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Curso de Riesgo Credito

OUTLINE:

1. Structural Approach
2. Hazard Process Approach
3. Hedging Defaultable Claims
4. Credit Default Swaps
5. Several Defaults

Credit Risk: Structural Approach

Tomasz R. Bielecki, IIT, Chicago
Monique Jeanblanc, University of Evry
Marek Rutkowski, University of New South Wales, Sydney

Defaultable Claims and Traded Assets

Defaultable Claims

Let us first describe a generic defaultable claim:

1. **Default** of a firm occurs at time τ. Default may be bankruptcy or other financial distress.
2. At maturity T the **promised payoff** X is paid only if the default did not occurred.
3. The **promised dividends** A are paid up to default time.
4. The **recovery claim** $\widetilde{X}$ is received at time T, if default occurs prior to or at the claim's maturity date T.
5. The **recovery process** Z specifies the recovery payoff at time of default, if default occurs prior to or at the maturity date T.

Traded Assets

- We postulate that a **risky asset** V, which represents the value of the firm, is traded. The **riskless asset** (the savings account B) satisfies

$$dB_t = r\, B_t\, dt.$$

- The market where the riskless asset and the asset V are traded is assumed to be **complete** and **arbitrage free**.

- Under the unique equivalent martingale measure $\mathbb{P}^*$, the value of the firm V satisfies a diffusion process, for instance, a **geometric Brownian motion** given as

$$dV_t = V_t \left(r\, dt + \sigma\, dW_t \right)$$

where W is a one-dimensional standard Brownian motion under the martingale measure $\mathbb{P}^*$.

Merton's Model of Corporate Debt

Merton's Model

Merton's model of a corporate debt postulates that:

1. A firm has a single **liability** with the promised payoff at maturity (nominal value) L. Firm's debt is interpreted as the zero-coupon bond with maturity T.

2. **Default** may occur at time T only. The default event corresponds to the event $\{V_T < L\}$ so that the default time τ equals

$$\tau = T\mathbb{1}_{\{V_T<L\}} + \infty\mathbb{1}_{\{V_T\geq L\}}.$$

3. At maturity T, the holder of the **corporate bond** with the nominal value L receives

$$D_T = \min(V_T, L) = L - \max(L - V_T, 0) = L - (L - V_T)^+.$$

Debt Valuation

- The value $D(V_t)$ of the firm's debt at time t is given by the risk-neutral valuation formula

$$D(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{P}^*}(D_T \,|\, \mathcal{F}_t)$$

where $B(t,T)$ is the price of the unit T-maturity **risk-free bond**, that is,

$$B(t,T) = e^{-r(T-t)}.$$

- We also have that at maturity T

$$D_T = L - P_T = L - (L - V_T)^+.$$

- Hence for any date $t \in [0,T]$

$$D(t,T) = B(t,T)\big(L - \mathbb{E}_{\mathbb{P}^*}\left((L - V_T)^+ \,|\, \mathcal{F}_t\right)\big) = LB(t,T) - P_t$$

where P_t is the price of a **put option** with strike L and expiry T.

Merton's Formula

Proposition 1 *The value $D(t,T)$ of the corporate bond equals for* $0 \leq t < T$

$$D(t,T) = V_t \mathcal{N}\big(-d_+(V_t, T-t)\big) + LB(t,T)\mathcal{N}\big(d_-(V_t, T-t)\big)$$

where

$$d_\pm(V_t, T-t) = \frac{1}{\sigma\sqrt{T-t}} \left(\ln(V_t/L) + \big(r \pm \frac{1}{2}\sigma^2\big)(T-t) \right) .$$

This follows from the equality

$$D(t,T) = LB(t,T) - P_t$$

and the Black-Scholes formula for the put option

$$P_t = LB(t,T)\mathcal{N}\big(-d_-(V_t, T-t)\big) - V_t \mathcal{N}\big(-d_+(V_t, T-t)\big).$$

Distance to Default

- The **real-world probability** of finishing below L at date T is

$$\mathbb{P}(V_T \leq L|\mathcal{F}_t) = \mathcal{N}(-d_-) = \mathcal{N}\left(-\frac{\ln(V_t/L) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

- Hence

$$\mathbb{P}(V_T > L|\mathcal{F}_t) = 1 - \mathcal{N}(-d_-) = \mathcal{N}\left(\frac{\ln(V_t/L) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Definition 1 *The* ***distance to default*** *is given by*

$$\frac{\ln(V_t/L) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\mathbb{E}_{\mathbb{P}}(\ln V_T|\mathcal{F}_t) - \ln L}{\sigma\sqrt{T-t}}.$$

Equity as a Call Option

- The **equity value** at T is given by the expression

$$\max(V_T - L, 0).$$

- It corresponds to the payoff of a **call option** on the assets of the firm V with strike given by the bond's face value L and maturity T.

Corollary 1 *The value $E(V_t)$ of equity at time zero is therefore given by the Black-Scholes (1973)* ***call option pricing formula***

$$E(V_t) = V_t \mathcal{N}\big(d_+(V_t, T-t)\big) - LB(t,T)\mathcal{N}\big(d_-(V_t, T-t)\big)$$

or briefly

$$E(V_t) = \mathcal{BS}\,(V_t, T-t, L, r, \sigma).$$

Estimation of Parameters

- The face value L can be estimated from balance sheet data.
- The rate r can be estimated from prices of default-free (Treasury) bonds.
- To estimate V_0 and σ indirectly, we first observe the equity value $E(V_0)$ and its volatility σ_E directly from the stock market.
- Using these quantities, we then solve a system of two equations for V_0 and σ where:
  - the first equation is provided by the equity pricing formula, relating assets, asset volatility and equity:

$$E(V_0) = \mathcal{BS}(V_0, T, L, r, \sigma).$$

  - the second equation can be obtained via Itô's formula applied to the equity value:

$$\sigma_E E(V_0) = \sigma V_0 \mathcal{N}\big(d_+(V_0, T)\big).$$

Credit Spread

- For $t < T$ the **credit spread** $S(t,T)$ of the corporate bond is defined as
$$S(t,T) = -\frac{1}{T-t} \ln \frac{L\,B(t,T)}{D(t,T)}.$$

- If we define the **forward short spread** at time T as
$$FSS_T(\omega) = \lim_{t \uparrow T} S(t,T)(\omega)$$
then one may check that:
$$FSS_T(\omega) = 0 \quad \text{if} \quad \omega \in \{V_T > L\},$$
and
$$FSS_T(\omega) = \infty \quad \text{if} \quad \omega \in \{V_T < L\}.$$

Drawbacks of Merton's Model

From the practical viewpoint, the classic Merton's approach have several drawbacks:

1. It postulates a simple capital structure.
2. Default is only possible at the debt's maturity.
3. Costless bankruptcy.
4. Perfect capital markets.
5. Risk-free interest rates constant.
6. Only applicable to publicly traded firms.
7. Empirically not plausible.

Black and Cox Model

Black and Cox Model

- In the Black and Cox model, the default occurs at the **first passage time** of the value process V to a deterministic default-triggering barrier.
- The default may thus occur at any time before or on the bond's maturity date T.
- More precisely, the default time equals

$$\tau = \inf \{ t \in [0, T] : V_t < L \}$$

Corporate Bond

- The **corporate bond** is defined as the following defaultable claim:
  - the payoff L is paid at maturity T if there is no default before maturity
  - If the default takes place at $\tau < T$, the recovery $\beta V_\tau = \beta L$ where β is a constant in $[0, 1]$ is paid at time τ.
- Similarly as in Merton's model, it is assumed that the **short-term interest rate** is deterministic and equal to a positive constant r.

Risk-Neutral Valuation

- For any $t < T$ the price $D(t,T)$ of the corporate bond has the following probabilistic representation

$$\begin{aligned} D(t,T) &= L\mathbb{E}_{\mathbb{P}^*}\Big(e^{-r(T-t)}\mathbb{1}_{\{\tau \geq T\}}\,\Big|\,\mathcal{F}_t\Big) \\ &\quad + \beta L\,\mathbb{E}_{\mathbb{P}^*}\Big(e^{-r(\tau-t)}\mathbb{1}_{\{t<\tau<T\}}\,\Big|\,\mathcal{F}_t\Big) \end{aligned}$$

  which is valid on the event $\{\tau > t\}$.

- It is clear that $D(t,T) = u(V_t, t)$ for some **pricing function u.**

Risk-Neutral Valuation

- After default – that is, on the set $\{\tau \le t\}$, we clearly have

$$D(t,T) = 0$$

- To evaluate the conditional expectation, it suffices to use the conditional probability distribution $\mathbb{P}^*(\tau \le s \,|\, \mathcal{F}_t)$ of the first passage time of the process V to the barrier L, for $s \ge t$.

First Passage Time

- Let the value process V obey the SDE

$$dV_t = V_t\big((r-\kappa)\,dt + \sigma\,dW_t\big)$$

with constant coefficients κ and $\sigma > 0$.

- For every $t < s \le T$, on the event $\{t < \tau\}$,

$$\mathbb{P}^*(\tau \le s \,|\, \mathcal{F}_t) = \mathcal{N}\left(\frac{\ln \frac{L}{V_t} - \nu(s-t)}{\sigma\sqrt{s-t}}\right) + \left(\frac{L}{V_t}\right)^{2b} \mathcal{N}\left(\frac{\ln \frac{L}{V_t} + \nu(s-t)}{\sigma\sqrt{s-t}}\right),$$

where

$$b = \frac{\nu}{\sigma^2} = \frac{r-\kappa-\frac{1}{2}\sigma^2}{\sigma^2}.$$

Zero Recovery Case

Let $\kappa = 0$ and let $D^0(t,T)$ be the value of a claim that delivers L at time T if $T < \tau$ and zero otherwise, i.e., the bond with **zero recovery**.

$$D^0(t,T) = e^{-r(T-t)} L\, \mathbb{P}^*\big(\tau \ge T \,\big|\, \mathcal{F}_t\big).$$

Proposition 2 *Let $\nu = r - \frac{1}{2}\sigma^2$. We have, on the event $\{\tau > t\}$,*

$$\begin{aligned} D^0(t,T) \quad &= \quad LB(t,T)\Big(\mathcal{N}\big(h_1(V_t, T-t)\big) - \left(\frac{L}{V_t}\right)^{2\nu} \mathcal{N}\big(h_2(V_t, T-t)\big)\Big), \\ & h_1(V_t, T-t) = \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\ & h_2(V_t, T-t) = \frac{\ln(L/V_t) + \nu(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

Black and Cox Formula: General Case

- In the Black and Cox model, the default occurs at the first passage time of the value process V to a **deterministic** default-triggering barrier.

- More precisely, the default time equals

$$\tau = \inf \{ t \in [0, T] : V_t < K e^{-\gamma(T-t)} \}$$

  for some constant $K \leq L$.

- We write

$$\bar{v}(t) = K e^{-\gamma(T-t)}.$$

Corporate Bond

- The **corporate bond** is defined as the following defaultable claim

$$X = L,\ C = 0,\ Z = \beta_2 V,\ \widetilde{X} = \beta_1 V_T,\ \tau = \bar{\tau} \wedge \widehat{\tau},$$

  where β_1, β_2 are constants in $[0,1]$ and the **early default time** $\bar{\tau}$ equals

$$\bar{\tau} = \inf\{\, t \in [0,T)\ : V_t \leq \bar{v}(t)\}$$

  and $\widehat{\tau}$ is Merton's default time: $\widehat{\tau} = T\mathbb{1}_{\{V_T < L\}} + \infty\mathbb{1}_{\{V_T \geq L\}}$.

- Similarly as in Merton's model, it is assumed that the **short-term interest rate** is deterministic and equal to a positive constant r.

- We postulate, in addition, that $\bar{v}(t) \leq LB(t,T)$ or, more explicitly,

$$Ke^{-\gamma(T-t)} \leq Le^{-r(T-t)}, \quad \forall\, t \in [0,T].$$

Recall that

$$dV_t = V_t \left((r - \kappa)\, dt + \sigma_V \, dW_t \right)$$

where W is a one-dimensional standard Brownian motion under the martingale measure $\mathbb{P}^*$.

We denote

$$\begin{aligned}
\nu &= r - \kappa - \frac{1}{2}\sigma_V^2, \\
m &= \nu - \gamma = r - \kappa - \gamma - \frac{1}{2}\sigma_V^2, \\
b &= m\sigma_V^{-2}.
\end{aligned}$$

For the sake of brevity, in the statement of the Black and Cox valuation result we shall write σ instead of σ_V.

First Passage Time

- For every $t < s \leq T$ and $x \geq L$, the following equality holds on the event $\{t < \tau\}$

$$\mathbb{P}^*(V_s \geq x,\, \tau \geq s \,|\, \mathcal{F}_t) = \mathcal{N}\left(\frac{\ln(V_t/x) + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right) - \left(\frac{L}{V_t}\right)^{2b} \mathcal{N}\left(\frac{\ln L^2 - \ln(xV_t) + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right),$$

  where $\nu = r - \kappa - \frac{1}{2}\sigma_V^2$.

- Both formulae follow from the well known properties of the Brownian motion (in particular, the reflection principle).

Basic Lemma

Let $\sigma > 0$ and $\nu \in \mathbb{R}$. Let $X_t = \nu t + \sigma W_t$ for every $t \in \mathbb{R}_+$ where W is a Brownian motion under $\mathbb{Q}$.

Lemma 1 *For every* $x > 0$

$$\mathbb{Q}\big(\sup_{0 \le u \le s} X_u \le x\big) = \mathcal{N}\left(\frac{x - \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu\sigma^{-2}x}\mathcal{N}\left(\frac{-x - \nu s}{\sigma\sqrt{s}}\right)$$

and for every $x < 0$

$$\mathbb{Q}\big(\inf_{0 \le u \le s} X_u \ge x\big) = \mathcal{N}\left(\frac{-x + \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu\sigma^{-2}x}\mathcal{N}\left(\frac{x + \nu s}{\sigma\sqrt{s}}\right).$$

Proof of the Lemma: 1

To derive the first equality, we combine Girsanov's theorem with reflection principle for the Brownian motion. Assume first that $\sigma = 1$.

Let $\mathbb{P}$ be the probability measure on $(\Omega, \mathcal{F}_s)$ given by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{-\nu W_s - \frac{\nu^2}{2} s}, \quad \mathbb{Q}\text{-a.s.}$$

so that the process $W_t^* := X_t = W_t + \nu t$, $t \in [0, s]$, is a standard Brownian motion under $\mathbb{P}$. Also

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\nu W_s^* - \frac{\nu^2}{2} s}, \quad \mathbb{P}\text{-a.s.}$$

Moreover, for $x > 0$,

$$\mathbb{Q}\Big(\sup_{0 \le u \le s} X_u > x,\, X_s \le x\Big) = \mathbb{E}_{\mathbb{P}}\Big(e^{\nu W_s^* - \frac{\nu^2}{2} s}\, \mathbb{1}_{\{\sup_{0 \le u \le s} W_u^* > x,\, W_s^* \le x\}}\Big).$$

Proof of the Lemma: 2

We set $\tau_x = \inf\{\, t \geq 0 : W^*_t = x\}$ and we define an auxiliary process $(\widetilde{W}_t,\, t \in [0, s])$ by setting

$$\widetilde{W}_t = W^*_t \mathbb{1}_{\{\tau_x \geq t\}} + (2x - W^*_t)\mathbb{1}_{\{\tau_x < t\}}.$$

By virtue of the reflection principle, $\widetilde{W}$ is a Brownian motion under $\mathbb{P}$.

Moreover, we have

$$\{\sup_{0 \leq u \leq s} \widetilde{W}_u > x,\, \widetilde{W}_s \leq x\} = \{W^*_s \geq x\} \subset \{\tau_x \leq s\}.$$

Let

$$J = \mathbb{Q}\big(\sup_{0 \leq u \leq s} X_u \leq x\big) = \mathbb{Q}\big(\sup_{0 \leq u \leq s} (W_u + \nu u) \leq x\big).$$

Proof of the Lemma: 3

$$
\begin{aligned}
J &= \mathbb{Q}(X_s \le x) - \mathbb{Q}\big(\sup_{0\le u\le s} X_u > x,\ X_s \le x\big) \\
&= \mathbb{Q}(X_s \le x) - \mathbb{E}_{\mathbb{P}}\left(e^{\nu W^*_s - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{\sup_{0\le u\le s} W^*_u > x,\, W^*_s \le x\}}\right) \\
&= \mathbb{Q}(X_s \le x) - \mathbb{E}_{\mathbb{P}}\left(e^{\nu \widetilde{W}_s - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{\sup_{0\le u\le s} \widetilde{W}_u > x,\, \widetilde{W}_s \le x\}}\right) \\
&= \mathbb{Q}(X_s \le x) - \mathbb{E}_{\mathbb{P}}\left(e^{\nu (2x - W^*_s) - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{W^*_s \ge x\}}\right) \\
&= \mathbb{Q}(X_s \le x) - e^{2\nu x}\,\mathbb{E}_{\mathbb{P}}\left(e^{\nu W^*_s - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{W^*_s \le -x\}}\right) \\
&= \mathbb{Q}(W_s + \nu s \le x) - e^{2\nu x}\,\mathbb{Q}(W_s + \nu s \le -x) \\
&= \mathcal{N}\left(\frac{x - \nu s}{\sqrt{s}}\right) - e^{2\nu x}\mathcal{N}\left(\frac{-x - \nu s}{\sqrt{s}}\right).
\end{aligned}
$$

Proof of the Lemma: 4

This ends the proof of the first equality for $\sigma = 1$.

We have, for any $\sigma > 0$,

$$\mathbb{Q}\Big(\sup_{0 \le u \le s} (\sigma W_u + \nu u) \le x \Big) = \mathbb{Q}\Big(\sup_{0 \le u \le s} (W_u + \nu \sigma^{-1} u) \le x\sigma^{-1} \Big)$$

and this implies the first formula for any $\sigma \neq 0$.

Since $-W$ is a standard Brownian motion under $\mathbb{Q}$, we also have that, for any $x < 0$,

$$\mathbb{Q}\Big(\inf_{0 \le u \le s} (\sigma W_u + \nu u) \ge x \Big) = \mathbb{Q}\Big(\sup_{0 \le u \le s} (\sigma W_u - \nu u) \le -x \Big)$$

and thus the second formula follows from the first one.

Black and Cox Formula

Proposition 3 *Assume that*

$$m^2 + 2\sigma^2(r-\gamma) > 0.$$

The price $D(t,T) = u(V_t,t)$ of the corporate bond equals, on the event $\{\tau > t\}$,

$$\begin{aligned} D(t,T) = {} & LB(t,T)\Big(\mathcal{N}\big(h_1(V_t,T-t)\big) - R_t^{2b}\mathcal{N}\big(h_2(V_t,T-t)\big)\Big) \\ & + \beta_1 V_t e^{-\kappa(T-t)}\Big(\mathcal{N}\big(h_3(V_t,T-t)\big) - \mathcal{N}\big(h_4(V_t,T-t)\big)\Big) \\ & + \beta_1 V_t e^{-\kappa(T-t)} R_t^{2b+2}\Big(\mathcal{N}\big(h_5(V_t,T-t)\big) - \mathcal{N}\big(h_6(V_t,T-t)\big)\Big) \\ & + \beta_2 V_t\Big(R_t^{\theta+\zeta}\mathcal{N}\big(h_7(V_t,T-t)\big) + R_t^{\theta-\zeta}\mathcal{N}\big(h_8(V_t,T-t)\big)\Big) \end{aligned}$$

where

$$R_t = \bar{v}(t)/V_t,\ \ \theta = b+1,\ \ \zeta = \sigma^{-2}\sqrt{m^2 + 2\sigma^2(r-\gamma)}.$$

Black and Cox Formula

$$
\begin{aligned}
h_1(V_t, T-t) &= \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\
h_2(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\
h_3(V_t, T-t) &= \frac{\ln(L/V_t) - (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_4(V_t, T-t) &= \frac{\ln(K/V_t) - (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_5(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(LV_t) + (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_6(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(KV_t) + (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_{7,8}(V_t, T-t) &= \frac{\ln(\bar{v}(t)/V_t) \pm \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}
$$

Proof of the Black and Cox Formula

Lemma 2 *For any $a \in \mathbb{R}$ and $b > 0$ we have, for every $y > 0$,*

$$\int_0^y x\, d\mathcal{N}\left(\frac{\ln x + a}{b}\right) = e^{\frac{1}{2}b^2 - a}\,\mathcal{N}\left(\frac{\ln y + a - b^2}{b}\right)$$

$$\int_0^y x\, d\mathcal{N}\left(\frac{-\ln x + a}{b}\right) = e^{\frac{1}{2}b^2 + a}\,\mathcal{N}\left(\frac{-\ln y + a + b^2}{b}\right).$$

Let a, b, $c \in \mathbb{R}$ satisfy $b < 0$ and $c^2 > 2a$. Then for $y > 0$

$$\int_0^y e^{ax}\, d\mathcal{N}\left(\frac{b - cx}{\sqrt{x}}\right) = \frac{d+c}{2d}\, g(y) + \frac{d-c}{2d}\, h(y),$$

where
$d = \sqrt{c^2 - 2a}$, $g(y) = e^{b(c-d)}\,\mathcal{N}\left(\frac{b-dy}{\sqrt{y}}\right)$, $h(y) = e^{b(c+d)}\,\mathcal{N}\left(\frac{b+dy}{\sqrt{y}}\right)$.

Drawbacks of Black and Cox Model

Black and Cox model inherits some drawbacks of the original Merton approach:

1. Simple capital structure.
2. Perfect capital markets.
3. Risk-free interest rates constant.
4. Only applicable to publicly traded firms.
5. Empirically not plausible.

Shortcomings of Structural Approach

1. Assumes the total value of firm assets can be easily observed.
2. Postulates that the total value of firm assets is a tradable security.
3. Generates low credit spreads for corporate bonds close to maturity.
4. Requires a judicious specification of the default barrier in order to get a good fit with the observed spread curves.
5. Defaults can be determined by factors other than assets and liabilities (for example, defaults could occur for reasons of illiquidity).

Further Developments

The first-passage-time approach was later developed by:

- Leland (1994), Hilberink and Rogers (2005), Decamps et al. (2008): optimal capital structure, bankruptcy costs, tax benefits,
- Longstaff and Schwartz (1995): constant barrier and random interest rates (Vasicek's model),
- Kou (2003) : First passage time, Lévy process, constant barrier
- Moraux (2003): Parisian default time,
- Coculescu et al. (2007), Herkommer (2007), Cetin (2008): Incomplete information
- and others.

Levy processes

In Kou's model

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \, ,$$

where the density of the law of Y_1 is

$$\nu(dx) = \left(p\eta_1 e^{-\eta_1 x} 1\!\!1_{\{x>0\}} + (1-p)\eta_2 e^{\eta_2 x} I_{\{x<0\}}\right) dx \, .$$

Here, η_i are positive real numbers, and $p \in [0,1]$.

The default time is

$$\tau = \inf\{t \, : \, X_t \leq b\}$$

Parisian Default Time

- For a continuous process V and a given $t > 0$, we introduce a random variable $g_t^b(V)$, representing the last moment before t when the process V was at a given level b

$$g_t^b(V) = \sup \{ 0 \leq s \leq t : V_s = b \}.$$

- The **Parisian stopping time** is the first time at which the process V is below the level b for a time period of length greater or equal to a constant D. Formally, the stopping time τ is given by the formula

$$\tau = \inf \{ t \in \mathbb{R}_+ : (t - g_t^b(V)) \mathbb{1}_{\{V_t < b\}} \geq D \}.$$

- In the case of V given by the Black-Scholes equation, it is possible to find the joint probability distribution of (τ, V_τ) by means of the Laplace transform.

Partial Observation

The investor has no full knowledge of the value of the firm

- The observed process is correlated with the value of the firm
- The value of the firm is observed with noise

In that case, one has to compute

$$\mathbb{Q}(\tau > t|\mathcal{G}_t)$$

where $\mathbb{G}$ is the filtration of the observation

References

- F. Black and M. Scholes (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81, 81-98.
- R.C. Merton (1974) On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *Journal of Finance* 29, 449-470.
- F. Black and J.C. Cox (1976) Valuing Corporate Securities: Some Bond Indenture Provisions. *Journal of Finance* 31, 351-367.
- KMV (1997) Modeling Default Risk. www.kmv.com
- JP Morgan (1997) CreditMetrics: Technical Document. www.riskmetrics.com