \infty[$, such that

$$\mathbb{E} [\exp(\lambda X_1)] = \exp(\Psi(\lambda))$$

is called the **Laplace exponent** of the Lévy process X .

It follows that, if $\Psi(\lambda)$ exists,

$$\mathbb{E} [\exp(iuX_t)] = \exp(-t\Phi(u)), \quad \mathbb{E} [\exp(\lambda X_t)] = \exp(t\Psi(\lambda))$$

and

$$\Psi(\lambda) = -\Phi(-i\lambda).$$

From LK formula, the characteristic exponent and the Laplace exponent can be computed as follows:

$$\begin{aligned}\Phi(u) &= -ium + \frac{1}{2}\sigma^2 u^2 - \int (e^{iux} - 1 - iux \mathbb{1}_{|x| \leq 1}) \nu(dx) \\ \Psi(\lambda) &= \lambda m + \frac{1}{2}\sigma^2 \lambda^2 + \int (e^{\lambda x} - 1 - \lambda x \mathbb{1}_{|x| \leq 1}) \nu(dx).\end{aligned}$$

Martingales

- If $\mathbb{E}(|X_t|) < \infty$, i.e., $\int_{|x| \geq 1} |x| \nu(dx) < \infty$;

$\mathbb{E}(X_t) = t(m + \int_{|x| \geq 1} |x| \nu(dx))$, the process $X_t - \mathbb{E}(X_t)$ is a martingale.

Martingales

- If $\mathbb{E}(|X_t|) < \infty$, i.e., $\int_{|x| \geq 1} |x| \nu(dx) < \infty$, the process $X_t - \mathbb{E}(X_t)$ is a martingale and $\mathbb{E}(X_t) = t(m + \int_{|x| \geq 1} |x| \nu(dx))$.
- If $\Psi(\alpha)$ exists (i.e., if $\int_{|x| > 1} e^x \nu(dx) < \infty$), the process

$$\frac{e^{\alpha X_t}}{\mathbb{E}(e^{\alpha X_t})} = e^{\alpha X_t - t\Psi(\alpha)}$$

is a martingale

More generally, for any bounded predictable process H

$$\mathbb{E} \left[\sum_{s \leq t} H_s f(\Delta X_s) \right] = \mathbb{E} \left[\int_0^t ds H_s \int f(x) d\nu(x) \right]$$

and if H is a predictable function (i.e. $H : \Omega \times \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ is $\mathcal{P} \times \mathcal{B}$ measurable)

$$\mathbb{E} \left[\sum_{s \leq t} H_s(\omega, \Delta X_s) \right] = \mathbb{E} \left[\int_0^t ds \int d\nu(x) H_s(\omega, x) \right].$$

Both sides are well defined and finite if

$$\mathbb{E} \left[\int_0^t ds \int d\nu(x) |H_s(\omega, x)| \right] < \infty$$

(Exponential formula.) Let X be a Lévy process and ν its Lévy measure. For all t and all Borel function f defined on $\mathbb{R}^+ \times \mathbb{R}^d$ such that $\int_0^t ds \int |1 - e^{f(s,x)}| \nu(dx) < \infty$, one has

$$\mathbb{E} \left[\exp \left(\sum_{s \leq t} f(s, \Delta X_s) \mathbb{1}_{\{\Delta X_s \neq 0\}} \right) \right] = \exp \left(- \int_0^t ds \int (1 - e^{f(s,x)}) \nu(dx) \right).$$

The above property does not extend to predictable functions.

Lévy-Itô's decomposition

If X is a R^d -valued Lévy process, it can be decomposed into $X = Y^{(0)} + Y^{(1)} + Y^{(2)} + Y^{(3)}$ where $Y^{(0)}$ is a affine function, $Y^{(1)}$ is a linear transform of a Brownian motion, $Y^{(2)}$ is a compound Poisson process with jump size greater than or equal to 1 and $Y^{(3)}$ is a Lévy process with jumps sizes smaller than 1. The processes $Y^{(i)}$ are independent.

More precisely

$$X_t = mt + \sigma W_t + X_t^1 + \lim_{\epsilon \rightarrow 0} \tilde{X}_t^\epsilon$$

where

$$X_t^1 = \int_0^t \int_{\{|x| \geq 1\}} x \mathbf{N}(dx, ds) = \sum_{s \leq t} \Delta X_s \mathbb{1}_{|\Delta X_s| \geq 1}$$

$$X_t^\epsilon = \int_0^t \int_{\{\epsilon \leq |x| < 1\}} x (\mathbf{N}(dx, ds) - \nu(dx) ds)$$

The processes X^1 is a compound Poisson process, the process \tilde{X}^ϵ is a compensated compound Poisson process, it is a martingale. Note that $\int_0^t \int_{\{\epsilon \leq |x| < 1\}} x \mathbf{N}(dx, ds)$ and $\int_0^t \int_{\{\epsilon \leq |x| < 1\}} x \nu(dx) ds$ are well defined. However, these quantities do not converge as ϵ goes to 0.

Path properties

- The Lévy process X is continuous iff $\nu = 0$
- The Lévy process X with piecewise constant paths iff it is a compound Poisson process or iff $\sigma = 0$ and $\int \nu(dx) < \infty$
- The Lévy process X is with finite variation path iff $\sigma = 0$ and $\int_{|x| \leq 1} |x| \nu(dx) < \infty$. In that case,

$$ium + \int (e^{iux} - 1 - iux \mathbb{1}_{|x| \leq 1}) \nu(dx)$$

can be written

$$ium_0 + \int (e^{iux} - 1) \nu(dx)$$

and

$$X_t = m_0 t + \sum_{s \leq t} \Delta X_s$$

- If $\int_{|x| \leq 1} |x| \nu(dx) = \infty$, the sum $\sum_{s \leq t} |\Delta X_s| \mathbf{1}_{|\Delta X_s| \leq \epsilon}$ diverges, however the compensated sum converges.

If X is a Lévy process with jumps bounded (by 1), it admits moments of any order, and, setting $Z_t = X_t - \mathbb{E}(X_t)$, $Z = Z^c + Z^d$ where Z^c is a continuous martingale,

$$Z_t^d = \int_{|x|<1} x (\mathbf{N}(dt, dx) - \nu(dx)dt)$$

and Z^c and Z^d are martingales and independent Lévy processes.

If X is a Lévy process with jumps bounded (by 1), it admits moments of any order, and, setting $Z_t = X_t - \mathbb{E}(X_t)$, $Z = Z^c + Z^d$ where Z^c is a continuous martingale,

$$Z_t^d = \int_{|x|<1} x (\mathbf{N}(dt, dx) - \nu(dx)dt)$$

and Z^c and Z^d are martingales and independent Lévy processes.

If X is a Lévy process, it admits a decomposition as

$$dX_t = \alpha dt + \sigma dB_t + \int_{|x|<1} x (\mathbf{N}(dt, dx) - \nu(dx)dt) + \int_{|x|\geq 1} x \mathbf{N}(dt, dx)$$

The Lévy process is a semi-martingale, hence $\sum_{0 < s \leq t} (\Delta X_s)^2 < \infty$

Some definitions on general stochastic processes

Local martingale

An adapted, right-continuous process M is an **F-local martingale** if there exists a sequence of stopping times (T_n) such that

- (i) The sequence T_n is increasing and $\lim_n T_n = \infty$, a.s.
- (ii) For every n , the stopped process $M^{T_n} \mathbb{1}_{\{T_n > 0\}}$ is an **F**-martingale.

Covariation of Martingales

- **Continuous martingales:** Let X be a continuous local martingale. The **predictable quadratic variation** process of X is the continuous increasing process $\langle X \rangle$ such that $X^2 - \langle X \rangle$ is a local martingale.

Let X and Y be two continuous local martingales. The predictable covariation process is the continuous finite variation process $\langle X, Y \rangle$ such that $XY - \langle X, Y \rangle$ is a local martingale. Note that $\langle X \rangle = \langle X, X \rangle$ and

$$\langle X + Y \rangle = \langle X \rangle + \langle Y \rangle + 2\langle X, Y \rangle.$$

- **General Local martingales:**

Let X and Y be two local martingales.

► The **covariation process** is the finite variation process $[X, Y]$ such that

$XY - [X, Y]$ is a local martingale

$$\Delta[X, Y]_t = \Delta X_t \Delta Y_t$$

The process $[X, X]$ is non-decreasing.

► If the martingales X and Y are continuous, $[X, Y] = \langle X, Y \rangle$.

► This covariation process is the limit in probability of

$\sum_{i=1}^{p(n)} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$, for $0 < t_1 < \dots < t_{p(n)} \leq t$ when $\sup_{i \leq p(n)} (t_i - t_{i-1})$ goes to 0.

- **General Local martingales:**

Let X and Y be two local martingales.

► The **covariation process** is the finite variation process $[X, Y]$ such that

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$$\Delta[X, Y]_t = \Delta X_t \Delta Y_t$$

The process $[X, X]$ is non-decreasing.

► If the martingales X and Y are continuous, $[X, Y] = \langle X, Y \rangle$.

► The covariation $[X, Y]$ of both processes X and Y can be also defined by polarisation

$$[X + Y, X + Y] = [X, X] + [Y, Y] + 2[X, Y]$$

Let us recall that, if W is a Brownian motion $\langle W \rangle_t = [W]_t = t$. If M is the compensated martingale of a Poisson process, $[M]_t = N_t$ and $\langle M \rangle_t = \lambda t$.

For any martingale $M \in \mathbf{H}^2$, we denote by M^c its projection on $\mathbf{H}^{2,c}$ and by M^d its projection on $\mathbf{H}^{2,d}$. Then, $M = M^c + M^d$ is the decomposition of any martingale in \mathbf{H}^2 into its continuous and purely discontinuous parts.

Covariation of Semi-martingales

A semi-martingale is a càdlàg process X such that $X_t = M_t + A_t$, M martingale, A bounded variation process. One can write

$X_t = M_t^c + M_t^d + A_t$ where M^c is continuous. We use the (usual) notation $X^c := M^c$.

► If X and Y are semi-martingales and if X^c, Y^c are their continuous martingale parts, their quadratic covariation is

$$[X, Y]_t = \langle X^c, Y^c \rangle_t + \sum_{s \leq t} (\Delta X_s)(\Delta Y_s).$$

► In the case where X, Y are continuous semi-martingales, their predictable covariation process is the predictable covariation process of their continuous martingale parts.

Stochastic Integral

Let N be a counting process. The **stochastic integral**

$$\int_0^t C_s dN_s$$

is defined pathwise as a Stieltjes integral for every bounded measurable process (not necessarily \mathbf{F}^N -adapted) $(C_t, t \geq 0)$ by

$$(C \star N)_t \stackrel{def}{=} \int_0^t C_s dN_s = \int_{]0, t]} C_s dN_s \stackrel{def}{=} \sum_{n=1}^{\infty} C_{T_n} \mathbb{1}_{\{T_n \leq t\}}.$$

We emphasize that the integral $\int_0^t C_s dN_s$ is here an integral over the time interval $]0, t]$, where the upper limit t is included and the lower limit 0 excluded. This integral is finite since there is a finite number of jumps during the time interval $]0, t]$. We shall also write

$$\int_0^t C_s dN_s = \sum_{s \leq t} C_s \Delta N_s$$

where the right-hand side contains only a finite number of non-zero terms. The integral $\int_0^\infty C_s dN_s$ is defined as $\int_0^\infty C_s dN_s = \sum_{n=1}^\infty C_{T_n}$, when the right-hand side converges.

We shall also use the differential notation $d(C \star N)_t \stackrel{def}{=} C_t dN_t$.

Doláns-Dade exponential of a finite variation process Let U be a càdlàg process with finite variation. The unique solution of

$$dY_t = Y_{t-} dU_t, \quad Y_0 = y$$

is the stochastic exponential of U (the Doléans-Dade exponential of U) equal to

$$\begin{aligned} Y_t &= y \exp(U_t^c - U_0^c) \prod_{s \leq t} (1 + \Delta U_s) \\ &= y \exp(U_t - U_0) \prod_{s \leq t} (1 + \Delta U_s) e^{-\Delta U_s}. \end{aligned}$$

PROOF: Applying the integration by parts formula shows that it is a solution to the equation $dY_t = Y_{t-}dU_t$. As for the uniqueness, if $Y^i, i = 1, 2$ are two solutions, then, setting $Z = Y^1 - Y^2$ we get $Z_t = \int_0^t Z_{s-}dU_s$. Let $M_t = \sup_{s \leq t} |Z_s|$, then, if V_t is the variation process of U_t

$$|Z_t| \leq M_t V_t$$

which implies that

$$|Z_t| \leq M_t \int_0^t V_{s-}dV_s = M_t \frac{V_t^2}{2}.$$

Iterating, we obtain $|Z_t| \leq M_t \frac{V_t^n}{n!}$ and the uniqueness follows by letting $n \rightarrow \infty$. △

Itô's formula

Poisson process

Integration by parts formula

Let $(x_t, t \geq 0)$ and $(y_t, t \geq 0)$ be two predictable processes and let $X_t = x + \int_0^t x_s dN_s$ and $Y_t = y + \int_0^t y_s dN_s$. The jumps of X (resp. of Y) occur at the same times as the jumps of N and

$\Delta X_s = x_s \Delta N_s, \Delta Y_s = y_s \Delta N_s$. Then

$$X_t Y_t = xy + \sum_{s \leq t} \Delta(XY)_s = xy + \sum_{s \leq t} X_{s-} \Delta Y_s + \sum_{s \leq t} Y_{s-} \Delta X_s + \sum_{s \leq t} \Delta X_s \Delta Y_s$$

The first equality is obvious, the second one is easy to check. Hence, from the definition of stochastic integrals

$$X_t Y_t = xy + \int_0^t Y_{s-} dX_s + \int_0^t X_{s-} dY_s + [X, Y]_t$$

where (note that $(\Delta N_t)^2 = \Delta N_t$)

$$[X, Y]_t := \sum_{s \leq t} \Delta X_s \Delta Y_s = \sum_{s \leq t} x_s y_s \Delta N_s = \int_0^t x_s y_s dN_s .$$

More generally, if $dX_t = \mu_t dt + x_t dN_t$ with $X_0 = x$ and $dY_t = \nu_t dt + y_t dN_t$ with $Y_0 = y$, one gets

$$X_t Y_t = xy + \int_0^t Y_{s-} dX_s + \int_0^t X_{s-} dY_s + [X, Y]_t$$

where $[X, Y]_t = \int_0^t x_s y_s dN_s$. In particular, if $dX_t = x_t dM_t$ and $dY_t = y_t dM_t$, the process $X_t Y_t - [X, Y]_t$ is a local martingale.

Itô's Formula For Poisson processes

Let N be a Poisson process and f a bounded Borel function. The decomposition

$$f(N_t) = f(N_0) + \sum_{0 < s \leq t} [f(N_s) - f(N_{s-})]$$

is trivial and is the main step to obtain Itô's formula for a Poisson process.

We can write the right-hand side as a stochastic integral:

$$\begin{aligned} \sum_{0 < s \leq t} [f(N_s) - f(N_{s-})] &= \sum_{0 < s \leq t} [f(N_{s-} + 1) - f(N_{s-})] \Delta N_s \\ &= \int_0^t [f(N_{s-} + 1) - f(N_{s-})] dN_s, \end{aligned}$$

hence, the canonical decomposition of $f(N_t)$ as the sum of a martingale and an absolute continuous adapted process is

$$f(N_t) = f(N_0) + \int_0^t [f(N_{s-} + 1) - f(N_{s-})] dM_s + \int_0^t [f(N_{s-} + 1) - f(N_{s-})] \lambda ds .$$

More generally, assume that N is an inhomogeneous Poisson process (i.e., N is a counting process and there exists a non-negative function λ such that $N_t - \int_0^t \lambda(s)ds$ is a martingale). Let h be an adapted process and g a predictable process such that $\int_0^t |h_s|ds < \infty$, $\int_0^t |g_s|\lambda_s ds < \infty$. Let $F \in C^{1,1}(\mathbb{R}^+ \times \mathbb{R})$ and

$$dX_t = h_t dt + g_t dM_t = (h_t - g_t \lambda_t) dt + g_t dN_t$$

Then

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_{s-}) (h_s - g_s \lambda(s)) ds \\ &\quad + \sum_{s \leq t} F(s, X_s) - F(s, X_{s-}) \\ &= F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\ &\quad + \sum_{s \leq t} [F(s, X_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) g_s \Delta N_s] . \end{aligned}$$

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_s) (h_s - g_s \lambda(s)) ds \\ &\quad + \int_0^t [F(s, X_s) - F(s, X_{s-})] dN_s \end{aligned}$$

PROOF: Indeed, between two jumps, $dX_t = (h_t - \lambda_t g_t)dt$, and for $T_n < s < t < T_{n+1}$,

$$F(t, X_t) = F(s, X_s) + \int_s^t \partial_t F(u, X_u) du + \int_s^t \partial_x F(u, X_u) (h_u - g_u \lambda_u) du.$$

At jump times, $F(T_n, X_{T_n}) = F(T_n, X_{T_n-}) + \Delta F(\cdot, X)_{T_n}$. \triangle

Remark that, in the “ ds ” integrals, we can write X_{s-} or X_s , since, for any bounded Borel function f ,

$$\int_0^t f(X_{s-})ds = \int_0^t f(X_s)ds.$$

Note that since dN_s a.s. $N_s = N_{s-} + 1$, one has

$$\int_0^t f(N_{s-})dN_s = \int_0^t f(N_s + 1)dN_s.$$

We shall use systematically use the form $\int_0^t f(N_{s-})dN_s$, even if the $\int_0^t f(N_s + 1)dN_s$ has a meaning.

The reason is that $\int_0^t f(N_{s-})dM_s = \int_0^t f(N_{s-})dN_s + \lambda \int_0^t f(N_{s-})ds$ is a martingale, whereas $\int_0^t f(N_s + 1)dM_s$ is not.

Check that the above formula can be written as

$$\begin{aligned}
& F(t, X_t) - F(0, X_0) \\
= & \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_s) (h_s - g_s \lambda(s)) ds \\
& + \int_0^t [F(s, X_s) - F(s, X_{s-})] dN_s \\
= & \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\
& + \int_0^t [F(s, X_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) g_s] dN_s
\end{aligned}$$

$$\begin{aligned}
& F(t, X_t) - F(0, X_0) \\
= & \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\
& + \int_0^t [F(s, X_{s-} + g_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) g_s] dN_s \\
= & \int_0^t (\partial_t F(s, X_s) + [F(s, X_{s-} + g_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) g_s] \lambda) ds \\
& + \int_0^t [F(s, X_{s-} + g_s) - F(s, X_{s-})] dM_s
\end{aligned}$$

Let X be a ν -compound Poisson process, and $Z_t = Z_0 + bt + X_t$. Then,

using that $\mathbf{N} = \sum_{n=1}^{\infty} \delta_{T_n, Y_n}$, Itô's formula

$$\begin{aligned}
 f(Z_t) - f(Z_0) &= b \int_0^t f'(Z_s) ds + \sum_{k, T_k \leq t} f(Z_{T_k}) - f(Z_{T_k-}) \\
 &= b \int_0^t f'(Z_s) ds + \int_0^t \int_{\mathbb{R}} [f(Z_{s-} + y) - f(Z_{s-})] \mathbf{N}(ds, dy) \\
 &= \int_0^t ds (\mathcal{L}f)(Z_s) + M(f)_t
 \end{aligned}$$

can be written as where $\mathcal{L}f(x) = bf'(x) + \int_{\mathbb{R}} (f(x+y) - f(x)) \nu(dy)$ is the infinitesimal generator of Z and

$$M(f)_t = \int_0^t \int_{\mathbb{R}} [f(Z_{s-} + y) - f(Z_{s-})] (\mathbf{N}(ds, dy) - ds \nu(dy))$$

is a local martingale.

Let X be a semi-martingale and $f \in C^{1,2}$ Then

$$\begin{aligned} df(t, X_t) &= \partial_t f(t, X_t)dt + \partial_x f(t, X_{t-})dX_t + \frac{1}{2}\partial_{xx} f(t, X_{t-})d[X^c]_t \\ &\quad + f(t, X_t) - f(t, X_{t-}) - \Delta X_t \partial_x f(t, X_{t-}) \end{aligned}$$

Back to Lévy processes

Covariation processes Let X be a (m, σ^2, ν) real valued Lévy process. Then, $X_t^c = \sigma W_t$ and

$$[X]_t = \sigma^2 t + \int_0^t \int x^2 \mathbf{N}(ds, dx)$$

If $\int x^2 \nu(dx) < \infty$,

$$\langle X \rangle_t = \sigma^2 t + t \int x^2 \nu(dx)$$

Itô's formula

Let $Y_t = f(t, X_t)$, with f bounded with bounded derivatives. Then, Y is a semi-martingale

Its martingale part is

$$\partial f_x(t, X_t)\sigma dB_t + \int (f(t, X_{t-} + x) - f(t, X_{t-})) (\mathbf{N}(dt, dx) - \nu(dx)dt)$$

Its finite variation part is

$$\begin{aligned} \partial_t f(t, X_t) &+ \partial_x f(t, X_t)\alpha + \frac{1}{2}\sigma^2 \partial_{xx} f(t, X_t) \\ &+ \int (f(t, X_{t-} + x) - f(t, X_{t-}) - x\partial_x f(t, X_{t-})\mathbb{1}_{x \leq 1}) \nu(dx) \end{aligned}$$

Itô's formula

Let X be a Lévy process with decomposition

$$dX_t = \alpha dt + \sigma dB_t + \int_{|x|<1} x (\mathbf{N}(dt, dx) - \nu(dx)dt) + \int_{|x|\geq 1} x \mathbf{N}(dt, dx)$$

Let $Y_t = f(t, X_t)$ where f is a $C^{1,2}$ function. Then, Y is a semi-martingale

$$\begin{aligned} dY_t &= \partial_t f(t, X_t)dt + \partial_x f(t, X_t) (\alpha dt + \sigma dB_t) + \frac{1}{2} \sigma^2 \partial_{xx} f(t, X_t) dt \\ &+ \int_{|x|<1} (f(t, X_{t-} + x) - f(t, X_{t-}) - x \partial_x f(t, X_{t-})) \nu(dx)dt \\ &+ \int_{|x|<1} (f(t, X_{t-} + x) - f(t, X_{t-})) (\mathbf{N}(dt, dx) - \nu(dx)dt) \\ &+ \int_{|x|\geq 1} (f(t, X_{t-} + x) - f(t, X_{t-})) \mathbf{N}(dt, dx) \end{aligned}$$

As a consequence of the semi-martingale property, if F is a C^2 function, then, the series

$$\sum_{s \leq t} |F(\Delta X_s) - F(0) - F'(0)\Delta X_s|$$

converges.

Representation Theorem

Let X be a \mathbb{R}^d -valued Lévy process and \mathbf{F}^X its natural filtration. Let M be a locally square integrable martingale with $M_0 = m$. Then, there exists a family (φ, ψ) of predictable processes such that

$$\int_0^t |\varphi_s^i|^2 ds < \infty, \text{ a.s.}$$

$$\int_0^t \int_{\mathbb{R}^d} |\psi_s(x)|^2 ds \nu(dx) < \infty, \text{ a.s.}$$

and

$$M_t = m + \sum_{i=1}^d \int_0^t \varphi_s^i dW_s^i + \int_0^t \int_{\mathbb{R}^d} \psi_s(x) (\mathbf{N}(ds, dx) - ds \nu(dx)).$$

Change of measure

Poisson Process

Let N be a Poisson process with intensity λ , and \mathbb{Q} be the probability defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = (1 + \beta)^{N_t} e^{-\lambda\beta t}$$

Then, the process N is a \mathbb{Q} -Poisson process with intensity equal to $(1 + \beta)\lambda$.

The process L defined as

$$L_t = (1 + \beta)^{N_t} e^{-\lambda\beta t}$$

is a strictly positive martingale with expectation equal to 1. Then, from the definition of \mathbb{Q} , for any sequence $0 = t_1 < t_2 < \dots < t_{n+1} = t$,

$$\mathbb{E}_{\mathbb{Q}} \left(\prod_{i=1}^n x_i^{N_{t_{i+1}} - N_{t_i}} \right) = \mathbb{E}_{\mathbb{P}} \left(e^{-\lambda\beta t} \prod_{i=1}^n ((1 + \beta)x_i)^{N_{t_{i+1}} - N_{t_i}} \right)$$

The right-hand side is computed using that, under \mathbb{P} , the process N is a Poisson process (hence with independent increments) and is equal to

$$\begin{aligned} e^{-\lambda\beta t} \prod_{i=1}^n \mathbb{E}_P \left(((1 + \beta)x_i)^{N_{t_{i+1}-t_i}} \right) &= e^{-\lambda\beta t} \prod_{i=1}^n e^{-\lambda(t_{i+1}-t_i)} e^{\lambda(t_{i+1}-t_i)(1+\beta)x_i} \\ &= \prod_{i=1}^n e^{(1+\beta)\lambda(t_{i+1}-t_i)(x_i-1)}. \end{aligned}$$

It follows that, for any j (take all the x_i 's, except the j th one, equal to 1)

$$\mathbb{E}_Q \left(x_j^{N_{t_{j+1}} - N_{t_j}} \right) = e^{(1+\beta)\lambda(t_{j+1}-t_j)(x_j-1)},$$

which establishes that, under \mathbb{Q} , the r.v. $N_{t_{j+1}} - N_{t_j}$ has a Poisson law with parameter $(1 + \beta)\lambda$, then that

$$\mathbb{E}_Q \left(\prod_{i=1}^n x_i^{N_{t_{i+1}} - N_{t_i}} \right) = \prod_{i=1}^n \mathbb{E}_Q \left(x_i^{N_{t_{i+1}} - N_{t_i}} \right)$$

which is equivalent to the independence of the increments.

Compound Poisson process

Let X be a ν -compound Poisson process under \mathbb{P} , we present some particular probability measures \mathbb{Q} equivalent to \mathbb{P} such that, under \mathbb{Q} , X is still a compound Poisson process.

Let $\tilde{\nu}$ a positive finite measure on \mathbb{R} absolutely continuous w.r.t. ν , and $\tilde{\lambda} = \tilde{\nu}(\mathbb{R}) > 0$. Let

$$L_t = \exp \left(t(\lambda - \tilde{\lambda}) + \sum_{s \leq t} \ln \left(\frac{d\tilde{\nu}}{d\nu} \right) (\Delta X_s) \right).$$

Applying the martingale property (0.1) for $f = \ln \left(\frac{d\tilde{\nu}}{d\nu} \right)$, the process L is a martingale. Set $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$. **Under \mathbb{Q} , the process X is a $\tilde{\nu}$ -compound Poisson process.**

PROOF: First we find the law of the r.v. X_t under \mathbb{Q} . From the definition of \mathbb{Q}

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(e^{iuX_t}) &= \mathbb{E}_P(e^{iuX_t} \exp \left(t(\lambda - \hat{\lambda}) + \sum_{k=1}^{N_t} f(Y_k) \right)) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} e^{t(\lambda - \hat{\lambda})} \left(\mathbb{E}_P(e^{iuY_1 + f(Y_1)}) \right)^n \\
&= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} e^{t(\lambda - \hat{\lambda})} \left(\mathbb{E}_P \left(\frac{d\hat{\nu}}{d\nu}(Y_1) e^{iuY_1} \right) \right)^n \\
&= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-t\hat{\lambda}} \left(\frac{1}{\lambda} \int e^{iuy} d\hat{\nu}(y) \right)^n = \exp t \int (e^{iuy} - 1) d\hat{\nu}(y)
\end{aligned}$$

It remains to check that X is with independent and stationary increments under \mathbb{Q} .

By Bayes formula, for $t > s$

$$\begin{aligned}\mathbb{E}_Q(e^{iu(X_t - X_s)} | \mathcal{F}_s) &= \frac{1}{L_s} \mathbb{E}_P(L_t e^{iu(X_t - X_s)} | \mathcal{F}_s) \\ &= \exp\left((t - s) \int (e^{iux} - 1) \tilde{\nu}(dx)\right).\end{aligned}$$

△

Esscher transform

We assume that $\mathbb{E}(e^{(\lambda \cdot X_t)}) < \infty$. We define a probability \mathbb{Q} , equivalent to \mathbb{P} by the formula

$$\mathbb{Q}|_{\mathcal{F}_t} = \frac{e^{(\lambda \cdot X_t)}}{\mathbb{E}(e^{(\lambda \cdot X_t)})} \mathbb{P}|_{\mathcal{F}_t} . (*)$$

This particular choice of measure transformation, (called an Esscher transform) preserves the Lévy process property.

Let X be a \mathbb{P} -Lévy process with parameters (m, A, ν) where $A = R^T R$. Let λ be such that $\mathbb{E}(e^{(\lambda \cdot X_t)}) < \infty$ and suppose \mathbb{Q} is defined by (*).

Then X is a Lévy process under \mathbb{Q} , then the decomposition of X under \mathbb{Q} is

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\exp(i(u \cdot X_1))) &= \exp \left(i(u \cdot m^{(\lambda)}) - \frac{1}{2}(u \cdot Au) \right. \\ &\quad \left. + \int_{\mathbb{R}^d} (e^{i(u \cdot x)} - 1 - i(u \cdot x) \mathbb{1}_{|x| \leq 1}) \nu^{(\lambda)}(dx) \right) \end{aligned}$$

with

$$\begin{aligned} m^{(\lambda)} &= m + R\lambda + \int_{|x| \leq 1} x(e^{\lambda x} - 1) \nu(dx) \\ \nu^{(\lambda)}(dx) &= e^{\lambda x} \nu(dx). \end{aligned}$$

The characteristic exponent of X under \mathbb{Q} is

$$\Phi^{(\lambda)}(u) = \Phi(u - i\lambda) - \Phi(-i\lambda).$$

If $\Psi(\lambda) < \infty$, $\Psi^{(\lambda)}(u) = \Psi(u + \lambda) - \Psi(\lambda)$ for $u \geq \min(-\lambda, 0)$.

General case

More generally, any density $(L_t, t \geq 0)$ which is a positive martingale can be used.

$$dL_t = \sum_{i=1}^d \tilde{\varphi}_t^i dW_t^i + \int \tilde{\psi}_t(x) [\mathbf{N}(dt, dx) - dt\nu(dx)].$$

From the strict positivity of L , there exists φ, ψ such that $\tilde{\varphi}_t = L_{t-} \varphi_t$, $\tilde{\psi}_t = L_{t-} (e^{\psi(t,x)} - 1)$, hence the process L satisfies

$$dL_t = L_{t-} \left(\sum_{i=1}^d \varphi_t^i dW_t^i + \int (e^{\psi(t,x)} - 1) [\mathbf{N}(dt, dx) - dt\nu(dx)] \right) (**)$$

Let $\mathbb{Q}|\mathcal{F}_t = L_t \mathbb{P}|\mathcal{F}_t$ where L is defined in (**). With respect to \mathbb{Q} ,

(i) $W_t^\varphi \stackrel{\text{def}}{=} W_t - \int_0^t \varphi_s ds$ is a Brownian motion

(ii) The process N is compensated by $e^{\psi(s,x)} ds\nu(dx)$ meaning that for any Borel function h such that

$$\int_0^T \int_{\mathbb{R}} |h(s,x)| e^{\psi(s,x)} ds\nu(dx) < \infty,$$

the process

$$\int_0^t \int_{\mathbb{R}} h(s,x) \left(\mathbf{N}(ds, dx) - e^{\psi(s,x)} ds\nu(dx) \right)$$

is a local martingale.

Fluctuation theory

Let $M_t = \sup_{s \leq t} X_s$ be the running maximum of the Lévy process X . The reflected process $M - X$ enjoys the strong Markov property.

Let θ be an exponential variable with parameter q , independent of X . Note that

$$\mathbb{E}(e^{iuX_\theta}) = q \int \mathbb{E}(e^{iuX_t})e^{-qt} dt = q \int e^{-t\Phi(u)}e^{-qt} dt.$$

Using excursion theory, **the random variables M_θ and $X_\theta - M_\theta$ can be proved to be independent**, hence

$$\mathbb{E}(e^{iuM_\theta})\mathbb{E}(e^{iu(X_\theta - M_\theta)}) = \frac{q}{q + \Phi(u)}.$$

This equality is known as the **Wiener-Hopf factorization**.

Let $m_t = \min_{s \leq t}(X_s)$. Then

$$m_\theta \stackrel{law}{=} X_\theta - M_\theta.$$

If $\mathbb{E}(e^{X_1}) < \infty$, using Wiener-Hopf factorization, Mordecki proves that

the boundaries for perpetual American options are given by

$$b_p = K\mathbb{E}(e^{m_\theta}), b_c = K\mathbb{E}(e^{M_\theta})$$

where $m_t = \inf_{s \leq t} X_s$ and θ is an exponential r.v. independent of X with parameter r , hence $b_c b_p = \frac{rK^2}{1 - \ln \mathbb{E}(e^{X_1})}$.

Pecherskii-Rogozin Identity

For $x > 0$, denote by T_x the first passage time above x defined as

$$T_x = \inf\{t > 0 : X_t > x\}$$

and by $K_x = X_{T_x} - x$ the so-called overshoot.

For every triple of positive numbers (α, β, q) ,

$$\int_0^\infty e^{-qx} \mathbb{E}(e^{-\alpha T_x - \beta K_x}) dx = \frac{\kappa(\alpha, q) - \kappa(\alpha, \beta)}{(q - \beta)\kappa(\alpha, q)}$$

where κ is the Laplace exponent of the ladder process defined as

$$e^{-\ell\kappa(\alpha, \beta)} = E(\exp(-\alpha\tau_\ell - \beta H_\ell)),$$

where H is defined in terms of M and the local time of $M - X$.

Exponential Lévy Processes

Let X be a Lévy process.

The process X is a martingale iff $\int_{|x|\geq 1} |x|\nu(dx) < \infty$ and $b + \int_{|x|\geq 1} x\nu(dx) = 0$.

The process e^X is a martingale iff $\int_{|x|\geq 1} e^x\nu(dx)$ and $b + \frac{1}{2}\sigma^2 + \int(e^x - 1 - x\mathbb{1}_{|x|\leq 1})\nu(dx) = 0$

Let $C(t, S)$ be a $C^{1,2}$ function and $S_t = S_0 e^{rt+X_t}$ where $\int_{|x|\geq 1} e^{2x}\nu(dx) < \infty$.

The process $e^{-rt}C(t, S_t)$ is a martingale iff

$$\begin{aligned} \frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} - rC \\ + \int \nu(dx) (C(t, Se^x) - C(t, S) - S(e^x - 1) \frac{\partial C}{\partial S}) = 0 \end{aligned}$$

Exponential and stochastic exponential of Lévy Processes

Doléans-Dade Exponential Let X be a real-valued (m, σ^2, ν) -Lévy process. The solution of

$$dZ_t = Z_{t-} dX_t, \quad Z_0 = 1$$

is

$$Z_t = Z_{t-} \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}$$

Proof: in a first step, we prove that

$$V_t = \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}$$

is well-defined and is a finite variation process.

$$V_t = \underbrace{\prod_{s \leq t, |\Delta X_s| \leq 1/2} (1 + \Delta X_s) e^{-\Delta X_s}}_{V_t^1} \underbrace{\prod_{s \leq t, |\Delta X_s| > 1/2} (1 + \Delta X_s) e^{-\Delta X_s}}_{V_t^2}$$

The product in V^2 contains a finite number of terms

The process V^1 is non-negative and

$$\ln(V_t^1) = \sum_{s \leq t, |\Delta X_s| \leq 1/2} (\ln(1 + \Delta X_s) - \Delta X_s)$$

Using

$$0 \geq (\ln(1 + \Delta X_s) - \Delta X_s) \geq -(\Delta X_s)^2$$

we check that V^1 is well defined and with bounded variation.

Then, we apply integration by parts formula. Let $Z_t = e^{X_t - \frac{1}{2}\sigma^2 t} V_t$.

Then

$$\begin{aligned}
 dZ_t &= -\frac{\sigma^2}{2} Z_{t-} dt + Z_{t-} dX_t + e^{X_{t-} - \frac{1}{2}\sigma^2 t} dV_t + \frac{\sigma^2}{2} Z_{t-} dt \\
 &\quad + (Z_t - Z_{t-}) - Z_{t-} \Delta X_t - e^{X_{t-} - \frac{1}{2}\sigma^2 t} \Delta V_t \\
 &= Z_{t-} dX_t + Z_{t-} dV_t + e^{X_{t-} - \frac{1}{2}\sigma^2 t} (V_t e^{\Delta X_t} - V_{t-} - V_{t-} \Delta X_t) \\
 &= Z_{t-} dX_t
 \end{aligned}$$

Exponentials of Lévy Processes

The solution of the SDE

$$dS_t = S_{t-}(b(t)dt + \sigma(t)dX_t)$$

is

$$S_t = S_0 \exp \left(\int_0^t \sigma(s)dX_s + \int_0^t \left(b(s) - \frac{\sigma^2(s)}{2} \right) ds \right) \\ \prod_{0 < s \leq t} (1 + \sigma(s)\Delta X_s) \exp(-\sigma(s)\Delta X_s).$$

Let X be a real-valued (m, σ^2, ν) -Lévy process.

Let $S_t = e^{X_t}$ be the ordinary exponential of the process X . The stochastic logarithm of S (i.e., the process Y which satisfies $S_t = \mathcal{E}(Y)_t$) is a Lévy process and is given by

$$Y_t := \mathcal{L}(S)_t = X_t + \frac{1}{2}\sigma^2 t - \sum_{0 < s \leq t} (1 + \Delta X_s - e^{\Delta X_s}) .$$

The Lévy characteristics of Y are

$$m_Y = m + \frac{1}{2}\sigma^2 + \int ((e^x - 1)\mathbb{1}_{\{|e^x - 1| \leq 1\}} - x\mathbb{1}_{\{|x| \leq 1\}}) \nu(dx)$$

$$\sigma_Y^2 = \sigma^2$$

$$\nu_Y(A) = \nu(\{x : e^x - 1 \in A\}) = \int \mathbb{1}_A(e^x - 1) \nu(dx) .$$

Let $Z_t = \mathcal{E}(X)_t$ the Doléans-Dade exponential of X . If $Z > 0$, the ordinary logarithm of Z is a Lévy process L given by

$$L_t := \ln(Z_t) = X_t - \frac{1}{2}\sigma^2 t + \sum_{0 < s \leq t} (\ln(1 + \Delta X_s) - \Delta X_s) .$$

Its Lévy characteristics are

$$m_L = m - \frac{1}{2}\sigma^2 + \int (\ln(1 + x) \mathbb{1}_{\{|\ln(1+x)| \leq 1\}} - x \mathbb{1}_{\{|x| \leq 1\}}) \nu(dx)$$

$$\sigma_L^2 = \sigma^2$$

$$\nu_L(A) = \nu(\{x : \ln(1 + x) \in A\}) = \int \mathbb{1}_A(\ln(1 + x)) \nu(dx)$$

Exponentials of Lévy Processes

Let $S_t = xe^{X_t}$ where X is a (m, σ^2, ν) real valued Lévy process.

Let us assume that $\mathbb{E}(e^{-\alpha X_1}) < \infty$, for $\alpha \in [-\epsilon, \epsilon]$. This implies that X has finite moments of all orders.

A particular case

Let W be a Brownian motion and Z be a ν -compound Poisson process independent of W of the form $Z_t = \sum_{n=1}^{N_t} Y_n$. Let

$$dS_t = S_{t-} (\mu dt + \sigma dW_t + dZ_t), \quad (0.1)$$

where μ and σ are constants. The process $(S_t e^{-rt}, t \geq 0)$ is a martingale if and only if $E(|Y_1|) < \infty$ and $\mu + \lambda E(Y_1) = r$.

If $Y_1 \geq -1$ a.s., the process S can be written in an exponential form as

$$S_t = S_0 e^{X_t}, \quad X_t = bt + \sigma W_t + V_t$$

where $b = \mu - \frac{1}{2}\sigma^2$, V is the (λ, \tilde{F}) -compound Poisson process

$$V_t = \sum_{n=1}^{N_t} \ln(1 + Y_n) = \sum_{n=1}^{N_t} U_n$$

with $\tilde{F}(u) = F(e^u - 1)$.

Option pricing with Esscher Transform

Let $S_t = S_0 e^{rt+X_t}$ where X is a Lévy process under the historical probability \mathbb{P} . Assume that $\Psi(\alpha) = \mathbb{E}(e^{\alpha X_1}) < \infty$ on some open interval (a, b) with $b - a > 1$ and that there exists a real number θ such that $\Psi(\theta) = \Psi(\theta + 1)$.

The process $e^{-rt} S_t = S_0 e^{X_t}$ is a martingale under the probability \mathbb{Q} defined as $\mathbb{Q} = Z_t \mathbb{P}$ where $Z_t = \frac{e^{\theta X_t}}{\Psi(\theta)}$

Hence, the value of a contingent claim $h(S_T)$ can be obtained, assuming that the emm chosen by the market is \mathbb{Q} as

$$V_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}(h(S_T) | \mathcal{F}_t) = e^{-r(T-t)} \frac{1}{\Psi(\theta)} \mathbb{E}_{\mathbb{P}}(h(y e^{r(T-t)+X_{T-t}} e^{\theta X_{T-t}}) |_{y=S_t})$$

A Differential Equation for Option Pricing

Let $S_t = \exp(rt + X_t)$ where X is a (m, σ^2, ν) Lévy process with $\int e^{2x} \nu(dx) < \infty$. Then

$$dS_t = rS_t dt + \sigma S_t dW_t + \int (e^x - 1) S_{t-} (\mathbf{N}(dt, dx) - dt\nu(dx))$$

$$\partial_t C(t, x) + rx \partial_x C + \frac{1}{2} \sigma^2 \partial_{xx} C + \int \nu(dy) (C(t, xe^y) - C(t, x) - x(e^y - 1) \partial_x C) = 0$$

and a terminal condition $C(T, x) = h(x)$

Put-call Symmetry

Let us study a financial market with a riskless asset with constant interest rate r , and a price process (a currency) $S_t = S_0 e^{X_t}$ where X is a Q -Lévy process such that $(Z_t = e^{-(r-\delta)t} S_t / S_0, t \geq 0)$ is a Q -strictly positive martingale with initial value equal to 1. The Q -characteristic triple (m, σ^2, ν) of X is such that

$$m = r - \delta - \sigma^2/2 - \int (e^y - 1 - y\mathbb{1}_{\{|y| \leq 1\}}) \nu(dy).$$

Then,

$$\begin{aligned} E_Q(e^{-rT}(S_T - K)^+) &= E_Q(e^{-\delta T} Z_T (S_0 - K S_0/S_T)^+) \\ &= E_{\widehat{Q}}(e^{-\delta T} (S_0 - K S_0/S_T)^+) \end{aligned}$$

with $\widehat{Q}|_{\mathcal{F}_t} = Z_t Q|_{\mathcal{F}_t}$. The process X is a \widehat{Q} -Lévy process, with characteristic exponent $\Psi(\lambda + 1) - \Psi(1)$. The process $S_0/S_t = e^{-X_t}$ is the exponential of the Lévy process, $Y = -X$ which is the dual of the Lévy process X and the characteristic exponent of Y is $\widetilde{\Psi}(\lambda) = \Psi(1 - \lambda) - \Psi(1)$. Hence, the following symmetry between call and put prices holds:

$$C_E(S_0, K, r, \delta, T, \Psi) = P_E(K, S_0, \delta, r, T, \widetilde{\Psi})$$

Subordinators

A spectrally negative Lévy process is a Lévy process with no positive jumps, its Lévy measure is supported by $(-\infty, 0)$. Then, X admits exponential moments

$$\mathbb{E}(\exp(\lambda X_t)) = \exp(t\Psi(\lambda)) < \infty, \quad \forall \lambda > 0$$

where

$$\Psi(\lambda) = \lambda m + \frac{1}{2}\sigma^2\lambda^2 + \int_{-\infty}^0 (e^{\lambda x} - 1 - \lambda x \mathbb{1}_{\{-1 < x < 0\}}) \nu(dx).$$

A Lévy process which takes values in $[0, \infty[$ (i.e. with increasing paths) is a subordinator.

$$X_t = bt + \int_0^t \int s \mathbf{N}(dx, ds)$$

In this case, the parameters in the Lévy-Khintchine decomposition are $m \geq 0, \sigma = 0$ and the Lévy measure ν is a measure on $]0, \infty[$ with

$$\int_{]0, \infty[} (1 \wedge x) \nu(dx) < \infty.$$

The Laplace exponent can be expressed as

$$\Phi(u) = \delta u + \int_{]0, \infty[} (1 - e^{-ux}) \nu(dx)$$

where $\delta \geq 0$.

Let Z be a subordinator and X an independent Lévy process. The process $\tilde{X}_t = X_{Z_t}$ is a Lévy process, called subordinated Lévy process.

Compound Poisson process. A compound Poisson process with $Y_k \geq 0$ is a subordinator.

Gamma process. The Gamma process $G(t; \gamma)$ is a subordinator which satisfies

$$G(t+h; \gamma) - G(t; \gamma) \stackrel{law}{=} \Gamma(h; \gamma).$$

Here $\Gamma(h; \gamma)$ follows the Gamma law. The Gamma process is an increasing Lévy process, hence a subordinator, with one sided Lévy measure

$$\frac{1}{x} \exp\left(-\frac{x}{\gamma}\right) \mathbb{1}_{x>0}.$$

Hitting times Let W be a BM, and

$$T_r = \inf\{t \geq 0 : W_t \geq r\}.$$

The process $(T_r, r \geq 0)$ is a stable $(1/2)$ subordinator, its Lévy measure is $\frac{1}{\sqrt{2\pi} x^{3/2}} \mathbb{1}_{x>0} dx$. Let B be a BM independent of W . The process B_{T_t} is a Cauchy process, its Lévy measure is $dx/(\pi x^2)$.

Changes of Lévy characteristics under subordination

Let X be a (a^X, A^X, ν^X) Lévy process and Z be a subordinator with drift β and Lévy measure ν^Z , independent of X . The process $\tilde{X}_t = X_{Z_t}$ is a Lévy process with characteristic exponent

$$\Phi(u) = i(\tilde{a} \cdot u) + \frac{1}{2} \tilde{A}(u) - \int (e^{i(u \cdot x)} - 1 - i(u \cdot x) \mathbb{1}_{|x| \leq 1}) \tilde{\nu}(dx)$$

with

$$\begin{aligned} \tilde{a} &= \beta a^X + \int \nu^Z(ds) \mathbb{1}_{|x| \leq 1} x \mathbb{P}(X_s \in dx) \\ \tilde{A} &= \beta A^X \\ \tilde{\nu}(dx) &= \beta \nu^X dx + \int \nu^Z(ds) \mathbb{P}(X_s \in dx). \end{aligned}$$

Variance-Gamma Model

The **variance Gamma process** is a Lévy process where X_t has a Variance Gamma law $\text{VG}(\sigma, \nu, \theta)$. Its characteristic function is

$$\mathbb{E}(\exp(iuX_t)) = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-t/\nu}.$$

The Variance Gamma process can be characterized as a **time changed BM** with drift as follows: let W be a BM, $\gamma(t)$ a $G(1/\nu, 1/\nu)$ process. Then

$$X_t = \theta\gamma(t) + \sigma W_{\gamma(t)}$$

is a $\text{VG}(\sigma, \nu, \theta)$ process.

The variance Gamma process is a finite variation process. Hence it is the difference of two increasing processes. Madan et al. showed that it is the difference of two independent Gamma processes

$$X_t = G(t; \mu_1, \gamma_1) - G(t; \mu_2, \gamma_2).$$

Indeed, the characteristic function can be factorized

$$\mathbb{E}(\exp(iuX_t)) = \left(1 - \frac{iu}{\nu_1}\right)^{-t/\gamma} \left(1 + \frac{iu}{\nu_2}\right)^{-t/\gamma}$$

with

$$\begin{aligned} \nu_1^{-1} &= \frac{1}{2} \left(\theta\nu + \sqrt{\theta^2\nu^2 + 2\nu\sigma^2} \right) \\ \nu_2^{-1} &= \frac{1}{2} \left(\theta\nu - \sqrt{\theta^2\nu^2 + 2\nu\sigma^2} \right) \end{aligned}$$

The Lévy density of X is

$$\begin{aligned} \frac{1}{\gamma} \frac{1}{|x|} \exp(-\nu_1|x|) & \quad \text{for } x < 0 \\ \frac{1}{\gamma} \frac{1}{x} \exp(-\nu_2x) & \quad \text{for } x > 0. \end{aligned}$$

The density of X_1 is

$$\frac{2e^{\frac{\theta x}{\sigma^2}}}{\gamma^{1/\gamma} \sqrt{2\pi} \sigma \Gamma(1/2)} \left(\frac{x^2}{\theta^2 + 2\sigma^2/\gamma} \right)^{\frac{1}{2\gamma} - \frac{1}{4}} K_{\frac{1}{\gamma} - \frac{1}{2}} \left(\frac{1}{\sigma^2} \sqrt{x^2(\theta^2 + 2\sigma^2/\gamma)} \right)$$

where K_α is the modified Bessel function.

Stock prices driven by a Variance-Gamma process have dynamics

$$S_t = S_0 \exp \left(rt + X(t; \sigma, \nu, \theta) + \frac{t}{\nu} \ln \left(1 - \theta\nu - \frac{\sigma^2\nu}{2} \right) \right)$$

From $\mathbb{E}(e^{X_t}) = \exp \left(-\frac{t}{\nu} \ln \left(1 - \theta\nu - \frac{\sigma^2\nu}{2} \right) \right)$, we get that $S_t e^{-rt}$ is a martingale. The parameters ν and θ give control on skewness and kurtosis.

The CGMY model, introduced by Carr et al. is an extension of the Variance-Gamma model. The Lévy density is

$$\begin{cases} \frac{C}{x^{Y+1}} e^{-Mx} & x > 0 \\ \frac{C}{|x|^{Y+1}} e^{Gx} & x < 0 \end{cases}$$

with $C > 0$, $M \geq 0$, $G \geq 0$ and $Y < 2$, $Y \notin \mathbb{Z}$.

If $Y < 0$, there is a finite number of jumps in any finite interval, if not, the process has infinite activity. If $Y \in [1, 2[$, the process is with infinite variation. This process is also called KoBol.

Double Exponential Model

The Model A particular Lévy model is the double exponential jumps model, introduced by Kou and Wang. In this model

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

where W is a Brownian motion independent of N and $\sum_{i=1}^{N_t} Y_i$ is a compound Poisson process. The r.v.'s Y_i are i.i.d., independent of N and W and the density of the law of Y_1 is

$$f(x) = p\eta_1 e^{-\eta_1 x} \mathbb{1}_{\{x>0\}} + (1-p)\eta_2 e^{\eta_2 x} \mathbb{1}_{\{x<0\}}.$$

The Lévy measure of X is $\nu(dx) = \lambda f(x)dx$.

Here, η_i are positive real numbers, and $p \in [0, 1]$. With probability p (resp. $(1-p)$), the jump size is positive (resp. negative) with exponential law with parameter η_1 (resp η_2).

It is easy to prove that

$$\mathbb{E}(Y_1) = \frac{p}{\eta_1} - \frac{1-p}{\eta_2}, \quad \text{var}(Y_1) = \frac{p}{\eta_1^2} + \frac{1-p}{\eta_2^2} + p(1-p) \left(\frac{1}{\eta_1} + \frac{1}{\eta_2} \right)^2$$

and that, for $\eta_1 > 1$, $\mathbb{E}(e^{Y_1}) = p \frac{\eta_1}{\eta_1-1} + (1-p) \frac{\eta_2}{1+\eta_2}$. Moreover

$$\mathbb{E}(e^{iuX_t}) = \exp \left(t \left\{ -\frac{1}{2} \sigma^2 u^2 + ibu + \lambda \left(\frac{p\eta_1}{\eta_1 - iu} + \frac{(1-p)\eta_2}{\eta_2 + iu} - 1 \right) \right\} \right),$$

where $b = \mu + \lambda \mathbb{E}(Y_1 \mathbb{1}_{|Y_1| \leq 1}) =$

$\mu + \lambda p \left(\frac{1-e^{-\eta_1}}{\eta_1} - e^{-\eta_1} \right) - \lambda(1-p) \left(\frac{1-e^{-\eta_2}}{\eta_2} - e^{-\eta_2} \right)$. The Laplace exponent of X , i.e., the function Ψ such that $\mathbb{E}(e^{\beta X_t}) = \exp(\Psi(\beta)t)$ is defined for $-\eta_2 < \beta < \eta_1$ as

$$\Psi(\beta) = \beta\mu + \frac{1}{2}\beta^2\sigma^2 + \lambda \left(\frac{p\eta_1}{\eta_1 - \beta} + \frac{(1-p)\eta_2}{\beta + \eta_2} - 1 \right).$$

Change of probability Let $S_t = S_0 e^{rt+X_t}$ where $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$. Then, setting $V_i = e^{Y_i}$, using an Escher transform, the process $S_t e^{-rt}$ will be a \mathbb{Q} martingale with $\mathbb{Q}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$ and $L_t = \frac{e^{\alpha X_t}}{\mathbb{E}(e^{\alpha X_t})}$, for α such that $\Psi(\alpha) = \Psi(\alpha + 1)$. Under \mathbb{Q} , the Lévy measure of X is $\widehat{\nu}(dx) = e^{\alpha x} \nu(dx) = e^{\alpha x} \lambda f(x) dx = \widehat{\lambda} \widehat{f}(x) dx$ where, after some standard computations

$$\begin{aligned} \widehat{f}(x) &= \left(\widehat{p} \widehat{\eta}_1 e^{-\widehat{\eta}_1 x} \mathbb{1}_{\{x>0\}} + (1 - \widehat{p}) \widehat{\eta}_2 e^{\widehat{\eta}_2 x} \mathbb{1}_{\{x<0\}} \right) . \\ \widehat{\eta}_1 &= \eta_1 - \alpha, \quad \widehat{\eta}_2 = \eta_2 + \alpha \\ \widehat{\lambda} &= \lambda \left(\frac{p\eta_1}{\eta_1 - \alpha} + \frac{(1-p)\eta_2}{\eta_2 + \alpha} \right) \\ \widehat{p} &= p\eta_1 \frac{\eta_2 + \alpha}{\alpha p\eta_1 + \eta_2(\eta_1 - \alpha + \alpha p\eta_1)} \end{aligned}$$

In particular, the process X is a double exponential process under \mathbb{Q} .

“Vous leur conseillerez donc de faire le calcul. Elles [les grandes personnes] adorent les chiffres: ça leur plaira. Mais ne perdez pas votre temps à ce pensum. C’est inutile. Vous avez confiance en moi.”

Le petit prince, A. de St Exupéry. Gallimard. 1946. p. 59.

Hitting times

Proposition 0.1 *For any $x > 0$*

$$\mathbb{P}(\tau_b \leq t, X_{\tau_b} - b \geq x) = e^{-\eta_1 x} \mathbb{P}(\tau_b \leq t, X_{\tau_b} - b \geq 0)$$

PROOF:

The infinitesimal generator of X is

$$\mathcal{L}f = \frac{1}{2}\sigma^2\partial_{xx}f + \mu\partial_xf + \lambda \int_{\mathbb{R}} (f(x+y) - f(x))\nu(dx)$$

Let $T_x = \inf\{t : X_t \geq x\}$. Then Kou and Wang [?] establish that, for $r > 0$ and $x > 0$,

$$\mathbb{E}(e^{-rT_x}) = \frac{\eta_1 - \beta_1}{\eta_1} \frac{\beta_2}{\beta_2 - \beta_1} e^{-x\beta_1} + \frac{\beta_2 - \eta_1}{\eta_1} \frac{\beta_1}{\beta_2 - \beta_1} e^{-x\beta_2}$$

$$\mathbb{E}(e^{-rT_x} \mathbb{1}_{X_{T_x} - x > y}) = e^{\eta_1 y} \frac{\eta_1 - \beta_1}{\eta_1} \frac{\beta_2 - \eta_1}{\beta_2 - \beta_1} (e^{-x\beta_1} - e^{-x\beta_2})$$

$$\mathbb{E}(e^{\theta X_{T_x} - rT_x}) = e^{\theta x} \left(\frac{\eta_1 - \beta_1}{\beta_2 - \beta_1} \frac{\beta_2 - \theta}{\eta_1 - \theta} e^{-x\beta_1} + \left(\frac{\beta_2 - \eta_1}{\beta_2 - \beta_1} \frac{\beta_1 - \theta}{\eta_1 - \theta} e^{-x\beta_2} \right) \right)$$

where $0 < \beta_1 < \eta_1 < \beta_2$ are roots of $G(\beta) = r$. The method is based on finding an explicit solution of $\mathcal{L}u = ru$ where \mathcal{L} is the infinitesimal generator of the process X .