

X Simposio de Probabilidad y Procesos Estocasticos

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Curso de Riesgo Credito

1. Structural Approach
2. Hazard Process Approach
3. Hedging Defaultable Claims
4. Credit Default Swaps
5. Several Defaults: Density Approach

Credit Default Swaps

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Objectives

1. Valuation of Credit Default Swaps
2. Hedging of Defaultable Claims with a CDS
3. Hedging of First-to-Default Claims with CDSs
4. Hedging of Basket Credit Derivatives with CDSs

Credit Default Swaps, Single name

Defaultable Claims

A generic **defaultable claim** (X, A, Z, τ) consists of:

1. A **promised contingent claim** X representing the payoff received by the owner of the claim at time T , if there was no default prior to or at maturity date T .
2. A process A representing the **dividends stream** prior to default.
3. A **recovery process** Z representing the recovery payoff at time of default, if default occurs prior to or at maturity date T .
4. A **default time** τ , where the use of the term **default** is merely a convention.

Dividend Process

The dividend process D describes all cash flows associated with a defaultable claim (except for the initial price of a claim at time 0).

Definition 1 The *dividend process* D of a defaultable claim (X, A, Z, τ) maturing at T equals, for every $t \in [0, T]$,

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T, \infty[}(t) + \int_{]0, t]} (1 - H_u) dA_u + \int_{]0, t]} Z_u dH_u.$$

Note that the process D has finite variation on $[0, T]$.

Ex-dividend Price

The ex-dividend price S_t of a defaultable claim is aimed to represent the current value at time t of all dividend payments occurring during the time period $]t, T]$.

Let the process B represent the savings account.

Definition 2 The *ex-dividend price process* S associated with the dividend process D satisfies, for every $t \in [0, T]$,

$$S_t = B_t \mathbb{E}_{\mathbb{Q}} \left(\int_{]t, T]} B_u^{-1} dD_u \mid \mathcal{G}_t \right)$$

where \mathbb{Q} is a spot martingale measure.

Cumulative Price

The cumulative price \widehat{S}_t is aimed to represent the current value at time t of all dividend payments occurring during the period $]t, T]$ under the convention that they were immediately reinvested in the savings account.

Definition 3 *The **cumulative price process** S associated with the dividend process D satisfies, for every $t \in [0, T]$,*

$$\widehat{S}_t = B_t \mathbb{E}_{\mathbb{Q}} \left(\int_{]0, T]} B_u^{-1} dD_u \mid \mathcal{G}_t \right) = S_t + \widehat{D}_t$$

where \widehat{D}_t equals

$$\widehat{D}_t = B_t \int_{]0, t]} B_u^{-1} dD_u, \quad \forall t \in [0, T].$$

Credit Default Swap

Definition 4 A (stylized) **credit default swap** with a constant rate κ and **recovery at default** is a claim $(0, A, Z, \tau)$, where $Z = \delta$ and $A_t = -\kappa t$.

- An \mathbb{F} -predictable process $\delta : [0, T] \rightarrow \mathbb{R}$ represents the **default protection stream**.
- A constant κ represents the **CDS spread**. It defines the fee leg, also known as the **survival annuity stream**.

Lemma 1 The *ex-dividend price* of a CDS maturing at T equals

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}} \left(\mathbf{1}_{\{t < \tau \leq T\}} \delta_{\tau} \mid \mathcal{G}_t \right) - \mathbb{E}_{\mathbb{Q}} \left(\mathbf{1}_{\{t < \tau\}} \kappa ((\tau \wedge T) - t) \mid \mathcal{G}_t \right)$$

where \mathbb{Q} is a spot martingale measure and $B = 1$.

Hazard Process Approach

Standing assumptions:

- The **default time** τ is a non-negative random variable on $(\Omega, \mathcal{G}, \mathbb{Q})$, where \mathbb{Q} is a spot martingale measure.
- The **default process** $H_t = \mathbb{1}_{\{\tau \leq t\}}$ generates the filtration \mathbb{H} .
- We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where \mathbb{F} is a **reference filtration**.
- We define the **risk-neutral survival process** G_t as

$$G_t = \mathbb{Q}\{\tau > t \mid \mathcal{F}_t\}.$$

- We **assume** that the **hazard process** Γ equals

$$\Gamma_t = -\ln G_t = \int_0^t \gamma_u du$$

where γ is the **default intensity**.

Ex-dividend Price of a CDS

Recall that the survival process G_t satisfies

$$G_t = \mathbb{Q}\{\tau > t \mid \mathcal{F}_t\} = \exp\left(-\int_0^t \gamma_u du\right).$$

We make the standing assumption that $\mathbb{E}_{\mathbb{Q}}|\delta_\tau| < \infty$.

Lemma 2 *The ex-dividend price at time $t \in [s, T]$ of a credit default swap started at s , with rate κ and protection payment δ_τ at default, equals*

$$S_t(\kappa) = \frac{1}{G_t} \mathbb{E}_{\mathbb{Q}} \left(-\int_t^T \delta_u dG_u - \kappa \int_t^T G_u du \mid \mathcal{F}_t \right) = \mathbb{1}_{\{t < \tau\}} \tilde{S}_t(\kappa)$$

where $\tilde{S}_t(\kappa)$ is the pre-default ex-dividend price.

Market CDS Rate

The **market CDS rate** is defined similarly as the **forward swap rate** in a (default-free) interest rate swap.

Definition 5 *The T -maturity **market CDS rate** $\kappa(t, T)$ at time t is the level of a CDS rate κ for which the values of the two legs of a CDS are equal at time t .*

By assumption, κ is an \mathcal{F}_t -measurable random variable.

The T -maturity market CDS rate $\kappa(t, T)$ is given by the formula

$$\kappa(t, T) = -\frac{\mathbb{E}_{\mathbb{Q}} \left(\int_t^T \delta_u dG_u \mid \mathcal{F}_t \right)}{\mathbb{E}_{\mathbb{Q}} \left(\int_t^T G_u du \mid \mathcal{F}_t \right)}, \quad \forall t \in [0, T].$$

We fix a maturity date T , and we shall frequently write κ_t instead of $\kappa(t, T)$.

Deterministic Default Intensity

Ex-dividend Price of a CDS

Standing assumptions:

- Assume that \mathbb{F} is trivial, and the **survival function** $G(t)$ satisfies

$$G(t) = \mathbb{Q}\{\tau > t\} = \exp\left(-\int_0^t \gamma(u) du\right).$$

- We postulate that the default protection $\delta : [0, T] \rightarrow \mathbb{R}$ is deterministic.

In that case, the ex-dividend price at time $t \in [0, T]$ of a CDS with the spread κ and protection payment $\delta(\tau)$ at default equals

$$S_t(\kappa) = \mathbb{1}_{\{t < \tau\}} \frac{1}{G(t)} \left(-\int_t^T \delta(u) dG(u) - \kappa \int_t^T G(u) du \right) = \mathbb{1}_{\{t < \tau\}} \tilde{S}_t(\kappa).$$

Market CDS Rate

- The T -maturity **market CDS rate** $\kappa(t, T)$ solves the following equation

$$\int_t^T \delta(u) dG(u) + \kappa(t, T) \int_t^T G(u) du = 0.$$

- We thus have, for every $t \in [0, T]$,

$$\kappa(t, T) = -\frac{\int_t^T \delta(u) dG(u)}{\int_t^T G(u) du}.$$

- We fix a maturity date T , and we write briefly $\kappa(t)$ instead of $\kappa(t, T)$.
- In addition, we assume that all CDSs with different starting dates have a common recovery function δ .

Market CDS Rate: Special Case

- Assume that $\delta(t) = \delta$ is constant, and $F(t) = 1 - e^{-\gamma t}$ for some constant default intensity $\gamma > 0$ under \mathbb{Q} .
- The ex-dividend price of a (spot) CDS with rate κ equals, for every $t \in [0, T]$,

$$S_t(\kappa) = \mathbb{1}_{\{t < \tau\}} (\delta\gamma - \kappa) \gamma^{-1} \left(1 - e^{-\gamma(T-t)} \right).$$

- The last formula yields $\kappa(s, T) = \delta\gamma$ for every $s < T$, so that the market rate $\kappa(s, T)$ is here independent of s .
- As a consequence, the ex-dividend price of a market CDS started at s equals zero not only at the inception date s , but indeed at any time $t \in [s, T]$, both prior to and after default).
- Hence, this process follows a trivial martingale under \mathbb{Q} .

Price Dynamics of a CDS

The following result furnishes risk-neutral dynamics of the ex-dividend price of a CDS with spread κ and maturity T .

Proposition 1 *The dynamics of the ex-dividend price $S_t(\kappa)$ on $[s, T]$ are*

$$dS_t(\kappa) = -S_{t-}(\kappa) dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t)) dt$$

where the \mathbb{H} -martingale M under a spot martingale measure \mathbb{Q} is given by the formula

$$M_t = H_t - \int_0^t (1 - H_u)\gamma(u) du, \quad \forall t \in \mathbb{R}_+.$$

Prior to default, we have

$$d\tilde{S}_t(\kappa) = \tilde{S}_t(\kappa)\gamma(t) dt + (\kappa - \delta(t)\gamma(t)) dt.$$

At default, the ex-dividend price process jumps to 0.

Replication of Defaultable Claims

We assume that the following two assets are traded:

- a CDS with maturity $U \geq T$,
- the constant savings account $B = 1$ (this is not restrictive).

Let ϕ^0, ϕ^1 be an \mathbb{H} -predictable processes and let $C : [0, T] \rightarrow \mathbb{R}$ be a function of finite variation with $C_0 = 0$.

Definition 6 *We say that $(\phi, C) = (\phi^0, \phi^1, C)$ is a **trading strategy with financing cost** C if the wealth process $V(\phi, C)$, defined as*

$$V_t(\phi, C) = \phi_t^0 + \phi_t^1 S_t(\kappa),$$

where $S_t(\kappa)$ is the ex-dividend price of a CDS at time t , satisfies

$$dV_t(\phi, C) = \phi_t^1 (dS_t(\kappa) + dD_t) - dC(t)$$

where D is the dividend process of a CDS.

Recall that a generic defaultable claim (X, A, Z, τ) consists of

1. a promised claim X ,
2. a function A representing dividends stream,
3. a recovery function Z ,
4. a default time τ .

Definition 7 *A trading strategy (ϕ, C) **replicates a defaultable claim** (X, A, Z, τ) if:*

1. *the processes $\phi = (\phi^0, \phi^1)$ and $V(\phi, C)$ are stopped at $\tau \wedge T$,*
2. *$C(\tau \wedge t) = A(\tau \wedge t)$ for every $t \in [0, T]$,*
3. *we have $V_{\tau \wedge T}(\phi, C) = Y$, where the random variable Y equals*

$$Y = X \mathbb{1}_{\{\tau > T\}} + Z(\tau) \mathbb{1}_{\{\tau \leq T\}}.$$

Risk-Neutral Valuation of a Defaultable Claim

- Let us denote, for every $t \in [0, T]$,

$$\tilde{Y}(t) = \frac{1}{G(t)} \left(XG(T) - \int_t^T Z(u) dG(u) \right)$$

and

$$\tilde{A}(t) = \frac{1}{G(t)} \int_{]t, T]} G(u) dA(u).$$

- Let π and $\tilde{\pi}$ be the **risk-neutral value** and the **pre-default risk-neutral value** of a defaultable claim, so that $\pi_t = \mathbb{1}_{\{t < \tau\}} \tilde{\pi}(t)$ for $t \in [0, T]$.
- Let $\hat{\pi}$ stand for the **risk-neutral cumulative value**

$$\hat{\pi}_t = B_t \mathbb{E}_{\mathbb{Q}} \left(\int_{]0, T]} B_u^{-1} dD_u \mid \mathcal{G}_t \right).$$

- It is clear that $\pi(0) = \tilde{\pi}(0) = \hat{\pi}(0)$.

Price Dynamics of a Defaultable Claim

Proposition 2 *The pre-default risk-neutral value of a defaultable claim (X, A, Z, τ) equals*

$$\tilde{\pi}(t) = \tilde{Y}(t) + \tilde{A}(t)$$

and thus

$$d\tilde{\pi}(t) = \gamma(t)(\tilde{\pi}(t) - Z(t)) dt - dA(t).$$

Moreover

$$d\pi_t = -\tilde{\pi}(t-) dM_t - \gamma(t)(1 - H_t)Z(t) dt - dA(t \wedge \tau)$$

and

$$d\hat{\pi}_t = (Z(t) - \tilde{\pi}(t-)) dM_t.$$

Replication of a Defaultable Claim

Proposition 3 *Assume that the inequality $\tilde{S}_t(\kappa) \neq \delta(t)$ holds for every $t \in [0, T]$.*

Let $\phi_t^1 = \tilde{\phi}_1(\tau \wedge t)$, where the function $\tilde{\phi}_1 : [0, T] \rightarrow \mathbb{R}$ is given by the formula

$$\tilde{\phi}_1(t) = \frac{Z(t) - \tilde{\pi}(t-)}{\delta(t) - \tilde{S}_t(\kappa)}, \quad \forall t \in [0, T],$$

and let $\phi_t^0 = V_t(\phi, A) - \phi_t^1 S_t(\kappa)$, where the process $V(\phi, A)$ is given by the formula

$$V_t(\phi, A) = \tilde{\pi}(0) + \int_{]0, \tau \wedge t]} \tilde{\phi}_1(u) d\hat{S}_u(\kappa) - A(t \wedge \tau).$$

Then the strategy (ϕ^0, ϕ^1, A) replicates the defaultable claim (X, A, Z, τ) .

Stochastic Default Intensity, immersion property

Valuation of Single-Name CDSs

Let us now examine the valuation of single-name CDSs under the assumption that the interest rate equals zero. The full filtration is $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ (we assume immersion property, and that \mathbb{F} is a Brownian filtration).

We consider the CDS

- with the constant spread κ ,
- which delivers $\delta(\tau)$ at time τ if $\tau < T$, where δ is a deterministic function.

The process

$$M_t = H_t - \int_0^{t \wedge \tau} \gamma_u du,$$

with $\gamma_t dt = \frac{dF_t}{1-F_t}$ is a \mathbb{G} -martingale.

The dynamics of the ex-dividend price $S_t(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa) dM_t + (1-H_t)B_t G_t^{-1} dm_t + (1-H_t)(r_t S_t(\kappa) + \kappa - \delta_t \gamma_t) dt,$$

where m is the (\mathbb{Q}, \mathbb{F}) -martingale given by

$$m_t = \mathbb{E}_{\mathbb{Q}} \left(\int_0^T B_u^{-1} \delta_u G_u \gamma_u du - \kappa \int_0^T B_u^{-1} G_u du \mid \mathcal{F}_t \right).$$

Our aim is to hedge

$$Y = \mathbb{1}_{\{T \geq \tau\}} Z_\tau + \mathbb{1}_{\{T < \tau\}} X.$$

using two CDS with maturities T_i , rates κ_i and protection payment δ^i (with $i = 1, 2$). We assume $r = 0$. Let $\zeta^i, i = 1, 2$ and ζ^Z defined as

$$m_t^i = \mathbb{E}_Q \left(\int_0^T \delta_u^i G_u \gamma_u du - \kappa_i \int_0^T G_u du \mid \mathcal{F}_t \right), \quad dm_t^i = \zeta_t^i dW_t$$

and

$$m_t^Y = \mathbb{E}_Q \left(- \int_0^\infty Z_u dG_u + G_T X \mid \mathcal{F}_t \right), \quad dm_t^Y = \zeta_t^Y dW_t$$

Assume that there exist \mathbb{F} -predictable processes ϕ^1, ϕ^2 such that

$$\begin{aligned}\phi_t^1(\delta_t^1 - \tilde{S}_t^1(\kappa_1)) + \phi_t^2(\delta_t^2 - \tilde{S}_t^2(\kappa_2)) &= Z_t - \tilde{Y}_t, \\ \phi_t^1 \zeta_t^1 + \phi_t^2 \zeta_t^2 &= \zeta_t^Y,\end{aligned}$$

where \tilde{Y} is given by

$$\tilde{Y}_t = \frac{1}{G_t} \mathbb{E}_Q \left(- \int_t^T Z_u dG_u + G_T X \mid \mathcal{F}_t \right).$$

Let $\phi_t^0 = V_t(\phi) - \phi_t^1 S_t^1(\kappa_1) - \phi_t^2 S_t^2(\kappa_2)$, where the process $V(\phi)$ is given by

$$dV_t(\phi) = \sum_{i=1}^2 \phi_t^i (dS_t^i(\kappa_i) + dD_t^i)$$

with the initial condition $V_0(\phi) = \mathbb{E}_Q(Y)$. Then the self-financing trading strategy $\phi = (\phi^0, \phi^1, \phi^2)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$.

Model with Density

The value of the CDS is

$$\begin{aligned} S_t &= \mathbb{1}_{t < \tau} \mathbb{E}(\delta(\tau) \mathbb{1}_{t < \tau \leq T} - \kappa((T \wedge \tau) - t) | \mathcal{G}_t) \\ &= \mathbb{1}_{t < \tau} \frac{1}{G_t(t)} \int_t^T (\delta(u) \alpha_t(u) - G_t(u) \kappa) du \end{aligned}$$

For any $\theta \in \mathbb{R}^+$, there exists a family of processes $g.(\theta)$ such that

$$G_t(\theta) = \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \mathbb{P}(\tau > \theta) + \int_0^t g_s(\theta) dW_s$$

Then

$$G_t = G_t(t) = G_0(t) + \int_0^t g_s(s) dW_s + \int_0^t \alpha_s(s) ds$$

The dynamics of a CDS can be obtained in a closed form

$$\begin{aligned}
dS_t(\kappa) &= r(t)S_t(\kappa) dt - S_{t-}(\kappa) dM_t + \left(\kappa - \frac{\alpha_t(t)}{G_t(t)}\delta(t)\right)(1 - H_t)dt \\
&\quad + (1 - H_t)\sigma_t(T) \left(dW_t - \frac{g_t(t)}{G_t(t)} dt\right).
\end{aligned}$$

where

$$\sigma_t(T) = \int_t^T (\delta(u)\partial_1 g_t(u) - \kappa g_t(u)) du + g_t(t)\tilde{S}_t(\kappa)$$

If immersion property holds, then $g_t(t) = 0$, otherwise

$$W_{t \wedge \tau} - \int_0^{t \wedge \tau} \frac{g_s(s)}{G_s(s)} ds$$

is a \mathbb{G} -martingale

Several Names: Deterministic Default Intensities

Valuation of a Single-Name CDS: Two defaults, trivial reference filtration

We assume that two default times are given: $\tau_i, i = 1, 2$

We introduce the *joint survival process* $G(u, v)$: for every $u, v \in \mathbb{R}_+$,

$$G(u, v) = \mathbb{Q}(\tau_1 > u, \tau_2 > v)$$

We write

$$\partial_1 G(u, v) = \frac{\partial G}{\partial u}(u, v), \quad \partial_{12} G(u, v) = \frac{\partial^2 G}{\partial u \partial v}(u, v).$$

We assume that the joint density $f(u, v) = \partial_{12} G(u, v)$ exists. In other words, we postulate that $G(u, v)$ can be represented as follows

$$G(u, v) = \int_u^\infty \left(\int_v^\infty f(x, y) dy \right) dx.$$

Let

$$\tilde{\lambda}^1(t) = -\frac{\partial_1 G(t, t)}{G(t, t)}, \quad \lambda^{1|2}(t, s) = -\frac{f(t, s)}{\partial_2 G(t, s)}$$

Then, the process

$$M_t^1 = H_t^1 - \int_0^{t \wedge \tau_1 \wedge \tau_2} \tilde{\lambda}^1(u) du - \int_{t \wedge \tau_1 \wedge \tau_2}^{t \wedge \tau_1} \lambda^{1|2}(u, \tau_2) du,$$

is a \mathbb{G} -martingale.

Let us now examine the valuation of single-name CDSs under the assumption that the interest rate equals zero. We consider a CDS with constant spread κ , which delivers $\delta(\tau_1)$ at time τ_1 if $\tau_1 < T$, where δ is a deterministic function. The value $S(\kappa)$ of this CDS, computed relative to the filtration \mathbb{G} , i.e., accounting for the information about the timing of τ_2 , is computed in two successive steps. Let us denote by $\tau_{(1)} = \tau_1 \wedge \tau_2$ the moment of the first default.

Then, $\mathbb{1}_{\{t < \tau_{(1)}\}} S_t(\kappa) = \mathbb{1}_{\{t < \tau_{(1)}\}} \tilde{S}_t(\kappa)$, where

$$\tilde{S}_t(\kappa) = \frac{1}{G(t, t)} \left(- \int_t^T \delta(u) \partial_1 G(u, t) du - \kappa \int_t^T G(u, t) du \right)$$

Hence the dynamics of the *pre-default ex-dividend value* \tilde{S}_t are

$$d\tilde{S}_t(\kappa) = \left((\tilde{\lambda}_1(t) + \tilde{\lambda}_2(t)) \tilde{S}_t(\kappa) + \kappa - \tilde{\lambda}_1(t) \delta(t) - \tilde{\lambda}_2(t) S_{t|2}(\kappa) \right) dt,$$

where for $i = 1, 2$ the function $\tilde{\lambda}_i(t) = -\frac{\partial_i G(t, t)}{G(t, t)}$ is the (deterministic) *pre-default intensity* of τ_i and $S_{t|2}(\kappa)$ is given by the expression

$$S_{t|2}(\kappa) = \frac{-1}{\partial_2 G(t, t)} \left(\int_t^T \delta(u) f(u, t) du + \kappa \int_t^T \partial_2 G(u, t) du \right).$$

In the financial interpretation, $S_{1|2}(t)$ is the ex-dividend price at time t of a CDS on the first credit name, under the assumption that the default τ_2 occurs at time t and the first name has not yet defaulted (recall that simultaneous defaults are excluded).

Let us now consider the event $\{\tau_2 \leq t < \tau_1\}$. The ex-dividend price of a CDS equals

$$\widehat{S}_t(\kappa) = \frac{1}{\partial_2 G(t, \tau_2)} \left(- \int_t^T \delta(u) f(u, \tau_2) du - \kappa \int_t^T \partial_2 G(u, \tau_2) du \right).$$

Consequently, on the event $\{\tau_2 \leq t < \tau_1\}$ we obtain

$$d\widehat{S}_t(\kappa) = \left(\lambda^{1|2}(t, \tau_2) \left(\widehat{S}_t(\kappa) - \delta(t) \right) + \kappa \right) dt,$$

where $\lambda^{1|2}(t, s) = -\frac{f(t, s)}{\partial_2 G(t, s)}$, evaluated at $s = \tau_2$, represents the value of the default intensity process of τ_1 with respect to the filtration \mathbb{H}^2 on the event $\{\tau_2 < t\}$.

$$\begin{aligned} dS_t &= (1 - H_t^1)(1 - H_t^2)d\widetilde{S}(t) + (1 - H_t^1)H_t^2d\widehat{S}_t \\ &\quad - S_{t-}dH_t^1 - (1 - H_t^1)(\widetilde{S}(t) - \widehat{S}_t)dH_t^2 \end{aligned}$$

First-to-Default Intensities and Martingales

Assumptions and Objectives

Let τ_1, \dots, τ_n be default times of n reference entities.

Assume that:

1. The joint distribution of default times (τ_1, \dots, τ_n) is known.
2. The protection payments at default are known functions of time, number of defaults and names of defaulted entities.
3. Single-name CDSs for n reference entities are traded.

We will argue that it is possible to replicate a **basket CDS** with single-name CDSs under mild technical assumptions of non-degeneracy (a system of linear equations).

It suffices to consider the case of a first-to-default claim and then to use the backward induction.

Default Times and Filtrations

- Let $\tau_1, \tau_2, \dots, \tau_n$ be the default times associated with n names, respectively.

- Let

$$F(t_1, t_2, \dots, t_n) = \mathbb{Q}(\tau_1 \leq t_1, \tau_2 \leq t_2, \dots, \tau_n \leq t_n)$$

denote the **joint distribution function** of the default times associated with the n names.

- For each $i = 1, 2, \dots, n$ we define the default indicator process for the i th credit name as $H_t^i = \mathbb{1}_{\{\tau_i \leq t\}}$ and the σ -field $\mathbb{H}_t^i = \sigma(H_u^i : u \leq t)$.

- We write

$$\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2 \vee \dots \vee \mathbb{H}^n$$

and

$$\mathbb{G}^i = \mathbb{H}^1 \vee \dots \vee \mathbb{H}^{i-1} \vee \mathbb{H}^{i+1} \vee \dots \vee \mathbb{H}^n$$

so that $\mathbb{G} = \mathbb{G}^i \vee \mathbb{H}^i$ for $i = 1, 2, \dots, n$.

First-to-Default Intensities

Definition 8 *The i th first-to-default intensity is the function*

$$\begin{aligned}\tilde{\lambda}_i(t) &= \lim_{h \downarrow 0} \frac{1}{h} \frac{\mathbb{Q}(t < \tau_i \leq t + h \mid \tau_1 > t, \dots, \tau_{i-1} > t, \tau_{i+1} > t, \dots, \tau_n > t)}{\mathbb{Q}(\tau_i > t \mid \tau_1 > t, \dots, \tau_{i-1} > t, \tau_{i+1} > t, \dots, \tau_n > t)} \\ &= \lim_{h \downarrow 0} \frac{1}{h} \mathbb{Q}(t < \tau_i \leq t + h \mid \tau_{(1)} > t).\end{aligned}$$

Definition 9 *The first-to-default intensity $\tilde{\lambda}$ is defined as the sum $\tilde{\lambda} = \sum_{i=1}^n \tilde{\lambda}_i$, or equivalently, as the intensity function of the random time $\tau_{(1)}$ modeling the moment of the first default.*

First-to-Default Martingales

- Let λ^i be the \mathbb{G}^i -intensity of the i th default time. The process M^i given by the formula

$$M_t^i = H_t^i - \int_0^t (1 - H_u^i) \lambda_u^i du, \quad \forall t \in \mathbb{R}_+,$$

is known to be a \mathbb{G} -martingale under \mathbb{Q} .

- A random time $\tau_{(1)}$ is manifestly a \mathbb{G} -stopping time. Therefore, for each $i = 1, 2, \dots, n$, the process \widehat{M}^i , given by the formula

$$\widehat{M}_t^i := M_{t \wedge \tau_{(1)}}^i = H_{t \wedge \tau_{(1)}}^i - \int_0^t \mathbb{1}_{\{\tau_{(1)} > u\}} \widetilde{\lambda}_i(u) du, \quad \forall t \in \mathbb{R}_+,$$

also follows a \mathbb{G} -martingale under \mathbb{Q} .

- Processes \widehat{M}^i are referred to as the **basic first-to-default martingales**.

Traded Credit Default Swaps

- As traded assets, we take the constant savings account and a family of single-name CDSs with default protections δ_i and rates κ_i .
- For convenience, we assume that the CDSs have the same maturity T , but this assumption can be easily relaxed. The i th traded CDS is formally defined by its dividend process

$$D_t^i = \int_{(0,t]} \delta_i(u) dH_u^i - \kappa_i(t \wedge \tau_i), \quad \forall t \in [0, T].$$

- Consequently, the price at time t of the i th CDS equals

$$S_t^i(\kappa_i) = \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{\{t < \tau_i \leq T\}} \delta_i(\tau_i) \mid \mathcal{G}_t) - \kappa_i \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{\{t < \tau_i\}} ((\tau_i \wedge T) - t) \mid \mathcal{G}_t).$$

- To replicate a first-to-default claim, we only need to examine the dynamics of each CDS on the interval $[0, \tau_{(1)} \wedge T]$.

The Value of a CDS at Default

- For any $j \neq i$, we define a function $S_{t|j}^i(\kappa_i), t \in [0, T]$, which represents the ex-dividend price of the i th CDS at time t on the event $\{\tau_{(1)} = \tau_j = t\}$.
- Formally, this quantity is defined as the unique function satisfying

$$\mathbb{1}_{\{\tau_{(1)} = \tau_j \leq T\}} S_{\tau_{(1)}|j}^i(\kappa_i) = \mathbb{1}_{\{\tau_{(1)} = \tau_j \leq T\}} S_{\tau_{(1)}}^i(\kappa_i)$$

so that

$$\mathbb{1}_{\{\tau_{(1)} \leq T\}} S_{\tau_{(1)}}^i(\kappa_i) = \sum_{j \neq i} \mathbb{1}_{\{\tau_{(1)} = \tau_j \leq T\}} S_{\tau_{(1)}|j}^i(\kappa_i).$$

- Let $m = 2$. Then the function $S_{t|2}(\kappa_1), t \in [0, T]$, is the price of the first CDS at time t on the event $\{\tau_{(1)} = \tau_2 = t\}$.

Lemma 3 *The function $S_{v|2}(\kappa_1), v \in [0, T]$, equals*

$$S_{v|2}(\kappa_1) = \frac{\int_v^T \delta_1(u) f(u, v) du}{\int_v^\infty f(u, v) du} - \kappa_1 \frac{\int_v^T du \int_u^\infty f(z, v) dz}{\int_v^\infty f(u, v) du}.$$

Price Dynamics of the i th CDS

Proposition 4 *The dynamics of the pre-default ex-dividend price $\tilde{S}_t^i(\kappa_i)$ are*

$$d\tilde{S}_t^i(\kappa_i) = \tilde{\lambda}_i(t) (\tilde{S}_t^i(\kappa_i) - \delta_i(t)) dt + \sum_{j \neq i} \tilde{\lambda}_j(t) (\tilde{S}_t^i(\kappa_i) - S_{t|j}^i(\kappa_i)) dt + \kappa_i dt.$$

The cumulative ex-dividend price of the i th CDS stopped at $\tau_{(1)}$ satisfies

$$\hat{S}_t^i(\kappa_i) = S_t^i(\kappa_i) + \int_0^t \delta_i(u) dH_{u \wedge \tau_{(1)}}^i + \sum_{j \neq i} \int_0^t S_{u|j}^i(\kappa_i) dH_{u \wedge \tau_{(1)}}^j - \kappa_i(\tau_{(1)} \wedge t),$$

and thus

$$d\hat{S}_t^i(\kappa_i) = (\delta_i(t) - \tilde{S}_{t-}^i(\kappa_i)) d\hat{M}_t^i + \sum_{j \neq i} (S_{t|j}^i(\kappa_i) - \tilde{S}_{t-}^i(\kappa_i)) d\hat{M}_t^j.$$

Replication of First-to-Default Claims

Definition 10 A *first-to-default claim* (an *FTDC*, for short) on a basket of n credit names is a defaultable claim $(X, A, Z, \tau_{(1)})$, where

1. X is a constant amount payable at maturity if no default occurs,
2. $A : [0, T] \rightarrow \mathbb{R}$ with $A_0 = 0$ is a function of bounded variation representing the dividend stream up to $\tau_{(1)}$,
3. $Z = (Z_1, Z_2, \dots, Z_n)$, where a function $Z_i : [0, T] \rightarrow \mathbb{R}$ specifies the recovery payment made at the time τ_i if the i th firm was the first defaulted firm, that is, on the event $\{\tau_i = \tau_{(1)} \leq T\}$.

Pricing of an FTDC

Proposition 5 *The pre-default risk-neutral value of an FTDC equals*

$$\tilde{\pi}(t) = \sum_{i=1}^n \frac{\Psi_i(t)}{G_{(1)}(t)} + \frac{1}{G_{(1)}(t)} \int_t^T G_{(1)}(u) dA(u) + X \frac{G_{(1)}(T)}{G_{(1)}(t)}$$

where

$$\Psi_i(t) = \int_{u_i=t}^T \int_{u_1=u_i}^{\infty} \cdots \int_{u_{i-1}=u_i}^{\infty} \int_{u_{i+1}=u_i}^{\infty} \cdots \int_{u_n=u_i}^{\infty} Z_i(u_i) F(du_1, \dots, du_{i-1}, du_i, du_{i+1}, \dots, du_n).$$

Price Dynamics of an FTDC

Proposition 6 *The pre-default risk-neutral value of an FTDC satisfies*

$$d\tilde{\pi}(t) = \sum_{i=1} \tilde{\lambda}_i(t) (\tilde{\pi}(t) - Z_i(t)) dt - dA(t).$$

Moreover, the risk-neutral value of an FTDC satisfies

$$d\pi_t = \sum_{i=1}^n (Z_i(t) - \tilde{\pi}(t-)) d\widehat{M}_u^i - dA(\tau_{(1)} \wedge t),$$

and the risk-neutral cumulative value $\widehat{\pi}$ of an FTDC satisfies

$$d\widehat{\pi}_t = \sum_{i=1}^n (Z_i(t) - \tilde{\pi}(t-)) d\widehat{M}_u^i.$$

Self-financing Strategies with CDSs

- Consider a family of single-name CDSs with default protections δ_i and rates κ_i .
- For convenience, we assume that they have the same maturity T ; this assumption can be easily relaxed.

Definition 11 *A trading strategy $\phi = (\phi^0, \phi^1, \dots, \phi^n)$, in assets $(B, S^1(\kappa_1), \dots, S^n(\kappa_n))$ is **self-financing** with financing cost C if its wealth process $V(\phi)$, defined as*

$$V_t(\phi) = \phi_t^0 + \sum_{i=1}^n \phi_t^i S_t^i(\kappa_i),$$

satisfies

$$dV_t(\phi) = \sum_{i=1}^n \phi_t^i (dS_t^i(\kappa_i) + dD_t^i) - dC_t,$$

where $S^i(\kappa_i)$ is the ex-dividend price of the i th CDS.

Standing Assumption

- We assume that $\det N(t) \neq 0$ for every $t \in [0, T]$, where

$$N(t) = \begin{bmatrix} \delta_1(t) - \tilde{S}_t^1(\kappa_1) & S_{t|1}^2(\kappa_2) - \tilde{S}_t^2(\kappa_2) & \cdot & S_{t|1}^n(\kappa_n) - \tilde{S}_t^n(\kappa_n) \\ S_{t|2}^1(\kappa_1) - \tilde{S}_t^1(\kappa_1) & \delta_2(t) - \tilde{S}_t^2(\kappa_2) & \cdot & S_{t|2}^n(\kappa_n) - \tilde{S}_t^n(\kappa_n) \\ \cdot & \cdot & \cdot & \cdot \\ S_{t|n}^1(\kappa_1) - \tilde{S}_t^1(\kappa_1) & S_{t|n}^2(\kappa_1) - \tilde{S}_t^2(\kappa_1) & \cdot & \delta_n(t) - \tilde{S}_t^n(\kappa_n) \end{bmatrix}$$

- Let $\tilde{\phi}(t) = (\tilde{\phi}_1(t), \tilde{\phi}_2(t), \dots, \tilde{\phi}_n(t))$ be the unique solution to the equation

$$N(t)\tilde{\phi}(t) = h(t)$$

where $h(t) = (h_1(t), h_2(t), \dots, h_n(t))$ with $h_i(t) = Z_i(t) - \tilde{\pi}(t-)$.

Replication of an FTDC

Proposition 7 *Let the functions $\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_n$ satisfy for $t \in [0, T]$*

$$\tilde{\phi}_i(t)(\delta_i(t) - \tilde{S}_t^i(\kappa_i)) + \sum_{j \neq i} \tilde{\phi}_j(t)(S_{t|_i}^j(\kappa_j) - \tilde{S}_t^j(\kappa_j)) = Z_i(t) - \tilde{\pi}(t-).$$

Let $\phi_t^i = \tilde{\phi}_i(\tau_{(1)} \wedge t)$ for $i = 1, 2, \dots, n$ and

$$\phi_t^0 = V_t(\phi, A) - \sum_{i=1}^n \phi_t^i S_t^i(\kappa_i), \quad \forall t \in [0, T],$$

where the process $V(\phi, A)$ is given by the formula

$$V_t(\phi, A) = \tilde{\pi}(0) + \sum_{i=1}^n \int_{]0, \tau_{(1)} \wedge t]} \tilde{\phi}_i(u) d\hat{S}_u^i(\kappa_i) - A(\tau_{(1)} \wedge t).$$

Then the trading strategy (ϕ, A) replicates an FTDC $(X, A, Z, \tau_{(1)})$.

Final Remarks

In a single-name case:

- we first considered the case of a default time with a deterministic intensity,
- we have shown that a generic defaultable claim can be replicated by dynamic trading in a CDS and the savings account,
- the extension to the case of non-trivial reference filtration was not presented.

In a multi-name case:

- we first considered the case of a finite family of default times with known joint distribution,
- the replicating strategy for a first-to-default claim was examined; the method can be extended to k th-to-default claims,
- in the next step, the approach was extended to the case of a reference filtration generated by a multi-dimensional Brownian motion.

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