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Curso de Riesgo Credito

Hedging of Defaultable Claims

Tomasz R. Bielecki, IIT, Chicago
Monique Jeanblanc, University of Evry
Marek Rutkowski, University of New South Wales, Sydney

The Model

In the sequel,

- $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ is a filtered probability space,
- The process W is a $\mathbb{G}$ Brownian motion with natural filtration $\mathbb{F}$,
- τ is a $\mathbb{G}$-stopping time,
- We assume $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$
- $M_t = H_t - \int_0^t (1 - H_s)\lambda_s ds$ is the compensated $(\mathbb{P}, \mathbb{G})$-martingale.

1. Two default free assets, one defaultable asset
    1.1 Two default free assets, one total default asset
    1.2 Two default free assets, one defaultable with recovery
2. Two defaultable assets

Two default-free assets, one defaultable asset

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- a defaultable asset

$$dY_t^3 = Y_{t-}^3(\mu_{3,t}dt + \sigma_{3,t}dW_t + \kappa_{3,t}dM_t)\,,$$

where the coefficients $\mu_3, \sigma_3, \kappa_3$ are $\mathbb{G}$-adapted processes with $\kappa_3 \geq -1$.

Our aim is to hedge defaultable claims. As we shall establish, the case of **total default** for the third asset (i.e. $\kappa_{3,t} \equiv -1$) is different from the others.

Two default-free assets, a total default asset

Assume that Y^3 is a defaultable asset with zero recovery, so that

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(\mu_2\,dt + \sigma_2\,dW_t\big), \\ dY_t^3 &= Y_{t-}^3\big(\mu_3\,dt + \sigma_3\,dW_t - \,dM_t\big). \end{aligned}$$

that $Y^{i,1} = Y^i/Y^1$ are martingales is

$$d\mathbb{Q}|_{\mathcal{G}_t} = L_t d\mathbb{P}|_{\mathcal{G}_t}\,,$$

where

$$dL_t = L_{t-}(\theta_t dW_t + \zeta_t dM_t)$$

The unknown processes θ and ζ in the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfy the following equations

$$\begin{aligned} \mu_2 - r + \sigma_2\theta_t &= 0, \\ \mu_3 - r + \sigma_3\theta_t - \lambda\zeta_t &= 0, \quad \text{for } t \le \tau. \end{aligned}$$

Hence, the unique solution is

$$\begin{aligned} \theta &= \frac{r - \mu_2}{\sigma_2} \\ \zeta\lambda &= \mu_3 - r + \sigma_3 \frac{r - \mu_2}{\sigma_2}, \quad \text{for } t \leq \tau \end{aligned}$$

as soon as $\zeta > -1$.

Under $\mathbb{Q}$, the processes

$$
\begin{aligned}
W_t^* &= W_t - \int_0^t \theta_s ds \\
M_t^* &= M_t - \int_0^t (1 - H_s)\lambda_s \zeta_s ds = H_t - \int_0^t (1 - H_s)\lambda_s^* ds
\end{aligned}
$$

where

$$\lambda_t^* = \lambda_t(1 + \zeta_t)$$

are $\mathbb{G}$-martingales.

Here, our aim is to hedge **survival claims** $(X, 0, \tau)$, i.e. contingent claims of the form $X \mathbb{1}_{T<\tau}$ where $X \in \mathcal{F}_T$.

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$$C_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}(X \mathbb{1}_{T<\tau} | \mathcal{G}_t)$$

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The hedging strategy consists of a triple ϕ^1, ϕ^1, ϕ^3 such that

$$\phi_t^3 Y_t^3 = C_t, \quad \phi_t^1 e^{rt} + \phi_t^2 Y_t = 0$$

and which satisfies the self financing condition.

PDE Approach

We are working in a model with constant (or Markovian) coefficients

$$\begin{aligned} dY_t &= Y_t r dt \\ dY_t^2 &= Y_t^2(\mu_2 dt + \sigma_2 dW_t) \\ dY_t^3 &= Y_{t-}^3(\mu_3 dt + \sigma_3 dW_t - dM_t)\,. \end{aligned}$$

In other terms, $\sigma_i = \sigma_i(t, Y_t^2, Y_t^3, H_t)$.

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Let $C(t, Y_t^2, Y_t^3, H_t)$ be the price of the contingent claim $G(Y_T^2, Y_T^3, H_T)$ and λ^* be the risk-neutral intensity of default.

Then,

$$\partial_t C(t, y_2, y_3; 0) + r y_2 \partial_2 C(t, y_2, y_3; 0) + \widehat{r} y_3 \partial_3 C(t, y_2, y_3; 0) - \widehat{r} C(t, y_2, y_3; 0)$$
$$+ \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 0) + \lambda^* C(t, y_2, 0; 1) = 0$$

where $\widehat{r} = r + \lambda^*$

Then,

$$\partial_t C(t, y_2, y_3; 0) + r y_2 \partial_2 C(t, y_2, y_3; 0) + \widehat{r} y_3 \partial_3 C(t, y_2, y_3; 0) - \widehat{r} C(t, y_2, y_3; 0)$$
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where $\widehat{r} = r + \lambda^*$ and

$$\partial_t C(t, y_2; 1) + r y_2 \partial_2 C(t, y_2; 1) + \frac{1}{2} \sigma_2^2 y_2^2 \partial_{22} C(t, y_2; 1) - r C(t, y_2; 1) = 0$$

Then,

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with the terminal conditions

$$C(T, y_2, y_3; 0) = G(y_2, y_3; 0), \quad C(T, y_2; 1) = G(y_2, 0; 1).$$

The ***replicating strategy*** ϕ *for* Y *is given by formulae*

$$
\begin{aligned}
\phi_t^3 Y_{t-}^3 &= -\Delta C(t) := -C(t, Y_t^2, 0; 1) + C(t, Y_t^2, Y_{t-}^3; 0) \\
\sigma_2 \phi_t^2 Y_t^2 &= -\Delta C(t) + \sum_{i=2}^{3} Y_{t-}^i \sigma_i \partial_i C(t) \\
\phi_t^1 Y_t^1 &= C(t) - \phi_t^2 Y_t^2 - \phi_t^3 Y_t^3 \,.
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Note that, in the case of survival claim, $C(t, Y_t^2, 0; 1) = 0$ and $\phi_t^3 Y_{t-}^3 = C(t, Y_{t-}^2, Y_{t-}^3; 0)$ for every $t \in [0, T]$.

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$$\phi_t^3 Y_t^3 = C(t, Y_t^2, Y_t^3; 0), \quad \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0.$$

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The last equality is a special case of the **balance condition**. It ensures that the wealth of a replicating portfolio falls to 0 at default time.

Example 1

Assume that λ^* is a constant. Consider a survival claim $Y = \mathbb{1}_{\{T<\tau\}} g(Y_T^2)$. Its pre-default pricing function $C(t, y_2, y_3\,; 0) = C^g(t, y_2)$ where C^g solves

$$\begin{aligned} \partial_t C^g(t,y;0) + ry\partial_2 C^g(t,y;0) + \frac{1}{2}\sigma_2^2 y^2 \partial_{22} C^g(t,y;0) - \widehat{r} C^g(t,y;0) &= 0 \\ C^g(T,y;0) &= g(y) \end{aligned}$$

The solution is

$$C^g(t,y) = e^{(\widehat{r}-r)(t-T)}\, C^{r,g,2}(t,y) = e^{\widehat{\lambda}(t-T)}\, C^{r,g,2}(t,y),$$

where $C^{r,g,2}$ is the price of an option with payoff $g(Y_T)$ in a BS model with interest rate r and volatility σ_2.

Example 2

Consider a survival claim of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T<\tau\}} g(Y_T^3).$$

Then the pre-default pricing function $C^g(\cdot\,;0)$ is

$$C^g(t, y_2, y_3; 0) = C^{\widehat{r},g,3}(t, y_3)$$

where $C^{\alpha,g,3}(t, y)$ is the price of the contingent claim $g(Y_T)$ in the Black-Scholes framework with the interest rate α and the volatility parameter equal to σ_3.

Two default-free assets, one defaultable asset with Recovery, PDE approach

Let the price processes Y^1, Y^2, Y^3 *satisfy*

$$\begin{aligned} dY_t^1 &= rY_t^1 dt \\ dY_t^2 &= Y_t^2(\mu_2 dt + \sigma_2 dW_t) \\ dY_t^3 &= Y_{t-}^3(\mu_3 dt + \sigma_3 dW_t + \kappa_3 dM_t) \end{aligned}$$

with $\sigma_2 \neq 0$. *Assume that* $\kappa_3 \neq 0, \kappa_3 > -1$.

We also assume that the model is Markov.

The martingale

$$dL_t = L_{t-}(\theta_t dW_t + \zeta_t dM_t)$$

is an e.m.m. if

$$\begin{aligned} \theta_t &= \sigma_2^{-1}(\mu_2 - r) \\ \mu_3 - r + \sigma_3\theta_t + \kappa_3\zeta_t &= 0, \quad \text{on } t < \tau \\ \mu_3 - r + \sigma_3\theta_t &= 0, \quad \text{on } t > \tau \end{aligned}$$

with the condition $\zeta > -1$. Hence

$$\frac{\mu_3 - r}{\sigma_3} = \frac{\mu_2 - r}{\sigma_2} \quad \text{on } t > \tau$$

In the particular case where the coefficients are deterministic functions of time, $\zeta = 0$

Under $\mathbb{Q}$,

$$H_t - \int_0^t \lambda_s^*(1 - H_s)ds$$

is a $\mathbb{G}$-martingale, with $\lambda_t^* = \lambda_t(1 + \zeta_t)$

Then the price of a contingent claim $Y = G(Y_T^2, Y_T^3, H_T)$ can be represented as $C(t, Y_t^2, Y_t^3; H_t)$, where the pricing functions $C(\cdot\,;0)$ and $C(\cdot\,;1)$ satisfy the following PDEs

$$\partial_t C(t, y_2, y_3; 1) + r y_2 \partial_2 C(t, y_2, y_3; 1) + r y_3 \partial_3 C(t, y_2, y_3; 1) - r C(t, y_2, y_3; 1)$$
$$+ \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 1) = 0$$

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$$+ \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 1) = 0$$

and

$$\begin{aligned} &\partial_t C(t, y_2, y_3; 0) + r y_2 \partial_2 C(t, y_2, y_3; 0) + y_3 \left(r - \kappa_3 \lambda^*\right) \partial_3 C(t, y_2, y_3; 0) \\ - \quad & r C(t, y_2, y_3; 0) + \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 0) \\ & + \lambda^* \big(C(t, y_2, y_3(1+\kappa_3); 1) - C(t, y_2, y_3; 0) \big) = 0 \end{aligned}$$

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subject to the terminal conditions

$$C(T, y_2, y_3; 0) = G(y_2, y_3, 0), \quad C(T, y_2, y_3; 1) = G(y_2, y_3, 1).$$

The replicating strategy equals $\phi = (\phi^1, \phi^2, \phi^3)$

$$\begin{aligned}
\phi_t^2 &= \frac{1}{\sigma_2 \kappa_3 Y_t^2} \Bigg(\kappa_3 \sum_{i=2}^{3} \sigma_i y_i \partial_i C(t, Y_t^2, Y_{t-}^3, H_{t-}) \\
&\quad - \sigma_3 \big(C(t, Y_t^2, Y_{t-}^3 (1+\kappa_3); 1) - C(t, Y_t^2, Y_{t-}^3; 0) \big) \Big) , \\
\phi_t^3 &= \frac{1}{\kappa_3 Y_{t-}^3} \big(C(t, Y_t^2, Y_{t-}^3 (1+\kappa_3); 1) - C(t, Y_t^2, Y_{t-}^3; 0) \big) ,
\end{aligned}$$

and where ϕ_t^1 *is given by* $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 = C_t$.

Example: constant coefficients Consider a survival claim of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T<\tau\}} g(Y_T^3).$$

Then the post-default pricing function $C^g(\cdot\,;1)$ vanishes identically, and the pre-default pricing function $C^g(\cdot\,;0)$ solves

$$\begin{aligned}
\partial_t C^g(\cdot\,;0) \quad &+ \quad r y_2 \partial_2 C^g(\cdot\,;0) + y_3\,(r - \kappa_3\lambda)\,\partial_3 C^g(\cdot\,;0) \\
&+ \quad \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C^g(\cdot\,;0) - (r+\lambda) C^g(\cdot\,;0) = 0 \\
C^g(T, y_2, y_3; 0) \quad &= \quad g(y_3)
\end{aligned}$$

Denote $\alpha = r - \kappa_3\lambda$ and $\beta = \lambda(1+\kappa_3)$.

Then, $C^g(t, y_2, y_3; 0) = e^{\beta(T-t)} C^{\alpha,g,3}(t, y_3)$ where $C^{\alpha,g,3}(t,y)$ is the price of the contingent claim $g(Y_T)$ in the Black-Scholes framework with the interest rate α and the volatility parameter equal to σ_3.

Let C_t be the current value of the contingent claim Y, so that

$$C_t = \mathbb{1}_{\{t<\tau\}} e^{\beta(T-t)} C^{\alpha,g,3}(t, y_3).$$

The hedging strategy of the survival claim is, on the event $\{t < \tau\}$,

$$\begin{aligned}
\phi_t^3 Y_t^3 &= -\frac{1}{\kappa_3} e^{-\beta(T-t)} C^{\alpha,g,3}(t, Y_t^3) = -\frac{1}{\kappa_3} C_t, \\
\phi_t^2 Y_t^2 &= \frac{\sigma_3}{\sigma_2} \left(Y_t^3 e^{-\beta(T-t)} \partial_y C^{\alpha,g,3}(t, Y_t^3) - \phi_t^3 Y_t^3 \right).
\end{aligned}$$

Hedging of a Recovery Payoff

The price C^g of the payoff $G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T \geq \tau\}} g(Y_T^2)$ solves

$$\begin{aligned} \partial_t C^g(\cdot\,;1) + ry\partial_y C^g(\cdot\,;1) + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy} C^g(\cdot\,;1) - rC^g(\cdot\,;1) &= 0 \\ C^g(T, y; 1) &= g(y) \end{aligned}$$

hence $C^g(t, y_2, y_3, 1) = C^{r,g,2}(t, y_2)$ is the price of $g(Y_T^2)$ in the model Y^1, Y^2. Prior to default, the price of the claim solves

$$\begin{aligned} \partial_t C^g(\cdot;0) &+ ry_2 \partial_2 C^g(\cdot\,;0) + y_3\left(r - \kappa_3\lambda\right)\partial_3 C^g(\cdot\,;0) \\ &+ \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C^g(\cdot\,;0) - (r+\lambda)C^g(\cdot\,;0) = -\lambda C^g(t, y_2; 1) \\ C^g(T, y_2, y_3; 0) &= 0 \end{aligned}$$

Hence $\quad C^g(t, y_2, y_3; 0) = (1 - e^{\lambda(t-T)})C^{r,g,2}(t, y_2).$

Two defaultable assets with total default

Assume that Y^1 and Y^2 are defaultable tradeable assets with zero recovery and a common default time τ.

$$dY_t^i = Y_{t-}^i(\mu_i dt + \sigma_i dW_t - dM_t), i = 1, 2$$

Then

$$Y_t^1 = 1\!\!1_{\{\tau > t\}}\widetilde{Y}_t^1, \quad Y_t^2 = 1\!\!1_{\{\tau > t\}}\widetilde{Y}_t^2$$

with

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i((\mu_i + \lambda_t)dt + \sigma_i dW_t), i = 1, 2$$

The wealth process V associated with the self-financing trading strategy (ϕ^1, ϕ^2) satisfies for $t \in [0, T]$

$$V_t = Y_t^1 \left(V_0^1 + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^{2,1} \right)$$

where $\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2 / \widetilde{Y}_t^1$.

Obviously, this market is **incomplete, however, some contingent claims are hedgeable**, as we present now.

Hedging Survival claim: martingale approach

A strategy (ϕ^1, ϕ^2) replicates a survival claim $C = X\mathbb{1}_{\{\tau>T\}}$ whenever we have

$$\widetilde{Y}_T^1\Big(\widetilde{V}_0^1 + \int_0^T \phi_t^2 \, d\widetilde{Y}_t^{2,1}\Big) = X$$

for some constant $\widetilde{V}_0^1$ and some $\mathbf{F}$-predictable process ϕ^2.

It follows that if $\sigma_1 \neq \sigma_2$, **any survival claim $C = X\mathbb{1}_{\{\tau>T\}}$ is attainable.**

Let $\widetilde{Q}$ be a probability measure such that $\widetilde{Y}_t^{2,1}$ is an $\mathbb{F}$-martingale under $\widetilde{Q}$. Then the pre-default value $\widetilde{U}_t(C)$ at time t of $(X, 0, \tau)$ equals

$$\widetilde{U}_t(C) = \widetilde{Y}_t^1 \, E_{\widetilde{Q}}\left(X(\widetilde{Y}_T^1)^{-1} \,|\, \mathcal{F}_t\right).$$

Example: Call option on a defaultable asset We assume that $Y_t^1 = D(t,T)$ represents a defaultable ZC-bond with zero recovery, and $Y_t^2 = \mathbb{1}_{\{t<\tau\}}\widetilde{Y}_t^2$ is a generic defaultable asset with total default. The payoff of a call option written on the defaultable asset Y^2 equals

$$C_T = (Y_T^2 - K)^+ = \mathbb{1}_{\{T<\tau\}}(\widetilde{Y}_T^2 - K)^+$$

The replication of the option reduces to finding a constant x and an $\mathbb{F}$-predictable process ϕ^2 that satisfy

$$x + \int_0^T \phi_t^2 \, d\widetilde{Y}_t^{2,1} = (\widetilde{Y}_T^2 - K)^+.$$

Assume that the volatility $\sigma_{1,t} - \sigma_{2,t}$ of $\widetilde{Y}^{2,1}$ is deterministic. Let $\widetilde{F}_2(t,T) = \widetilde{Y}_t^2(\widetilde{D}(t,T))^{-1}$

The credit-risk-adjusted forward price of the option written on Y^2 equals

$$\widetilde{C}_t = \widetilde{Y}_t^2 \mathcal{N}\big(d_+(\widetilde{F}_2(t,T),t,T)\big) - K\widetilde{D}(t,T)\mathcal{N}\big(d_-(\widetilde{F}_2(t,T),t,T)\big),$$

where

$$d_\pm(\widetilde{f},t,T) = \frac{\ln \widetilde{f} - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_{1,u} - \sigma_{2,u})^2 \, du.$$

Moreover the replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi_t^1 = -K\mathcal{N}\big(d_-(\widetilde{F}_2(t,T),t,T)\big), \quad \phi_t^2 = \mathcal{N}\big(d_+(\widetilde{F}_2(t,T),t,T)\big).$$

Hedging Survival claim: PDE approach

Assume that $\sigma_1 \neq \sigma_2$. Then the pre-default pricing function v satisfies the PDE

$$\partial_t C + y_1 \left(\mu_1 + \lambda - \sigma_1 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) \partial_1 C + y_2 \left(\mu_2 + \lambda - \sigma_2 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) \partial_2 C$$
$$+ \frac{1}{2} \left(y_1^2 \sigma_1^2 \partial_{11} C + y_2^2 \sigma_2^2 \partial_{22} C + 2 y_1 y_2 \sigma_1 \sigma_2 \partial_{12} C \right) = \left(\mu_1 + \lambda - \sigma_1 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) C$$

with the terminal condition $C(T, y_1, y_2) = G(y_1, y_2)$.

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