X Simposio de Probabilidad y Procesos Estocasticos

1ra Reunión Franco Mexicana de Probabilidad

Guanajuato, 3 al 7 de noviembre de 2008

Curso de Riesgo Credito

1. Structural Approach
2. Hazard Process Approach
3. Hedging Defaultable Claims
4. Credit Default Swaps
5. Several Defaults: Density Approach

Hazard Process Approach

Tomasz R. Bielecki, IIT, Chicago
Monique Jeanblanc, University of Evry
Marek Rutkowski, University of New South Wales, Sydney

In a financial market built on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, a default occurs at some random time τ.

The filtration $\mathbb{F}$ is called the reference filtration

The random time τ is a non-negative random variable on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and we denote by $H_t = \mathbb{1}_{\tau \leq t}$ the default process.

OUTLINE:

- **Model for single default**
- **Intensity approach**
- **Density approach**

Model for single default

Hazard Process

Hazard Process of a Random Time

- We set $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ where $\mathcal{H}_t$ is the natural filtration of H
- We shall write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ to denote the **full filtration**.
- We denote $G_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t)$ the **conditional survival probability**.
- It is easily seen that G is a bounded, non-negative, $\mathbb{F}$-super-martingale.
- We assume that $G_t > 0$ for every $t \in \mathbb{R}_+$ and we set $\Gamma_t = -\ln G_t$. The process Γ is called the **hazard process**.
- Any $\mathcal{G}_t$-measurable random variable Y_t writes

$$Y_t \mathbb{1}_{t<\tau} = \widetilde{Y}_t \mathbb{1}_{t<\tau}$$

where $\widetilde{Y}_t$ is $\mathcal{F}_t$-measurable

Properties of the supermartingale G

- Let $G_t = m_t - A_t$ be the Doob-Meyer decomposition of the super-martingale $(G_t, t \geq 0)$.

- The process

$$M_t = H_t - \int_0^t (1 - H_s) \frac{dA_s}{G_{s-}} = H_t - \Lambda_{t \wedge \tau}$$

is a $\mathbb{G}$-martingale. The process $\Lambda_t = \int_0^t \frac{dA_s}{G_{s-}}$ is called the $\mathbb{F}$-intensity.

- The multiplicative decomposition of the supermartingale G is $G_t = n_t K_t$ where n is an $\mathbb{F}$-martingale and K a predictable non increasing process. One has

$$dn_t = e^{\Lambda_t} dM_t, \; K_t = e^{-\Lambda_t}$$

- If G is **non increasing**, then τ is a pseudo-stopping time. A random time τ is a pseudo-stopping time if for any bounded $\mathbb{F}$-martingale m, one has $\mathbb{E}(m_\tau) = m_0$ or, equivalently if $m_{t\wedge\tau}$ is a $\mathbb{G}$-martingale
- If any $\mathbb{F}$ martingale is continuous and if τ avoids $\mathbb{F}$-stopping times, then G is continuous
- G is a continuous non increasing process if and only if τ is a pseudo-stopping time that avoids stopping times.
- If G is continuous and **non increasing**, then $\Lambda = \Gamma$ and the process $M_t = H_t - \Gamma_{t\wedge\tau}$ is a $\mathbb{G}$-martingale.

Conditional Expectations

- (*Dellacherie*) For any $\mathcal{G}$**-measurable** random variable Y we have

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}Y \,|\, \mathcal{F}_t)}{\mathbb{P}(\tau > t \,|\, \mathcal{F}_t)}.$$

- If, in addition, Y is $\mathcal{F}_s$**-measurable** for $s \geq t$, then

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}}Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{P}}(Y G_s \,|\, \mathcal{F}_t).$$

- Let G be continuous and let Z be an $\mathbb{F}$-predictable process. Then for any $t \leq s$, we have

$$\mathbb{E}_{\mathbb{P}}(Z_\tau \mathbb{1}_{\{t<\tau\leq s\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{P}}\Big(-\int_t^s Z_u \, dG_u \,\Big|\, \mathcal{F}_t \Big).$$

Immersion property

- (*Brémaud-Yor*) If

$$G_t = \mathbb{P}(\tau > t | \mathcal{F}_t) = \mathbb{P}(\tau > t | \mathcal{F}_\infty)$$

  then any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.

- (*Kusuoka*) In that case, if the filtration $\mathbb{F}$ is generated by a Brownian motion W, then, any $\mathbb{G}$-martingale Z admits a representation

$$Z_t = z + \int_0^t \widehat{z}_s dW_s + \int_0^t \widetilde{z}_s dM_s$$

  where $\widehat{z}$ and $\widetilde{z}$ are $\mathbb{G}$-predictable processes

Interpretation of the Intensity Process

- We now restrict our attention to the case where $\Lambda_t = \int_0^t \lambda_u \, du$ where λ represents the $\mathbb{F}$-intensity rate of τ.

- Intuitively

$$\mathbb{P}\{\tau \in [t, t+dt] \,|\, \mathcal{F}_t \vee \mathcal{H}_t\} = \mathbb{1}_{\{\tau > t\}} \lambda_t \, dt$$

that is

$$\mathbb{P}\{\tau \in [t, t+dt] \,|\, \mathcal{F}_t \vee \{\tau > t\}\} = \lambda_t \, dt.$$

Canonical Construction

- Let Λ be an $\mathbb{F}$-adapted, increasing, continuous processes, defined on a probability space $(\widehat{\Omega}, \mathbb{F}, \mathbb{P})$. We assume that $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$.
- Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ be an auxiliary probability space with a random variable U uniformly distributed on $[0,1]$. Hence $\zeta = -\ln U$ has the unit exponential probability distribution
- We set, on $(\Omega, \mathcal{F}, \ \mathbb{P}) = (\widehat{\Omega} \times \widetilde{\Omega}, \widehat{\mathcal{F}} \otimes \widetilde{\mathcal{F}}, \widehat{\mathbb{P}} \times \widetilde{\mathbb{P}})$

$$\tau = \inf \{ \, t \in \mathbb{R}_+ \ : \ \Lambda_t(\widehat{\omega}) \geq - \ln U(\widetilde{\omega}) \, \}$$

- The random variable U is independent of the hazard process Λ, the r.v. $-\ln U$ has exponential law.
- Then

$$\mathbb{P}(\tau > t | \mathcal{F}_t) = \exp(-\Lambda_t) = \mathbb{P}(\tau > t | \mathcal{F}_\infty)$$

In that model, any $\mathbb{F}$-martingale in a $\mathbb{G}$-martingale.

Immersion property

It can be proved that, if

$$G_t = \mathbb{P}(\tau > t | \mathcal{F}_t) = \mathbb{P}(\tau > t | \mathcal{F}_\infty)$$

is continuous and strictly increasing, then there exists a random variable Θ, independent of $\mathcal{F}_\infty$ such that

$$\tau = \inf\{\, t \in \mathbb{R}_+ \,:\, -\ln G_t \geq \Theta \,\} = \inf\{\, t \in \mathbb{R}_+ \,:\, \Gamma_t \geq -\ln U \,\}$$

Immersion property (2)

Assume that

- $\mathbb{F}$ is the filtration generated by default-free assets with prices $S_t, t \geq 0$
- this market is complete and arbitrage free
- using strategies which are $\mathbb{G}$-adapted does not give arbitrage opportunities
- the prices of default free assets remain $(S_t, t \geq 0)$ in the filtration $\mathbb{G}$,

then immersion property holds true under the unique e.m.m.

Valuation of Defaultable Claims

- In order to value a defaultable claim we need also to specify a discount factor (for instance, the savings account).
- Here we have assumed that $B = 1$, that is, $r = 0$.
- **We assume that immersion property holds under the e.m.m. $\mathbb{Q}$**

Valuation of the Terminal Payoff

To value the terminal payoff we shall use the following result.

Proposition 1

If λ^Q is the default intensity rate under $\mathbb{Q}$ then

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{Q}}(Y e^{-\int_t^s \lambda_u^Q \, du} \,|\, \mathcal{F}_t).$$

Valuation of Recovery Process

The following result appears to be useful in the valuation of the recovery payoff Z_τ which occurs at time τ.

Proposition 2 *If λ^Q is the default intensity under $\mathbb{Q}$ then*

$$\mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{t<\tau\le s\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{Q}}\Big(\int_t^s Z_u e^{-\int_t^u \lambda_v^Q\, dv}\, \lambda_u^Q\, du \,\Big|\, \mathcal{F}_t \Big).$$

Valuation of Promised Dividends

To value the **promised dividends** A that are paid prior to τ we shall make use of the following result.

Proposition 3 *Assume that Λ^Q is a continuous process and let A be an $\mathbb{F}$-predictable bounded process of finite variation. Then for every $t \leq s$*

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{(t,s]} (1 - H_u)\, dA_u \,\Big|\, \mathcal{G}_t \Big) = \mathbb{1}_{\{\tau > t\}}\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{(t,s]} e^{\Lambda_t^Q - \Lambda_u^Q}\, dA_u \,\Big|\, \mathcal{F}_t \Big).$$

Defaultable Assets

Let $B(t,T)$ be the price at time t of a default-free bond paying 1 at maturity T satisfies

$$B(t,T) = \mathbb{E}_{\mathbb{Q}}\Big(\exp\big(-\int_t^T r_s\, ds\big) \,\Big|\, \mathcal{F}_t \Big).$$

The market price $D(t,T)$ of a defaultable zero-coupon bond with maturity T is

$$\begin{aligned} D(t,T) &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{T<\tau\}} \exp\big(-\int_t^T r_s\, ds\big) \,\Big|\, \mathcal{G}_t \Big) \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\Big(\exp\big(-\int_t^T [r_s + \lambda_s^Q]\, ds\big) \,\Big|\, \mathcal{F}_t \Big). \end{aligned}$$

Recovery paid at Maturity

We consider a contract which pays Z_τ at date T, if $\tau \leq T$ where Z is an $\mathbb{F}$-adapted process and no payment in the case $\tau > T$. We also assume that **the interest rate is null**.

Recovery paid at Maturity

We consider a contract which pays Z_τ at date T, if $\tau \leq T$ where Z is an $\mathbb{F}$-adapted process and no payment in the case $\tau > T$. We also assume that the interest rate is null. The price at time t of this contract is

$$S_t = \mathbb{E}(Z_\tau \mathbb{1}_{\tau \leq T} | \mathcal{G}_t) = Z_\tau \mathbb{1}_{\tau \leq t} + \mathbb{1}_{t < \tau} \mathbb{E}(Z_\tau \mathbb{1}_{t < \tau \leq T} | \mathcal{G}_t)$$

Recovery paid at Maturity

We consider a contract which pays Z_τ at date T, if $\tau \leq T$ where Z is an $\mathbb{F}$-adapted process and no payment in the case $\tau > T$. We also assume that the interest rate is null. The price at time t of this contract is

$$\begin{aligned} S_t &= \mathbb{E}(Z_\tau 1\!\!1_{\tau \leq T} | \mathcal{G}_t) = Z_\tau 1\!\!1_{\tau \leq t} + 1\!\!1_{t<\tau} \mathbb{E}(Z_\tau 1\!\!1_{t<\tau\leq T} | \mathcal{G}_t) \\ &= Z_\tau 1\!\!1_{\tau \leq t} + 1\!\!1_{t<\tau}\, e^{\Lambda_t}\, \mathbb{E}(\int_t^T Z_u e^{-\Lambda_u} \lambda_u du | \mathcal{F}_t) \end{aligned}$$

Recovery paid at Maturity

We consider a contract which pays Z_τ at date T, if $\tau \leq T$ where Z is an $\mathbb{F}$-adapted process and no payment in the case $\tau > T$. We also assume that the interest rate is null. The price at time t of this contract is

$$\begin{aligned} S_t &= \mathbb{E}(Z_\tau 1\!\!1_{\tau \leq T} | \mathcal{G}_t) = Z_\tau 1\!\!1_{\tau \leq t} + 1\!\!1_{t \leq \tau} \mathbb{E}(Z_\tau 1\!\!1_{t < \tau \leq T} | \mathcal{G}_t) \\ &= Z_\tau 1\!\!1_{\tau \leq t} + 1\!\!1_{t < \tau}\, e^{\Lambda_t}\, \mathbb{E}(\int_t^T Z_u e^{-\Lambda_u} \lambda_u du | \mathcal{F}_t) \\ &= \int_0^t Z_u dH_u + L_t \left(m_t^Z - \int_0^t Z_u e^{-\Lambda_u} \lambda_u du \right) \end{aligned}$$

where $m_t^Z = \mathbb{E}(\int_0^T Z_u e^{-\Lambda_u} \lambda_u du | \mathcal{F}_t)$ and $L_t = 1\!\!1_{t < \tau}\, e^{\Lambda_t}$

Recovery paid at Maturity

We consider a contract which pays Z_τ at date T, if $\tau \le T$ where Z is an $\mathbb{F}$-adapted process and no payment in the case $\tau > T$. We also assume that the interest rate is null. The price at time t of this contract is

$$
\begin{aligned}
S_t &= \mathbb{E}(Z_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) = Z_\tau 1\!\!1_{\tau<t} + 1\!\!1_{t<\tau}\mathbb{E}(Z_\tau 1\!\!1_{t<\tau<T}|\mathcal{G}_t) \\
&= Z_\tau 1\!\!1_{\tau<t} + 1\!\!1_{t<\tau}\, e^{\Lambda_t}\, \mathbb{E}(\int_t^T Z_u e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t) \\
&= \int_0^t Z_u dH_u + L_t \left(m_t^Z - \int_0^t Z_u e^{-\Lambda_u}\lambda_u du \right)
\end{aligned}
$$

where $m_t^Z = \mathbb{E}(\int_0^T Z_u e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t)$ and $L_t = 1\!\!1_{t<\tau}\, e^{\Lambda_t}$ are $\mathbb{G}$ martingales.

We assume here that $\mathbb{F}$-martingales are continuous. From $dL_t = -L_{t-} dM_t$ and integration by parts formula we deduce that

$$
\begin{aligned}
dS_t &= Z_t(dH_t - \lambda_t(1 - H_t)dt) - S_{t-} dM_t + L_t dm_t^Z \\
&= (Z_t - S_{t-})\, dM_t + L_t dm_t^Z
\end{aligned}
$$

Recovery paid at Default

If the payment Z is done at time τ

$$
\begin{aligned}
S_t &= \mathbb{1}_{t<\tau}\mathbb{E}(Z_\tau \mathbb{1}_{t<\tau<T}|\mathcal{G}_t) \\
&= L_t\left(m_t^Z - \int_0^t Z_u e^{-\Lambda_u}\lambda_u du\right)
\end{aligned}
$$

Recovery paid at Default

If the payment Z is done at time τ

$$\begin{aligned} S_t &= \mathbb{1}_{t<\tau}\mathbb{E}(Z_\tau \mathbb{1}_{t<\tau<T}|\mathcal{G}_t) \\ &= L_t \left(m_t^Z - \int_0^t Z_u e^{-\Lambda_u} \lambda_u du \right) \end{aligned}$$

Recovery paid at Default

If the payment Z is done at time τ

$$\begin{aligned} S_t &= \mathbb{1}_{t<\tau}\mathbb{E}(Z_\tau \mathbb{1}_{t<\tau<T}|\mathcal{G}_t) \\ &= L_t\left(m_t^Z - \int_0^t Z_u e^{-\Lambda_u}\lambda_u du\right) \end{aligned}$$

where $m_t^Z = \mathbb{E}(\int_0^T Z_u e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t)$.

Recovery paid at Default

If the payment Z is done at time τ

$$\begin{aligned} S_t &= \mathbb{1}_{t<\tau}\mathbb{E}(Z_\tau \mathbb{1}_{t<\tau<T}|\mathcal{G}_t) \\ &= L_t\left(m_t^Z - \int_0^t Z_u e^{-\Lambda_u}\lambda_u du\right) \end{aligned}$$

where $m_t^Z = \mathbb{E}(\int_0^T Z_u e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t)$. In that case $S_t = 0$ after default.

$$dS_t = -Z_t\lambda_t(1-H_t)dt - S_{t-}dM_t + L_t dm_t^Z\,.$$

Recovery paid at Default

If the payment Z is done at time τ

$$\begin{aligned} S_t &= \mathbb{1}_{t<\tau}\mathbb{E}(Z_\tau \mathbb{1}_{t<\tau<T}|\mathcal{G}_t) \\ &= L_t \left(m_t^Z - \int_0^t Z_u e^{-\Lambda_u} \lambda_u du \right) \end{aligned}$$

where $m_t^Z = \mathbb{E}(\int_0^T Z_u e^{-\Lambda_u} \lambda_u du|\mathcal{F}_t)$. In that case $S_t = 0$ after default.

$$dS_t = -Z_t \lambda_t (1 - H_t)dt - S_{t-} dM_t + L_t dm_t^Z \,.$$

The process $S_t + \int_0^t Z_s(1 - H_s)\lambda_s ds$ is a martingale.

Price and Hedging a defaultable call

The savings account $Y^0_t = 1$, a risky asset with risk-neutral dynamics $dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t,T)$ are traded.

Price and Hedging a defaultable call

The savings account $Y_t^0 = 1$, a risky asset with risk-neutral dynamics $dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t,T)$ are traded. The reference filtration is that of the Brownian motion W.

Price and Hedging a defaultable call

The savings account $Y_t^0 = 1$, a risky asset with risk-neutral dynamics $dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t,T)$ are traded. The reference filtration is that of the Brownian motion W. Then,

$$D(t,T) = L_t \mathbb{Q}(\tau > T | \mathcal{F}_t) = L_t m_t$$

with $m_t = \mathbb{Q}(\tau > T | \mathcal{F}_t) = \mathbb{E}(e^{-\Lambda_T} | \mathcal{F}_t)$, $L_t = \mathbb{1}_{t<\tau}\, e^{\Lambda_t}$.

Price and Hedging a defaultable call

The savings account $Y_t^0 = 1$, a risky asset with risk-neutral dynamics $dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t,T)$ are traded. The reference filtration is that of the Brownian motion W. Then,

$$D(t,T) = L_t \mathbb{Q}(\tau > T | \mathcal{F}_t) = L_t m_t$$

with $m_t = \mathbb{Q}(\tau > T|\mathcal{F}_t) = \mathbb{E}(e^{-\Lambda_T}|\mathcal{F}_t)$, $L_t = 1\!\!1_{t<\tau}\, e^{\Lambda_t}$.

The price of a defaultable call with payoff $1\!\!1_{T<\tau}(Y_T - K)^+$ is

$$C_t = \mathbb{E}(1\!\!1_{T<\tau}(Y_T - K)^+ | \mathcal{G}_t) = 1\!\!1_{t<\tau} e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+ | \mathcal{F}_t)$$

Price and Hedging a defaultable call

The savings account $Y_t^0 = 1$, a risky asset with risk-neutral dynamics $dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t,T)$ are traded. The reference filtration is that of the Brownian motion W. Then,

$$D(t,T) = L_t \mathbb{Q}(\tau > T | \mathcal{F}_t) = L_t m_t$$

with $m_t = \mathbb{Q}(\tau > T|\mathcal{F}_t) = \mathbb{E}(e^{-\Lambda_T}|\mathcal{F}_t)$, $L_t = \mathbb{1}_{t<\tau}\, e^{\Lambda_t}$.
The price of a defaultable call with payoff $\mathbb{1}_{T<\tau}(Y_T - K)^+$ is

$$\begin{aligned} C_t &= \mathbb{E}(\mathbb{1}_{T<\tau}(Y_T - K)^+|\mathcal{G}_t) = \mathbb{1}_{t<\tau} e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+|\mathcal{F}_t) \\ &= L_t m_t^Y \end{aligned}$$

with $m_t^Y = \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+|\mathcal{F}_t)$.

Price and Hedging a defaultable call

The savings account $Y_t^0 = 1$, a risky asset with risk-neutral dynamics $dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t,T)$ are traded. The reference filtration is that of the Brownian motion W. Then,

$$D(t,T) = L_t \mathbb{Q}(\tau > T | \mathcal{F}_t) = L_t m_t$$

with $m_t = \mathbb{Q}(\tau > T|\mathcal{F}_t) = \mathbb{E}(e^{-\Lambda_T}|\mathcal{F}_t)$, $L_t = \mathbb{1}_{t<\tau} e^{\Lambda_t}$.
The price of a defaultable call with payoff $\mathbb{1}_{T<\tau}(Y_T - K)^+$ is

$$\begin{aligned} C_t &= \mathbb{E}(\mathbb{1}_{T<\tau}(Y_T - K)^+|\mathcal{G}_t) = \mathbb{1}_{t<\tau} e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+|\mathcal{F}_t) \\ &= L_t m_t^Y \end{aligned}$$

with $m_t^Y = \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+|\mathcal{F}_t)$, hence

$$dC_t = L_t dm_t^Y - m_t^Y L_{t-} dM_t$$

In the particular case where λ is deterministic, $m_t = e^{-\Lambda_T}$ and $dm_t = 0$. Therefore, $D(t,T) = m_t L_t = L_t e^{-\Lambda_T}$ and

$$dD(t,T) = m_t dL_t = -m_t L_{t-} dM_t = -e^{-\Lambda_T} L_{t-} dM_t \, .$$

In the particular case where λ is deterministic, $m_t = e^{-\Lambda_T}$ and $dm_t = 0$. Therefore, $D(t,T) = m_t L_t = L_t e^{-\Lambda_T}$ and

$$dD(t,T) = m_t dL_t = -m_t L_{t-} dM_t = -e^{-\Lambda_T} L_{t-} dM_t \,.$$

Furthermore,

$$m_t^Y = e^{-\Lambda_T} \mathbb{E}((Y_T - K)^+ | \mathcal{F}_t) = e^{-\Lambda_T} C_t^Y$$

where C^Y is the price of a call in the Black Scholes model.

In the particular case where λ is deterministic, $m_t = e^{-\Lambda_T}$ and $dm_t = 0$. Therefore, $D(t,T) = m_t L_t = L_t e^{-\Lambda_T}$ and

$$dD(t,T) = m_t dL_t = -m_t L_{t-} dM_t = -e^{-\Lambda_T} L_{t-} dM_t \, .$$

Furthermore,

$$m_t^Y = e^{-\Lambda_T} \mathbb{E}((Y_T - K)^+ | \mathcal{F}_t) = e^{-\Lambda_T} C_t^Y$$

where C^Y is the price of a call in the Black Scholes model.
This quantity is $C_t^Y = C^Y(t, Y_t)$ and satisfies $dC_t^Y = \Delta_t dY_t$ where Δ_t is the Delta-hedge ($\Delta_t = \partial_y C^Y(t, Y_t)$).

In the particular case where λ is deterministic, $m_t = e^{-\Lambda_T}$ and $dm_t = 0$. Therefore, $D(t,T) = m_t L_t = L_t e^{-\Lambda_T}$ and

$$dD(t,T) = m_t dL_t = -m_t L_{t-} dM_t = -e^{-\Lambda_T} L_{t-} dM_t \,.$$

Furthermore,

$$m_t^Y = e^{-\Lambda_T} \mathbb{E}((Y_T - K)^+ | \mathcal{F}_t) = e^{-\Lambda_T} C_t^Y$$

where C^Y is the price of a call in the Black Scholes model.
This quantity is $C_t^Y = C^Y(t, Y_t)$ and satisfies $dC_t^Y = \Delta_t dY_t$ where Δ_t is the Delta-hedge ($\Delta_t = \partial_y C^Y(t, Y_t)$).

$$\begin{aligned} C_t &= L_t m_t^Y = \mathbb{1}_{t<\tau} e^{\Lambda_t} e^{-\Lambda_T} C^Y(t, Y_t) \\ &= L_t e^{-\Lambda_T} C^Y(t, Y_t) = D(t,T) C^Y(t, Y_t) \end{aligned}$$

From

$$C_t = D(t,T)C^Y(t,Y_t)$$

we deduce

$$\begin{aligned} dC_t &= e^{-\Lambda_T}(L_t dC_t^Y + C_t^Y dL_t) = e^{-\Lambda_T}(L_t \Delta_t dY_t - C_t^Y L_{t-} dM_t) \\ &= e^{-\Lambda_T}(L_t \Delta_t dY_t - C_t^Y L_{t-} dM_t) \end{aligned}$$

From

$$C_t = D(t,T)C^Y(t, Y_t)$$

we deduce

$$\begin{aligned} dC_t &= e^{-\Lambda_T}(L_t dC_t^Y + C_t^Y dL_t) = e^{-\Lambda_T}(L_t \Delta_t dY_t - C_t^Y L_{t-} dM_t) \\ &= e^{-\Lambda_T}(L_t \Delta_t dY_t - C_t^Y L_{t-} dM_t) \end{aligned}$$

Therefore, using that $dD(t,T) = m_t dL_t = -e^{-\Lambda_T} L_{t-} dM_t$ we get

$$dC_t = e^{-\Lambda_T} L_t \Delta_t dY_t + C_t^Y dD(t,T) = e^{-\Lambda_T} L_t \Delta_t dY_t + \frac{C_t}{D(t,T)} dD(t,T)$$

From

$$C_t = D(t,T)C^Y(t,Y_t)$$

we deduce

$$\begin{aligned} dC_t &= e^{-\Lambda_T}(L_t dC_t^Y + C_t^Y dL_t) = e^{-\Lambda_T}(L_t \Delta_t dY_t - C_t^Y L_{t-} dM_t) \\ &= e^{-\Lambda_T}(L_t \Delta_t dY_t - C_t^Y L_{t-} dM_t) \end{aligned}$$

Therefore, using that $dD(t,T) = m_t dL_t = -e^{-\Lambda_T} L_{t-} dM_t$ we get

$$dC_t = e^{-\Lambda_T} L_t \Delta_t dY_t + C_t^Y dD(t,T) = e^{-\Lambda_T} L_t \Delta_t dY_t + \frac{C_t}{D(t,T)} dD(t,T)$$

hence, an hedging strategy consists of holding $\frac{C_{t-}}{D(t,T)}$ DZCs.

In the general case, one obtains

$$dC_t = \frac{C_{t-}}{D(t,T)} dD(t,T) + L_{t-} \frac{m_t^Y}{m_t} dM_t + L_{t-} dm_t^Y$$

An hedging strategy consists of holding $\frac{C_{t-}}{D(t,T)}$ DZCs.

Intensity approach

In intensity based models, the default time τ is a stopping time in a given filtration $\mathbb{G}$, representing the full information of the market.

Definition of the intensity process

- The process $(H_t = 1\!\!1_{\tau \leq t},\, t \geq 0)$ is a $\mathbb{G}$-adapted increasing càdlàg process, hence a $\mathbb{G}$-submartingale, and there exists a unique $\mathbb{G}$-predictable increasing process $\Lambda^{\mathbb{G}}$, called the $\mathbb{G}$-compensator, such that the process
$$M_t = H_t - \Lambda_t^{\mathbb{G}}$$
is a $\mathbb{G}$-martingale. The compensator satisfies $\Lambda_t^{\mathbb{G}} = \Lambda_{t\wedge\tau}^{\mathbb{G}}$.
- The process $\Lambda^{\mathbb{G}}$ is continuous if and only if τ is a $\mathbb{G}$-totally inaccessible stopping time.
- If τ is predictable, $M_t = H_t - H_t = 0$

A predictable stopping time T is a stopping time such that there exists a sequence of stopping times T_n so that $T_n < T$ and $T_n \to T$

A totally inaccessible stopping time is a stopping time so that $\mathbb{P}(T = S) = 0$ for any predictable stopping time S.

- In intensity based models, it is generally assumed that $\Lambda^{\mathbb{G}}$ is absolutely continuous with respect to Lebesgue measure, i.e., that there exists a non-negative $\mathbb{G}$-adapted process $(\lambda_t^{\mathbb{G}}, t \geq 0)$ such that

$$M_t = H_t - \int_0^t \lambda_s^{\mathbb{G}} ds$$

is a $\mathbb{G}$-martingale.

- This process $\lambda^{\mathbb{G}}$ is called the $\mathbb{G}$-intensity rate and vanishes after time τ, i.e.,

$$M_t = H_t - \int_0^{t \wedge \tau} \lambda_s^{\mathbb{G}} ds = H_t - \int_0^t (1 - H_s) \lambda_s^{\mathbb{G}} ds.$$

- One gets, under some regularity assumption,

$$\lambda_t^{\mathbb{G}} = \lim_{h \to 0} \frac{1}{h} \mathbb{Q}(t < \tau \leq t + h | \mathcal{G}_t) = \lim_{h \to 0} \frac{1}{h} \mathbb{1}_{\{t < \tau\}} \mathbb{Q}(\tau \leq t + h | \mathcal{G}_t),$$

when the limit (a.s.) exists.

Pricing rule for conditional claims

For $X \in \mathcal{G}_T$, integrable,

$$\mathbb{E}_{\mathbb{Q}}(X \mathbb{1}_{T<\tau} | \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \left(V_t - \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau \leq T\}} \Delta V_\tau | \mathcal{G}_t) \right)$$

where the process V is defined by:

$$V_t = e^{\Lambda_t^{\mathbb{G}}} \mathbb{E}_{\mathbb{Q}}(X e^{-\Lambda_T^{\mathbb{G}}} | \mathcal{G}_t) = e^{\Lambda_{t \wedge \tau}^{\mathbb{G}}} \mathbb{E}_{\mathbb{Q}}(X e^{-\Lambda_{T \wedge \tau}^{\mathbb{G}}} | \mathcal{G}_t).$$

and where ΔV_τ denotes the jump of V at τ, i.e., $\Delta V_\tau = V_\tau - V_{\tau^-}$.

Using the intensity rate, the pricing rule becomes:

$$\mathbb{E}_{\mathbb{Q}}(X \mathbb{1}_{T<\tau} | \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}} \left(X e^{-\int_t^T \lambda_s^{\mathbb{G}} ds} \middle| \mathcal{G}_t \right) - \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau \leq T\}} \Delta V_\tau | \mathcal{G}_t).$$

Proof: Apply the integration by parts formula to the product $U = VL$ (remark $U_T = 1\!\!1_{\{T<\tau\}} X$), with $L_t = 1 - H_t$

$$dU_t = (\Delta V_\tau)\, dL_t + (L_{t-} dm_t - V_{t-} dM_t)\,,$$

(where $dm_t = e^{\Lambda_t} dY_t$, for $Y_t = e^{-\Lambda_t} V_t$), which yields to $U_t = \mathbb{E}_{\mathbb{Q}}(1\!\!1_{t<\tau\leq T}\, \Delta V_\tau + U_T | \mathcal{G}_t)$.

For example, whereas the price of a zero-coupon bond writes (if $\beta_t = \exp\left(-\int_0^t r_s ds\right)$ denotes the savings account):

$$B\left(t,T\right) = \beta_t \mathbb{E}_{\mathbb{Q}}\left(\left.\frac{1}{\beta_T}\right| \mathcal{G}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(\left.e^{-\int_t^T r_s ds}\right| \mathcal{G}_t\right),$$

the price of a defaultable zero-coupon bond with no recovery and notional 1 is:

$$D\left(t,T\right) = \beta_t \mathbb{E}_{\mathbb{Q}}\left(\left.\frac{1\!\!1_{T<\tau}}{\beta_T}\right| \mathcal{G}_t\right)$$

$$= 1\!\!1_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}\left(\left.e^{-\int_t^T \left(r_s + \lambda_s^{\mathbb{G}}\right) ds}\right| \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{Q}}(1\!\!1_{\{t<\tau\leq T\}}\, \Delta V_\tau^D | \mathcal{G}_t)$$

where $V_t^D = \mathbb{E}_{\mathbb{Q}}(\exp - \int_t^{\tau\wedge T} \lambda_s ds | \mathcal{G}_t)$.

Density Approach

We assume here that there exists a family of processes $(\alpha(u), u \in \mathbb{R}^+)$ such that

$$G_t(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty \alpha_t(u) du$$

Note that, for any u, the process $(\alpha_t(u), t \geq 0)$ is an $\mathbb{F}$-martingale such that $\int_0^\infty \alpha_t(u) du = 1$. One has

$$G_t = G_t(t) = \mathbb{P}(\tau > t | \mathcal{F}_t)$$

We assume that G and α does not vanish.

- The Doob-Meyer decomposition of G is given by

$$G_t = Z_t - \int_0^t \alpha_u(u)du$$

where Z is the square-integrable martingale defined as

$$Z_t = 1 - \int_0^t \big(\alpha_t(u) - \alpha_u(u)\big)du = \mathbb{E}[\int_0^\infty \alpha_u(u)du|\mathcal{F}_t].$$

- We define $\lambda_t := \frac{\alpha_t(t)}{G_{t-}}$. The process

$$H_t - \int_0^t (1 - H_s)\lambda_s ds$$

is a $\mathbb{G}$-martingale

- The multiplicative decomposition of G is given by

$$G_t = n_t e^{-\int_0^t \lambda_s ds} \quad \text{where} \quad dn_t = e^{\int_0^t \lambda_s ds} dZ_t.$$

The $\mathcal{G}_t$-conditional expectation of $f(\tau)$ is given by

$$\mathbb{E}[f(\tau)|\mathcal{G}_t] = \alpha_t^{\mathrm{bd}}(f)\,1\!\!1_{\{\tau>t\}} + f(\tau)\,1\!\!1_{\{\tau\le t\}}$$

where

$$\alpha_t^{\mathrm{bd}}(f) = \frac{1}{G_t}\int_t^\infty \alpha_t(u) f(u)du$$

More generally, the $\mathcal{G}_t$-conditional expectation of any integrable $\mathcal{F}_t \otimes \sigma(\tau)$-measurable r.v. $Y_t(\tau)$, is given by

$$\mathbb{E}[Y_t(\tau)|\mathcal{G}_t] = \alpha_t^{\mathrm{bd}}(Y_t)\, \mathbb{1}_{\{\tau > t\}} + Y_t(\tau)\, \mathbb{1}_{\{\tau \leq t\}}$$

where

$$\alpha_t^{\mathrm{bd}}(Y_t) = \frac{\int_t^\infty Y_t(u)\alpha_t(u)\eta(du)}{S_t}$$

For any $T \geq t$, let $Y_T(\theta)$ be a function $\mathcal{F}_T \otimes \mathcal{B}$-measurable

Then,

$$\mathbb{E}[Y_T(\tau)|\mathcal{G}_t]\mathbb{1}_{\{\tau\leq t\}} = Y_t^{ad}(T,\tau)\mathbb{1}_{\{\tau\leq t\}} \quad d\mathbb{P}-a.s..$$

where

$$Y_t^{ad}(T,\theta) := \frac{\mathbb{E}\big[Y_T(\theta)\alpha_T(\theta)\big|\mathcal{F}_t\big]}{\alpha_t(\theta)} \quad d\mathbb{P}-a.s..$$

Assume that $\widetilde{S} = (\widetilde{S}_t, t \leq T)$ is an $\mathbb{R}^{n+2}$ valued process constructed on $(\Omega, \mathcal{A}, \mathbb{P})$, S^0 denoting the saving accounts, and $\mathbb{G}$ is the natural filtration generated by $\widetilde{S}$.

We emphasize that $\mathbb{P}$ is a probability measure defined on $\mathcal{A}$.

We denote by $\Theta^{\mathbb{G}}_{\mathbb{P}}(\widetilde{S})$ the set of $\mathbb{G}$-e.m.ms, i.e., the set of probability measures $\mathbb{Q}$ defined on $\mathcal{A}$, equivalent to $\mathbb{P}$ on $\mathcal{A}$, such that the discounted process $(\widetilde{S}_t/S^0_t, t \leq T)$ is a $(\mathbb{G}, \mathbb{Q})$-local martingale.

In what follows, we assume that $S^0 \equiv 1$.

Assume that $\mathbb{F}$ is the natural filtration of the $\mathbb{R}^{n+1}$-valued process S and that this market is complete. Let $\mathbb{P}^*$ be an e.m.m. (the restriction of $\mathbb{P}^*$ to $\mathbb{F}$ is unique) For every $X \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$, there exists two $\mathbb{G}$-predictable process β and γ such that

$$dX_t = \gamma_t d\widehat{S}_t + \beta_t dM_t.$$

There exists a probability $\mathbb{Q} \in \Theta^{\mathbb{G}}_{\mathbb{P}^*}(S)$ such that immersion property holds under $\mathbb{Q}$

If the market generated by S is incomplete, we assume that the market chooses an e.m.m. $\mathbb{P}^*$. We assume that a default sensitive asset S^{n+2} is traded.

There exists a unique $\mathbb{G}$-e.m.m. $\mathbb{Q} \in \Theta_{\mathbb{P}^*}^{\mathbb{G}}\left(\widetilde{S}\right)$, that preserves $\mathcal{F}_T$, i.e.,

$$\mathbb{E}^{\mathbb{Q}}\left(X_T\right) = \mathbb{E}^*\left(X_T\right),$$

for any $X_T \in L^2\left(\mathcal{F}_T\right)$.