

Credit Risk IV

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IV. Hazard process Approach

1. General case
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General case

The model

Two kinds of information :

the information from the asset's prices, denoted as $(\mathcal{F}_t, t \geq 0)$

the information from the default time τ modeled by the filtration \mathcal{H}_t

generated by the default process $H_t = \mathbb{1}_{\tau \leq t}$.

We denote by $\mathcal{G}_t \stackrel{def}{=} \mathcal{F}_t \vee \mathcal{H}_t$.

Key lemma

Any \mathcal{G}_t -random variable is equal, on the set $\{\tau > t\}$, to an \mathcal{F}_t -measurable random variable.

We denote by $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$ the conditional law of τ given the information \mathcal{F}_t , and $G_t = 1 - F_t$.

Let X be an \mathcal{F}_T -measurable integrable r.v. Then,

$$\mathbb{E}(X \mathbb{1}_{T < \tau} | \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}(X \mathbb{1}_{\{\tau > T\}} | \mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau > t\}} | \mathcal{F}_t)} = \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}(X e^{-\Gamma_T} | \mathcal{F}_t).$$

where $\Gamma_t \stackrel{def}{=} -\ln(1 - F_t) = -\ln G_t$

Let h be an \mathbf{F} -predictable process. Then,

$$\mathbb{E}(h_\tau \mathbb{1}_{\tau < T} | \mathcal{G}_t) = h_\tau \mathbb{1}_{\{\tau < t\}} + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E} \left(\int_t^T h_u dF_u | \mathcal{F}_t \right).$$

Martingales

We assume for simplicity that F is continuous.

(i) The process $L_t \stackrel{def}{=} (1 - H_t)e^{\Gamma_t}$ is a **G-martingale**.

(ii) The process $M_t \stackrel{def}{=} H_t - \Gamma_{t \wedge \tau}$ is a **G**-martingale as soon as F (or Γ) is increasing.

The submartingale F admits a **decomposition** as $F = Z + A$ where Z is a **martingale** and A a **predictable increasing** process.

(iii) The process

$$M_t \stackrel{def}{=} H_t - \int_0^{t \wedge \tau} \frac{dA_u}{1 - F_u}$$

is a **G-martingale**.

Proofs: The process $L_t = (1 - H_t)e^{\Gamma_t}$ is a **G**-martingale.

From the key lemma, for $t > s$

$$\begin{aligned}\mathbb{E}(L_t|\mathcal{G}_s) &= \mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{G}_s) = \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}\mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{F}_s) \\ &= \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}\mathbb{E}(\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)e^{\Gamma_t}|\mathcal{F}_s) = \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}\mathbb{E}(e^{-\Gamma_t}e^{\Gamma_t}|\mathcal{F}_s) \\ &= \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s} = L_s\end{aligned}$$

The process $M_t = H_t - \Gamma_{t \wedge \tau}$ is a \mathbf{G} -martingale as soon as F (or Γ) is increasing.

From integration by parts formula :

$$dL_t = (1 - H_t)e^{\Gamma_t} d\Gamma_t - e^{\Gamma_t} dH_t$$

and the process $M_t = H_t - \Gamma(t \wedge \tau)$ can be written

$$M_t \stackrel{def}{=} \int_{]0,t]} dH_u - \int_{]0,t]} (1 - H_u) d\Gamma_u = - \int_{]0,t]} e^{-\Gamma_u} dL_u$$

and is a \mathbf{G} -martingale since L is \mathbf{G} -martingale.

The process

$$M_t = H_t - \int_0^{t \wedge \tau} \frac{dA_u}{1 - F_u}$$

is a \mathbf{G} -martingale.

Let $s < t$. We give the proof in two steps, using the Doob-Meyer decomposition of F as $F_t = Z_t + A_t$.

First step: we prove

$$\mathbb{E}(H_t|\mathcal{G}_s) = H_s + \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} \mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

Indeed,

$$\begin{aligned} \mathbb{E}(H_t|\mathcal{G}_s) &= 1 - \mathbb{P}(t < \tau|\mathcal{G}_s) = 1 - \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} \mathbb{E}(1 - F_t|\mathcal{F}_s) \\ &= 1 - \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} \mathbb{E}(1 - Z_t - A_t|\mathcal{F}_s) \\ &= 1 - \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} (1 - Z_s - A_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s)) \\ &= 1 - \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} (1 - F_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s)) \\ &= \mathbb{1}_{\tau \leq s} + \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} \mathbb{E}(A_t - A_s|\mathcal{F}_s) \end{aligned}$$

In a second step, we prove that, setting $\Lambda_t = \int_0^t (1 - H_s) \frac{dA_s}{1 - F_s}$,

$$\mathbb{E}(\Lambda_{t \wedge \tau} | \mathcal{G}_s) = \Lambda_{s \wedge \tau} + \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} \mathbb{E}(A_t - A_s | \mathcal{F}_s)$$

From the key formula,

$$\begin{aligned} \mathbb{E}(\Lambda_{t \wedge \tau} | \mathcal{G}_s) &= \Lambda_{s \wedge \tau} \mathbb{1}_{\tau \leq s} + \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} \mathbb{E} \left(\int_s^\infty \Lambda_{t \wedge u} dF_u | \mathcal{F}_s \right) \\ &= \Lambda_{s \wedge \tau} \mathbb{1}_{\tau \leq s} + \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} \mathbb{E} \left(\int_s^t \Lambda_u dF_u + \int_t^\infty \Lambda_t dF_u | \mathcal{F}_s \right) \\ &= \Lambda_{s \wedge \tau} \mathbb{1}_{\tau \leq s} + \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} \mathbb{E} \left(\int_s^t \Lambda_u dF_u + \Lambda_t (1 - F_t) | \mathcal{F}_s \right) \end{aligned}$$

We now use IP formula, using that Λ is bounded variation and continuous

$$d(\Lambda_t(1 - F_t)) = -\Lambda_t dF_t + (1 - F_t)d\Lambda_t = -\Lambda_t dF_t + dA_t$$

hence

$$\begin{aligned} \int_s^t \Lambda_u dF_u + \Lambda_t(1 - F_t) &= -\Lambda_t(1 - F_t) + \Lambda_s(1 - F_s) + A_t - A_s + \Lambda_t(1 - F_t) \\ &= \Lambda_s(1 - F_s) + A_t - A_s \end{aligned}$$

From

$$\mathbb{E}(\Lambda_{t \wedge \tau} | \mathcal{G}_s) = \Lambda_{s \wedge \tau} \mathbb{1}_{\tau \leq s} + \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} \mathbb{E} \left(\int_s^t \Lambda_u dF_u + \Lambda_t (1 - F_t) | \mathcal{F}_s \right)$$

and

$$\int_s^t \Lambda_u dF_u + \Lambda_t (1 - F_t) = \Lambda_s (1 - F_s) + A_t - A_s$$

it follows that

$$\begin{aligned} \mathbb{E}(\Lambda_{t \wedge \tau} | \mathcal{G}_s) &= \Lambda_{s \wedge \tau} \mathbb{1}_{\tau \leq s} + \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} \mathbb{E} (\Lambda_s (1 - F_s) + A_t - A_s | \mathcal{F}_s) \\ &= \Lambda_{s \wedge \tau} + \mathbb{1}_{s < \tau} \frac{1}{1 - F_s} \mathbb{E} (A_t - A_s | \mathcal{F}_s) . \end{aligned}$$

If A is absolutely continuous wrt the Lebesgue measure, there exists an \mathbf{F} -adapted process γ , called the intensity such that the process

$$H_t - \int_0^{t \wedge \tau} \gamma_u du = H_t - \int_0^t (1 - H_u) \gamma_u du$$

is a \mathbf{G} -martingale. The process γ satisfies

$$\gamma_t = \lim_{h \rightarrow 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t + h | \mathcal{F}_t)}{\mathbb{P}(t < \tau | \mathcal{F}_t)}$$

Let \tilde{V} and R be \mathbf{F} -predictable processes. The process

$$V_t = \tilde{V}_t \mathbb{1}_{\{t < \tau\}} + R_\tau \mathbb{1}_{\{\tau \leq t\}}$$

is a \mathbf{G} -martingale if and only if the process

$$\tilde{V}_t e^{-\Gamma_t} + \int_0^t R_u e^{-\Gamma_u} d\Gamma_u$$

is an \mathbf{F} -martingale

PROOF: The direct part comes from the fact that if V is a \mathbf{G} -martingale, then $\mathbb{E}_{\mathbb{Q}}(V_t|\mathcal{F}_t)$ is an \mathbf{F} -martingale. The converse is an immediate application of the Key Lemma △

Let P be the ex-dividend price process of a claim which delivers R_τ at default time and pays a cumulative coupon C till the default time, i.e. the discounted cum-dividend process

$$\beta_t P_t + \mathbb{1}_{\{\tau \leq t\}} \beta_\tau R_\tau + \int_0^{t \wedge \tau} \beta_u dC_u$$

is a \mathbf{G} -martingale. Let \tilde{P}_t be the predefault price of the process P , i.e., \tilde{P} is \mathbf{F} -predictable and $P_t = \mathbb{1}_{\{t < \tau\}} \tilde{P}_t$. Then the process

$$P_t^* = \alpha_t \tilde{P}_t + \int_0^t \alpha_s dC_s + \int_0^t R_u \alpha_u d\Gamma_u$$

is an \mathbf{F} -martingale.

Conversely, if \tilde{V} is an \mathbf{F} -predictable process such that the process $\alpha_t \tilde{V}_t + \int_0^t \alpha_s dC_s + \int_0^t R_u \alpha_u d\Gamma_u$ is an \mathbf{F} -martingale, then (the discounted cum-dividend) process

$$\beta_t \tilde{V}_t \mathbf{1}_{\{t < \tau\}} + \mathbf{1}_{\{\tau \leq t\}} \beta_\tau R_\tau + \int_0^{t \wedge \tau} \beta_u dC_u$$

is a \mathbf{G} -martingale.

PROOF: This is an application of the previous Lemma .



Computation in a restricted filtration

Let $\tilde{\mathbf{F}} \subset \mathbf{F}$ and $\tilde{\mathcal{G}}_t = \tilde{\mathcal{F}}_t \vee \mathcal{H}_t$.

From

$$F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$$

we deduce

$$\tilde{F}_t = \mathbb{P}(\tau \leq t | \tilde{\mathcal{F}}_t) = \mathbb{E}(F_t | \tilde{\mathcal{F}}_t)$$

The computation of the intensity is more difficult, the $\tilde{\mathbf{F}}$ -intensity in the restricted filtration is not the conditional expectation of the \mathbf{F} -intensity

(\mathcal{H}) Hypothesis

Complete model case

Let S be a semi-martingale on $(\Omega, \mathcal{G}, \mathbb{P})$ such that there exists a **unique** probability \mathbb{Q} , equivalent to \mathbb{P} on \mathcal{F}_T , where $\mathcal{F}_t = \mathcal{F}_t^S = \sigma(S_s, s \leq t)$ such that $(\tilde{S}_t = S_t R_t, 0 \leq t \leq T)$ is an \mathbf{F}^S -martingale under the probability \mathbb{Q} . We assume that there exists a probability $\tilde{\mathbb{Q}}$, equivalent to \mathbb{P} on \mathcal{G}_T such that $(\tilde{S}_t, 0 \leq t \leq T)$ is a \mathbf{G} -martingale under the probability $\tilde{\mathbb{Q}}$. Then, **any (\mathbf{F}, \mathbb{Q}) -martingale is a (\mathbf{G}, \mathbb{Q}) -martingale** and the restriction of $\tilde{\mathbb{Q}}$ to \mathcal{F}_T is equal \mathbb{Q} .

Definition and Properties of immersion

We shall now examine the immersion property (or (\mathcal{H}) -hypothesis) which reads:

(\mathcal{H}) **Every \mathbf{F} square-integrable martingale is a \mathbf{G} square integrable martingale.**

This hypothesis implies that the \mathbf{F} -Brownian motion remains a Brownian motion in the enlarged filtration and that every \mathbf{F} -local martingale is a \mathbf{G} -local martingale .

Assume that $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$, where \mathbf{F} is an arbitrary filtration and \mathbf{H} is generated by the process $H_t = \mathbb{1}_{\{\tau \leq t\}}$. Then the following conditions are equivalent to the hypothesis (\mathcal{H}) .

(i) For any $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(\tau \leq t \mid \mathcal{F}_t) = \mathbb{P}(\tau \leq t \mid \mathcal{F}_\infty).$$

(ii) For any $t \in \mathbb{R}_+$, the σ -fields \mathcal{F}_∞ and \mathcal{G}_t are conditionally independent given \mathcal{F}_t under \mathbb{P} , that is,

$$\mathbb{E}_{\mathbb{P}}(\xi \eta \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi \mid \mathcal{F}_t) \mathbb{E}_{\mathbb{P}}(\eta \mid \mathcal{F}_t)$$

for any bounded, \mathcal{F}_∞ -measurable random variable ξ and bounded, \mathcal{G}_t -measurable random variable η .

(iii) For any $t \in \mathbb{R}_+$ and any bounded, \mathcal{F}_∞ -measurable random variable ξ :
 $\mathbb{E}_{\mathbb{P}}(\xi \mid \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\xi \mid \mathcal{F}_t)$.

Change of a probability measure

Kusuoka shows, by means of a counter-example, that the hypothesis (\mathcal{H}) is not invariant with respect to an equivalent change of the underlying probability measure, in general.

Let \mathbb{Q} be a probability measure equivalent to \mathbb{P} on (Ω, \mathcal{G}_t) for every $t \in \mathbb{R}_+$, with the associated Radon-Nikodým density process η . If the **density process η is \mathbf{F} -adapted** then we have

$$\mathbb{Q}(\tau \leq t \mid \mathcal{F}_t) = \mathbb{P}(\tau \leq t \mid \mathcal{F}_t)$$

for every $t \in \mathbb{R}_+$. Hence, the hypothesis (\mathcal{H}) is also valid under \mathbb{Q} and the \mathbf{F} -intensities of τ under \mathbb{Q} and under \mathbb{P} coincide.

PROOF:

$$\begin{aligned} \mathbb{Q}(\tau \leq t | \mathcal{F}_t) &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbf{1}_{\{\tau \leq t\}} | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t | \mathcal{F}_t)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbf{1}_{\{\tau \leq t\}} | \mathcal{F}_{\infty})}{\mathbb{E}_{\mathbb{P}}(\eta_t | \mathcal{F}_{\infty})} \\ &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_{\infty} \mathbf{1}_{\{\tau \leq t\}} | \mathcal{F}_{\infty})}{\mathbb{E}_{\mathbb{P}}(\eta_{\infty} | \mathcal{F}_{\infty})} = \mathbb{P}(\tau \leq t | \mathcal{F}_{\infty}). \end{aligned}$$

Stochastic Barrier

Suppose that

$$P(\tau \leq t | \mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is an arbitrary continuous strictly increasing \mathbf{F} -adapted process. There exists a random variable Θ , independent of \mathcal{F}_∞ , with exponential law of parameter 1, such that $\tau \stackrel{law}{=} \inf \{t \geq 0 : \Gamma_t > \Theta\}$. In fact $\Theta \stackrel{def}{=} \Gamma_\tau$.

PROOF: : Suppose that

$$P(\tau \leq t | \mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is an arbitrary continuous strictly increasing \mathbf{F} -adapted process.

Let us set $\Theta \stackrel{def}{=} \Gamma_\tau$. Then

$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of Γ , so that $\Gamma_{C_t} = t$. Therefore

$$P(\Theta > u | \mathcal{F}_\infty) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established the required properties, namely, the probability law of Θ and its independence of the σ -field \mathcal{F}_∞ . Furthermore,

$$\tau = \inf\{t : \Gamma_t > \Gamma_\tau\} = \inf\{t : \Gamma_t > \Theta\}.$$

Representation theorem

Kusuoka establishes the following representation theorem. Under (\mathcal{H}) , any \mathbf{G} -square integrable martingale admits a representation as the sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale M .

Suppose that hypothesis (\mathcal{H}) holds under \mathbb{P} and that any \mathbf{F} -martingale is continuous. Then, the martingale $M_t^h = \mathbb{E}_{\mathbb{P}}(h_\tau | \mathcal{G}_t)$, where h is an \mathbf{F} -predictable process such that $\mathbb{E}(h_\tau) < \infty$, admits the following decomposition

$$M_t^h = m_0^h + \int_0^{t \wedge \tau} e^{\Gamma_u} dm_u^h + \int_{]0, t \wedge \tau]} (h_u - J_u) dM_u,$$

where m^h is the continuous \mathbf{F} -martingale

$$m_t^h = \mathbb{E}_{\mathbb{P}} \left(\int_0^\infty h_u dF_u \mid \mathcal{F}_t \right),$$

$J_t = e^{\Gamma_t} (m_t^h - \int_0^t h_u dF_u)$ and M is the discontinuous \mathbf{G} -martingale

$$M_t = H_t - \Gamma_{t \wedge \tau}.$$

PROOF: : We know that

$$\begin{aligned}
M_t^h &= \mathbb{E}(h_\tau | \mathcal{G}_t) \\
&= \mathbb{1}_{\{\tau \leq t\}} h_\tau + \mathbb{1}_{\{\tau > t\}} e^{\Gamma t} \mathbb{E} \left(\int_t^\infty h_u dF_u \mid \mathcal{F}_t \right) \\
&= \mathbb{1}_{\{\tau \leq t\}} h_\tau + \mathbb{1}_{\{\tau > t\}} e^{\Gamma t} \left(m_t^h - \int_0^t h_u dF_u \right) \\
&= \int_0^t h_u dH_u + \mathbb{1}_{\{\tau > t\}} J_t.
\end{aligned}$$

From the facts that Γ is an increasing process

m^h a continuous martingale

and using the integration by parts formula, we deduce that

$$dJ_t = e^{\Gamma t} dm_t^h + (J_t - h_t) \frac{dF_t}{1 - F_t}$$

Partial information

As pointed out by Jamshidian, “*one may wish to apply the general theory perhaps as an intermediate step, to a subfiltration that is not equal to the default-free filtration. In that case, \mathbf{F} rarely satisfies hypothesis (\mathcal{H})* ”.

Information at discrete times

Assume that

$$dV_t = V_t(\mu dt + \sigma dW_t), \quad V_0 = v$$

i.e., $V_t = ve^{\sigma(W_t + \nu t)} = ve^{\sigma X_t}$. The default time is assumed to be the first hitting time of α with $\alpha < v$, i.e.,

$$\tau = \inf\{t : V_t \leq \alpha\} = \inf\{t : X_t \leq a\}$$

where $a = \sigma^{-1} \ln(\alpha/v)$.

Here, \mathbf{F} is the filtration of the observations of V at discrete times t_1, \dots, t_n where $t_n \leq t < t_{n+1}$, i.e.,

$$\mathcal{F}_t = \sigma(V_{t_1}, \dots, V_{t_n}, t_i \leq t)$$

The process $F_t = P(\tau \leq t | \mathcal{F}_t)$ is **continuous and increasing in** $[t_i, t_{i+1}[$ but is **not increasing**.

Lemma 0.1 *The process ζ defined by*

$$\zeta_t = \sum_{i, t_i \leq t} \Delta F_{t_i}.$$

is an \mathbf{F} -martingale.

The Doob-Meyer decomposition of F is

$$F_t = \zeta_t + (F_t - \zeta_t),$$

where ζ is an \mathbf{F} -martingale and $F_t - \zeta_t$ is a predictable increasing process.

From

$$P(\inf_{s \leq t} X_s > z) = \Phi(\nu, t, z),$$

where

$$\Phi(\nu, t, z) = \mathcal{N}\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z} \mathcal{N}\left(\frac{z + \nu t}{\sqrt{t}}\right), \quad \text{for } z < 0, t > 0,$$

$$= 0, \quad \text{for } z \geq 0, t \geq 0,$$

$$\Phi(\nu, 0, z) = 1, \quad \text{for } z < 0$$

we obtain (we skip the parameter ν in the definition of Φ) for $t_1 < t < t_2$ and $X_{t_1} > a$

$$F_t = 1 - \Phi(t - t_1, a - X_{t_1}) \left[1 - \exp\left(-\frac{2a}{t_1}(a - X_{t_1})\right) \right].$$

The case $X_{t_1} \leq a$ corresponds to default: for $X_{t_1} \leq a$, $F_t = 1$.

Another example, related with Parisian stopping times is presented in Çetin et al.

Delayed information

Guo et al. suggested to start from a structural model with delayed information, i.e. the reference filtration is $\mathcal{F}_t = \sigma(S_u, u \leq t - \delta)$. In that case, (H) hypothesis is not satisfied.

Intensity approach

In the so-called intensity approach, the default time τ is a \mathbf{G} -stopping time. The intensity is defined as any non-negative process λ , such that

$$M_t \stackrel{\text{def}}{=} H_t - \int_0^{t \wedge \tau} \lambda_s ds$$

is a \mathbf{G} -martingale.

The existence of the intensity relies on the fact that H is a sub-martingale and can be written as $M + A$ where M is a martingale M and A a predictable increasing process. The increasing process A is such that

$$A_t \mathbb{1}_{t \geq \tau} = A_\tau \mathbb{1}_{t \geq \tau}.$$

The intensity exists only if τ is a totally inaccessible stopping time.

We emphasize that, in that setting the intensity is not well defined after time τ , i.e., if λ is an intensity, for any non-negative predictable process g the process $\tilde{\lambda}_t = \lambda_t \mathbb{1}_{t \leq \tau} + g_t \mathbb{1}_{\{t > \tau\}}$ is also an intensity.

If the process $Y_t = \mathbb{E} \left(X \exp \left(- \int_0^T \lambda_u du \right) \mid \mathcal{G}_t \right)$ is continuous at time τ , then, setting $L_t = \mathbb{1}_{\{t < \tau\}} e^{\Gamma_t}$

$$\mathbb{E}(X \mathbb{1}_{\{T < \tau\}} \mid \mathcal{G}_t) = \mathbb{1}_{\{t < \tau\}} \mathbb{E} \left(X \exp \left(- \int_t^T \lambda_u du \right) \mid \mathcal{G}_t \right) = L_t Y_t$$

If Y is not continuous

$$\mathbb{E}(X \mathbb{1}_{\{T < \tau\}} \mid \mathcal{G}_t) = L_t Y_t - \mathbb{E}(\Delta Y_\tau \mathbb{1}_{\tau < T} \mid \mathcal{G}_t).$$

It can be mentioned that the continuity of the process depends on the choice of λ after time τ .

If the process Y is not continuous, then setting

$$U_t = L_t Y_t = \mathbf{1}_{t < \tau} \exp\left(\int_0^t \lambda_s ds\right) \mathbb{E}\left(X \exp\left(-\int_0^T \lambda_u du\right) \mid \mathcal{G}_t\right)$$

we have $U_T = X \mathbf{1}_{\{T < \tau\}}$ and

$$dU_t = L_{t-} dY_t + Y_{t-} dL_t + d[L, Y]_t = L_{t-} dY_t + Y_{t-} dL_t + \Delta L_t \Delta Y_t$$

and

$$\mathbb{E}(U_T \mid \mathcal{G}_t) = \mathbb{E}(X \mathbf{1}_{\{T < \tau\}} \mid \mathcal{G}_t) = U_t - \mathbb{E}(\Delta Y_\tau \mathbf{1}_{\tau < T} \mid \mathcal{G}_t).$$

Then, for any $X \in \mathcal{G}_T$:

$$\mathbb{E}(X \mathbf{1}_{T < \tau} | \mathcal{G}_t) = \mathbf{1}_{\tau > t} (e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T} X | \mathcal{G}_t) - \mathbb{E}(e^{\Lambda_\tau} \Delta Y_\tau \mathbf{1}_{\tau < T} | \mathcal{G}_t))$$

where $Y_t = \mathbb{E}(X \exp(-\Lambda_T) | \mathcal{G}_t)$ and $\Lambda_t = \int_0^t \lambda_u du$

CDS Price, General case

The ex-dividend price of a credit default swap, with a rate process κ and a protection payment δ_τ at default, equals, for every $t \in [s, T]$,

$$\begin{aligned} S_t(\kappa) &= \mathbb{E}_Q \left(\mathbb{1}_{\{t < \tau \leq T\}} \delta_\tau \mid \mathcal{G}_t \right) - \mathbb{E}_Q \left(\mathbb{1}_{\{t < \tau\}} \int_t^{\tau \wedge T} \kappa_s ds \mid \mathcal{G}_t \right), \\ &= \mathbb{1}_{\{t < \tau\}} \frac{1}{G_t} \mathbb{E}_Q \left(- \int_t^T \delta_u dG_u + \int_t^\infty dG_u \int_t^{u \wedge T} \kappa_v dv \mid \mathcal{F}_t \right). \end{aligned}$$

We now assume that **(H) hypothesis holds** between \mathbf{F} and \mathbf{G} , that is \mathbf{F} -martingales are \mathbf{G} -martingales. Then, F is increasing and the process

$$M_t = H_t - \int_0^{t \wedge \tau} \gamma_u du,$$

with $\gamma_t dt = \frac{dF_t}{G_t}$ is a \mathbf{G} -martingale.

The dynamics of the ex-dividend price $S_t(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa) dM_t + (1 - H_t) B_t G_t^{-1} dm_t + (1 - H_t)(r_t S_t(\kappa) + \kappa - \delta_t \gamma_t) dt,$$

where m is the (Q, \mathbf{F}) -martingale given by

$$m_t = \mathbb{E}_Q \left(\int_0^T B_u^{-1} \delta_u G_u \gamma_u du - \kappa \int_0^T B_u^{-1} G_u du \mid \mathcal{F}_t \right).$$

Hedging defaultable claims

Our aim is to hedge

$$Y = \mathbb{1}_{\{T \geq \tau\}} Z_\tau + \mathbb{1}_{\{T < \tau\}} X.$$

using two CDS with maturities T_i , rates κ_i and protection payment δ^i . We assume $r = 0$. Let ζ_t^i defined as

$$m_t^i = \mathbb{E}_Q \left(\int_0^T \delta_u^i G_u \gamma_u du - \kappa_i \int_0^T G_u du \mid \mathcal{F}_t \right), \quad dm_t^i = \zeta_t^i dW_t$$

and

$$m_t^Z = \mathbb{E}_Q \left(- \int_0^\infty Z_u dG_u + G_T X \mid \mathcal{F}_t \right), \quad dm_t^Z = \zeta_t^Z dW_t$$

Assume that there exist \mathbf{F} -predictable processes ϕ^1, ϕ^2 such that

$$\sum_{i=1}^2 \phi_t^i (\delta_t^i - \tilde{S}_t^i(\kappa_i)) = Z_t - \tilde{y}_t, \quad \sum_{i=1}^2 \phi_t^i \zeta_t^i = \zeta_t,$$

where \tilde{y} is given by

$$\tilde{y}_t = \frac{1}{G_t} \mathbb{E}_Q \left(- \int_t^T Z_u dG_u + G_T X \mid \mathcal{F}_t \right).$$

Let $\phi_t^0 = V_t(\phi) - \sum_{i=1}^2 \phi_t^i S_t^i(\kappa_i)$, where the process $V(\phi)$ is given by

$$dV_t(\phi) = \sum_{i=1}^2 \phi_t^i (dS_t^i(\kappa_i) + dD_t^i)$$

with the initial condition $V_0(\phi) = \mathbb{E}_Q(Y)$. Then the self-financing trading strategy $\phi = (\phi^0, \phi^1, \phi^2)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$.