
Credit Risk V.

T.R. Bielecki, M. Jeanblanc and M. Rutkowski

MLV, M2 Janvier 2007

V. Hedging of credit derivatives

1. Two default free assets, one defaultable asset
 - 1.1 Two default free assets, one total default asset
 - 1.2 Two default free assets, one defaultable with recovery
2. Two defaultable assets

Two default-free assets, one defaultable asset

We present a particular case where there are two default-free assets

- the savings account with constant interest rate r
- An asset with dynamics

$$dY_t^2 = Y_t^2(\mu_t dt + \sigma_t dW_t)$$

where the coefficients μ_t, σ_t are \mathbf{F} -adapted processes

- a defaultable asset

$$dY_t^3 = Y_{t-}^3(\mu_{3,t} dt + \sigma_{3,t} dW_t + \kappa_{3,t} dM_t),$$

where the coefficients $\mu_3, \sigma_3, \kappa_3$ are \mathbf{G} -adapted processes with $\kappa_3 \geq -1$.

Our aim is to hedge defaultable claims. As we shall establish, the case of **total default** for the third asset (i.e. $\kappa_{3,t} \equiv -1$) is really different from the others.

Two default-free assets, a total default asset

We assume that

$$dY_t^3 = Y_{t-}^3 (\mu_{3,t} dt + \sigma_{3,t} dW_t - dM_t).$$

$$Y_t^3 = \tilde{Y}_t^3 \mathbb{1}_{t < \tau}.$$

Here, our aim is to hedge **survival claims** $(X, 0, \tau)$, i.e. contingent claims of the form $X \mathbb{1}_{T < \tau}$ where $X \in \mathcal{F}_T$.

The price of the contingent claim is

$$C_t = e^{-r(T-t)} E_Q(X \mathbb{1}_{T < \tau} | \mathcal{G}_t)$$

The hedging strategy consists of a triple ϕ^1, ϕ^2, ϕ^3 such that

$$\phi_t^3 Y_t^3 = C_t, \quad \phi_t^1 e^{rt} + \phi_t^2 Y_t = 0$$

and which satisfies the self financing condition.

PDE Approach

We are working in a model with constant (or Markovian) coefficients

$$dY_t = Y_t r dt$$

$$dY_t^2 = Y_t^2 (\mu_2 dt + \sigma_2 dW_t)$$

$$dY_t^3 = Y_{t-}^3 (\mu_3 dt + \sigma_3 dW_t - dM_t).$$

Let $C(t, Y_t^2, Y_t^3, H_t)$ be the price of the contingent claim $G(Y_T^2, Y_T^3, H_T)$ and γ^* be the risk-neutral intensity of default.

Then,

$$\begin{aligned} & \partial_t C(t, y_2, y_3; 0) + r y_2 \partial_2 C(t, y_2, y_3; 0) + \hat{r} y_3 \partial_3 C(t, y_2, y_3; 0) - \hat{r} C(t, y_2, y_3; 0) \\ & + \frac{1}{2} \sum_{i,j=2}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 0) + \gamma^* C(t, y_2, 0; 1) = 0 \end{aligned}$$

where $\hat{r} = r + \gamma^*$ and

$$\partial_t C(t, y_2; 1) + r y_2 \partial_2 C(t, y_2; 1) + \frac{1}{2} \sigma_2^2 y_2^2 \partial_{22} C(t, y_2; 1) - r C(t, y_2; 1) = 0$$

with the terminal conditions

$$C(T, y_2, y_3; 0) = G(y_2, y_3; 0), \quad C(T, y_2; 1) = G(y_2, 0; 1).$$

The *replicating strategy* ϕ for Y is given by formulae

$$\begin{aligned}\phi_t^3 Y_{t-}^3 &= -\Delta C(t) = -C(t, Y_t^2, 0; 1) + C(t, Y_t^2, Y_{t-}^3; 0) \\ \sigma_2 \phi_t^2 Y_t^2 &= -\Delta C(t) + \sum_{i=2}^3 Y_{t-}^i \sigma_i \partial_i C(t) \\ \phi_t^1 Y_t^1 &= C(t) - \phi_t^2 Y_t^2 - \phi_t^3 Y_t^3.\end{aligned}$$

Note that, in the case of survival claim, $C(t, Y_t^2, 0; 1) = 0$ and $\phi_t^3 Y_{t-}^3 = C(t, Y_{t-}^2, Y_{t-}^3; 0)$ for every $t \in [0, T]$. Hence, the following relationships holds, for every $t < \tau$,

$$\phi_t^3 Y_t^3 = C(t, Y_t^2, Y_t^3; 0), \quad \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0.$$

The last equality is a special case of the **balance condition**. It ensures that the wealth of a replicating portfolio falls to 0 at default time.

Example 1

Consider a survival claim $Y = \mathbb{1}_{\{T < \tau\}} g(Y_T^2)$. Its pre-default pricing function $C(t, y_2, y_3; 0) = C^g(t, y_2)$ where C^g solves

$$\begin{aligned} \partial_t C^g(t, y; 0) + ry \partial_2 C^g(t, y; 0) + \frac{1}{2} \sigma_2^2 y^2 \partial_{22} C^g(t, y; 0) - \widehat{r} C^g(t, y; 0) &= 0 \\ C^g(T, y; 0) &= g(y) \end{aligned}$$

The solution is

$$C^g(t, y) = e^{(\widehat{r} - r)(t - T)} C^{r, g, 2}(t, y) = e^{\widehat{\gamma}(t - T)} C^{r, g, 2}(t, y),$$

where $C^{r, g, 2}$ is the price of an option with payoff $g(Y_T)$ in a BS model with interest rate r and volatility σ_2 .

Example 2

Consider a survival claim of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T < \tau\}} g(Y_T^3).$$

Then the post-default pricing function $C^g(\cdot; 1)$ vanishes identically, and the pre-default pricing function $C^g(\cdot; 0)$ is

$$C^g(t, y_2, y_3; 0) = \widehat{C}^{\hat{r}, g, 3}(t, y_3)$$

where $C^{\alpha, g, 3}(t, y)$ is the price of the contingent claim $g(Y_T)$ in the Black-Scholes framework with the interest rate α and the volatility parameter equal to σ_3 .

Two default-free assets, one defaultable asset with Recovery, PDE approach

Let the price processes Y^1, Y^2, Y^3 satisfy

$$dY_t^1 = rY_t^1 dt$$

$$dY_t^2 = Y_t^2(\mu_2 dt + \sigma_2 dW_t)$$

$$dY_t^3 = Y_{t-}^3(\mu_3 dt + \sigma_3 dW_t + \kappa_3 dM_t)$$

with $\sigma_2 \neq 0$. Assume that the relationship $\sigma_2(r - \mu_3) = \sigma_3(r - \mu_2)$ holds and $\kappa_3 \neq 0, \kappa_3 > -1$. Then the price of a contingent claim $Y = G(Y_T^2, Y_T^3, H_T)$ can be represented as $\pi_t(Y) = C(t, Y_t^2, Y_t^3; H_t)$, where the pricing functions $C(\cdot; 0)$ and $C(\cdot; 1)$ satisfy the following PDEs

$$\begin{aligned} & \partial_t C(t, y_2, y_3; 1) + ry_2 \partial_2 C(t, y_2, y_3; 1) + ry_3 \partial_3 C(t, y_2, y_3; 1) - rC(t, y_2, y_3; 1) \\ & + \frac{1}{2} \sum_{i,j=2}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 1) = 0 \end{aligned}$$

and

$$\begin{aligned} & \partial_t C(t, y_2, y_3; 0) + ry_2 \partial_2 C(t, y_2, y_3; 0) + y_3 (r - \kappa_3 \gamma) \partial_3 C(t, y_2, y_3; 0) \\ & - rC(t, y_2, y_3; 0) + \frac{1}{2} \sum_{i,j=2}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 0) \\ & + \gamma (C(t, y_2, y_3(1 + \kappa_3); 1) - C(t, y_2, y_3; 0)) = 0 \end{aligned}$$

subject to the terminal conditions

$$C(T, y_2, y_3; 0) = G(y_2, y_3, 0), \quad C(T, y_2, y_3; 1) = G(y_2, y_3, 1).$$

The replicating strategy equals $\phi = (\phi^1, \phi^2, \phi^3)$

$$\begin{aligned}\phi_t^2 &= \frac{1}{\sigma_2 \kappa_3 Y_t^2} \left(\kappa_3 \sum_{i=2}^3 \sigma_i y_i \partial_i C(t, Y_t^2, Y_{t-}^3, H_{t-}) \right. \\ &\quad \left. - \sigma_3 (C(t, Y_t^2, Y_{t-}^3 (1 + \kappa_3); 1) - C(t, Y_t^2, Y_{t-}^3; 0)) \right), \\ \phi_t^3 &= \frac{1}{\kappa_3 Y_{t-}^3} (C(t, Y_t^2, Y_{t-}^3 (1 + \kappa_3); 1) - C(t, Y_t^2, Y_{t-}^3; 0)),\end{aligned}$$

and where ϕ_t^1 is given by $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 = C_t$.

Example Consider a survival claim of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T < \tau\}} g(Y_T^3).$$

Then the post-default pricing function $C^g(\cdot; 1)$ vanishes identically, and the pre-default pricing function $C^g(\cdot; 0)$ solves

$$\begin{aligned} \partial_t C^g(\cdot; 0) &+ r y_2 \partial_2 C^g(\cdot; 0) + y_3 (r - \kappa_3 \gamma) \partial_3 C^g(\cdot; 0) \\ &+ \frac{1}{2} \sum_{i,j=2}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} C^g(\cdot; 0) - (r + \gamma) C^g(\cdot; 0) = 0 \end{aligned}$$

$$C^g(T, y_2, y_3; 0) = g(y_3)$$

Denote $\alpha = r - \kappa_3 \gamma$ and $\beta = \gamma(1 + \kappa_3)$.

Then, $C^g(t, y_2, y_3; 0) = e^{\beta(T-t)} C^{\alpha, g, 3}(t, y_3)$ where $C^{\alpha, g, 3}(t, y)$ is the price of the contingent claim $g(Y_T)$ in the Black-Scholes framework with the interest rate α and the volatility parameter equal to σ_3 .

Let C_t be the current value of the contingent claim Y , so that

$$C_t = \mathbb{1}_{\{t < \tau\}} e^{\beta(T-t)} C^{\alpha, g, 3}(t, y_3).$$

The hedging strategy of the survival claim is, on the event $\{t < \tau\}$,

$$\begin{aligned} \phi_t^3 Y_t^3 &= -\frac{1}{\kappa_3} e^{-\beta(T-t)} C^{\alpha, g, 3}(t, Y_t^3) = -\frac{1}{\kappa_3} C_t, \\ \phi_t^2 Y_t^2 &= \frac{\sigma_3}{\sigma_2} \left(Y_t^3 e^{-\beta(T-t)} \partial_y C^{\alpha, g, 3}(t, Y_t^3) - \phi_t^3 Y_t^3 \right). \end{aligned}$$

Hedging of a Recovery Payoff

The price C^g of the payoff $G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T \geq \tau\}} g(Y_T^2)$ solves

$$\begin{aligned} \partial_t C^g(\cdot; 1) + r y \partial_y C^g(\cdot; 1) + \frac{1}{2} \sigma_2^2 y^2 \partial_{yy} C^g(\cdot; 1) - r C^g(\cdot; 1) &= 0 \\ C^g(T, y; 1) &= g(y) \end{aligned}$$

hence $C^g(t, y_2, y_3, 1) = C^{r,g,2}(t, y_2)$ is the price of $g(Y_T^2)$ in the model Y^1, Y^2 . Prior to default, the price of the claim solves

$$\begin{aligned} \partial_t C^g(\cdot; 0) + r y_2 \partial_2 C^g(\cdot; 0) + y_3 (r - \kappa_3 \gamma) \partial_3 C^g(\cdot; 0) \\ + \frac{1}{2} \sum_{i,j=2}^3 \sigma_i \sigma_j y_i y_j \partial_{ij} C^g(\cdot; 0) - (r + \gamma) C^g(\cdot; 0) &= -\gamma C^g(t, y_2; 1) \\ C^g(T, y_2, y_3; 0) &= 0 \end{aligned}$$

Hence $C^g(t, y_2, y_3; 0) = (1 - e^{\gamma(t-T)}) C^{r,g,2}(t, y_2)$.

Two defaultable assets with total default

Assume that Y^1 and Y^2 are defaultable tradeable assets with zero recovery and a common default time τ .

$$dY_t^i = Y_{t-}^i (\mu_i dt + \sigma_i dW_t - dM_t), i = 1, 2$$

Then

$$Y_t^1 = \mathbb{1}_{\{\tau > t\}} \tilde{Y}_t^1, \quad Y_t^2 = \mathbb{1}_{\{\tau > t\}} \tilde{Y}_t^2$$

with

$$d\tilde{Y}_t^i = \tilde{Y}_t^i ((\mu_i + \gamma_t) dt + \sigma_i dW_t), i = 1, 2$$

The wealth process V associated with the self-financing trading strategy (ϕ^1, ϕ^2) satisfies for $t \in [0, T]$

$$V_t = Y_t^1 \left(V_0^1 + \int_0^t \phi_u^2 d\tilde{Y}_u^{2,1} \right)$$

where $\tilde{Y}_t^{2,1} = \tilde{Y}_t^2 / \tilde{Y}_t^1$.

Obviously, this market is **incomplete, however, some contingent claims are hedgeable**, as we present now.

Hedging Survival claim: martingale approach

A strategy (ϕ^1, ϕ^2) replicates a survival claim $C = X \mathbb{1}_{\{\tau > T\}}$ whenever we have

$$\tilde{Y}_T^1 \left(\tilde{V}_0^1 + \int_0^T \phi_t^2 d\tilde{Y}_t^{2,1} \right) = X$$

for some constant \tilde{V}_0^1 and some \mathbf{F} -predictable process ϕ^2 .

It follows that if $\sigma_1 \neq \sigma_2$, **any survival claim $C = X \mathbb{1}_{\{\tau > T\}}$ is attainable.**

Let \tilde{Q} be a probability measure such that $\tilde{Y}_t^{2,1}$ is an \mathbf{F} -martingale under \tilde{Q} . Then the pre-default value $\tilde{U}_t(C)$ at time t of $(X, 0, \tau)$ equals

$$\tilde{U}_t(C) = \tilde{Y}_t^1 E_{\tilde{Q}} \left(X (\tilde{Y}_T^1)^{-1} \mid \mathcal{F}_t \right).$$

Example: Call option on a defaultable asset We assume that $Y_t^1 = D(t, T)$ represents a defaultable ZC-bond with zero recovery, and $Y_t^2 = \mathbb{1}_{\{t < \tau\}} \tilde{Y}_t^2$ is a generic defaultable asset with total default. The payoff of a call option written on the defaultable asset Y^2 equals

$$C_T = (Y_T^2 - K)^+ = \mathbb{1}_{\{T < \tau\}} (\tilde{Y}_T^2 - K)^+$$

The replication of the option reduces to finding a constant x and an \mathbf{F} -predictable process ϕ^2 that satisfy

$$x + \int_0^T \phi_t^2 d\tilde{Y}_t^{2,1} = (\tilde{Y}_T^2 - K)^+.$$

Assume that the volatility $\sigma_{1,t} - \sigma_{2,t}$ of $\tilde{Y}^{2,1}$ is deterministic. Let $\tilde{F}_2(t, T) = \tilde{Y}_t^2 (\tilde{D}(t, T))^{-1}$

The credit-risk-adjusted forward price of the option written on Y^2 equals

$$\tilde{C}_t = \tilde{Y}_t^2 \mathcal{N}(d_+(\tilde{F}_2(t, T), t, T)) - K \tilde{D}(t, T) \mathcal{N}(d_-(\tilde{F}_2(t, T), t, T)),$$

where

$$d_{\pm}(\tilde{f}, t, T) = \frac{\ln \tilde{f} - \ln K \pm \frac{1}{2}v^2(t, T)}{v(t, T)}$$

and

$$v^2(t, T) = \int_t^T (\sigma_{1,u} - \sigma_{2,u})^2 du.$$

Moreover the replicating strategy ϕ in the spot market satisfies for every $t \in [0, T]$, on the set $\{t < \tau\}$,

$$\phi_t^1 = -K \mathcal{N}(d_-(\tilde{F}_2(t, T), t, T)), \quad \phi_t^2 = \mathcal{N}(d_+(\tilde{F}_2(t, T), t, T)).$$

Hedging Survival claim: PDE approach

Assume that $\sigma_1 \neq \sigma_2$. Then the pre-default pricing function v satisfies the PDE

$$\begin{aligned} \partial_t C + y_1 \left(\mu_1 + \gamma - \sigma_1 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) \partial_1 C + y_2 \left(\mu_2 + \gamma - \sigma_2 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) \partial_2 C \\ + \frac{1}{2} \left(y_1^2 \sigma_1^2 \partial_{11} C + y_2^2 \sigma_2^2 \partial_{22} C + 2y_1 y_2 \sigma_1 \sigma_2 \partial_{12} C \right) = \left(\mu_1 + \gamma - \sigma_1 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) C \end{aligned}$$

with the terminal condition $C(T, y_1, y_2) = G(y_1, y_2)$.