EMS SCHOOL

Risk Theory and Related Topics

Bedlewo, Poland. september 29th- october 8th, 2008

Credit Risk: Reduced Form Approach

Tomasz R. Bielecki, IIT, Chicago
Monique Jeanblanc, University of Evry
Marek Rutkowski, University of New South Wales, Sydney

In a financial market built on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, a default occurs at some random time τ.

The filtration $\mathbb{F}$ is called the reference filtration

OUTLINE:

1. Hazard function approach
2. Hazard process approach
3. Hedging defaultable claims
4. Credit Default Swaps
5. Enlargement of filtration results

HAZARD FUNCTION APPROACH

- **Model for single default**
- **Several Defaults**

Model for single default

Definition and Properties of the Hazard Function

Set-up

- We assume that the only information available is the probability distribution of default time.
- Hence we do not take into account the uncertainty of conditional default probabilities.
- Formally, we assume that the reference filtration is trivial, or that the default time is independent of the reference filtration.
- This approach can also be used in the multi-name set-up.

Random Time

- Let τ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, referred to as a **random time**.
- We assume that $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$ for any $t \in \mathbb{R}_+$ so that the c.d.f. F satisfies, for every $t \in \mathbb{R}_+$,

$$F(t) = \mathbb{P}(\tau \leq t) < 1.$$

  This means that τ is an unbounded random variable.
- We introduce the associated **default process**

$$H_t = \mathbb{1}_{\{\tau \leq t\}}$$

  and we write $\mathbb{H} = (\mathcal{H}_t)_{t \in \mathbb{R}_+}$ to denote the filtration generated by H.
- Of course, τ is an **$\mathbb{H}$-stopping time**, that is, the event $\{\tau \leq t\}$ is in $\mathcal{H}_t$ for any $t \in \mathbb{R}_+$.

Conditional Expectation

We shall assume throughout that all random variables and processes satisfy suitable integrability conditions.

Lemma 1 *For any $\mathcal{G}$-**measurable random variable** Y we have*

$$\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} \mathbb{E}_{\mathbb{P}}(Y \,|\, \tau) + \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y)}{\mathbb{P}(\tau > t)}.$$

*For any $\mathcal{H}_t$-**measurable random variable** Y we have*

$$Y = \mathbb{1}_{\{\tau \le t\}} \mathbb{E}_{\mathbb{P}}(Y \,|\, \tau) + \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y)}{\mathbb{P}(\tau > t)},$$

that is, $Y = h(\tau \wedge t)$ for some function $h : \mathbb{R}^+ \to \mathbb{R}$.

Hazard Function

- The notion of the **hazard function** of a random time τ is closely related to the cumulative distribution function F of τ.
- Recall that the c.d.f. of τ equals

$$F(t) = \mathbb{P}(\tau \leq t), \quad \forall\, t \in \mathbb{R}_+.$$

- Let G stand for the tail: $G(t) = 1 - F(t)$ for $t \in \mathbb{R}_+$.

Definition 1 *The function $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula*

$$\Gamma(t) = -\ln\left(1 - F(t)\right) = -\ln G(t), \quad \forall\, t \in \mathbb{R}_+,$$

is called the ***hazard function*** *of a random time τ.*

Intensity of Default

- If the distribution function F is an **absolutely continuous function**, that is,

$$F(t) = \int_0^t f(u)\,du$$

for some function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ then we have

$$F(t) = 1 - e^{-\Gamma(t)} = 1 - e^{-\int_0^t \gamma(u)\,du}$$

where we denote

$$\gamma(t) = \frac{f(t)}{1 - F(t)}\,.$$

- $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a non-negative function and $\int_0^\infty \gamma(u)\,du = \infty$.
- γ is called the **intensity function** or the **hazard rate** of τ.

Conditional Expectations

Corollary 1

- *In terms of the hazard function* Γ *of* τ, *we have*

$$\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} \mathbb{E}_{\mathbb{P}}(Y \,|\, \tau) + \mathbb{1}_{\{\tau > t\}}\, e^{\Gamma(t)}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y).$$

- *If* $Y = h(\tau)$ *for some function* $h : \mathbb{R}_+ \to \mathbb{R}$ *then*

$$\mathbb{E}_{\mathbb{P}}(h(\tau) \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} \int_t^\infty h(u) e^{\Gamma(t) - \Gamma(u)}\, d\Gamma(u).$$

- *If, in addition, the random time* τ *has intensity* γ *then*

$$\mathbb{E}_{\mathbb{P}}(h(\tau) \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} \int_t^\infty h(u) \gamma(u) e^{-\int_t^u \gamma(v)\, dv}\, du.$$

Conditional Survival Probabilities

- For any $t \leq T$, the last formula yields

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\tau>T} \,|\, \mathcal{H}_t) = \mathbb{P}(\tau > T \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau>t\}}\, e^{-\int_t^T \gamma(v)\,dv}.$$

In particular

$$\mathbb{P}(\tau > T \,|\, \tau > t) = e^{-\int_t^T \gamma(v)\,dv}.$$

- We also have that

$$\mathbb{P}(t < \tau < T \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau>t\}}\Big(1 - e^{-\int_t^T \gamma(v)\,dv}\Big)$$

and thus

$$\mathbb{P}(t < \tau < T \,|\, \tau > t) = 1 - e^{-\int_t^T \gamma(v)\,dv}.$$

Interpretation of Intensity

- Let us observe that

$$\mathbb{P}\{\tau \in [t, t+dt] \,|\, \mathcal{H}_t\} = \mathbb{1}_{\{\tau > t\}} \gamma(t)\, dt$$

that is

$$\lim_{h \to 0} \frac{1}{h} \mathbb{P}\{\tau \in [t, t+h] \,|\, \tau > t\} = \gamma(t).$$

- Recall that

$$\mathbb{P}\{\tau \in [t, t+dt]\} = f(t)\, dt.$$

and

$$\gamma(t) = \frac{f(t)}{1 - F(t)}\,.$$

Martingales

Martingale L

A first martingale can be associated with any random time, that is, the c.d.f. F may be discontinuous.

Proposition 1 *The process L given by the formula*

$$L_t = \frac{1 - H_t}{1 - F(t)} = (1 - H_t)e^{-\Gamma(t)}$$

is an $\mathbb{H}$***-martingale****:* $\mathbb{E}_{\mathbb{P}}(L_s \,|\, \mathcal{H}_t) = L_t$ *for* $s \geq t$.

Martingale M

In the next result, the c.d.f. F of a random time τ is assumed to be **continuous**.

Proposition 2

- *Assume that F (and thus also Γ) is a continuous function. Then the process*

$$M_t = H_t - \Gamma(t \wedge \tau) = H_t - \int_0^t (1 - H_s) \frac{dF(s)}{1 - F(s)}$$

  is an $\mathbb{H}$-martingale.

- *If a random time τ admits the intensity function γ then the process*

$$M_t = H_t - \int_0^{\tau \wedge t} \gamma(u)\, du$$

  follows an $\mathbb{H}$-martingale.

Martingale M

In the general case, the process

$$M_t = H_t - \int_0^{t \wedge \tau} \frac{dF(s)}{1 - F(s-)}$$

is an $\mathbb{H}$-martingale.

Equivalent Probability Measure

Change of a Probability Measure

- Let $\mathbb{P}^*$ be any probability measure on $(\Omega, \mathcal{H}_\infty)$, which is equivalent to $\mathbb{P}$, that is: for any event $A \in \mathcal{H}_\infty$ we have $\mathbb{P}^*(A) = 0$ if and only if $\mathbb{P}(A) = 0$.

  Then there exists a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

  $$\mathbb{E}_{\mathbb{P}}(h(\tau)) = \int_0^\infty h(u)\, dF(u) = 1$$

  and the Radon-Nikodým density of $\mathbb{P}^*$ with respect to $\mathbb{P}$ equals

  $$\eta = \frac{d\mathbb{P}^*}{d\mathbb{P}} = h(\tau) > 0, \quad \mathbb{P}\text{-a.s.}$$

- In the financial interpretation, $\mathbb{P}$ is the real-world probability and $\mathbb{P}^*$ is a spot martingale measure (pricing probability).

Assumptions and Notation

- Assume that $\mathbb{P}\{\tau = 0\} = 0$ and $\mathbb{P}\{\tau > t\} > 0$ for $t \in \mathbb{R}_+$.
- Note that for every $t \in \mathbb{R}_+$

$$\mathbb{P}^*\{\tau > t\} = 1 - F^*(t) = \int_{(t,\infty)} h(u)\, dF(u) > 0$$

where F^* is the c.d.f. of τ under $\mathbb{P}^*$. Equivalently

$$F^*(t) = \mathbb{P}^*\{\tau \le t\} = \int_{(0,t]} h(u)\, dF(u).$$

- Let

$$g(t) = e^{\Gamma(t)}\, \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau > t\}} h(\tau)\big) = e^{\Gamma(t)} \int_{(t,\infty)} h(u)\, dF(u)$$

and let $h^* : \mathbb{R}_+ \to \mathbb{R}$ be given by $h^*(t) = h(t) g^{-1}(t)$.

Hazard Function under $\mathbb{P}^*$

- If F (and thus F^*) is continuous then the hazard function Γ^* of τ under $\mathbb{P}^*$ satisfies

$$d\Gamma^*(t) = \frac{dF^*(t)}{1 - F^*(t)}$$

  and thus

$$d\Gamma^*(t) = h^*(t)\, d\Gamma(t).$$

- Let us denote

$$\kappa(t) = h^*(t) - 1 = h(t)g^{-1}(t) - 1 > -1.$$

Proposition 3 *Let $\mathbb{P}^*$ and $\mathbb{P}$ be two equivalent probabilities on $(\Omega, \mathcal{H})$. If the hazard function Γ of τ under $\mathbb{P}$ is continuous then the hazard function Γ^* of τ under $\mathbb{P}^*$ is continuous and*

$$d\Gamma^*(t) = (1 + \kappa(t))\, d\Gamma(t)$$

In case where the intensity exists $\gamma^*(t) = (1 + \kappa(t))\gamma(t)$.

Valuation of Defaultable Claims

A defaultable claim consists of:

- the **promised contingent claim** X, representing the payoff received by the owner of the claim at time T, if there was no default prior to or at time T,
- the process A representing the **promised dividends** – that is, the stream of (continuous or discrete) cash flows received by the owner of the claim prior to default; we assume that $A_0 = 0$,
- the **recovery process** Z, representing the recovery payoff at time of default, if default occurs prior to or at time T,
- the **recovery claim** $\widetilde{X}$, which represents the recovery payoff at time T if default occurs prior to or at the maturity date T.

Dividend Process

- A defaultable claim can be represented as $(X, A, \widetilde{X}, Z, \tau)$.
- The **dividend process** D of a defaultable claim $(X, A, \widetilde{X}, Z, \tau)$ equals

$$D_t = X^d(T)\mathbb{1}_{\{t \geq T\}} + \int_{(0,t]} (1 - H_u)\, dA_u + \int_{(0,t]} Z_u\, dH_u$$

or equivalently

$$D_t = X^d(T)\mathbb{1}_{\{t \geq T\}} + A_{\tau \wedge t} + Z_\tau \mathbb{1}_{\{\tau \leq t\}}.$$

- The random variable

$$X^d(T) = X\mathbb{1}_{\{\tau > T\}} + \widetilde{X}\mathbb{1}_{\{\tau \leq T\}}$$

represents the payoff occurring at maturity T.

Ex-Dividend Price

Definition 2 *The* ***ex-dividend price*** S *of a defaultable claim* $(X, A, \widetilde{X}, Z, \tau)$ *which settles at time* T *is given as*

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}^*} \Big(\int_{(t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big)$$

where $\mathbb{Q}^*$ *is the* ***spot martingale measure*** *for our model and* B *represents the savings account*

$$B_t = \exp \Big(\int_0^t r(u) \, du \Big).$$

- This expression is known as the risk-neutral valuation formula.
- Note that $S_T = 0$ and, in general, the value of S_t depends only on the future cash flows occurring after time t.

Defaultable Bonds

We assume that

- the default time admits the intensity function γ^* under $\mathbb{Q}^*$,
- the short-term interest rate r is deterministic.

In view of the latter assumption, the price at time t of the unit **default-free zero-coupon bond** (ZCB) of maturity T equals

$$B(t,T) = e^{-\int_t^T r(u)\,du}.$$

- A defaultable bond is an example of a defaultable claim with the promised payoff $X = L$ where L is the face value of a bond.
- We assume no coupons so that $A = 0$.
- Hence we only need to specify the recovery value of a bond.

Zero Recovery Scheme

- A corporate ZCB with zero recovery at default can be represented as a defaultable claim $(L, 0, 0, 0, \tau)$.
- Let $D^0(t,T)$ be the price of a bond with zero recovery.
- It is easily seen that $D^0(t,T) = \mathbb{1}_{\{\tau > t\}} \widetilde{D}^0(t,T)$ for any $t \in [0,T]$.

Lemma 2 *The* ***pre-default value*** $\widetilde{D}^0(t,T)$ *of such a bond equals (per unit of the face value L)*

$$\widetilde{D}^0(t,T) = e^{-\int_t^T (r(v)+\gamma^*(v))\,dv} = e^{-\int_t^T \widetilde{r}(v)\,dv}$$

where $\widetilde{r} = r + \gamma^*$ *is the* ***default-risk-adjusted interest rate****.*
Equivalently

$$\widetilde{D}^0(t,T) = B(t,T)e^{-\int_t^T \gamma^*(v)\,dv}.$$

Fractional Recovery of Par Value – FRPV

Let $Z_t = \delta L$ for some constant recovery rate $0 \leq \delta \leq 1$, so that the corporate bond is given as a defaultable claim $(L, 0, 0, \delta L, \tau)$.

Lemma 3 *The pre-default value $\widetilde{D}^\delta(t,T)$ of this bond equals (per unit of the face value L)*

$$\widetilde{D}^\delta(t,T) = \left(\delta \int_t^T e^{-\int_t^u \widetilde{r}(v)\,dv} \gamma^*(u)\,du + e^{-\int_t^T \widetilde{r}(v)\,dv}\right)$$

where $\widetilde{r} = r + \gamma^*$. Equivalently

$$\widetilde{D}^\delta(t,T) = \left(\delta \int_t^T \widetilde{D}^0(t,u)\gamma^*(u)\,du + \widetilde{D}^0(t,T)\right).$$

Fractional Recovery of Treasury Value – FRTV

- Let $Z_t = \delta L B(t,T)$ so that the corporate bond is given as a defaultable claim $(L, 0, 0, \delta L B(t,T), \tau)$.
- The price $D^\delta(t,T)$ can be expressed as follows

$$D^\delta(t,T) = \mathbb{1}_{\{\tau>t\}} B(t,T)\Big(\delta\,\mathbb{Q}^*(t<\tau\le T\,|\,\mathcal{H}_t) + \mathbb{Q}^*(\tau>T\,|\,\mathcal{H}_t)\Big).$$

Lemma 4 *The pre-default value* $\widehat{D}^\delta(t,T)$ *equals*

$$\widehat{D}^\delta(t,T) = \Big(\int_t^T \delta B(t,T) e^{-\int_t^u \gamma^*(v)dv}\gamma^*(u)\,du + e^{-\int_t^T \widetilde{r}(v)dv}\Big)$$

that is

$$\widehat{D}^\delta(t,T) = B(t,T)\Big(\delta\Big(1 - e^{-\int_t^T \gamma^*(v)\,dv}\Big) + e^{-\int_t^T \gamma^*(v)\,dv}\Big).$$

Extensions

- Similar representations can be derived under the assumption that the market risk and the credit risk are independent. Specifically, we assume that
  - the default time admits the $\mathbb{F}$-intensity process γ^* under $\mathbb{Q}^*$,
  - the short-term interest rate r follows a stochastic process independent of the filtration $\mathbb{F}$.
- Another popular convention regarding recovery at default is the fractional recovery of the market value scheme. Under this convention, the value of a corporate bond at default is equal to a fixed fraction of its pre-default value.

Several Defaults

General case

We assume that two default times are given: $\tau_i, i = 1, 2$

We introduce the *joint survival process* $G(u, v)$: for every $u, v \in \mathbb{R}_+$,

$$G(u, v) = \mathbb{Q}(\tau_1 > u, \tau_2 > v)$$

We write

$$\partial_1 G(u, v) = \frac{\partial G}{\partial u}(u, v), \quad \partial_{12} G(u, v) = \frac{\partial^2 G}{\partial u \partial v}(u, v).$$

We assume that the joint density $f(u, v) = \partial_{12} G(u, v)$ exists. In other words, we postulate that $G(u, v)$ can be represented as follows

$$G(u, v) = \int_u^\infty \Big(\int_v^\infty f(x, y)\, dy \Big)\, dx.$$

We compute conditional expectation in the filtration $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$:
For $t < T$

$$\mathbb{P}(T < \tau_1 | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) = 1\!\!1_{t<\tau_1} \frac{\mathbb{P}(T < \tau_1 | \mathcal{H}_t^2)}{\mathbb{P}(t < \tau_1 | \mathcal{H}_t^2)}$$

$$\begin{aligned} &= 1\!\!1_{t<\tau_1} \left(1\!\!1_{t<\tau_2} \frac{\mathbb{P}(T < \tau_1, t < \tau_2)}{\mathbb{P}(t < \tau_1, t < \tau_2)} + 1\!\!1_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) \\ &= 1\!\!1_{t<\tau_1} \left(1\!\!1_{t<\tau_2} \frac{G(T,t)}{G(t,t)} + 1\!\!1_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) \end{aligned}$$

• The computation of $\mathbb{P}(T < \tau_1 | \tau_2)$ can be done as follows:

$$\mathbb{P}(T < \tau_1 | \tau_2 = v) = \frac{\mathbb{P}(T < \tau_1, \tau_2 \in dv)}{\mathbb{P}(\tau_2 \in dv)} = \frac{\partial_2 G(T, v)}{\partial_2 G(0, v)}$$

hence, on the set $\tau_2 < T$,

$$\mathbb{P}(T < \tau_1 | \tau_2) = \frac{\partial_2 G(T, \tau_2)}{\partial_2 G(0, \tau_2)}$$

Value of credit derivatives

We introduce different credit derivatives

A **defaultable zero-coupon** related to the default time τ_i delivers 1 monetary unit if τ_i is greater that T: $D^i(t,T) = \mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{T<\tau_i\}}|\mathcal{H}_t^1 \vee \mathcal{H}_t^2)$

We obtain

$$D^1(t,T) \quad = \quad \mathbb{1}_{\{\tau_1>t\}} \left(\mathbb{1}_{\{\tau_2 \le t\}} \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + \mathbb{1}_{\{\tau_2>t\}} \frac{G(T,t)}{G(t,t)} \right)$$

A contract which pays R_1 is one default occurs before T and R_2 if the two defaults occur before T:

$$\begin{aligned}
CD_t &= \mathbb{E}_{\mathbb{Q}^*}(R_1 1\!\!1_{\{0<\tau_{(1)}\leq T\}} + R_2 1\!\!1_{\{0<\tau_{(2)}\leq T\}} | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) \\
&= R_1 1\!\!1_{\{\tau_{(1)}>t\}} \left(\frac{G(t,t) - G(T,T)}{G(t,t)} \right) + R_2 1\!\!1_{\{\tau_{(2)}\leq t\}} + R_1 1\!\!1_{\{\tau_{(1)}\leq t\}} \\
&\quad +R_2 1\!\!1_{\{\tau_{(2)}>t\}} \left\{ I_t(0,1) \left(1 - \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} \right) + I_t(1,0) \left(1 - \frac{\partial_1 G(\tau_1,T)}{\partial_1 G(\tau_1,t)} \right) \right. \\
&\qquad \left. +I_t(0,0) \left(1 - \frac{G(t,T) + G(T,t) - G(T,T)}{G(t,t)} \right) \right\}
\end{aligned}$$

where by

$$\begin{aligned}
I_t(1,1) &= 1\!\!1_{\{\tau_1\leq t, \tau_2\leq t\}} \ , & I_t(0,0) &= 1\!\!1_{\{\tau_1> t, \tau_2> t\}} \\
I_t(1,0) &= 1\!\!1_{\{\tau_1\leq t, \tau_2> t\}} \ , & I_t(0,1) &= 1\!\!1_{\{\tau_1> t, \tau_2\leq t\}}
\end{aligned}$$

More generally, some easy computation leads to

$$\mathbb{E}_{\mathbb{Q}^*}(h(\tau_1,\tau_2)|\mathcal{H}_t) = I_t(1,1)h(\tau_1,\tau_2)+I_t(1,0)\Psi_{1,0}(\tau_1)+I_t(0,1)\Psi_{0,1}(\tau_2)+I_t(0,0)\Psi_{0,0}$$

where

$$\begin{aligned}
\Psi_{1,0}(u) &= -\frac{1}{\partial_1 G(u,t)}\int_t^\infty h(u,v)\partial_1 G(u,dv)\\
\Psi_{0,1}(v) &= -\frac{1}{\partial_2 G(t,v)}\int_t^\infty h(u,v)\partial_2 G(du,v)\\
\Psi_{0,0} &= \frac{1}{G(t,t)}\int_t^\infty\int_t^\infty h(u,v)G(du,dv)
\end{aligned}$$

Copula

Copula Function

The concept of a **copula function** allows to produce various multidimensional probability distributions with the same univariate marginal probability distributions.

Definition 3 *A function $C : [0,1]^n \to [0,1]$ is a copula function if:*

- $C(1, \ldots, 1, v_i, 1, \ldots, 1) = v_i$ *for any i and any $v_i \in [0,1]$,*
- *C is an n-dimensional cumulative distribution function.*

Examples of copulae:

- product copula: $\Pi(v_1, \ldots, v_n) = \Pi_{i=1}^n v_i$,
- Gumbel copula: for $\theta \in [1, \infty)$ we set

$$C(v_1, \ldots, v_n) = \exp\left(- \left[\sum_{i=1}^n (-\ln v_i)^\theta \right]^{1/\theta} \right).$$

Sklar's Theorem

Theorem 1

- *For any cumulative distribution function F on $\mathbb{R}^n$ there exists a copula function C such that*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

where F_i is the i^{th} marginal cumulative distribution function.

If, in addition, F is continuous then C is unique.

- *Conversely, if C is an n-dimensional copula and $F_1, F_2, \ldots, F_n$ are the distribution functions, then the function*

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))$$

is a n-dimensional distribution function with marginals $F_1, F_2, \ldots, F_n$.

Survival Copula

- We can represent the joint survival function as some copula as well. Since for standard uniform random variables $U_1, U_2, \dots, U_n$, the random variables $\widetilde{U_1} = 1 - U_1, \widetilde{U_2} = 1 - U_2, \dots, \widetilde{U_n} = 1 - U_n$ are also uniform random variables.

- Hence we have

$$
\begin{aligned}
& G(x_1, x_2, \dots, x_n) \\
&= \mathbb{P}(X_1 \geq x_1, X_2 \geq x_2, \dots, X_n \geq x_n) \\
&= \mathbb{P}(F_1(X_1) \geq F_1(x_1), \dots, F_n(X_n) \geq F_n(x_n)) \\
&= \mathbb{P}(1 - F_1(X_1) \leq 1 - F_1(x_1), \dots, 1 - F_n(X_n) \leq 1 - F_n(x_n)) \\
&= \mathbb{P}(\widetilde{U}_1 \leq G_1(x_1), \widetilde{U}_2 \leq G_2(x_2), \dots, \widetilde{U}_n \leq G_n(x_n)) \\
&= \widetilde{C}(G_1(x_1), G_2(x_2), \dots, G_n(x_n))
\end{aligned}
$$

Gaussian Copula

- Gaussian copulae have become an industry standard for CDO and credit portfolio modelling, despite of several drawbacks.
- Assume that the marginal cumulative distribution functions $F_1, F_2, \ldots, F_n$ of default times $\tau_1, \tau_2, \ldots, \tau_n$ are known.
- The default times $\tau_1, \tau_2, \ldots, \tau_n$ are modelled from a **Gaussian vector** $(X_1, X_2, \ldots, X_n)$ with zero means, unit variances, and covariance matrix Σ.
- Specifically, $\tau_i = F_i^{-1}(\Phi(X_i))$ for $i = 1, \ldots, n$, where F_i^{-1} denotes the generalized inverse of F_i and Φ is the standard Gaussian distribution function, so that

$$\mathbb{P}(\tau_i \leq t) = \mathbb{P}(\Phi(X_i) \leq F_i(t)) = F_i(t)$$

Multivariate Gaussian Copula

Let R be an $n \times n$ symmetric, positive definite matrix with $R_{ii} = 1$ for $i = 1, 2, \ldots, n$, and let Φ_R be the standardized multivariate normal distribution with correlation matrix R

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} |R|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{x}' R^{-1} \mathbf{x}\right).$$

Definition 4 *The multivariate Gaussian copula* C_R *is defined as:*

$$C_R(u_1, u_2, \ldots, u_n) = \Phi_R(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_n))$$

where $\Phi^{-1}(u)$ *represents the inverse of the normal cumulative distribution function.*

One-Factor Gaussian Copula

- A **one-factor Gaussian copula** is the multivariate Gaussian copula corresponding to the joint distribution of the vector $(X_1, X_2, \dots, X_n)$ where

$$X_i = \rho_i V + \sqrt{1 - \rho_i^2}\, Y_i$$

  where V and $Y_1, Y_2, \dots, Y_n$ are independent standard Gaussian random variables and $0 \le \rho_i \le 1$ for $i = 1, 2, \dots, n$.

- Then we can get (recall that $\tau_i = F_i^{-1}(\Phi(X_i)))$

$$\mathbb{P}(\tau_i \le t \,|\, V) = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1 - \rho_i^2}} \right).$$

- The case $\rho_1 = \ldots = \rho_n = 0$ corresponds to independent defaults, whereas $\rho_1 = \ldots = \rho_n = 1$ represents the co-monotonic case.

Default Times

- We assume that a default has occurred by time t, in case a non-decreasing function χ_i has crossed the **trigger level** X_i prior to or at t.
- Formally, the default times are given by

$$\tau_i = \inf\{t \in \mathbb{R}_+ : \chi_i(t) \geq X_i\}, \quad i = 1, 2, \ldots, n,$$

where $\chi_i(t) = \Phi^{-1}(F_i(t))$ (and $\mathbb{P}(\tau_i \leq t) = F_i(t)$).

- This construction of dependent default times $\tau_1, \tau_2, \ldots, \tau_n$ is referred to as the **one-factor copula model**.
- We shall now compare this approach with the intensity-based approach to correlated defaults.

Comparison with Intensity-Based Model

- If F_{X_i} is a continuous function for every i then

  $$\tau_i = \inf \{t \in \mathbb{R}_+ : F_{X_i}(\chi_i(t)) \geq F_{X_i}(X_i)\} = \inf \{t \in \mathbb{R}_+ : G_i(t) \leq \widetilde{U}_i\}$$

  where $(\widetilde{U}_1, \widetilde{U}_2, \ldots, \widetilde{U}_n)$ with $\widetilde{U}_i = 1 - F_{X_i}(X_i)$ are random variables with uniform marginal distributions (not independent) and $G_i(t) = 1 - F_{X_i}(\chi_i(t)) = 1 - \mathbb{P}\{\tau_i \leq t\}$.

- This representation of the one-factor copula model allows for easy comparison with the intensity-based model in which

  $$\tau_i = \inf \{t \in \mathbb{R}_+ : G_t^i \leq U_i\}$$

  where $(U_1, U_2, \ldots, U_n)$ are independent uniformly distributed random variables and $G^1, G^2, \ldots, G^n$ are non-increasing default countdown processes (not independent, in general).

Student t Copula

- Let us denote $V_i = \sqrt{W} X_i$ and $X_i = \rho_i V + \sqrt{1-\rho_i^2}\, Y_i$ where $V, Y_1, Y_2, \ldots, Y_n$ are independent $N(0,1)$ random variables. W is independent of $X_1, X_2, \ldots, X_n$ and has the inverse gamma distribution with parameter $\frac{\nu}{2}$.
- Let t_ν denote the c.d.f. of the Student t distribution with ν degrees of freedom.
- We set $\tau_i = F_i^{-1}(t_\nu(V_i))$, so that

$$\mathbb{P}(\tau_i \leq t \,|\, V, W) = \Phi\left(\frac{-\rho_i V + W^{-\frac{1}{2}} t_\nu^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}} \right).$$

- The default times $\tau_1, \tau_2, \ldots, \tau_n$ are thus modelled from the vector $(V_1, V_2, \ldots, V_n)$ with marginal distributions governed by a Student t distribution with ν degrees of freedom.
- The Gaussian copula can be seen as the limit of Student t copulae when ν tends to infinity.

Archimedean Copulae

- Let f be the density of a positive random variable V, which is called the mixing variable, and let

$$\psi(s) = \int_0^\infty e^{-sv} f(v)\, dv$$

be the Laplace transform of f. Let F_i be the c.d.f. of τ_i.

- We define the function D_i as

$$D_i(t) = \exp\big(- \psi^{-1}(F_i(t))\big).$$

- Then D_i and F_i satisfy

$$F_i(t) = \psi(-\ln D_i(t)) = \int_0^\infty (D_i(t))^v f(v)\, dv.$$

The function $(D_i)^v$ is a c.d.f. for any $v \geq 0$.

Archimedean Copulae

- The last formula shows that, conditionally on $V = v$, the cumulative distribution function of τ_i is $(D_i)^v$.
- Now we can define the joint cumulative distribution function of default times $\tau_1, \tau_2, \ldots, \tau_n$ by

$$F(t_1, t_2, \ldots, t_n) = \mathbb{P}(\tau_1 \leq t_1, \tau_2 \leq t_2, \ldots, \tau_n \leq t_n) = \int_0^\infty \prod_{i=1}^n (D_i)^v(t_i) f(v)\, dv$$

so that for any $t_1, t_2, \ldots, t_n$

$$\mathbb{P}(\tau_1 \leq t_1, \tau_2 \leq t_2, \ldots, \tau_n \leq t_n \,|\, V = v) = \prod_{i=1}^n (D_i)^v(t_i) = \prod_{i=1}^n \mathbb{P}(\tau_i \leq t_i \,|\, V = v).$$

- The last equality shows that the default times are conditionally independent given $V = v$.

Archimedean Copulae

- Since

$$(D_i)^v(t_i) = \exp(-v\psi^{-1}(F_i(t)))$$

  we conclude that

$$F(t_1, t_2, \ldots, t_n) \quad = \quad \int_0^\infty \prod_{i=1}^n (D_i)^v(t_i) f(v)\, dv = \psi\Big(\sum_{i=1}^n \psi^{-1}(F_i(t_i))\Big)$$

- The copula of default times $\tau_1, \tau_2, \ldots, \tau_n$ defined above is given by

$$C(u_1, u_2, \ldots, u_n) = \psi(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \cdots, +\psi^{-1}(u_n)).$$

- The function C is called an Archimedean copula with generator $\phi = \psi^{-1}$.

Archimedean Copulae: Examples

- A standard example of an Archimedean copula is the Clayton copula, where the mixing variable V has a Gamma distribution with parameter $1/\theta$, where $\theta > 0$.

- Hence we have
$$f(x) = \frac{1}{\Gamma(1/\theta)} e^{-x} x^{(1-\theta)/\theta}$$
and $\psi^{-1}(s) = s^{-\theta} - 1$ so that $\psi(s) = (1+s)^{-1/\theta}$.

- Now we can find
$$C(u_1, u_2, \ldots, u_n) = (u_1^{-\theta} + u_2^{-\theta} + \cdots + u_2^{-\theta} - n + 1)^{-1/\theta}$$
and $D_i(t) = \exp(1 - F_i(t)^{-\theta})$.

- Another classic example of an Archimedean copula is the Gumbel copula, which is generated by $\psi(s) = \exp(-s^{1/\theta})$.

Lévy Copulae

Let $X, Y^{(i)}$ be independent Lévy processes with same law and such that

$$\mathbb{E}(X_1) = 0, \mathrm{Var}(X_1) = 1$$

We set $X_i = X_\rho + Y^{(i)}_{1-\rho}$.

By properties of Lévy processes, X_i has the same law as X_1 and

$$\mathrm{Cor}(X_i, X_j) = \rho$$

Loss Process

Let $L_t = \sum_{i=1}^{n} (1 - R_i) 1\!\!1_{\tau_i \leq t}$ be the loss process.

Questions:

- Law of L_t?
- Hedging?
- The *top-down* approach starts from *top*, that is, it starts with modeling of evolution of the portfolio loss process subject to information structure $\mathbb{G}$. Then, it attempts to "decompose" the dynamics of the portfolio loss process *down* on the individual constituent names of the portfolio, so to deduce the dynamics of processes H^i.
- The *bottom-up* approach takes as $\mathbb{G}$ the filtration generated by process $H = (H^1, \ldots, H^n)$ and by a factor process Z.

HAZARD PROCESS APPROACH

- **Model for single default**
- **Intensity approach**
- **Several Defaults**

Model for single default

Properties of the Hazard Process

Hazard Process of a Random Time

- Let τ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We set $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for some **reference filtration** $\mathbb{F}$.
- We shall write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ to denote the **full filtration**.
- We denote $F_t = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t)$, so that

$$G_t = 1 - F_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t)$$

  is the **conditional survival probability**.
- It is easily seen that F is a bounded, non-negative, $\mathbb{F}$-submartingale.

Definition 5 *Assume that* $F_t < 1$ *for every* $t \in \mathbb{R}_+$. *Then the* $\mathbb{F}$-***hazard process*** Γ *of* τ *is defined through the equality* $1 - F_t = e^{-\Gamma_t}$.

Properties of the Hazard Process

- Let $F_t = m_t + A_t$ be the Doob-Meyer decomposition of the sub-martingale $(F_t, t \geq 0)$.
- Assuming that F is continuous, the process

$$M_t = H_t - \int_0^t (1 - H_s) \frac{dA_s}{1 - F_s} = H_t - \Lambda_{t \wedge \tau}$$

is a $\mathbb{G}$-martingale.

- The multiplicative decomposition of the supermartingale G is

$$G_t = n_t e^{-\Lambda_t}$$

- If F (hence Γ) is continuous and increasing, the process $M_t = H_t - \Gamma_{t \wedge \tau}$ is a $\mathbb{G}$-martingale.

Conditional Expectations

- For any $\mathcal{G}$-measurable random variable Y we have

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}Y \,|\, \mathcal{F}_t)}{\mathbb{P}(\tau > t \,|\, \mathcal{F}_t)}.$$

- If, in addition, Y is $\mathcal{F}_s$-measurable for $s \geq t$, then

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}}Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}(Y e^{\Gamma_t - \Gamma_s} \,|\, \mathcal{F}_t).$$

- Let Γ be a continuous process and let Z be an $\mathbb{F}$-predictable process. Then for any $t \leq s$ we have

$$\mathbb{E}_{\mathbb{P}}(Z_\tau \mathbb{1}_{\{t<\tau\leq s\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}\Big(\int_t^s Z_u e^{\Gamma_t - \Gamma_u} \, d\Gamma_u \,\Big|\, \mathcal{F}_t \Big).$$

Interpretation of the Hazard Process

- We now restrict our attention to the case where Γ is an $\mathbb{F}$-adapted, increasing, continuous process.
- If $\Gamma_t = \int_0^t \gamma_u \, du$ then γ represents the $\mathbb{F}$-intensity of τ.
- Intuitively

$$\mathbb{P}\{\tau \in [t, t+dt] \,|\, \mathcal{F}_t \vee \mathcal{H}_t\} = \mathbb{1}_{\{\tau > t\}} \gamma_t \, dt$$

that is

$$\mathbb{P}\{\tau \in [t, t+dt] \,|\, \mathcal{F}_t \vee \{\tau > t\}\} = \gamma_t \, dt.$$

Canonical Construction

- Let Γ be an $\mathbb{F}$-adapted, increasing, continuous processes, defined on a probability space $(\widehat{\Omega}, \mathbb{F}, \mathbb{P})$. We assume that $\Gamma_0 = 0$ and $\Gamma_\infty = \infty$.
- Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ be an auxiliary probability space with a random variable U uniformly distributed on $[0,1]$. Hence $\zeta = -\ln U$ has the unit exponential probability distribution
- We set, on $(\Omega, \mathcal{F}, \ \mathbb{P}) = (\widehat{\Omega} \times, \widehat{\mathcal{F}} \otimes \widetilde{\mathcal{F}}, \widehat{\mathbb{P}} \times \widetilde{\mathbb{P}})$

$$\tau = \inf \{ \, t \in \mathbb{R}_+ \ : \ \Gamma_t(\widehat{\omega}) \geq - \ln U(\widetilde{\omega}) \, \}$$

- The random variable U is independent of the hazard process Γ, the r.v. $-\ln U$ has exponential law.
- Then

$$\mathbb{P}(\tau > t | \mathcal{F}_t) = \exp(-\Gamma_t) = \mathbb{P}(\tau > t | \mathcal{F}_\infty)$$

- In that model, any $\mathbb{F}$-martingale in a $\mathbb{G}$-martingale.

It can be proved that, if

$$G_t = \mathbb{P}(\tau > t | \mathcal{F}_t) = \mathbb{P}(\tau > t | \mathcal{F}_\infty)$$

then, there exists a random variable Θ, independent of $\mathcal{F}_\infty$ such that

$$\tau = \inf \{ t \in \mathbb{R}_+ \, : \, - \ln G_t \geq \Theta \} = \inf \{ t \in \mathbb{R}_+ \, : \, \Gamma_t \geq - \ln U \}$$

Valuation of Defaultable Claims

- In order to value a defaultable claim we need also to specify a discount factor (for instance, the savings account).
- Here we have assumed that $B = 1$, that is, $r = 0$.

Valuation of the Terminal Payoff

To value the terminal payoff we shall use the following result.

Proposition 4

If γ^* is the default intensity under $\mathbb{Q}^*$ then

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{\tau>s\}}Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}_{\mathbb{Q}^*}(Y e^{-\int_t^s \gamma_u^*\,du} \,|\, \mathcal{F}_t).$$

Valuation of Recovery Process

The following result appears to be useful in the valuation of the recovery payoff Z_τ which occurs at time τ.

Proposition 5 *If γ^* is the default intensity under $\mathbb{Q}^*$ then*

$$\mathbb{E}_{\mathbb{Q}^*}\big(Z_\tau \mathbb{1}_{\{t<\tau\leq s\}} \,\big|\, \mathcal{G}_t\big) = \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^s Z_u e^{-\int_t^u \gamma_v^*\, dv}\, \gamma_u^*\, du \,\Big|\, \mathcal{F}_t\Big).$$

Valuation of Promised Dividends

To value the **promised dividends** A that are paid prior to τ we shall make use of the following result.

Proposition 6 *Assume that* Γ^* *is a continuous process and let* A *be an* $\mathbb{F}$*-predictable bounded process of finite variation. Then for every* $t \leq s$

$$\mathbb{E}_{\mathbb{Q}^*}\Big(\int_{(t,s]} (1 - H_u)\, dA_u \,\Big|\, \mathcal{G}_t \Big) = \mathbb{1}_{\{\tau > t\}}\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{(t,s]} e^{\Gamma^*_t - \Gamma^*_u}\, dA_u \,\Big|\, \mathcal{F}_t \Big).$$

Intensity approach

In intensity based models, the default time τ is a stopping time in a given filtration $\mathbb{G}$, representing the full information of the market.

Definition of the intensity process

- The process $(H_t = \mathbb{1}_{\tau \leq t},\, t \geq 0)$ is a $\mathbb{G}$-adapted increasing càdlàg process, hence a $\mathbb{G}$-submartingale, and there exists a unique $\mathbb{G}$-predictable increasing process $\Lambda^{\mathbb{G}}$, called the $\mathbb{G}$-compensator, such that the process

$$M_t = H_t - \Lambda_t^{\mathbb{G}}$$

  is a $\mathbb{G}$-martingale.

- The compensator satisfies $\Lambda_t^{\mathbb{G}} = \Lambda_{t\wedge\tau}^{\mathbb{G}}$.

- The process $\Lambda^{\mathbb{G}}$ is continuous if and only if τ is a $\mathbb{G}$-totally inaccessible stopping time.

A predictable stopping time T is a stopping time such that there exists a sequence of stopping times T_n so that $T_n < T$ and $T_n \to T$

A totally inaccessible stopping time is a stopping time so that $\mathbb{P}(T = S) = 0$ for any predictable stopping time S.

- In intensity based models, it is generally assumed that $\Lambda^{\mathbb{G}}$ is absolutely continuous with respect to Lebesgue measure, i.e., that there exists a non-negative $\mathbb{G}$-adapted process $(\lambda_t^{\mathbb{G}}, t \geq 0)$ such that

$$M_t = H_t - \int_0^t \lambda_s^{\mathbb{G}} ds$$

is a $\mathbb{G}$-martingale.

- This process $\lambda^{\mathbb{G}}$ is called the $\mathbb{G}$-intensity rate and vanishes after time τ, i.e.,

$$M_t = H_t - \int_0^{t\wedge\tau} \lambda_s^{\mathbb{G}} ds = H_t - \int_0^t (1 - H_s)\lambda_s^{\mathbb{G}} ds.$$

- One gets, under some regularity assumption,

$$\lambda_t^{\mathbb{G}} = \lim_{h\to 0} \frac{1}{h}\mathbb{P}(t < \tau \leq t + h|\mathcal{G}_t) = \lim_{h\to 0} \frac{1}{h}\mathbb{1}_{\{t<\tau\}}\mathbb{P}(\tau \leq t + h|\mathcal{G}_t),$$

when the limit (a.s.) exists.

Pricing rule for conditional claims

For $X \in \mathcal{G}_T$, integrable,

$$\mathbb{E}_{\mathbb{Q}^*}(X 1\!\!1_{T<\tau} | \mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} \left(V_t - \mathbb{E}_{\mathbb{Q}^*}(1\!\!1_{\{\tau \le T\}} \Delta V_\tau | \mathcal{G}_t) \right)$$

where the process V is defined by:

$$V_t = e^{\Lambda_t^{\mathbb{G}}} \mathbb{E}_{\mathbb{Q}^*}(X e^{-\Lambda_T^{\mathbb{G}}} | \mathcal{G}_t) = e^{\Lambda_{t\wedge\tau}^{\mathbb{G}}} \mathbb{E}_{\mathbb{Q}^*}(X e^{-\Lambda_{T\wedge\tau}^{\mathbb{G}}} | \mathcal{G}_t).$$

and where ΔV_τ denotes the jump of V at τ, i.e., $\Delta V_\tau = V_\tau - V_{\tau^-}$.

Using the intensity rate, the pricing rule becomes:

$$\mathbb{E}_{\mathbb{Q}^*}(X 1\!\!1_{T<\tau} | \mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}^*} \left(X e^{-\int_t^T \lambda_s^{\mathbb{G}} ds} \middle| \mathcal{G}_t \right) - \mathbb{E}_{\mathbb{Q}^*}(1\!\!1_{\{t<\tau\le T\}} \Delta V_\tau | \mathcal{G}_t).$$

Proof: Apply the integration by parts formula to the product $U = VL$ (remark $U_T = 1\!\!1_{\{T<\tau\}} X$), with $L_t = 1 - H_t$

$$dU_t = (\Delta V_\tau)\, dL_t + (L_{t-} dm_t - V_{t-} dM_t)\,,$$

(where $dm_t = e^{\Lambda_t} dY_t$, for $Y_t = e^{-\Lambda_t} V_t$), which yields to $U_t = \mathbb{E}_{\mathbb{Q}^*}(1\!\!1_{t<\tau\leq T}\, \Delta V_\tau + U_T | \mathcal{G}_t)$.

For example, whereas the price of a zero-coupon bond writes (if $\beta_t = \exp\left(-\int_0^t r_s ds\right)$ denotes the savings account):

$$B(t,T) = \beta_t \mathbb{E}_{\mathbb{Q}^*}\left(\left.\frac{1}{\beta_T}\right| \mathcal{G}_t\right) = \mathbb{E}_{\mathbb{Q}^*}\left(\left. e^{-\int_t^T r_s ds}\right| \mathcal{G}_t\right),$$

the price of a defaultable zero-coupon bond with no recovery and notional 1 is:

$$D(t,T) = \beta_t \mathbb{E}_{\mathbb{Q}^*}\left(\left.\frac{\mathbb{1}_{T<\tau}}{\beta_T}\right| \mathcal{G}_t\right)$$

$$= \mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}^*}\left(\left. e^{-\int_t^T \left(r_s+\lambda_s^{\mathbb{G}}\right) ds}\right| \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{t<\tau\leq T\}} \Delta V_\tau^D | \mathcal{G}_t)$$

where $V_t^D = \mathbb{E}_{\mathbb{Q}^*}(\exp - \int_t^{\tau\wedge T} \lambda_s ds | \mathcal{G}_t)$.

Several Defaults

Conditionally Independent Defaults

Canonical Construction

- Let Γ^i, $i = 1, \ldots, n$ be a given family of $\mathbb{F}$-adapted, increasing, continuous processes, defined on a probability space $(\widehat{\Omega}, \mathbb{F}, \mathbb{P})$, with $\Gamma^i_0 = 0$ and $\Gamma^i_\infty = \infty$.
- Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ be an auxiliary probability space with U_i, $i = 1, \ldots, n$ mutually independent r.v's uniformly distributed on $[0, 1]$.
- We set

$$\tau_i = \inf \{ t \in \mathbb{R}_+ \, : \, \Gamma^i_t(\widehat{\omega}) \geq - \ln U_i(\widetilde{\omega}) \}$$

on the product space

$$(\Omega, \mathcal{G}, \mathbb{Q}) = (\widehat{\Omega} \times \widetilde{\Omega}, \mathcal{F}_\infty \otimes \widetilde{\mathcal{F}}, \mathbb{P} \otimes \widetilde{\mathbb{P}}).$$

- We endow the space $(\Omega, \mathcal{G}, \mathbb{Q})$ with the full filtration $\mathbb{G}$ given as

$$\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n.$$

Conditional Independence

- Default times $\tau_1, \ldots, \tau_n$ defined in this way are conditionally independent with respect to $\mathbb{F}$ under $\mathbb{Q}$.

  This means that we have, for any $t > 0$ and any $t_1, \ldots, t_n \in [0, t]$,

$$\mathbb{Q}\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_t\} = \prod_{i=1}^{n} \mathbb{Q}\{\tau_i > t_i \,|\, \mathcal{F}_t\}.$$

- The process Γ^i is the $\mathbb{F}$-hazard process of τ_i, for any $s \geq t$,

$$\mathbb{Q}\{\tau_i > s \,|\, \mathcal{F}_t \vee \mathcal{H}_t^i\} = \mathbb{1}_{\{\tau_i > t\}} \, \mathbb{E}_{\mathbb{Q}}\big(e^{\Gamma_t^i - \Gamma_s^i} \,|\, \mathcal{F}_t\big).$$

- We have $\mathbb{Q}\{\tau_i = \tau_j\} = 0$ for every $i \neq j$ (no simultaneous defaults).

Interpretation of Conditional Independence

- Intuitive meaning of conditional independence:
  - the reference credits (credit names) are subject to common risk factors that may trigger credit (default) events,
  - in addition, each credit name is subject to idiosyncratic risks that are specific for this name.
- Conditional independence of default times means that once the common risk factors are fixed then the idiosyncratic risk factors are independent of each other.
- Conditional independence is not invariant with respect to an equivalent change of a probability measure.

Correlated Stochastic Intensities

- Let the process for the default intensity of name i be given by

$$\gamma_t^i = \rho_i \, h_0(t) + h_i(t)$$

where

$$h_0(t) = h_0(\widetilde{X}_t^0)$$

and for $i = 1, 2, \ldots, n$

$$h_j(t) = h_i(\widetilde{X}_t^i)$$

- The processes $\widetilde{X}^0, \widetilde{X}^1, \ldots, \widetilde{X}^n$ are independent components of the factor process $\widetilde{X} = (\widetilde{X}^0, \widetilde{X}^1, \ldots, \widetilde{X}^n)$.

- Then the process h_0 is referred to as the common intensity factor, and the processes h_i are called idiosyncratic intensity factors, since they only affect the credit worthiness of a single obligor.

Examples of Stochastic Intensities

- We can postulate that

$$\gamma_t^i = \widetilde{\rho}_i \, h_0(t) + h_i(t)$$

  - where h_i follows Vasicek's dynamics

$$dh_i(t) = \kappa_i(\theta_i - h_i(t)) \, dt + \sigma_i \, dW_t^i$$

  - or better, the CIR dynamics

$$dh_i(t) = \kappa_i(\theta_i - h_i(t)) \, dt + \sigma_i \sqrt{h_i(t)} \, dW_t^i .$$

- Note that we do not assume that $\widetilde{\rho}_i$ belongs to $[-1, 1]$.

Combined Approach

- We adopt the intensity-based approach, but we no longer assume that the random variables $U_1, \ldots, U_n$ are independent.
- Assume that the c.d.f. of $(U_1, \ldots, U_n)$ is an n-dimensional copula C.
- Then the univariate marginal laws are uniform on $[0,1]$, but the random variables $U_1, \ldots, U_n$ are not necessarily mutually independent.
- We still postulate that they are independent of $\mathbb{F}$, and we set

$$\tau_i = \inf \{ \, t \in \mathbb{R}_+ \; : \; \Gamma^i_t(\widehat{\omega}) \geq - \ln U_i(\widetilde{\omega}) \, \}.$$

If we drop independence condition, then immersion property does not hold, the intensity is no more obtained via Γ

Combined Approach

- The case of default times conditionally independent with respect to $\mathbb{F}$ corresponds to the choice of the product copula Π.

  In this case, for $t_1, \ldots, t_n \leq T$ we have

  $$\mathbb{Q}^*\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = \Pi(G^1_{t_1}, \ldots, G^n_{t_n})$$

  where we set $G^i_t = e^{-\Gamma^i_t}$.

- In general, for $t_1, \ldots, t_n \leq T$ we obtain

  $$\mathbb{Q}^*\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = C(G^1_{t_1}, \ldots, G^n_{t_n})$$

  where C is the copula function that was used in the construction of $\tau_1, \ldots, \tau_n$.

Survival Intensities

- Schönbucher and Schubert (2001) show that for arbitrary $s \leq t$, on the event $\{\tau_1 > s, \ldots, \tau_n > s\}$,

$$\mathbb{Q}^*\{\tau_i > t \,|\, \mathcal{G}_s\} = \mathbb{E}_{\mathbb{Q}^*}\left(\frac{C(G_s^1, \ldots, G_t^i, \ldots, G_{t_n}^n)}{C(G_s^1, \ldots, G_s^n)} \,\Big|\, \mathcal{F}_s \right).$$

- Consequently, the i^{th} intensity of survival equals, on $\{\tau_1 > t, \ldots, \tau_n > t\}$,

$$\lambda_t^i = \gamma_t^i \, G_t^i \, \frac{\partial}{\partial v_i} \ln C(G_t^1, \ldots, G_t^n).$$

  Here λ_t^i is understood as the limit

$$\lambda_t^i = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}^*\{t < \tau_i \leq t + h \,|\, \mathcal{F}_t, \tau_1 > t, \ldots, \tau_n > t\}.$$

Double Correlation

- We can postulate that

$$\gamma_t^i = \widetilde{\rho}_i \, h_0(t) + h_i(t)$$

where h_i are governed by Vasicek's dynamics

$$dh_i(t) = \kappa_i(\theta_i - h_i(t))\, dt + \sigma_i \, dW_t^i,$$

or by CIR dynamics

$$dh_i(t) = \kappa_i(\theta_i - h_i(t))\, dt + \sigma_i \sqrt{h_i(t)}\, dW_t^i.$$

- We can combine this with the one-factor Gaussian copula for $U_1, \ldots, U_n$.

- The first case was studied by Van der Voort (2004) in the context of basket CDSs and CDOs. The effect of intensity correlation is much smaller then the effect of the default correlation.