X Simposio de Probabilidad y Procesos Estocasticos

1ra Reunión Franco Mexicana de Probabilidad

Guanajuato, 3 al 7 de noviembre de 2008

Curso de Riesgo Credito

Begin at the beginning,

L. Carroll, Alice's Adventures in Wonderland , *Begin at the beginning, and go on till you come to the end. Then, stop.*

L. Carroll, Alice's Adventures in Wonderland ,

1. Structural Approach
2. Hazard Process Approach
3. Hedging Defaultable Claims
4. Credit Default Swaps
5. Several Defaults

Several Defaults

Several Defaults, no reference filtration

General setting

We assume that two default times are given: $\tau_i, i = 1, 2$

We introduce the *joint survival process* $G(u, v)$: for every $u, v \in \mathbb{R}_+$,

$$G(u, v) = \mathbb{Q}(\tau_1 > u, \tau_2 > v)$$

We write

$$\partial_1 G(u, v) = \frac{\partial G}{\partial u}(u, v), \quad \partial_{12} G(u, v) = \frac{\partial^2 G}{\partial u \partial v}(u, v).$$

We assume that the joint density $f(u, v) = \partial_{12} G(u, v)$ exists. In other words, we postulate that $G(u, v)$ can be represented as follows

$$G(u, v) = \int_u^\infty \Big(\int_v^\infty f(x, y)\, dy \Big)\, dx.$$

We compute conditional expectation in the filtration $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$:

- For $t < T$

$$\begin{aligned}
\mathbb{P}(T < \tau_1 | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) &= \mathbb{1}_{t<\tau_1} \frac{\mathbb{P}(T < \tau_1 | \mathcal{H}_t^2)}{\mathbb{P}(t < \tau_1 | \mathcal{H}_t^2)} \\
&= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{\mathbb{P}(T < \tau_1, t < \tau_2)}{\mathbb{P}(t < \tau_1, t < \tau_2)} + \mathbb{1}_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) \\
&= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{G(T,t)}{G(t,t)} + \mathbb{1}_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right)
\end{aligned}$$

- The computation of $\mathbb{P}(T < \tau_1 | \tau_2)$ can be done as follows:

$$\mathbb{P}(T < \tau_1 | \tau_2 = v) = \frac{\mathbb{P}(T < \tau_1, \tau_2 \in dv)}{\mathbb{P}(\tau_2 \in dv)} = \frac{\partial_2 G(T, v)}{\partial_2 G(0, v)}$$

hence, on the set $\tau_2 < T$,

$$\mathbb{P}(T < \tau_1 | \tau_2) = \frac{\partial_2 G(T, \tau_2)}{\partial_2 G(0, \tau_2)}$$

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Value of credit derivatives

We introduce different credit derivatives

A **defaultable zero-coupon** related to the default time τ_i delivers 1 monetary unit if τ_i is greater that T: $D^i(t,T) = \mathbb{E}_{\mathbb{Q}^*}(1\!\!1_{\{T<\tau_i\}} | \mathcal{H}_t^1 \vee \mathcal{H}_t^2)$

We obtain

$$D^1(t,T) \quad = \quad 1\!\!1_{\{\tau_1>t\}} \left(1\!\!1_{\{\tau_2 \le t\}} \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + 1\!\!1_{\{\tau_2>t\}} \frac{G(T,t)}{G(t,t)} \right)$$

A contract which pays R_1 is one default occurs before T and R_2 if the two defaults occur before T:

$$
\begin{aligned}
CD_t &= \mathbb{E}_{\mathbb{Q}^*}(R_1 1\!\!1_{\{0<\tau_{(1)}\le T\}} + R_2 1\!\!1_{\{0<\tau_{(2)}\le T\}} | \mathcal{H}^1_t \vee \mathcal{H}^2_t) \\
&= R_1 1\!\!1_{\{\tau_{(1)}>t\}} \left(\frac{G(t,t)-G(T,T)}{G(t,t)} \right) + R_2 1\!\!1_{\{\tau_{(2)}\le t\}} + R_1 1\!\!1_{\{\tau_{(1)}\le t\}} \\
&\quad + R_2 1\!\!1_{\{\tau_{(2)}>t\}} \left\{ I_t(0,1)\left(1-\frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)}\right) + I_t(1,0)\left(1-\frac{\partial_1 G(\tau_1,T)}{\partial_1 G(\tau_1,t)}\right) \right. \\
&\qquad\qquad \left. + I_t(0,0)\left(1-\frac{G(t,T)+G(T,t)-G(T,T)}{G(t,t)}\right) \right\}
\end{aligned}
$$

where by

$$
\begin{aligned}
I_t(1,1) &= 1\!\!1_{\{\tau_1\le t, \tau_2\le t\}} \ , \qquad & I_t(0,0) &= 1\!\!1_{\{\tau_1> t, \tau_2> t\}} \\
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\end{aligned}
$$

More generally, some easy computation leads to

$$\mathbb{E}_{\mathbb{Q}^*}(h(\tau_1,\tau_2)|\mathcal{H}_t) = I_t(1,1)h(\tau_1,\tau_2)+I_t(1,0)\Psi_{1,0}(\tau_1)+I_t(0,1)\Psi_{0,1}(\tau_2)+I_t(0,0)\Psi_{0,0}$$

where

$$\begin{aligned}
\Psi_{1,0}(u) &= -\frac{1}{\partial_1 G(u,t)}\int_t^\infty h(u,v)\partial_1 G(u,dv) \\
\Psi_{0,1}(v) &= -\frac{1}{\partial_2 G(t,v)}\int_t^\infty h(u,v)\partial_2 G(du,v) \\
\Psi_{0,0} &= \frac{1}{G(t,t)}\int_t^\infty\int_t^\infty h(u,v)G(du,dv)
\end{aligned}$$

The process

$$M_t^{1,1} = H_t^1 - \int_0^{t\wedge\tau_1} \lambda(s)ds$$

where $\lambda(s) = \frac{f(s)}{G(s)} = \frac{\partial_1 G(s,0)}{G(s,0)}$ is an $\mathbb{H}^1$-martingale.

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Let

$$\widetilde{\lambda}^1(t) = -\frac{\partial_1 G(t,t)}{G(t,t)}, \quad \lambda^{1|2}(t,s) = -\frac{f(t,s)}{\partial_2 G(t,s)}$$

Then, the process

$$M_t^{1,\mathbb{G}} = H_t^1 - \int_0^{t\wedge\tau_1\wedge\tau_2} \widetilde{\lambda}^1(u)\,du - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \lambda^{1|2}(u,\tau_2)\,du,$$

is a $\mathbb{G}$-martingale.

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Copula

Copula Function

The concept of a **copula function** allows to produce various multidimensional probability distributions with the same univariate marginal probability distributions.

Definition 1 *A function* $C : [0,1]^n \to [0,1]$ *is a copula function if:*

- $C(1,\ldots,1,v_i,1,\ldots,1) = v_i$ *for any* i *and any* $v_i \in [0,1]$,
- C *is an* n*-dimensional cumulative distribution function.*

Examples of copulae:

- product copula: $\Pi(v_1,\ldots,v_n) = \Pi_{i=1}^n v_i$,
- Gumbel copula: for $\theta \in [1,\infty)$ we set

$$C(v_1,\ldots,v_n) = \exp\left(-\left[\sum_{i=1}^n (-\ln v_i)^\theta\right]^{1/\theta}\right).$$

Sklar's Theorem

Theorem 1

- *For any cumulative distribution function F on $\mathbb{R}^n$ there exists a copula function C such that*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

where F_i is the i^{th} marginal cumulative distribution function.

If, in addition, F is continuous then C is unique.

- *Conversely, if C is an n-dimensional copula and $F_1, F_2, \ldots, F_n$ are the distribution functions, then the function*

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))$$

is a n-dimensional distribution function with marginals $F_1, F_2, \ldots, F_n$.

Survival Copula

- We can represent the joint survival function as some copula as well. Since for standard uniform random variables $U_1, U_2, \ldots, U_n$, the random variables $\widetilde{U_1} = 1 - U_1, \widetilde{U_2} = 1 - U_2, \ldots, \widetilde{U_n} = 1 - U_n$ are also uniform random variables.

- Hence we have

$$\begin{aligned}
&G(x_1, x_2, \ldots, x_n) \\
&\quad = \mathbb{P}(X_1 \geq x_1, X_2 \geq x_2, \ldots, X_n \geq x_n) \\
&\quad = \mathbb{P}(F_1(X_1) \geq F_1(x_1), \ldots, F_n(X_n) \geq F_n(x_n)) \\
&\quad = \mathbb{P}(1 - F_1(X_1) \leq 1 - F_1(x_1), \ldots, 1 - F_n(X_n) \leq 1 - F_n(x_n)) \\
&\quad = \mathbb{P}(\widetilde{U}_1 \leq G_1(x_1), \widetilde{U}_2 \leq G_2(x_2), \ldots, \widetilde{U}_n \leq G_n(x_n)) \\
&\quad = \widetilde{C}(G_1(x_1), G_2(x_2), \ldots, G_n(x_n))
\end{aligned}$$

Multivariate Gaussian Copula

Let R be an $n \times n$ symmetric, positive definite matrix with $R_{ii} = 1$ for $i = 1, 2, \ldots, n$, and let Φ_R be the standardized multivariate normal distribution with correlation matrix R

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} |R|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{x}' R^{-1}\mathbf{x}\right).$$

Definition 2 *The multivariate Gaussian copula C_R is defined as:*

$$C_R(u_1, u_2, \ldots, u_n) = \Phi_R(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_n))$$

where $\Phi^{-1}(u)$ represents the inverse of the normal cumulative distribution function.

Archimedean Copulae

- Let f be the density of a positive random variable V, which is called the mixing variable, and let

$$\psi(s) = \int_0^\infty e^{-sv} f(v)\, dv$$

be the Laplace transform of f. Let F_i be the c.d.f. of τ_i.

- We define the function D_i as

$$D_i(t) = \exp\big(-\psi^{-1}(F_i(t))\big).$$

- Then D_i and F_i satisfy

$$F_i(t) = \psi(-\ln D_i(t)) = \int_0^\infty (D_i(t))^v f(v)\, dv.$$

The function $(D_i)^v$ is a c.d.f. for any $v \geq 0$.

Archimedean Copulae

- The last formula shows that, conditionally on $V = v$, the cumulative distribution function of τ_i is $(D_i)^v$.

- Now we can define the joint cumulative distribution function of default times $\tau_1, \tau_2, \ldots, \tau_n$ by

$$F(t_1, t_2, \ldots, t_n) = \mathbb{P}(\tau_1 \leq t_1, \tau_2 \leq t_2, \ldots, \tau_n \leq t_n) = \int_0^{\infty} \prod_{i=1}^{n} (D_i)^v (t_i) f(v)\, dv$$

so that for any $t_1, t_2, \ldots, t_n$

$$\mathbb{P}(\tau_1 \leq t_1, \tau_2 \leq t_2, \ldots, \tau_n \leq t_n \,|\, V = v) = \prod_{i=1}^{n} (D_i)^v (t_i) = \prod_{i=1}^{n} \mathbb{P}(\tau_i \leq t_i \,|\, V = v).$$

- The last equality shows that the default times are conditionally independent given $V = v$.

Archimedean Copulae

- Since
$$(D_i)^v(t_i) = \exp(-v\psi^{-1}(F_i(t)))$$
we conclude that
$$F(t_1, t_2, \ldots, t_n) \quad = \quad \int_0^\infty \prod_{i=1}^n (D_i)^v(t_i) f(v)\, dv = \psi\Big(\sum_{i=1}^n \psi^{-1}(F_i(t_i))\Big)$$

- The copula of default times $\tau_1, \tau_2, \ldots, \tau_n$ defined above is given by
$$C(u_1, u_2, \ldots, u_n) = \psi(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \cdots, +\psi^{-1}(u_n)).$$

- The function C is called an Archimedean copula with generator $\phi = \psi^{-1}$.

Archimedean Copulae: Examples

- A standard example of an Archimedean copula is the Clayton copula, where the mixing variable V has a Gamma distribution with parameter $1/\theta$, where $\theta > 0$.

- Hence we have
$$f(x) = \frac{1}{\Gamma(1/\theta)} e^{-x} x^{(1-\theta)/\theta}$$
and $\psi^{-1}(s) = s^{-\theta} - 1$ so that $\psi(s) = (1+s)^{-1/\theta}$.

- Now we can find
$$C(u_1, u_2, \ldots, u_n) = (u_1^{-\theta} + u_2^{-\theta} + \cdots + u_2^{-\theta} - n + 1)^{-1/\theta}$$
and $D_i(t) = \exp(1 - F_i(t)^{-\theta})$.

- Another classic example of an Archimedean copula is the Gumbel copula, which is generated by $\psi(s) = \exp(-s^{1/\theta})$.

Gaussian Copula

- Gaussian copulae have become an industry standard for CDO and credit portfolio modelling, despite of several drawbacks.

- Assume that the marginal cumulative distribution functions $F_1, F_2, \ldots, F_n$ of default times $\tau_1, \tau_2, \ldots, \tau_n$ are known.

- Let $(X_1, X_2, \ldots, X_n)$ **Gaussian vector** with zero means, unit variances, and covariance matrix Σ, and set $\tau_i = F_i^{-1}(\Phi(X_i))$ for $i = 1, \ldots, n$, where F_i^{-1} denotes the generalized inverse of F_i and Φ is the standard Gaussian distribution function, so that

$$\mathbb{P}(\tau_i \leq t) = \mathbb{P}(\Phi(X_i) \leq F_i(t)) = F_i(t)$$

or

$$\tau_i = \inf\{t \in \mathbb{R}_+ : \chi_i(t) \geq X_i\}, \quad i = 1, 2, \ldots, n,$$

where $\chi_i(t) = \Phi^{-1}(F_i(t))$ (and $\mathbb{P}(\tau_i \leq t) = F_i(t)$).

Comparison with Intensity-Based Model

- If F_{X_i} is a continuous function for every i then

  $$\tau_i = \inf\{t \in \mathbb{R}_+ : F_{X_i}(\chi_i(t)) \geq F_{X_i}(X_i)\} = \inf\{t \in \mathbb{R}_+ : G_i(t) \leq \widetilde{U}_i\}$$

  where $(\widetilde{U}_1, \widetilde{U}_2, \ldots, \widetilde{U}_n)$ with $\widetilde{U}_i = 1 - F_{X_i}(X_i)$ are random variables with uniform marginal distributions (not independent) and $G_i(t) = 1 - F_{X_i}(\chi_i(t)) = 1 - \mathbb{P}\{\tau_i \leq t\}$.

- This representation of the one-factor copula model allows for easy comparison with the intensity-based model in which

  $$\tau_i = \inf\{t \in \mathbb{R}_+ : G_t^i \leq U_i\}$$

  where $(U_1, U_2, \ldots, U_n)$ are independent uniformly distributed random variables and $G^1, G^2, \ldots, G^n$ are non-increasing default countdown processes (not independent, in general).

One-Factor Gaussian Copula

- A **one-factor Gaussian copula** is the multivariate Gaussian copula corresponding to the joint distribution of the vector $(X_1, X_2, \ldots, X_n)$ where

$$X_i = \rho_i V + \sqrt{1-\rho_i^2}\, Y_i$$

where V and $Y_1, Y_2, \ldots, Y_n$ are independent standard Gaussian random variables and $0 \le \rho_i \le 1$ for $i = 1, 2, \ldots, n$.

- Then we can get (recall that $\tau_i = F_i^{-1}(\Phi(X_i))$)

$$\mathbb{P}(\tau_i \le t \,|\, V) = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right).$$

- The case $\rho_1 = \ldots = \rho_n = 0$ corresponds to independent defaults, whereas $\rho_1 = \ldots = \rho_n = 1$ represents the co-monotonic case.

Student t Copula

- Let us denote $V_i = \sqrt{W} X_i$ and $X_i = \rho_i V + \sqrt{1-\rho_i^2}\, Y_i$ where $V, Y_1, Y_2, \ldots, Y_n$ are independent $N(0,1)$ random variables. W is independent of $X_1, X_2, \ldots, X_n$ and has the inverse gamma distribution with parameter $\frac{\nu}{2}$.

- Let t_ν denote the c.d.f. of the Student t distribution with ν degrees of freedom.

- We set $\tau_i = F_i^{-1}(t_\nu(V_i))$, so that

$$\mathbb{P}(\tau_i \leq t \,|\, V, W) = \Phi\left(\frac{-\rho_i V + W^{-\frac{1}{2}} t_\nu^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}} \right).$$

- The default times $\tau_1, \tau_2, \ldots, \tau_n$ are thus modelled from the vector $(V_1, V_2, \ldots, V_n)$ with marginal distributions governed by a Student t distribution with ν degrees of freedom.

- The Gaussian copula can be seen as the limit of Student t copulae when ν tends to infinity.

Lévy Copulae

Let $X, Y^{(i)}$ be independent Lévy processes with same law and such that

$$\mathbb{E}(X_1) = 0, \mathrm{Var}(X_1) = 1$$

We set $X_i = X_\rho + Y^{(i)}_{1-\rho}$.

By properties of Lévy processes, X_i has the same law as X_1 and

$$\mathrm{Cor}(X_i, X_j) = \rho$$

Loss Process

Let $L_t = \sum_{i=1}^{n} (1 - R_i) \mathbb{1}_{\tau_i \le t}$ be the loss process.

Questions:

- Law of L_t?
- The *top-down* approach starts from *top*, that is, it starts with modeling of evolution of the portfolio loss process subject to information structure $\mathbb{G}$. Then, it attempts to "decompose" the dynamics of the portfolio loss process *down* on the individual constituent names of the portfolio, so to deduce the dynamics of processes H^i.
- The *bottom-up* approach takes as $\mathbb{G}$ the filtration generated by process $H = (H^1, \ldots, H^n)$ and possibly a factor process Z.

Collateralized debt obligations (CDO)

The loss process is

$$L_t = \sum_{i=1}^{n} H_t^i$$

Let $A_0 = 0 < \cdots < A_k < \cdots < A_\ell = n$.
We denote by $D_k = A_k$ and $U_k = A_{k+1}$ the lower and upper attachment points for the kth tranche and by κ_0^k the corresponding spread. It is convenient to introduce the *percentage loss* process

$$Q_t = \frac{1}{n} \sum_{i=1}^{n} H_t^i = \frac{N_0 - N_t}{N_0},$$

where $N_0 = n$ is the number of credit names in the reference portfolio and $N_t = N_0 - \sum_{i=1}^{n} H_t^i$ is the residual protection. Finally, denote by $C_k = A_{k+1} - A_k = U_k - D_k$ the width of the kth tranche.

Purchasing one unit of the kth tranche at time 0 generates the following discounted cash flows

$$\text{Premium leg} = \kappa_0^k \sum_{j=1}^{J} \frac{B_0}{B_{t_j}} N_{t_j}^k,$$

where N_t^k is the *residual tranche protection* at time t, that is,

$$N_t^k = N_0 \Big(C_k - \min \left(C_k, \max \left(Q_t - D_k, 0 \right) \right) \Big).$$

The discounted cash flows of the protection leg are

$$\text{Protection leg} = (1 - \delta) \sum_{i=1}^{n} \frac{B_0}{B_{\tau_i}} H_T^i \mathbb{1}_{\{D_k < Q_{\tau_i} \leq U_k\}}.$$

Several Defaults: reference filtration

Conditionally Independent Defaults

Canonical Construction

- Let Γ^i, $i = 1, \ldots, n$ be a given family of $\mathbb{F}$-adapted, increasing, continuous processes, defined on a probability space $(\widehat{\Omega}, \mathbb{F}, \mathbb{P})$, with $\Gamma^i_0 = 0$ and $\Gamma^i_\infty = \infty$.
- Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ be an auxiliary probability space with U_i, $i = 1, \ldots, n$ mutually independent r.v's uniformly distributed on $[0, 1]$.
- We set

$$\tau_i = \inf \{ t \in \mathbb{R}_+ \, : \, \Gamma^i_t(\widehat{\omega}) \geq -\ln U_i(\widetilde{\omega}) \}$$

on the product space

$$(\Omega, \mathcal{G}, \mathbb{Q}) = (\widehat{\Omega} \times \widetilde{\Omega}, \mathcal{F}_\infty \otimes \widetilde{\mathcal{F}}, \mathbb{P} \otimes \widetilde{\mathbb{P}}).$$

- We endow the space $(\Omega, \mathcal{G}, \mathbb{Q})$ with the full filtration $\mathbb{G}$ given as

$$\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n.$$

Conditional Independence

- Default times $\tau_1, \ldots, \tau_n$ defined in this way are conditionally independent with respect to $\mathbb{F}$ under $\mathbb{Q}$.

  This means that we have, for any $t > 0$ and any $t_1, \ldots, t_n \in [0, t]$,

$$\mathbb{Q}\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_t\} = \prod_{i=1}^{n} \mathbb{Q}\{\tau_i > t_i \,|\, \mathcal{F}_t\}.$$

- The process Γ^i is the $\mathbb{F}$-hazard process of τ_i, for any $s \geq t$,

$$\mathbb{Q}\{\tau_i > s \,|\, \mathcal{F}_t \vee \mathcal{H}_t^i\} = \mathbb{1}_{\{\tau_i > t\}} \, \mathbb{E}_{\mathbb{Q}}\big(e^{\Gamma_t^i - \Gamma_s^i} \,|\, \mathcal{F}_t\big).$$

- We have $\mathbb{Q}\{\tau_i = \tau_j\} = 0$ for every $i \neq j$ (no simultaneous defaults).

Interpretation of Conditional Independence

- Intuitive meaning of conditional independence:
  - the reference credits (credit names) are subject to common risk factors that may trigger credit (default) events,
  - in addition, each credit name is subject to idiosyncratic risks that are specific for this name.
- Conditional independence of default times means that once the common risk factors are fixed then the idiosyncratic risk factors are independent of each other.
- Conditional independence is not invariant with respect to an equivalent change of a probability measure.

Correlated Stochastic Intensities

- Let the process for the default intensity of name i be given by

$$\gamma_t^i = \rho_i \, h_0(t) + h_i(t)$$

where

$$h_0(t) = h_0(\widetilde{X}_t^0)$$

and for $i = 1, 2, \ldots, n$

$$h_j(t) = h_i(\widetilde{X}_t^i)$$

- The processes $\widetilde{X}^0, \widetilde{X}^1, \ldots, \widetilde{X}^n$ are independent components of the factor process $\widetilde{X} = (\widetilde{X}^0, \widetilde{X}^1, \ldots, \widetilde{X}^n)$.

- Then the process h_0 is referred to as the common intensity factor, and the processes h_i are called idiosyncratic intensity factors, since they only affect the credit worthiness of a single obligor.

Examples of Stochastic Intensities

- We can postulate that

$$\gamma_t^i = \widetilde{\rho}_i \, h_0(t) + h_i(t)$$

  - where h_i follows Vasicek's dynamics

$$dh_i(t) = \kappa_i(\theta_i - h_i(t)) \, dt + \sigma_i \, dW_t^i$$

  - or better, the CIR dynamics

$$dh_i(t) = \kappa_i(\theta_i - h_i(t)) \, dt + \sigma_i \sqrt{h_i(t)} \, dW_t^i.$$

- Note that we do not assume that $\widetilde{\rho}_i$ belongs to $[-1, 1]$.

Combined Approach

- We adopt the intensity-based approach, but we no longer assume that the random variables $U_1, \dots, U_n$ are independent.
- Assume that the c.d.f. of $(U_1, \dots, U_n)$ is an n-dimensional copula C.
- Then the univariate marginal laws are uniform on $[0, 1]$, but the random variables $U_1, \dots, U_n$ are not necessarily mutually independent.
- We still postulate that they are independent of $\mathbb{F}$, and we set

$$\tau_i = \inf \{ t \in \mathbb{R}_+ \, : \, \Gamma_t^i(\widehat{\omega}) \geq - \ln U_i(\widetilde{\omega}) \}.$$

If we drop independence condition, then immersion property does not hold, the intensity is no more obtained via Γ

Combined Approach

- The case of default times conditionally independent with respect to $\mathbb{F}$ corresponds to the choice of the product copula Π.

  In this case, for $t_1, \ldots, t_n \le T$ we have

  $$\mathbb{Q}^*\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = \Pi(G^1_{t_1}, \ldots, G^n_{t_n})$$

  where we set $G^i_t = e^{-\Gamma^i_t}$.

- In general, for $t_1, \ldots, t_n \le T$ we obtain

  $$\mathbb{Q}^*\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = C(G^1_{t_1}, \ldots, G^n_{t_n})$$

  where C is the copula function that was used in the construction of $\tau_1, \ldots, \tau_n$.

Survival Intensities

- Schönbucher and Schubert (2001) show that for arbitrary $s \leq t$, on the event $\{\tau_1 > s, \ldots, \tau_n > s\}$,

$$\mathbb{Q}^*\{\tau_i > t \,|\, \mathcal{G}_s\} = \mathbb{E}_{\mathbb{Q}^*}\left(\frac{C(G_s^1, \ldots, G_t^i, \ldots, G_{t_n}^n)}{C(G_s^1, \ldots, G_s^n)} \,\Big|\, \mathcal{F}_s \right).$$

- Consequently, the i^{th} intensity of survival equals, on $\{\tau_1 > t, \ldots, \tau_n > t\}$,

$$\lambda_t^i = \gamma_t^i \, G_t^i \, \frac{\partial}{\partial v_i} \ln C(G_t^1, \ldots, G_t^n).$$

  Here λ_t^i is understood as the limit

$$\lambda_t^i = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}^*\{t < \tau_i \leq t + h \,|\, \mathcal{F}_t, \tau_1 > t, \ldots, \tau_n > t\}.$$

Double Correlation

- We can postulate that

$$\gamma_t^i = \widetilde{\rho}_i \, h_0(t) + h_i(t)$$

where h_i are governed by Vasicek's dynamics

$$dh_i(t) = \kappa_i(\theta_i - h_i(t))\, dt + \sigma_i \, dW_t^i,$$

or by CIR dynamics

$$dh_i(t) = \kappa_i(\theta_i - h_i(t))\, dt + \sigma_i \sqrt{h_i(t)}\, dW_t^i.$$

- We can combine this with the one-factor Gaussian copula for $U_1, \ldots, U_n$.

- The first case was studied by Van der Voort (2004) in the context of basket CDSs and CDOs. The effect of intensity correlation is much smaller than the effect of the default correlation.

Density approach

A general framework is to assume that

$$\mathbb{P}(\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_t) = \int_{t_1}^{\infty} \ldots \int_{t_n}^{\infty} \alpha_t(u_1, \ldots, u_n) du_1 \ldots du_n$$

Let $n = 2$ and $\tau = \tau_1 \wedge \tau_2$, $\sigma = \tau_1 \vee \tau_2$. $\mathbb{F}$ be the reference filtration and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. We also introduce $\mathbb{G}^i = \mathbb{F} \vee \mathbb{H}^i$, $(i = 1, 2)$.

Let us consider the ordered default times

$$\tau = \tau^{(1)} := \min(\tau_1, \tau_2) \quad \text{and} \quad \sigma = \tau^{(2)} := \max(\tau_1, \tau_2).$$

Let $\mathbb{D}^{(1)}$ and $\mathbb{D}^{(2)}$ be the associated filtrations of τ and σ respectively. Let $\mathbb{G}^{(1)}$ be the filtration $\mathbb{F} \vee \mathbb{D}^{(1)}$ made right continuous and complete. Let $\mathbb{G}^{(2)}$ be obtained from $\mathbb{F} \vee \mathbb{D}^{(1)} \vee \mathbb{D}^{(2)}$ in a similar way. It is important to note that $\mathbb{G}^{(2)}$ is strictly included in $\mathbb{G}$.

The survival distribution of τ w.r.t $\mathbb{F}$ is given by

$$S_t^{\tau|\mathbb{F}}(\theta_1) := \mathbb{P}(\tau > \theta_1|\mathcal{F}_t) = \int_{\theta_1}^{\infty}\int_{u_1}^{\infty} \alpha_t(u_1, u_2)du_1du_2 = S_t(\theta, \theta)$$

The $\mathbb{F}$-density of τ is given by

$$\alpha_t^{\tau|\mathbb{F}}(\theta_1) = \int_{\theta_1}^{\infty} \alpha_t(\theta_1, u_2)du_2, \quad a.s..$$

For any $\theta_2, t \geq 0$, the $\mathbb{G}^{(1)}$-density of σ is given by

$$\alpha_t^{\sigma|\mathbb{G}^{(1)}}(\theta_2) = 1\!\!1_{\{\tau>t\}} \frac{\int_t^\infty \alpha_t(u,\theta_2)\eta_1(du)}{S_t^{\tau|\mathbb{F}}(t)} + 1\!\!1_{\{\tau\leq t\}} \frac{\alpha_t(\tau,\theta_2)}{\alpha_t^{\tau|\mathbb{F}}(\tau)}, \quad a.s.. \tag{0.1}$$

Let $Y_T(t_1, t_2; \omega)$ be $\mathcal{F}_T \times \mathcal{B}(\mathbb{R}^2)$ measurable. Then

$$\mathbb{E}\big[Y_T(\tau,\sigma)\,|\,\mathcal{G}_t^{(2)}\big] = \mathbb{1}_{\{\tau>t\}}\, q_t^1(T,Y_T) + \mathbb{1}_{\{\tau\le t<\sigma\}}\, q_t^2(T,\tau,Y_T) + \mathbb{1}_{\{\sigma\le t\}}\, q_t^3(T,\tau,\sigma,Y_T)$$

where

$$q_t^1(T,Y_T) = \frac{\mathbb{E}\big[\alpha_{T,t}(Y_T)\,|\,\mathcal{F}_t\big]}{\alpha_{t,t}(1)}, \quad q_t^2(T,\tau,Y_T) = \frac{\mathbb{E}\big[\alpha_{T,t}^{(u_1)}(Y_T)\,|\,\mathcal{F}_t\big]}{\alpha_{t,t}^{(u_1)}(1)}\Bigg|_{u_1=\tau}$$

and

$$q_t^3(T,\tau,\sigma,Y_T) = \frac{\mathbb{E}\big[\alpha_T^{(u_1,u_2)}(Y_T)\,|\,\mathcal{F}_t\big]}{\alpha_t^{(u_1,u_2)}(1)}\Bigg|_{\substack{u_1=\tau\\ u_2=\sigma}}$$

$$\begin{aligned}
\alpha_{T,t}(Y_T) &:= \int_t^\infty du_1 \int_{u_1}^\infty du_2 Y_T(u_1,u_2)\alpha_T(u_1,u_2)\\
\alpha_{T,t}^{(u_1)}(Y_T) &:= \int_t^\infty du_2 Y_T(u_1,u_2)\alpha_T(u_1,u_2),\\
\alpha_t^{(u_1,u_2)}(Y_t) &:= Y_t(u_1,u_2)\alpha_t(u_1,u_2).
\end{aligned}$$