

Credit Risk, II.

Summer School in Financial Mathematics

7-20 September 2009

Ljubljana

II. Cox Processes and Extensions

1. Construction of Default Time with a given Intensity
2. Properties
 - 2.1 Conditional expectation
 - 2.2. Choice of filtration
 - 2.3. Key Lemma
3. Defaultable Assets

Default Time with a given Intensity

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We define the random time τ as the first time when the process

$\Lambda_t = \int_0^t \lambda_s ds$ is above the random level Θ , i.e.,

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$\Lambda_t = \int_0^t \lambda_s ds$ is above the random level Θ , i.e.,

$$\tau = \inf \{t \geq 0 : \Lambda_t \geq \Theta\}.$$

In particular, $\{\tau > s\} = \{\Lambda_s < \Theta\}$.

Properties

Conditional Expectations

The conditional distribution function of τ given the σ -field \mathcal{F}_t is for $t \geq s$

$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \exp(-A_s).$$

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$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \exp(-\Lambda_s).$$

Proof : For $s \leq t$,

$$\begin{aligned} \mathbb{P}(\tau > s | \mathcal{F}_t) &= \mathbb{P}(\Lambda_s < \Theta | \mathcal{F}_t) \\ &= \Psi(\Lambda_s) \end{aligned}$$

where $\Psi(x) = \mathbb{P}(x < \Theta)$.

Choice of filtration

We write as before $H_t = \mathbb{1}_{\{\tau \leq t\}}$ and $\mathcal{H}_t = \sigma(H_s : s \leq t)$. We introduce the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, that is, the enlarged filtration generated by the underlying filtration \mathbb{F} and the process H . (We denote by \mathbb{F} the original filtration and by \mathbb{G} the enlarged one.)

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If $G_t \in \mathcal{G}_t$, then $G_t \cap \{\tau > t\} = B_t \cap \{\tau > t\}$ for some event $B_t \in \mathcal{F}_t$.

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If $G_t \in \mathcal{G}_t$, then $G_t \cap \{\tau > t\} = B_t \cap \{\tau > t\}$ for some event $B_t \in \mathcal{F}_t$.

Therefore any \mathcal{G}_t -measurable random variable Y_t satisfies

$\mathbb{1}_{\{\tau > t\}} Y_t = \mathbb{1}_{\{\tau > t\}} y_t$, where y_t is an \mathcal{F}_t -measurable random variable.

Key lemma

Let Y be an integrable r.v. Then,

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}(Y|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}(Y\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}}e^{\Lambda_t}\mathbb{E}(Y\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t).$$

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If $X \in \mathcal{F}_T$

$$\mathbb{E}(X \mathbb{1}_{\{\tau>T\}} | \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} e^{\Delta_t} \mathbb{E}(X e^{-\Delta_T} | \mathcal{F}_t).$$

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$$\mathbb{E}(X \mathbb{1}_{\{\tau>T\}} | \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} e^{\Lambda_t} \mathbb{E}(X e^{-\Lambda_T} | \mathcal{F}_t).$$

The process λ is called **the intensity** of τ .

In particular, one can check that

- (i) The process $L_t = \mathbb{1}_{t < \tau} e^{\Lambda_t} = (1 - H_t)e^{\Lambda_t}$ is a martingale
- (ii) Let X be an \mathcal{F}_∞ -measurable r.v.. Then

$$\mathbb{E}(X | \mathcal{G}_t) = \mathbb{E}(X | \mathcal{F}_t).$$

PROOF: (ii) Let X be an \mathcal{F}_∞ -measurable r.v. It suffices to check that

$$\mathbb{E}(B_t h(\tau \wedge t) X) = \mathbb{E}(B_t h(\tau \wedge t) \mathbb{E}(X | \mathcal{F}_t))$$

for any $B_t \in \mathcal{F}_t$ and any $h = \mathbb{1}_{[0, a]}$.

For $t \leq a$, the equality is obvious, since $h(\tau \wedge t) = 1$.

For $t > a$, we have $h(\tau \wedge t) = \mathbb{1}_{\{\tau \leq a\}}$ and

$$\begin{aligned} \mathbb{E}(B_t \mathbb{1}_{\{\tau \leq a\}} \mathbb{E}(X | \mathcal{F}_t)) &= \mathbb{E}(B_t \mathbb{E}(X | \mathcal{F}_t) \mathbb{E}(\mathbb{1}_{\{\tau \leq a\}} | \mathcal{F}_\infty)) \\ &= \mathbb{E}(\mathbb{E}(B_t X | \mathcal{F}_t) \mathbb{E}(\mathbb{1}_{\{\tau \leq a\}} | \mathcal{F}_t)) \\ &= \mathbb{E}(X B_t \mathbb{E}(\mathbb{1}_{\{\tau \leq a\}} | \mathcal{F}_t)) = \mathbb{E}(B_t X \mathbb{1}_{\{\tau \leq a\}}) \end{aligned}$$

as expected. \triangle

We now compute the expectation of the value at time τ of a predictable process.

(i) If h is an \mathbb{F} -predictable (bounded) process then

$$\mathbb{E}(h_\tau | \mathcal{G}_t) = e^{\Delta_t} \mathbb{E} \left(\int_t^\infty h_u dF_u \mid \mathcal{F}_t \right) \mathbb{1}_{\{\tau > t\}} + h_\tau \mathbb{1}_{\{\tau \leq t\}}$$

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- (ii) The process $(M_t := H_t - \int_0^{t \wedge \tau} \lambda_s ds, t \geq 0)$ is a \mathbb{G} -martingale.
- (iii) The martingale $L_t = \mathbb{1}_{t < \tau} e^{\Lambda_t}$ satisfies $dL_t = -L_{t-} dM_t$.

Defaultable Assets

Let $B(t, T)$ be the price at time t of a default-free bond paying 1 at maturity T satisfies

$$B(t, T) = \mathbb{E}_{\mathbb{Q}} \left(\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right).$$

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The market price $D(t, T)$ of a defaultable zero-coupon bond with maturity T is

$$\begin{aligned} D(t, T) &= \mathbb{E}_{\mathbb{Q}} \left(\mathbb{1}_{\{T < \tau\}} \exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{G}_t \right) \\ &= \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}} \left(\exp \left(- \int_t^T [r_s + \lambda_s^{\mathbb{Q}}] ds \right) \middle| \mathcal{F}_t \right). \end{aligned}$$

Promised payoff:

Let $X \in \mathcal{F}_T$

$$\mathbb{E}_{\mathbb{Q}} \left(X \mathbb{1}_{T < \tau} \exp - \int_t^T r_s ds \mid \mathcal{G}_t \right) = \mathbb{1}_{t < \tau} \mathbb{E}_{\mathbb{Q}} \left(X \exp - \int_t^T (r_s + \lambda_s) ds \mid \mathcal{F}_t \right)$$

λ is also called the **spread**.

Recovery paid at Maturity

We consider a contract which pays Z_τ at date T , if $\tau \leq T$ where Z is an \mathbb{F} -adapted process and no payment in the case $\tau > T$. We also assume that the interest rate is null.

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$$S_t = \mathbb{E}(Z_\tau \mathbb{1}_{\tau \leq T} | \mathcal{G}_t) = Z_\tau \mathbb{1}_{\tau \leq t} + \mathbb{1}_{t < \tau} \mathbb{E}(Z_\tau \mathbb{1}_{t < \tau \leq T} | \mathcal{G}_t)$$

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 &= \int_0^t Z_u dH_u + L_t \left(-\int_0^t Z_u e^{-\Lambda_u} \lambda_u du + m_t^Z\right)
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where $m_t^Z = \mathbb{E}\left(\int_0^T Z_u e^{-\Lambda_u} \lambda_u du | \mathcal{F}_t\right)$

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 \end{aligned}$$

where $m_t^Z = \mathbb{E} \left(\int_0^T Z_u e^{-\Lambda_u} \lambda_u du | \mathcal{F}_t \right)$ is an \mathbb{F} , hence a \mathbb{G} martingale.

We assume here that \mathbb{F} -martingales are continuous. From $dL_t = -L_{t-}dM_t$ and integration by parts formula we deduce that

$$dS_t = Z_t(dH_t - \lambda_t(1 - H_t)dt) - S_{t-}dM_t + L_t dm_t^Z$$

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In the case Z_t is a constant Z and with a deterministic interest rate

$$S_t = 1 - \mathbb{1}_{\{\tau > t\}}(Z - 1) \left(1 - \mathbb{E} \left(\exp - \int_t^T \lambda_s ds \mid \mathcal{F}_t \right) \right).$$

Recovery paid at Default

If the payment Z is done at time τ

$$S_t = \mathbb{1}_{t < \tau} \mathbb{E}(Z_\tau \mathbb{1}_{t < \tau < T} | \mathcal{G}_t) = \mathbb{1}_{t < \tau} e^{\Lambda_t} \mathbb{E} \left(\int_t^T Z_u dF_u | \mathcal{F}_t \right)$$

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$$dS_t = -Z_t \lambda_t (1 - H_t) dt - S_{t-} dM_t + L_t dm_t^Z .$$

The process $S_t + \int_0^t Z_s (1 - H_s) \lambda_s ds$ is a martingale.

Price and Hedging a defaultable call

The savings account $Y_t^0 = 1$, a risky asset with risk-neutral dynamics

$dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t, T)$ are traded.

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$$D(t, T) = L_t \mathbb{Q}(\tau > T | \mathcal{F}_t) = L_t m_t$$

with $m_t = \mathbb{Q}(\tau > T | \mathcal{F}_t) = \mathbb{E}(e^{-\Lambda T} | \mathcal{F}_t)$.

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The price of a defaultable call with payoff $\mathbb{1}_{T < \tau} (Y_T - K)^+$ is

$$C_t = \mathbb{E}(\mathbb{1}_{T < \tau} (Y_T - K)^+ | \mathcal{G}_t) = \mathbb{1}_{t < \tau} e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T} (Y_T - K)^+ | \mathcal{F}_t)$$

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with $m_t^Y = \mathbb{E}(e^{-\Lambda_T} (Y_T - K)^+ | \mathcal{F}_t)$.

Price and Hedging a defaultable call

The savings account $Y_t^0 = 1$, a risky asset with risk-neutral dynamics $dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t, T)$ are traded.

The reference filtration is that of the Brownian motion W . Then,

$$D(t, T) = L_t \mathbb{Q}(\tau > T | \mathcal{F}_t) = L_t m_t$$

with $m_t = \mathbb{Q}(\tau > T | \mathcal{F}_t) = \mathbb{E}(e^{-\Lambda T} | \mathcal{F}_t)$.

The price of a defaultable call with payoff $\mathbb{1}_{T < \tau} (Y_T - K)^+$ is

$$\begin{aligned} C_t &= \mathbb{E}(\mathbb{1}_{T < \tau} (Y_T - K)^+ | \mathcal{G}_t) = \mathbb{1}_{t < \tau} e^{\Lambda t} \mathbb{E}(e^{-\Lambda T} (Y_T - K)^+ | \mathcal{F}_t) \\ &= L_t m_t^Y \end{aligned}$$

with $m_t^Y = \mathbb{E}(e^{-\Lambda T} (Y_T - K)^+ | \mathcal{F}_t)$, hence

$$dC_t = L_t dm_t^Y - m_t^Y L_{t-} dM_t$$

In the particular case where λ is deterministic, $m_t = e^{-\lambda t}$ and $dm_t = 0$. Therefore, $D(t, T) = m_t L_t = L_t e^{-\lambda t}$ and

$$dD(t, T) = m_t dL_t = -m_t L_t dM_t = -e^{-\lambda t} L_t dM_t.$$

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Therefore, $D(t, T) = m_t L_t = e^{-\lambda t} L_t$ and

$$dD(t, T) = m_t dL_t = -m_t L_t dM_t = -e^{-\lambda t} L_t dM_t.$$

Furthermore,

$$m_t^Y = e^{-\lambda t} \mathbb{E}((Y_T - K)^+ | \mathcal{F}_t) = e^{-\lambda t} C_t^Y$$

where C_t^Y is the price of a call in the Black Scholes model.

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This quantity is $C_t^Y = C^Y(t, Y_t)$ and satisfies $dC_t^Y = \Delta_t dY_t$ where Δ_t is the Delta-hedge ($\Delta_t = \partial_y C^Y(t, Y_t)$).

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$$\begin{aligned} C_t &= L_t m_t^Y = \mathbb{1}_{t < \tau} e^{\lambda t} e^{-\lambda t} C_t^Y(t, Y_t) \\ &= L_t e^{-\lambda t} C_t^Y(t, Y_t) = D(t, T) C_t^Y(t, Y_t) \end{aligned}$$

From

$$C_t = D(t, T)C^Y(t, Y_t)$$

we deduce

$$\begin{aligned} dC_t &= e^{-\Lambda_T} (L_t dC_t^Y + C_t^Y dL_t) = e^{-\Lambda_T} (L_t \Delta_t dY_t - C_t^Y L_{t-} dM_t) \\ &= e^{-\Lambda_T} (L_t \Delta_t dY_t - C_t^Y L_{t-} dM_t) \end{aligned}$$

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Therefore, using that $dD(t, T) = m_t dL_t = -e^{-\Lambda_T} L_{t-} dM_t$ we get

$$dC_t = e^{-\Lambda_T} L_t \Delta_t dY_t + C_t^Y dD(t, T) = e^{-\Lambda_T} L_t \Delta_t dY_t + \frac{C_t}{D(t, T)} dD(t, T)$$

From

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hence, an hedging strategy consists of holding $\frac{C_{t-}}{D(t, T)}$ DZCs.

In the general case, one obtains

$$dC_t = \frac{C_{t-}}{D(t, T)} dD(t, T) - L_t \frac{m_t^Y}{m_t} dm_t + L_t dm_t^Y$$

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An hedging strategy consists of holding $\frac{C_{t-}}{D(t, T)}$ DZCs.